↓ **Lecture 7** [08.05.25]

13 | Finally, we can relate our findings to the geometrical quantities introduced in Section 1.3:

i | Define the Berry connection of band n:

$$\boxed{\mathcal{A}_i^{[n]}(\boldsymbol{k}) := -i\,\langle u_{n\boldsymbol{k}}|\tilde{\partial}_i u_{n\boldsymbol{k}}\rangle} \tag{1.94}$$

This is a U(1) connection on the Brillouin zone which is the compact 2D manifold T^2. The parameters are the momenta ($\boldsymbol{\Gamma} = \boldsymbol{k}$) and the local Hilbert spaces are one dimensional: $\mathcal{V}^{[n]}(\boldsymbol{k}) = \mathrm{span}\,\{|u_{n\boldsymbol{k}}\rangle\}$; these are the non-degenerate eigenspaces (no band crossings!) of the Hamiltonian family $\tilde{H}_0(\boldsymbol{k})$ with discrete spectrum $\varepsilon_n(\boldsymbol{k})$ (fix $\boldsymbol{k}$ as a parameter!). Thus $n = 1$ and $k = 2$ in the context of our general discussion in Section 1.3; in the present context, n denotes the band index.

ii | → Berry curvature of band n:

$$\boxed{\begin{aligned}\mathcal{F}_{ij}^{[n]}(\boldsymbol{k}) &= \tilde{\partial}_j \mathcal{A}_i^{[n]} - \tilde{\partial}_i \mathcal{A}_j^{[n]} \\ &= -i\,\langle \tilde{\partial}_j u_{n\boldsymbol{k}}|\tilde{\partial}_i u_{n\boldsymbol{k}}\rangle + i\,\langle \tilde{\partial}_i u_{n\boldsymbol{k}}|\tilde{\partial}_j u_{n\boldsymbol{k}}\rangle\end{aligned}} \tag{1.95}$$

The cross terms cancel.

iii | → Chern number of band n:

$$\boxed{\begin{aligned}C^{[n]} &= \frac{1}{2\pi}\int_{T^2} \mathcal{F}_{ij}\,\mathrm{d}\sigma^{ij} = -\frac{1}{2\pi}\int_{T^2} \mathcal{F}_{xy}\,\mathrm{d}^2k \\ &= \frac{i}{2\pi}\int_{T^2} \left\{\langle \tilde{\partial}_y u_{n\boldsymbol{k}}|\tilde{\partial}_x u_{n\boldsymbol{k}}\rangle - \langle \tilde{\partial}_x u_{n\boldsymbol{k}}|\tilde{\partial}_y u_{n\boldsymbol{k}}\rangle\right\}\mathrm{d}^2k\end{aligned}} \tag{1.96}$$

The integral is best evaluated with differential forms where $\mathcal{F} = \mathrm{d}\mathcal{A}$ is a 2-form and $\mathcal{A} = A_x\mathrm{d}k_x + A_y\mathrm{d}k_y$ is a 1-form. Then

$$C = \frac{1}{2\pi}\int_{T^2}\mathcal{F} = \frac{1}{2\pi}\int_{T^2}\left(\tilde{\partial}_y A_x\,\mathrm{d}k_y \wedge \mathrm{d}k_x + \tilde{\partial}_x A_y\,\mathrm{d}k_x \wedge \mathrm{d}k_y\right) \tag{1.97a}$$

$$= -\frac{1}{2\pi}\int_{T^2}\underbrace{\left(\tilde{\partial}_y A_x - \tilde{\partial}_x A_y\right)}_{\mathcal{F}_{xy}}\underbrace{\mathrm{d}k_x \wedge \mathrm{d}k_y}_{\mathrm{d}^2k} \tag{1.97b}$$

where we used $\mathrm{d}k_i \wedge \mathrm{d}k_j = -\mathrm{d}k_j \wedge \mathrm{d}k_i$.

14 | Compare Eq. (1.93) with Eq. (1.96) →

¡! Important: TKNN formula

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\sum_{n:\varepsilon_n<E_F} C^{[n]} = \frac{e^2}{h}\nu \quad \text{with} \quad \nu := \sum_{n:\varepsilon_n<E_F} C^{[n]} \in \mathbb{Z} \tag{1.98}$$

- In summary: The Hall conductivity of a system with non-degenerate bands that are either completely filled or completely empty is an integer multiple ν of $e^2/2\pi\hbar = e^2/h$, where ν is the sum of the Chern numbers of the filled bands. This quantization is robust and independent of microscopic details because the Chern numbers are topological invariants that are necessarily integer, as long as they are well-defined (= no gaps close).
- ¡! If the Fermi energy lies *within* a (then partially filled) band, our proof of the quantization of the Hall conductivity breaks down (where?). In this situation, we cannot make any statements about the value of σ_{xy}.
- ¡! You might wonder: Where is the magnetic field? In our derivation of the TKNN formula we didn't use it. But in experiments, the quantized Hall plateaus arise when tuning the magnetic flux through the sample. The answer is that the quantization of the Hall conductivity itself has nothing to do with a magnetic field. The statement is very clear: Whenever the Fermi energy lies within a gap, the Hall conductivity is quantized and given by the sum of Chern numbers of the filled bands. Note that our result is perfectly consistent with these Chern numbers (and thereby the Hall conductivity) being *zero*! In that sense we didn't prove the exact "staircase" shape of the Hall resistance observed in 2DEGs penetrated by a magnetic field. We only showed that *if* the Hall conductivity happens to be non-zero, then it must come in steps. The role of the magnetic field is twofold: First, it opens gaps $\hbar\omega_B$ between the Landau levels, so that the conditions for a quantization of σ_{xy} are met (namely when all Landau levels are either full or empty). Second, and this is both crucial and not obvious, it makes the Landau levels "topological" in that their Chern number is $C^{[n]} = \pm 1$ (the same for all n, the sign depends on conventions and the direction of the perpendicular magnetic field). This then explains the exact structure of the famous Hall resistance plots. One can study the emergence of Landau levels and their Chern numbers in the ↑ *Hofstadter model* [17, 78] (➔ Problemset 4). Two different approaches to explicitly compute the Chern numbers of Landau levels are discussed by Fradkin [63, Chapter 12].)
- In our proof, we explicitly used that the many-body ground state is given by a Fermi sea. This description is invalidated by interactions between the fermions (e.g. Coulomb interactions). Similarly, our use of Bloch wave functions is invalidated by disorder in the system. Remarkably, it can be shown that the quantization Eq. (1.98) remains robust under general perturbations (that break translation invariance and/or add interactions) if these perturbations are not too strong [76, 79].
- Another subtlety is that all our calculations refer to *bulk properties* (namely the linear response of the bulk to a homogeneous electric field). This is *not* what one measures in experiments where one attaches point contacts to the *boundary* of a "Hall bar" (which hosts the 2DEG). The conductivity (both longitudinal and transversal) is then determined by the properties of the system boundary and not the bulk. However, due to the → *bulk-boundary correspondence*, the topological nature of the bulk directly influences the property of the edge (→ *below*); in particular, the total Chern number of the bulk (= filled Landau levels) correlates one-to-one with gapless chiral edge modes on the boundary. It is the scattering-free transport in these edge modes that one measures in actual experiments, and the quantized Hall resistance is due to the number of edge modes that contribute (= are partially filled). Formally, this is described by the ↑ *Landauer-Büttiger formalism* [80].
- This formula was first derived by **T**houless, **K**ohmoto, **N**ightingale, and **N**ijs in Ref. [17]; hence the name. It is one of the achievements that earned D. J. Thouless the 2016 Nobel Prize in Physics. Since Thouless got a half-share of the prize, and the Nobel Committee

cited both his description of the KT phase transition and the TKNN result as motivation, one can put a Prize tag on Eq. (1.98): 1/4 of a Nobel Prize. I hope you are duly impressed (you can also be a bit proud of having followed the derivation to this point ☺).

- One can show that, without adding additional symmetry constraints, the TKNN invariant (Chern number) is the *only* quantized topological invariant that can be used to distinguish gapped bands [81].
- Historically, the first convincing (but more heuristic) argument for the quantization of the Hall plateaus was already given by Robert Laughlin in 1981 [82]. However, from this derivation one cannot establish a connection to the Chern number as a topological invariant.

15 | Closing remarks:

The salient feature of the integer quantum Hall effect is that a quantity that describes a macroscopic response of system (the Hall conductivity) is exactly quantized and hence impervious to microscopic disorder. This magic turns into comprehension when we go back [to Eq. (1.70)] and realize that we only showed that the *antisymmetric* part of the conductivity tensor has a topological character (remember that we argued the symmetric part away to evade a divergence in the DC limit). Note that in a conventional conductor (w/o magnetic field) the conductivity tensor is *not* antisymmetric but symmetric. So in general we should start with the decomposition

$$\sigma = \sigma_s + \sigma_a \tag{1.99}$$

with $\sigma_s^T = \sigma_s$ and $\sigma_a^T = -\sigma_a$. W/o magnetic field σ_a vanishes (this is an example of an *Onsager relation* [83]). Strictly speaking, we have only shown that the contribution of this antisymmetric part is topologically quantized. But this contribution is also special in another way. The current $\boldsymbol{J}$ is the response due to an external electric field: $\boldsymbol{J} = \sigma \boldsymbol{E}$. The power that is dissipated in an equilibrium setting (through bumps of the charge carriers with the crystal structure) is then $P = \boldsymbol{J} \cdot \boldsymbol{E}$ (if $\boldsymbol{J}$ is the current *density* this is of course the power *density*); this is known as *Joule's law*. Putting everything together, we find

$$P = \boldsymbol{E}^T \sigma \boldsymbol{E} = \boldsymbol{E}^T \sigma_s \boldsymbol{E} \tag{1.100}$$

since $\boldsymbol{E}^T \sigma_a \boldsymbol{E} = (\boldsymbol{E}^T \sigma_a \boldsymbol{E})^T = \boldsymbol{E}^T \sigma_a^T \boldsymbol{E} = -\boldsymbol{E}^T \sigma_a \boldsymbol{E} = 0$. Thus only the *symmetric* part of the conductivity tensor plays a role for dissipation! But we didn't show that this part is quantized, only the "non-dissipative" contribution σ_a is. So our intuition that a *dissipative* quantity should depend on microscopic details and hence *not* be quantized was right, after all. What we missed is that not everything about the conductivity *tensor* is dissipative; there is also a topological (or geometric) contribution that has nothing to do with microscopic physics. It is this contribution that gives rise to the integer quantum Hall effect.

There is much more to be said about the physics of the integer quantum Hall effect. Since this a course on the broader topic of topological phases, we should not linger too long, though. However, there are three last topics that must be mentioned to prevent misconceptions and embed the IQHE into the Big Picture. For students who want to dig deeper into quantum Hall physics, I can highly recommend the lecture notes by David Tong [64].

1.5. The role of disorder

The above derivation is based on *non-interacting* fermions in a translation invariant potential (= *w/o disorder*). However, the quantization of the Hall response is more general than that and prevails in the presence of disorder and/or interactions that do not close the spectral gap above the many-body ground state [76, 79].

This statement is based on a more general expression for the Hall conductivity that does not rely on the Brillouin zone (and therefore translation invariance). This approach can also be used to compute the Hall conductivity of the Landau levels of a continuum system on a torus, see Chapter 12.7 of Fradkin's textbook [63].

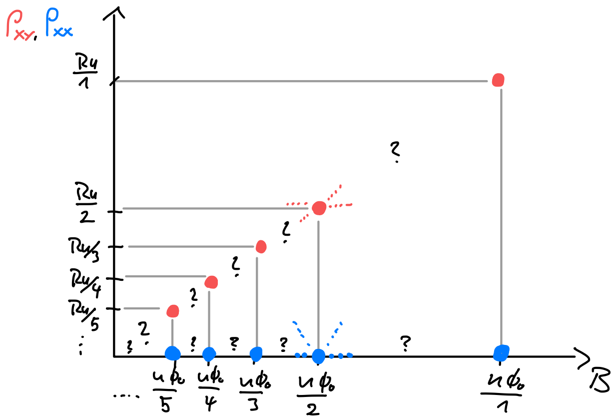
However, even if we take these statements for granted, there is still a problem that is sometimes swept under the rug in superficial discussions of the IQHE:

1 | $\sphericalangle$ System with fixed electron density n (= fixed chemical potential)

Recall Eq. (1.17): Number of states per LL: $N = \frac{AB}{\Phi_0} \stackrel{!}{=} \frac{An}{\nu}$

$\rightarrow$ Lowest $\nu \in \mathbb{N}$ LLs *exactly* filled for $B_\nu = \frac{\Phi_0 n}{\nu}$

$\rightarrow$ Only for the *discrete* B_ν the Hall response σ_{xy} is topological and thus quantized:
(Here we use that $C^{[n]} = \pm 1$ for Landau levels, which we did not derive explicitly.)



- For the longitudinal resistivity ρ_{xx} we used that systems with only completely filled/empty bands are ↓ *band insulators*, i.e., $\sigma_{xx} = 0 = \sigma_{yy}$ ($\Leftrightarrow$ $\rho_{xx} = 0 = \rho_{yy}$). This can be rigorously shown via a calculation very similar to our derivation in Section 1.4.2, i.e., starting from the Kubo formula.
- Note that $\sigma_{xx} = 0 = \sigma_{yy}$ and $\sigma_{xy} \neq 0$ translates to $\rho_{xx} = 0 = \rho_{yy}$ (!) and $\rho_{xy} = -1/\sigma_{xy}$:

$$\rho = \begin{bmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{yy} \end{bmatrix} \stackrel{\text{def}}{=} \sigma^{-1} = \begin{bmatrix} 0 & \sigma_{xy} \\ -\sigma_{xy} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -1/\sigma_{xy} \\ 1/\sigma_{xy} & 0 \end{bmatrix}. \tag{1.101}$$

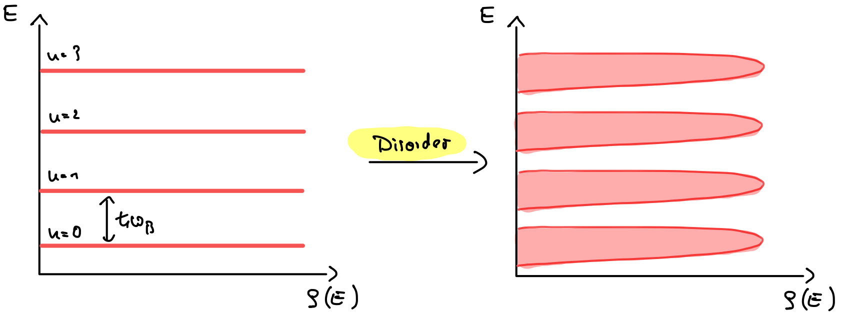
  This is not true in general, recall Eq. (1.4).

$\rightarrow$ This does not explain the observed plateaus!
Recall the experimental data shown previously to motivate our discussion of the IQHE.

The situation is a bit strange: Our hard-earned result (the TKNN formula) explains the quantization of the *height* of the plateaus, but not their *existence* (= finite width).

Solution: Disorder …

2 | First effect of disorder: LLs are *broadened*: [$\rho(E)$ denotes the ↓ *density of states*]



→ This does *still not* explain the observed plateaus!
The problem stays the same, whether the LLs are perfectly flat or not.

3 | Second effect of disorder:

- (Most) single-electron states are *localized* and *pinned* at local potential peaks/dips

  → Do not contribute to conductivity

  This pinning of free electron states due to disorder is known as ↑ *Anderson localization*.

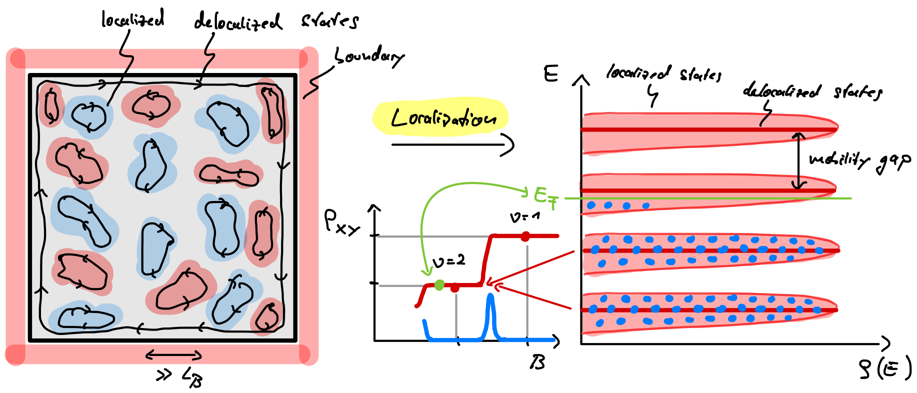
- At least one mode along the edge cannot be localized

  → Contributes to conductivity

  The existence of these non-localized "edge states" is a topological consequence of the non-zero Chern number of the LLs: the chirality makes backscattering along the edge impossible and prevents the edge modes from acquiring a gap [note → *below*].

  A characterization of "Chern bands" (bands with non-zero Chern number) is therefore that they prevent complete Anderson localization: even with disorder, some states must always remain delocalized.

→ Mobility gap:



→ Filling/depletion of broadened LLs for $B \lessgtr B_\nu$ does *not* affect conductivity as long as E_F is in the mobility gap

→ Explains *extended* Hall plateaus around B_ν with quantized height R_K/ν

4 | In a nutshell:

- *Topology* fixes the *height* of the plateaus but
- *disorder* gives them their finite *width* (= makes them visible).

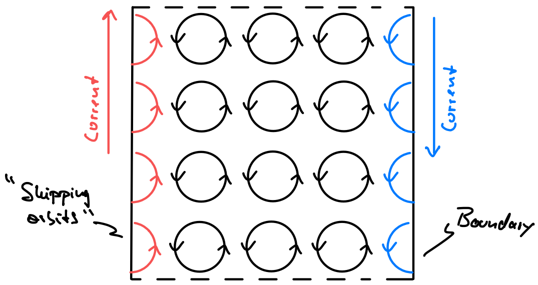
¡! This implies that in a (hypothetical) perfectly clean sample, the Hall plateaus cannot be observed.

1.6. Edge states

So far, we focused on the Hall conductivity σ_{xy}, a linear response function of the system; it is a property of the *bulk* and does not depend on the presence or absence of boundaries.

Above we have argued that in systems with boundary, there are delocalized single-particle modes running along the boundary in one direction (determined by the sign of the magnetic field and the sign of the charge carriers). These *edge states* on the 1D "surface" of the 2D system cannot be removed by disorder – they are topologically protected. We will encounter this phenomenon again in our discussion of topological insulators → *later*.

1 | Classical picture:

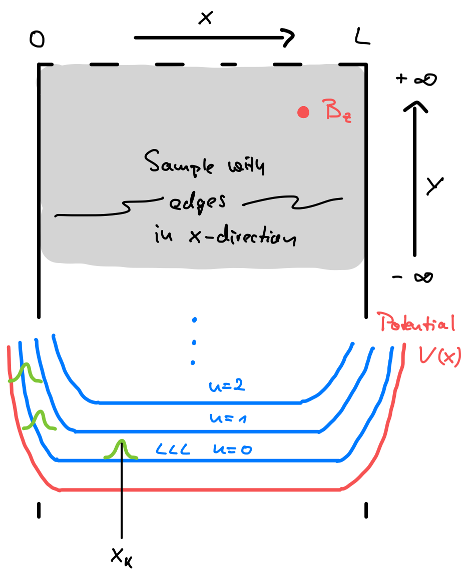


→ ⁂ *Skipping orbits* → Chiral currents along edges

2 | Quantum picture:

The following discussion provides a *heuristic* quantum mechanical picture for the emergence of edge states, the quantization of the Hall conductivity, and its robustness against disorder:

i | ◁ Strip geometry:



ii | Hamiltonian in Landau gauge: [recall Eq. (1.14)]

$$H_k = \frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x+kl_B^2)^2 + V(x) \tag{1.102}$$

$V(x)$: Potential that varies on length scales $\gg l_B$

iii | LL wavefunctions $\Psi_{n,k}$ [recall Eq. (1.16)] still eigenfunctions (with shifted energies):
⊲ Lowest Landau Level:

$$\text{Eq. (1.16)} \quad \rightarrow \quad \Psi_{0,k}(x,y) = \mathcal{N}\, e^{iky}\, e^{-\frac{(x+kl_B^2)^2}{2l_B^2}} \tag{1.103}$$

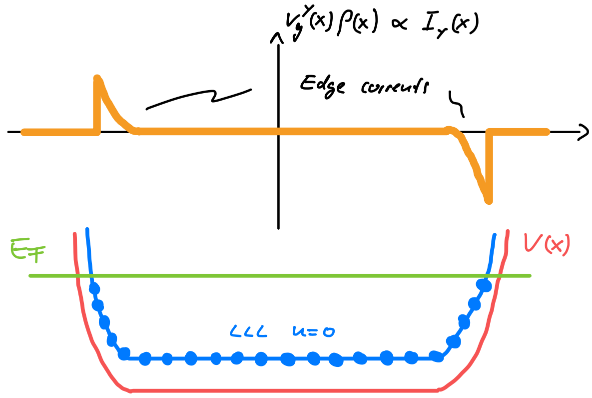
→ localized at $X_k = -kl_B^2$ (with y-momentum k)
→ Eigenenergy $E_k = \frac{1}{2}\hbar\omega_B + V(X_k)$

iv | ⊲ Group velocity in y-direction: ($l_B = \sqrt{\hbar/eB}$)

$$v_g^y(X) = \frac{1}{\hbar}\frac{\partial E_k}{\partial k} = \frac{1}{\hbar}\frac{\partial E_k}{\partial X_k}\frac{\partial X_k}{\partial k} = -\frac{l_B^2}{\hbar}\frac{\partial V(X)}{\partial X} = -\frac{1}{eB}\frac{\partial V(X)}{\partial X} \tag{1.104}$$

→ Current density $I_y(x) = -e\, v_g^y(x)\, \rho(x)$

$\rho(x)$: density of occupied states for fixed Fermi energy E_F



Note that the system is gapped with $\hbar\omega_B$ in the bulk but *gapless* on the edges!

→ Gapless, chiral edge modes

- The *chirality* of these modes (i.e., the fact that electrons can move only in one direction along the edge) is a consequence of time-reversal symmetry breaking (due to the magnetic field. It makes the charge transport robust against disorder since backscattering is impossible (there are no counterpropagating modes in which to scatter).
- This robustness prevents the generation of a gap on the edge (even in the presence of disorder and/or weak interactions). In the language of field theory, the low-energy physics on the edge is described by a ↑ *chiral Luttinger liquid*. Due to the missing counterpropagating modes, there are no relevant operators that can open a gap.
- The existence of these edge modes is deeply rooted in topology and a consequence of the non-zero Chern number of the Landau levels. The general statement that topologically non-trivial bulk insulators give rise to gapless modes on their boundary is known as ↑ *bulk-boundary correspondence* [84–86] and one of the striking features of systems with topological bands.

v | Consistency check: The current along the strip *vanishes* (at $T = 0$):

$$I_y = \int_{-\infty}^{\infty} I_y(x)\mathrm{d}x = -e\int_{-\infty}^{\infty} v_g^y(x)\rho(x)\mathrm{d}x \overset{1.104}{=} \frac{1}{B}\int_{-\infty}^{\infty}\frac{\partial V(x)}{\partial x}\rho(x)\mathrm{d}x \tag{1.105a}$$

$$\stackrel{\circ}{=} \frac{e}{2\pi\hbar}\int_{x_L}^{x_R}\frac{\partial V(x)}{\partial x}\mathrm{d}x = \frac{e}{2\pi\hbar}[\underbrace{V(x_R)}_{\mu_R} - \underbrace{V(x_L)}_{\mu_L}] \overset{V\text{ symmetric}}{=} 0 \tag{1.105b}$$

$\mu_i \equiv V(x_i)$: Chemical potential on edge i

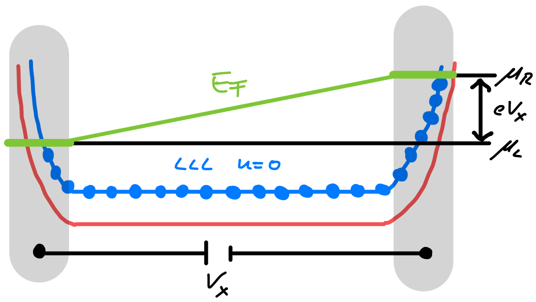
That's good news because there is no voltage applied!

Here we used Eq. (1.17) to show that the electron density of a homogeneous 2DEG with filled lowest Landau level is given by $\rho = N/A = 1/(2\pi l_B^2) = eB/(2\pi\hbar)$ so that $\rho(x) = \frac{eB}{2\pi\hbar}\,\mathbf{1}_{[x_L,x_R]}(x)$ where $\mathbf{1}_{[x_L,x_R]}(x)$ denotes the indicator function on $[x_L, x_B]$.

vi | ⊲ Hall conductivity:

Apply electric field in x-direction: $V(x) \mapsto V(x) + eEx \to \mu_R - \mu_L = eV_x$

V_x: Hall voltage between left and right boundary



→ Hall current:

$$I_y \overset{1.105b}{=} \frac{e}{2\pi\hbar}(\mu_R - \mu_L) = \frac{e^2}{2\pi\hbar} V_x \tag{1.106}$$

→ Hall conductivity per filled LL:

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} \tag{1.107}$$

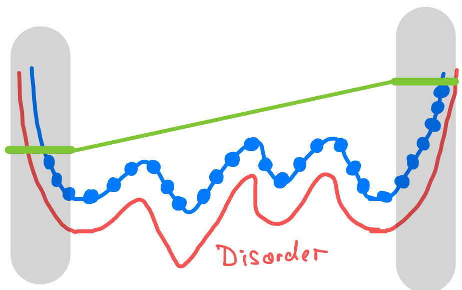
If the ν lowest Landau levels are filled, each contributes Eq. (1.107) to the total conductivity such that

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu\,, \tag{1.108}$$

consistent with the TKNN formula Eq. (1.98) and our (unproven) claim that $C^{[n]} = \pm 1$ for Landau levels.

vii | ⊲ Disorder:

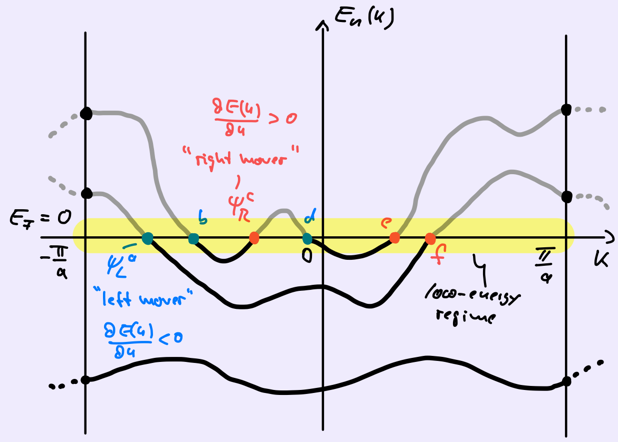
For weak disorder in the potential $V(x)$ (that does not cross the local Fermi energy), the above calculation of the Hall current remains correct as it only depends on the chemical potential at the left and right boundary, but not the behavior of E_k [or equivalently, $V(x)$] in between:



→ The result for the Hall conductivity Eq. (1.107) is robust to disorder!

3 | Chiral edge modes are special:

i | Let us first cite (the special case of) a no-go theorem with important consequences:

> **† Note: Nielsen-Ninomiya-Theorem in 1D**
>
> ⊲ Non-interacting fermions on a lattice in 1D:
>
> 
>
>
> → Brillouin zone = Circle S^1 (= bands must be periodic!)
>
> → Equal number of left (ψ_L) and right movers (ψ_R) in low-energy theories of lattice models
>
> This insight was formalized by Nielsen and Ninomiya in 1981 [87, 88] for higher-dimensional (and more important) cases, especially 3 + 1 dimensions. Then the fact that every chiral ↑ *Weyl fermion* must have a partner when discretized on a lattice is known as ↑ *fermion doubling problem*, which is inherent to lattice formulations of quantum field theories. The no-go theorem prevents lattice discretizations of chiral theories like the weak sector of the standard model. This implies in particular that there is (currently) no way to formulate the Standard Model of particle physics completely and consistently on a lattice! For more details see David Tong's lecture on gauge theory [89, Chapter 4].

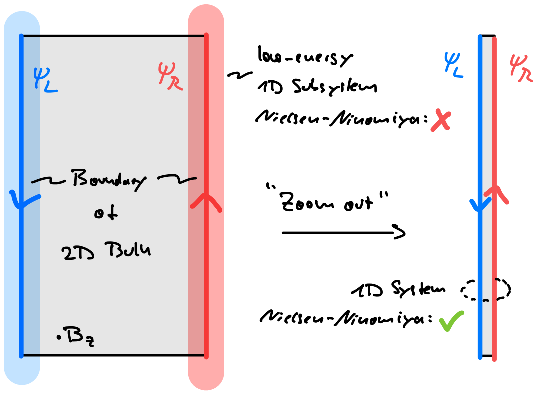
ii | → Chiral 1D modes can only appear on the boundary of a 2D bulk material!

Strictly speaking, the argument above applies only to *lattice formulations* of the IQHE (e.g. the ↑ *Hofstadter model*, ➲ Problemset 4) which, however, feature similar chiral edge modes as the IQHE in its continuum formulation. In the continuum, the proper line of arguments uses the concept of ↑ *gauge anomalies* (↑ Ref. [64, Chapter 5 & 6]).

This is an observation that goes deep with far-reaching ramifications: Effective low-energy theories that describe the gapless $D-1$-dimensional boundaries of gapped D-dimensional systems can have properties that are – under reasonable assumptions – impossible for "true" $D-1$-dimensional systems (i.e., systems that are *not* the boundary of some larger system).

iii | Intuitive explanation how to "circumvent" the Nielsen-Ninomiya theorem:

The magnetic field spatially separates left- and right movers:



iv | Comments

- In bands with non-zero Chern number, no single-particle basis exists where *all* wave functions are localized – this is known as a ↑ *topological obstruction* [90, 91]. Localized bases constructed from the Bloch wave functions are called ↑ *Wannier bases*; a non-zero Chern number therefore forbids the existence of a basis with completely localized Wannier states.

  → Delocalized edge modes

- To proper way to show the existence (and robustness) of the chiral edge modes is to construct a low-energy effective quantum field theory (QFT). This QFT turns out to be a gauge theory known as ↑ *Chern-Simons (CS) theory* (of the "abelian variety" and with "integer level"). In the presence of a boundary, the gauge invariance of the CS theory *requires* the existence of gapless physical degrees of freedom at the edge of the sample (gauge invariance demands a "chiral Luttinger liquid" on the boundary).

  → Robust edge modes

  The neat thing about the QFT approach is that it can be directly generalized to the *fractional* quantum Hall effect (then the CS theory can become "non-abelian" and is of "fractional level"). For details see Ref. [64, Chapter 5 & 6].