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Problem 2.1: The integer quantum Hall effect in the symmetric gauge [**Written** | 10 pt(s)]

ID: ex_integer_quantum_hall_effect_symmetric_gauge:tqp25

Learning objective

In this task you revisit the Hamiltonian of a charged particle in two dimensions in a magnetic field. In contrast to the lecture, here you work in the *symmetric gauge* to construct an alternative set of basis functions for the Landau levels. The benefit of this approach is that the basis wave functions can be naturally expressed in complex coordinates, a feature that is handy especially in the context of the *fractional* quantum Hall effect. This exercise is also nice application of methods from complex analysis to quantum mechanics.

We consider an electron in 2D in a perpendicular magnetic field $B \equiv B_z = \partial_x A_y - \partial_y A_x$. In the lecture it was shown that the Hamiltonian can be written as a harmonic oscillator:

$$H = \frac{(\boldsymbol{p} + e\boldsymbol{A})^2}{2m} = \frac{\pi_x^2 + \pi_y^2}{2m} = \hbar\omega_B \left(a^\dagger a + \frac{1}{2} \right) . \tag{1}$$

Here $\boldsymbol{\pi} = \boldsymbol{p} + e\boldsymbol{A}$ is the (gauge independent) kinetic momentum operator and $\omega_B = eB/m$ is the cyclotron frequency. The ladder operators are defined by

$$a := \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \qquad \text{and} \qquad a^\dagger := \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y) \quad \text{with} \quad \left[a, a^\dagger\right] = 1 . \tag{2}$$

Note: Recall that $\boldsymbol{p}$ is the *canonical* momentum whereas $\boldsymbol{\pi}$ is the *kinetic* momentum. In particular, different components of the *kinetic* momentum do not necessarily commute due to the magnetic field!

So far these results are gauge invariant (we did not yet fix a gauge). The result in Eq. (1) already determines the spectrum of the problem, but leaves open the question for *eigenstates* and the degeneracy of the eigenenergies (the Landau levels). The goal of this exercise is to answer these questions.

a) As a first step, define the additional “momentum operators” **1**pt(s)

$$\tilde{\boldsymbol{\pi}} := \boldsymbol{p} - e\boldsymbol{A} \tag{3}$$

and show that $[\tilde{\pi}_x, \tilde{\pi}_y] = ie\hbar B$. Is $\tilde{\boldsymbol{\pi}}$ gauge invariant?

b) Compute all commutators $[\pi_i, \tilde{\pi}_j]$ (for $i, j \in \{x, y\}$) of the kinetic momentum $\boldsymbol{\pi}$ with the additional momentum $\tilde{\boldsymbol{\pi}}$. **1**pt(s)

What does the result tell you about the possibility to construct good quantum numbers to label eigenstates?

c) Show that by fixing the *symmetric gauge* 1pt(s)

$$\boldsymbol{A} := -\frac{yB}{2}\boldsymbol{e}_x + \frac{xB}{2}\boldsymbol{e}_y \tag{4}$$

all commutators $[\pi_i, \tilde{\pi}_j]$ vanish. Then define the additional ladder operators

$$b := \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x + i\tilde{\pi}_y) \qquad \text{and} \qquad b^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x - i\tilde{\pi}_y) \tag{5}$$

and show that $\left[b, b^\dagger\right] = 1$ and $[a, b] = \left[a, b^\dagger\right] = 0$.

d) Write down the eigenstates of Hamiltonian (1) and label them with appropriate quantum numbers $n \in \mathbb{N}_0$ for $\hat{n} := a^\dagger a$ and $m \in \mathbb{N}_0$ for $\hat{m} := b^\dagger b$. Which states are energetically degenerate? 1pt(s)

Now that we formally derived a full basis set for each Landau level $n = 0, 1, 2, \ldots$, we can construct these states in real space. To do so, it is convenient to switch to complex coordinates:

$$z := x - iy \qquad \text{and} \qquad \bar{z} := x + iy. \tag{6}$$

With the complex (Wirtinger) derivatives

$$\partial := \frac{1}{2}(\partial_x + i\partial_y) \qquad \text{and} \qquad \bar{\partial} := \frac{1}{2}(\partial_x - i\partial_y) \tag{7}$$

it follows $\partial z = \bar{\partial}\bar{z} = 1$ and $\partial\bar{z} = \bar{\partial}z = 0$. A function f of complex variables is then holomorphic (= satisfies the Cauchy-Riemann equations) if and only if $\bar{\partial}f = 0$, i.e. $f = f(z)$.

Note: The unconventional sign of z and $\bar{z}$ makes the functions below holomorphic instead of antiholomorphic.

e) Use $\boldsymbol{p} = -i\hbar\nabla$ and the symmetric gauge for $\boldsymbol{A}$ to express the ladder operators a, $a^\dagger$, b, $b^\dagger$ in terms of ∂, $\bar{\partial}$, z, and $\bar{z}$. 1pt(s)

f) Show that the wave functions in the lowest Landau level (LLL) $n = 0$ take the form 1pt(s)

$$\langle z, \bar{z}|n = 0\rangle = \Psi_{n=0}(z, \bar{z}) = f(z)\, e^{-z\bar{z}/4l_B^2} \tag{8}$$

with an arbitrary holomorphic function $f(z)$. Here $l_B = \sqrt{\hbar/eB}$ denotes the magnetic length.

g) Derive now the real space representation $\langle z, \bar{z}|n = 0, m = 0\rangle$ for the LLL state with $m = 0$. 1pt(s)

Use the result to show that

$$\langle z, \bar{z}|n = 0, m\rangle = \Psi_{n=0,m}(z, \bar{z}) \propto \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4l_B^2} \tag{9}$$

for $m \in \mathbb{N}_0$. How does this relate to the result in Eq. (8)?

Hint: Recall from your course on complex analysis the properties of holomorphic functions.

h) Show that the angular momentum operator for rotations in the xy-plane can be written as 2pt(s)

$$J = \hbar(\hat{n} - \hat{m}). \tag{10}$$

Is the angular momentum a good quantum number? What are its eigenvalues for the LLL?

Why is a single quantum number sufficient to specify the angular momentum eigenstates?

Now that we know the real space wave functions for the LLL, we can finally derive its degeneracy:

i) Show that the probability densities of the LLL wave functions (9) are rotationally symmetric with a peak at radius $r = \sqrt{2m}\, l_B$. Use this to show that in the LLL there exists (approximately) one independent state per quantum of flux. **1**pt(s)

Compare your result to the one derived in the lecture using the Landau gauge.

Hint: Consider the LLL wave functions with maximum within a disc-shaped region of radius R.

Problem 2.2: Berry phase of a spin in a magnetic field [**Oral** | 8 (+2 bonus) pt(s)]

ID: ex_berry_phase_spin_magnetic_field:tqp25

Learning objective

In the lecture we (will) introduce on very general grounds the concepts of *Berry connection*, *Berry curvature*, *Berry phase* and the *Chern number*. The simplest model to observe these concepts in action is a single spin-$\frac{1}{2}$ in a slowly varying magnetic field. In this exercise you study this example in detail.

Consider a spin-$\frac{1}{2}$ with Hilbert space $\mathcal{H} = \mathbb{C}^2$ in an external magnetic field $\boldsymbol{B}$. The Hamiltonian is

$$H = -\boldsymbol{B} \cdot \boldsymbol{\sigma} + B \qquad \text{with} \qquad B = |\boldsymbol{B}| \tag{11}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$ the vector of Pauli matrices. We consider the magnetic field

$$\boldsymbol{B} = B \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \tag{12}$$

as tunable parameters (in the lecture called $\boldsymbol{\Gamma}$), given in spherical coordinates. The parameter space (= "the space of magnetic fields at the position of the spin") is then isomorphic to $\mathbb{R}^3$.

We are interested in the evolution of an initial ground state of the Hamiltonian (11) under a slow variation of the magnetic field $\boldsymbol{B} = \boldsymbol{B}(t)$. The **adiabatic theorem** ascertains that the physical system remains in its instantaneous eigenstate (here: ground state) if there is a gap to the rest of the spectrum and the time evolution is slow compared to this gap.

a) Write the Hamiltonian (11) as a 2×2-matrix that parametrically depends on θ, ϕ and B. Compute its spectrum and show that it is gapped and the ground state is non-degenerate for $B > 0$. **2**pt(s)

Show that the eigenstates can be written as

$$|n_+(\boldsymbol{B})\rangle = \begin{pmatrix} e^{-i\phi}\cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix} \quad \text{and} \quad |n_-(\boldsymbol{B})\rangle = \begin{pmatrix} e^{-i\phi}\sin(\theta/2) \\ -\cos(\theta/2) \end{pmatrix} . \tag{13}$$

For which parameters $\boldsymbol{B}$ is the ground state $|n_+(\boldsymbol{B})\rangle$ well-defined?

In the lecture we show that for a closed path $\Gamma : t \mapsto \boldsymbol{B}(t)$ with $t \in [0, T]$ in parameter space, an initial ground state $|\Psi(0)\rangle = |n_+(\boldsymbol{B}(0))\rangle$ picks up a *Berry phase*,

$$|\Psi(T)\rangle = e^{i\gamma(\Gamma)}\,|\Psi(0)\rangle \qquad \text{with} \qquad \gamma(\Gamma) = -\oint_\Gamma \boldsymbol{\mathcal{A}}^{[+]} \cdot \mathrm{d}\boldsymbol{B}\,, \tag{14}$$

where $\mathcal{A}_l^{[+]}(\boldsymbol{B}) := -i \langle n_+(\boldsymbol{B})| \partial_l |n_+(\boldsymbol{B})\rangle$ is called the *Berry connection*. Due to the spherical symmetry, it is convenient to use spherical coordinates $l \in \{B, \theta, \phi\}$ in the following, but the physical results are independent of your choice of coordinates on the parameter manifold (e.g. Cartesian coordinates $l \in \{B_x, B_y, B_z\}$).

Note that the choice of a (differentiable) basis vector $|n_+(\boldsymbol{B})\rangle$ for the ground state is a *gauge*. The Berry connection can be interpreted as a gauge field in parameter space that depends on this basis choice.

The *Berry curvature* is the corresponding (gauge invariant) field strength tensor, defined via

$$\mathcal{F}_{lm} := \partial_m \mathcal{A}_l - \partial_l \mathcal{A}_m \,. \tag{15}$$

b) Consider only parameters for which the ground state $|n_+(\boldsymbol{B})\rangle$ is well-defined and compute the Berry connection $\mathcal{A}_l^{[+]} = -i \langle n_+(\boldsymbol{B})| \partial_l |n_+(\boldsymbol{B})\rangle$ in spherical coordinates $l \in \{B, \theta, \phi\}$. **2pt(s)**

Show that the only non-vanishing term of the Berry curvature is

$$\mathcal{F}_{\theta\phi}^{[+]} = -\frac{\sin(\theta)}{2} \,. \tag{16}$$

c) Make the gauge transformation $|\tilde{n}_+(\boldsymbol{B})\rangle := e^{i\phi} |n_+(\boldsymbol{B})\rangle$. **2pt(s)**

For which parameters is this gauge well-defined?

Compute again the Berry connection and the Berry curvature and compare your results with b).

*d) Show that the Berry curvature of the ground state manifold reads in *Cartesian coordinates* B_i with labels $i, j \in \{x, y, z\}$ **+2pt(s)**

$$\mathcal{F}_{B_i B_j}^{[+]}(\boldsymbol{B}) = \varepsilon_{ijk} g \frac{B_k}{B^3} \qquad \text{with} \qquad g = -\frac{1}{2} \,. \tag{17}$$

This is the field strength of a *magnetic monopole* at $\boldsymbol{B} = \boldsymbol{0}$ with charge g.

Note: Note that $\mathcal{F}_{B_i B_j}$ is the field strength in *parameter space*, not in real space!

e) Compute the Berry phase $\gamma(\Gamma)$ for a closed path $\Gamma : t \mapsto \boldsymbol{B}(t)$ that avoids the origin, $\boldsymbol{B}(t) \neq \boldsymbol{0}$ for all $t \in [0, T]$. Show that it can be written as **1pt(s)**

$$e^{i\gamma(\Gamma)} = \exp\left(i \frac{\Omega_\Gamma}{2}\right) \tag{18}$$

where $0 \leq \Omega_\Gamma \leq 4\pi$ is the *solid angle* that is traced out by the path.

Hint: Employ Stokes' theorem and use that $\mathcal{F}_{ij}^{[+]} \mathrm{d}\sigma^{ij} = g/B^3 \, \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{\sigma}$ with the surface element $\mathrm{d}\boldsymbol{\sigma}$.

For a compact, closed *two*-dimensional parameter space $\mathcal{M}$, the *Chern number* is defined as

$$C = \frac{1}{2\pi} \int_{\mathcal{M}} \mathcal{F}_{lm} d\sigma^{lm} \,. \tag{19}$$

If there exists a gauge that is continuous on the complete parameter space, the Chern number is zero. In the lecture, we show that the Chern number is always an integer, $C \in \mathbb{Z}$.

We can use this mathematical fact, in conjunction with the physical system under study, to draw conclusions on the "magnetic monopole charge" g:

f) Consider the sphere $\mathcal{M} = S^2 \subset \mathbb{R}^3$ as a two-dimensional submanifold of magnetic fields and compute the Chern number C for the Berry curvature in Eq. (17). **1**pt(s)

What follows in this particular case for $g = -1/2$? Is it important that $\mathcal{M} = S^2$ is a sphere?

What follows for the magnetic monopole charge g in general?

Note: This result is mathematically equivalent to the famous *Dirac quantization condition* for magnetic monopoles.