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April 8th, 2025
SS 2025

Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Topological Quantum Many-Body Physics"***:

- The C@MPUS-ID of this course is 045500002.
- You can find detailed information on lecture and tutorials on the website of our institute:

  https://itp3.info/tqp25
- You can also find detailed information on lecture and tutorials on ILIAS:

  https://ilias3.uni-stuttgart.de/go/crs/4023335
- **Written** problems have to be handed in via ILIAS or during the tutorial and will be corrected by the tutors. You must earn at least **66 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **2** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Fermions, bosons and hard-core bosons [**Oral** | 9 pt(s)]

ID: ex_hardcore_boson:tqp25

Learning objective

In this exercise, we review the fermionic and bosonic creation and annihilation operators, which are essential to describe systems of many identical particles in quantum mechanics. Beyond that, we introduce the concept of *hard-core bosons* and compare them to fermions and bosons. Finally, as an application of the formalism, we solve the simple model of free fermions and bosons hopping on a one-dimensional lattice. Most of these concepts should be already familiar to you and will be used throughout this course.

Recall that the annihilation (creation) operators b_i ($b_i^\dagger$) that operate on a *bosonic* Fock space fulfill the commutation relations

$$[b_i, b_j^\dagger] = \delta_{i,j} \qquad \text{and} \qquad [b_i, b_j] = 0\,. \tag{1}$$

Similarly, on a *fermionic* Fock space the annihilation (creation) operators a_i ($a_i^\dagger$) fulfill the *anti*-commutation relations

$$\{a_i, a_j^\dagger\} = \delta_{i,j} \qquad \text{and} \qquad \{a_i, a_j\} = 0\,. \tag{2}$$

By contrast, *hard-core bosons* behave like bosons on different sites but as fermions on the same site. Formally, they are defined via the algebra

$$\{c_i, c_i^\dagger\} = 1, \qquad \{c_i, c_i\} = 0 \qquad \text{and} \qquad [c_i, c_j^{(\dagger)}] = 0 \text{ for } i \neq j\,, \tag{3}$$

where c_i ($c_i^\dagger$) are the annihilation (creation) operators (we do *not* assume Einstein summation!). The occupation number operator of the i-th mode is given as usual by $\hat{n}_i = c_i^\dagger c_i$. Let $|\boldsymbol{n}\rangle = |n_1, n_2, \dots\rangle$ denote the common eigenbasis of $\hat{n}_1, \hat{n}_2, \dots$ with eigenvalues $n_1, n_2, \dots$ respectively.

a) Let us focus on the representation (= Hilbert space) that can be constructed from the hard-core boson algebra Eq. (3). Using these (anti)commutation relations, show that for hard-core bosons **2**pt(s)

$$c_i|n_1, n_2, \dots\rangle = \sqrt{n_i}|n_1, \dots, 1-n_i, \dots\rangle, \tag{4a}$$
$$c_i^\dagger|n_1, n_2, \dots\rangle = \sqrt{1-n_i}|n_1, \dots, 1-n_i, \dots\rangle. \tag{4b}$$

Furthermore, show that for hard-core bosons there exists a state $|G\rangle$ with $c_i|G\rangle = 0$ and a state $|H\rangle$ with $c_i^\dagger|H\rangle = 0$. Use this to show

$$|n_1, n_2, \dots\rangle = \left[\prod\nolimits_i (c_i^\dagger)^{n_i}\right] |G\rangle. \tag{5}$$

Compare with the fermionic and the bosonic case (known from your quantum mechanics course).

Hint: Use the fact that the norm is positive definite.

b) Show that the hard-core boson algebra Eq. (3) can be realized in a system of spin-$\frac{1}{2}$ degrees of freedom via the identification **1**pt(s)

$$c_i \leftrightarrow \sigma_i^- \quad \text{and} \quad c_i^\dagger \leftrightarrow \sigma_i^+ \tag{6}$$

where $\sigma_i^\pm = \frac{1}{2}(\sigma_i^x \pm i\sigma_i^y)$ with Pauli matrices $\sigma_i^{x,y,z}$.

Note: This shows that you can always interpret a local model of hard-core bosons as a spin-$\frac{1}{2}$ system. In this picture, the states $|G\rangle$ and $|H\rangle$ correspond to the spin-polarized states $|\downarrow\downarrow\dots\rangle$ and $|\uparrow\uparrow\dots\rangle$, respectively. The Hilbert space you constructed in subtask a) is therefore not a Fock space but simply the tensor product $\mathcal{H} = \bigotimes_i \mathbb{C}_i^2$.

Let us now consider a general unitary transformation

$$\tilde{x}_i = \sum\nolimits_j U_{ij} x_j \qquad \text{with} \qquad U^\dagger U = UU^\dagger = 1, \tag{7}$$

where $x \in \{a, b, c\}$ is an annihilation operator.

c) Show that the transformed *bosonic* operators $\tilde{b}$ and $\tilde{b}^\dagger$ still fulfill the bosonic/CCR[1] algebra (1). Also show that the transformed *fermionic* operators $\tilde{a}$ and $\tilde{a}^\dagger$ still fulfill the fermionic/CAR[2] algebra (2). **1**pt(s)

d) Now calculate the (anti-)commutation relations for the transformed operators $\tilde{c}_i$ and $\tilde{c}_i^\dagger$ of *hard-core bosons*. Under which class of unitary transformations remains the hard-core boson algebra valid? **1**pt(s)

[1]Canonical Commutation Relations
[2]Canonical Anti-commutation Relations

We now consider $N \gg 1$ spinless particles in a deep, one-dimensional lattice with sites $i \in \{1, \dots L\}$, lattice spacing a and periodic boundary conditions. The Hamiltonian for this system is given by

$$H = -t \sum_{\langle i,j \rangle} x_i^\dagger x_j. \tag{8}$$

Here x_i ($x_i^\dagger$) is the annihilation (creation) operator of a particle localized at lattice site i. The kinetic term t describes the particle hopping from a lattice site i to an adjacent site j, indicated by $\langle i, j \rangle$.

e) Let us first consider the *fermionic* case where $x^{(\dagger)} = a^{(\dagger)}$. The Hamiltonian (8) then describes free fermions and its ground state is given by the Fermi sea. Diagonalize the Hamiltonian and determine the ground state (and its eigenenergy) of the system at half-filling $N = L/2$. **2**pt(s)

   Hint: In subtask c) you showed that the fermionic algebra is conserved under unitary transformations U. Show that the basis transformation $U_{l,j} = e^{ik_l x_j}/\sqrt{L}$ with $l, j \in \{1, \dots L\}$ is unitary and use it to diagonalize the Hamiltonian. Note that this unitary transformation corresponds to a Fourier transform from the site basis $x_j = ja$ to plane waves given by their momenta $k_l = 2\pi l/La$.

f) Now consider the *bosonic* case with $x^{(\dagger)} = b^{(\dagger)}$ and again diagonalize the Hamiltonian Eq. (8). What is the many-body ground state and ground state energy of the system at half-filling $N = L/2$? **1**pt(s)

g) Finally consider the *hard-core bosonic* case with $x^{(\dagger)} = c^{(\dagger)}$. Retrace your steps from subtasks e) and f) and convince yourself that the Hamiltonian Eq. (8) can no longer be diagonalized by a simple Fourier transform. **1**pt(s)

   Note: The particular Hamiltonian Eq. (8) *can* still be exactly diagonalized for hard-core bosons by means of a *Jordan-Wigner transformation* (which you will encounter later in this course). This only works in one dimension, though.

In conclusion, you have shown that *quadratic* Hamiltonians of fermions ($x_i = a_i$) and bosons ($x_i = b_i$) can be exactly solved and their many-body eigenstates (in particular their ground states) can be constructed from their single-particle eigenstates (either as a Fermi sea or a Bose-Einstein condensate). By contrast, quadratic Hamiltonians of hard-core bosons *cannot* be exactly diagonalized in general, and their many-body eigenstates do not derive from their single-particle eigenstates. We say that hard-core bosons are intrinsically *interacting* particles, whereas quadratic Hamiltonians of fermions or bosons describe *non-interacting* particles. This has important consequences for our discussion of symmetry-protected topological phases of interacting bosons later in this course.

Problem 1.2: Wigner's Theorem

[**Written** | 6 (+1 bonus) pt(s)]

ID: ex_wigners_theorem_kramers_theorem:tqp25

Learning objective

Wigner's theorem (Eugene Wigner, 1931) is a central insight in quantum mechanics that characterizes how physical symmetries – such as rotations, translations or time-reversal – are represented on the Hilbert space. The proof of Wigner's theorem provides a repetition of concepts like ray spaces, transition probabilities and Born's rule. A crucial result is that symmetries can be represented by *antiunitary* operators, a fact that will be used throughout this course.

In quantum mechanics, *physical* states are represented by *rays* in a Hilbert space. A ray is an equivalence class of normalized vectors with $|\Psi\rangle$ and $|\Psi'\rangle$ belonging to the same ray if $|\Psi\rangle = e^{i\varphi}|\Psi'\rangle$ for some phase $\varphi \in \mathbb{R}$.

Consider the system in a state represented by a ray $\mathcal{R}$ with $|\Psi\rangle \in \mathcal{R}$. If a measurement is performed to test whether the system is in a state represented by another ray $\mathcal{R}_k$ with $|\Psi_k\rangle \in \mathcal{R}_k$, then by *Born's rule* the probability is given by the expectation value

$$P(\mathcal{R} \mapsto \mathcal{R}_k) = \langle \mathbb{P}_k \rangle_\Psi = |\langle \Psi_k | \Psi \rangle|^2 \tag{9}$$

of the projection operator $\mathbb{P}_k = |\Psi_k\rangle\langle\Psi_k|$.

We define a *symmetry transformation* as a mathematical transformation between two equivalent descriptions (= observers) of the same physical situation; *equivalent* here means that the two descriptions lead to the same results for all possible experiments: If one observer O describes a system by a state represented by a ray $\mathcal{R}$ (or $\mathcal{R}_k$), then an equivalent observer O' describes the same system by a different state represented by a different ray $\mathcal{R}'$ (or $\mathcal{R}'_k$). Crucially, since these observers describe the *same* physical process, they must find the same transition probabilities for all possible measurements:

$$T : \mathcal{R} \mapsto \mathcal{R}' \text{ is a symmetry} \quad :\Leftrightarrow \quad \underbrace{P(\mathcal{R} \mapsto \mathcal{R}_k)}_{\text{Observer } O} = \underbrace{P(\mathcal{R}' \mapsto \mathcal{R}'_k)}_{\text{Observer } O'} \quad \text{for all rays } \mathcal{R}, \mathcal{R}_k \tag{10}$$

Wigner's theorem ascertains that for any such symmetry transformation $T: \mathcal{R} \mapsto \mathcal{R}'$ acting on a ray space, there exists a *compatible* operator $U : |\Psi\rangle \mapsto |\Psi'\rangle$ acting on the Hilbert space that is either *unitary* (and linear) or *antiunitary* (and antilinear). Furthermore, U is unique up to a phase factor.

- In this context, an operator U acting on the Hilbert space is called *compatible* with the transformation T acting on the ray space, if it fulfills $U|\Psi\rangle \in T(\mathcal{R})$ for every state-vector $|\Psi\rangle \in \mathcal{R}$ in the Hilbert space.
- The *adjoint* of an (anti-)linear operator A is defined by $\langle \Phi | A\Psi \rangle = \langle A^\dagger \Phi | \Psi \rangle^{(*)}$. (Anti-)Unitary operators are defined by $\langle U\Phi | U\Psi \rangle = \langle \Phi | \Psi \rangle^{(*)}$, they fulfill $U^\dagger = U^{-1}$.

This exercise guides you step-by-step through the proof of Wigner's theorem:

a) Consider a complete orthonormal set of state vectors $|\Psi_k\rangle \in \mathcal{R}_k$. Let $|\Psi'_k\rangle \in T(\mathcal{R}_k)$ be an arbitrary choice of state vectors belonging to the symmetry transformed rays. Show that the state vectors $|\Psi'_k\rangle$ again form a complete set of orthonormal state vectors. **1**pt(s)

b) Consider now an arbitrary state given by a ray $\mathcal{R}$ with some state vector $|\Psi\rangle = \sum_m C_m |\Psi_m\rangle \in \mathcal{R}$. Under a symmetry transformation T the ray is transformed to $T(\mathcal{R})$, represented by some state vector $|\Psi'\rangle = \sum_m C'_m |\Psi'_m\rangle \in T(\mathcal{R})$. **2**pt(s)

Show that the coefficients of the new state vector $|\Psi'\rangle$ must obey

$$\text{either} \qquad C'_k = \left[e^{i(\varphi_k - \varphi_l)} \frac{C'_l}{C_l}\right] C_k \qquad \text{or} \qquad C'_k = \left[e^{i(\varphi_k - \varphi_l)} \frac{C'_l}{C^*_l}\right] C^*_k \,, \tag{11}$$

where *w.l.o.g.* we assume $C_l \neq 0$. Note that $|C'_m| = |C_m|$ for all m.

The phase factors $e^{i\varphi_m}$ are determined by the phases chosen for the basis vectors $|\Psi'_m\rangle \in T(\mathcal{R}_m)$, i.e., they are independent of the state-vector $|\Psi\rangle$.

Hint: Compare the coefficients $C_m = \langle \Psi_m | \Psi \rangle$ with the amplitudes $\langle \Phi_k | \Psi \rangle$ with respect to $|\Phi_k\rangle = (|\Psi_l\rangle + |\Psi_k\rangle)/\sqrt{2} \in \mathcal{S}_k$ for $k \neq l$.

c) Show that for a given symmetry transformation the *same* choice in (11) must be made for *all* coefficients C'_k of a state-vector $|\Psi'\rangle$. **1**pt(s)

This leaves essentially two choices for the transformation of a state vector: Either all its coefficients are complex conjugated or they are not.

*d) Now consider two arbitrary state vectors $|A\rangle = \sum_k A_k |\Psi_k\rangle$ and $|B\rangle = \sum_k B_k |\Psi_k\rangle$. Show that for both state vectors the same choice in (11) must be made. **+1**pt(s)

This leaves us with only two choices for the transformation on the Hilbert space: either all state vectors are complex conjugated or they are not.

Hint: Assume that $|A\rangle$ and $|B\rangle$ transform under different choices in (11). Consider the scalar product $\langle B|A\rangle$ and show that the coefficients must fulfill $\sum_{k,l} \mathrm{Im}\,(A^*_k A_l)\,\mathrm{Im}\,(B^*_k B_l) = 0$. Then argue that there always exists a third state-vector $|C\rangle$ which transforms under the same choice in (11) as both $|A\rangle$ and $|B\rangle$.

At this point, we have proven that all state-vectors $|\Psi'\rangle$ in the Hilbert space must fulfill

$$\text{either} \quad |\Psi'\rangle = \frac{C'_l}{C_l} \sum_k C_k \left[e^{i(\varphi_k - \varphi_l)} |\Psi'_k\rangle\right] \quad \text{or} \quad |\Psi'\rangle = \frac{C'_l}{C^*_l} \sum_k C^*_k \left[e^{i(\varphi_k - \varphi_l)} |\Psi'_k\rangle\right] \,. \tag{12}$$

The phase factors $e^{i(\varphi_k - \varphi_l)}$ compensate for the fact that the relative phases of $|\Psi'_k\rangle$ are still arbitrary. Similarly, the coefficients $C'_l/C_l^{(*)}$ compensate for the fact that the phase of $|\Psi'\rangle$ relative to the basis vectors is still arbitrary. With this knowledge, we can finalize the proof of Wigner's theorem:

e) For a given symmetry transformation $T : \mathcal{R} \mapsto \mathcal{R}'$ acting on the ray space, show that we can always define a compatible operator $U : |\Psi\rangle \mapsto U|\Psi\rangle$ acting on the Hilbert space which is either unitary (and linear) or antiunitary (and antilinear). **2**pt(s)

What are the degrees of freedom left in the choice of U?

Hint: Which choices for the relative phase factors of the state vectors must be made such that their transformation becomes (anti-)linear?