↓**Lecture 6** [02.05.25]

5 | ⊲ Special case: Coupling to uniform electric field $\boldsymbol{E}(t) = \boldsymbol{E}e^{-i\omega t}$

i | Choose gauge such that $\boldsymbol{E}(t) = -\partial_t \boldsymbol{A}(t)$ (i.e. $A_t = \phi = \text{const}$)

Remember that in general $\boldsymbol{E} = -\nabla\phi - \partial_t \boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

$\to \boldsymbol{A}(t) = \boldsymbol{E}e^{-i\omega t}/(i\omega)$

ii | ⊲ Perturbation Hamiltonian:

$$\Delta H_I(t) = -\boldsymbol{J}(t)\cdot\boldsymbol{A}(t) \tag{1.63}$$

with (total) current operator $\boldsymbol{J}(t)$

- At this point we do not want to fix the unperturbed Hamiltonian H_0 that describes the charge carriers without the field. Hence we do not know the form of $\boldsymbol{J}(t)$ in the interaction picture. We therefore play it safe and carry a potential time-dependence along.
- This is a linearized version of the true coupling Hamiltonian that describes the effect of the electromagnetic field on electrical charges. For instance, a free particle with charge q (and with $\phi = \text{const} = 0$) is described by the Hamiltonian

$$H = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 = \underbrace{\frac{\boldsymbol{p}^2}{2m}}_{\sim H_0} \underbrace{-\overbrace{\frac{q\boldsymbol{p}}{m}}^{\sim \boldsymbol{J}}\cdot\boldsymbol{A}}_{\sim\Delta H(t)} + \cancel{\mathcal{O}(\boldsymbol{A}^2)}\,. \tag{1.64}$$

There is also a quadratic term $\boldsymbol{A}^2$ which does not contribute to the Hall conductance (so we can safely drop it).

- In therms of the ↓ *current density* $\boldsymbol{j}(\boldsymbol{r},t)$ the Hamiltonian reads

$$\Delta H_I(t) = -\int \mathrm{d}^2r\, \boldsymbol{j}(\boldsymbol{r},t)\cdot\boldsymbol{A}(\boldsymbol{r},t) \tag{1.65}$$

with the usual current density $\boldsymbol{j} = \frac{q}{2m}\sum_i [\boldsymbol{p}_i\delta(\boldsymbol{r}-\boldsymbol{r}_i) + \delta(\boldsymbol{r}-\boldsymbol{r}_i)\boldsymbol{p}_i]$ for many particles indexed by i. With a homogeneous electric field, this becomes

$$\Delta H_I(t) = -\boldsymbol{J}(t)\cdot\boldsymbol{A}(t) \quad \text{with total current} \quad \boldsymbol{J}(t) = \int \mathrm{d}^2r\, \boldsymbol{j}(\boldsymbol{r},t)\,. \tag{1.66}$$

For a homogeneous current, the total current is $\boldsymbol{J} = L_x L_y \boldsymbol{j} = A\boldsymbol{j}$ where $A = L_x L_y$ denotes the area of the sample.

iii | ⊲ Current as observable: $\mathcal{O} = J_i \to$

(Remember that we set the static expectation value to zero: $\langle 0|J_i|0\rangle = 0$.)

$$\langle J_i(t)\rangle \overset{1.62}{=} -\frac{1}{\hbar\omega}\int_{-\infty}^{t} \langle 0|\left[J_j(t'), J_i(t)\right]|0\rangle E_j e^{-i\omega t'}\mathrm{d}t' \tag{1.67a}$$

Time-translation invariance of H_0; Substitution $t'' = t - t'$

$$\stackrel{\circ}{=} \underbrace{\left\{-\frac{1}{\hbar\omega}\int_0^{\infty} \langle 0|\left[J_j(0), J_i(t'')\right]|0\rangle\, e^{i\omega t''}\mathrm{d}t''\right\}}_{=:\sigma_{ij}(\omega)\,A} E_j e^{-i\omega t} \tag{1.67b}$$

with ⁂ *conductivity tensor* $\sigma_{ij}(\omega)$

The sample area $A = L_x L_y$ shows up because the conductivity tensor relates, by definition, the current *density* j_i to the electric field, and not the total current $J_i = A j_i$.

To show the second equality, use that $J_j(t') = e^{\frac{i}{\hbar}H_0 t'} J_j e^{-\frac{i}{\hbar}H_0 t'}$ [and similar for $J_i(t)$] and that $|0\rangle$ is an eigenstate of H_0.

iv | → Hall conductivity:

$$\sigma_{xy}(\omega) = -\frac{1}{\hbar\omega A}\int_0^\infty \langle 0|\left[J_y(0), J_x(t)\right]|0\rangle\, e^{i\omega t}\,\mathrm{d}t \tag{1.68}$$

This is the *AC Hall conductivity* as it is still frequency dependent.

v | Set $t_0 = 0$ and use $U_0(t) = \sum_n e^{-iE_n t/\hbar}|n\rangle\langle n|$ and $J_i(t) = U_0^\dagger(t) J_i U_0(t)$:

→

$$\sigma_{xy}(\omega) = -\frac{1}{\hbar\omega A}\int_0^\infty \sum_n \left\{\begin{matrix} \langle 0|J_y|n\rangle\langle n|J_x|0\rangle e^{i(E_n-E_0)t/\hbar} \\ -\langle 0|J_x|n\rangle\langle n|J_y|0\rangle e^{i(E_0-E_n)t/\hbar}\end{matrix}\right\} e^{i\omega t}\,\mathrm{d}t \tag{1.69a}$$

Integrate (using a regularization $\omega + i\varepsilon$ to make the integral convergent)

$$= -\frac{i}{\omega A}\sum_{n\neq 0}\left\{\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle}{\hbar\omega + E_n - E_0} - \frac{\langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{\hbar\omega + E_0 - E_n}\right\} \tag{1.69b}$$

vi | Take DC limit $\omega \to 0$ and use $\frac{1}{\hbar\omega+E_n-E_0} = \frac{1}{E_n-E_0} - \frac{\hbar\omega}{(E_n-E_0)^2} + \mathcal{O}(\omega^2)$:

(Note the i/ω that must be canceled to render the expression finite!)

$$\sigma_{xy} \overset{\circ}{=} \frac{i\hbar}{A}\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle - \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{(E_n-E_0)^2} \tag{1.70}$$

This is the Hall conductivity expressed in terms of current matrix elements. Our → *next* project will be a (quite tedious) reformulation of this expansion with the goal to re-express it in terms of a topological invariant, namely the ← *Chern number*.

vii | Comment on the constant term:

For the derivation of Eq. (1.70) it is crucial that

$$\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle + \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{E_n-E_0} = 0 \tag{1.71}$$

which makes the constant terms of the Taylor expansion cancel (this avoids the divergence for $\omega \to 0$!).

One way to see this is from *rotation invariance* of the system in the x-y-plane (a quantum Hall system should be rotation invariant about the axis of the magnetic field). In particular, σ_{xy} should be invariant under the $\pi/2$-rotation $J_x \mapsto J_y$ and $J_y \mapsto -J_x$ (note that $\boldsymbol{J}$ is a vector operator). This means that

$$\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle + \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{E_n-E_0} \overset{!}{=} -\sum_{n\neq 0}\frac{\langle 0|J_x|n\rangle\langle n|J_y|0\rangle + \langle 0|J_y|n\rangle\langle n|J_x|0\rangle}{E_n-E_0} \tag{1.72}$$

which implies Eq. (1.71) so that only the *antisymmetric* part of σ_{xy} survives.

Note that this is a quite general argument: If we decompose the 2D conductivity tensor into symmetric and antisymmetric parts, $\sigma = \sigma_s + \sigma_a$, and demand rotational invariance of the tensor, i.e., $\sigma = R\sigma R^T$ for a 2D rotation matrix R, we have $\sigma_s = R\sigma_s R^T$ and $\sigma_a = R\sigma_a R^T$ separately. The only *symmetric* matrix invariant under rotations is proportional to the identity, $\sigma_s = \sigma_{xx} \cdot \mathbb{1}$, so that there cannot be a symmetric contribution to the off-diagonals (that is, the Hall conductivity σ_{xy}). Thus the most general form of a *rotation invariant* conductivity tensor is

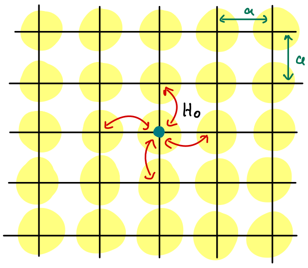
$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} . \tag{1.73}$$

1.4.2. The TKNN invariant

Here we want to connect the Hall conductivity [given by the Kubo formula Eq. (1.70)] to the Chern number and thereby explain the quantization of the former. To do so, we consider non-interacting electrons in a two-dimensional periodic potential, so that the momentum space is a torus.

The rationale of the following discussion is similar to the original approach by Thouless *et al.* [17].

1 | ⊲ Single electron in a periodic potential with Hamiltonian H_0:



System size: $L_x \times L_y$ & periodic boundaries

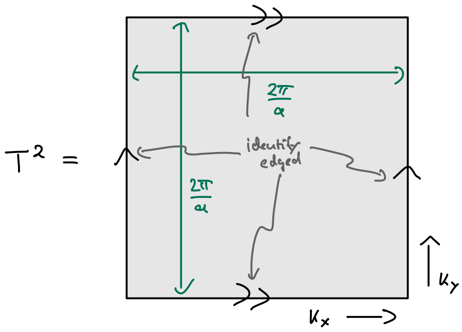
We take the thermodynamic limit $L_x, L_y \to \infty$ later.

2 | ↓ *Bloch theorem*:

- Eigenfunctions: $\Psi_{n\boldsymbol{k}} = e^{i\boldsymbol{k}\boldsymbol{x}} u_{n\boldsymbol{k}}(\boldsymbol{x})$
  with $u_{n\boldsymbol{k}}(\boldsymbol{x} + \boldsymbol{R}) = u_{n\boldsymbol{k}}(\boldsymbol{x})$ for lattice vectors $\boldsymbol{R}$ and band index $n = 1, 2, \ldots$
- Eigenenergies $\varepsilon_n(\boldsymbol{k})$ continuous in $\boldsymbol{k}$ → "Bands"
- $\Psi_{n\boldsymbol{k}+\boldsymbol{K}} = \Psi_{n\boldsymbol{k}}$ for reciprocal lattice vectors $\boldsymbol{K}$

  If $\boldsymbol{R} = an_x\boldsymbol{e}_x + an_y\boldsymbol{e}_y$ describes a square lattice with lattice constant a, the reciprocal lattice is $\boldsymbol{K} = m_1\boldsymbol{k}_1 + m_2\boldsymbol{k}_2$ with $\boldsymbol{k}_i = \frac{2\pi}{a}\boldsymbol{e}_i$.

  → Brillouin zone = Torus T^2



Since our system is finite, momenta are discrete. The size of the Brillouin zone is determined by the inverse lattice constant and remains fixed in the following.

3 | ⊲ Many-body Fock states with Fermi energy E_F:

¡! While we can understand the integer quantum Hall effect within the framework of non-interacting fermions, the quantization of the Hall conductivity is a genuine quantum many-body phenomenon. It is crucial that you understand the difference (and relation) between these concepts.

$$\text{Ground state} = |0\rangle \mapsto |\mathbf{0}\rangle = \text{Filled Fermi sea} \tag{1.74a}$$

$$\text{Excited states} = |n\rangle \mapsto |\mathbf{n}\rangle = \text{Fermi sea with particle-hole excitations} \tag{1.74b}$$

$$\text{Current operator} = J_i \mapsto \mathfrak{J}_i = \text{Second-quantized current operator} \tag{1.74c}$$

In the following, **bold states** live in the fermionic Fock space (= many-body states), whereas states in normal font live in the single-particle Hilbert space.

4 | Eq. (1.70) → Hall conductivity of fermionic many-body system:

$$\sigma_{xy} \stackrel{\circ}{=} \frac{i\hbar}{A} \sum_{\mathbf{n}\neq\mathbf{0}} \frac{\langle\mathbf{0}|\mathfrak{J}_y|\mathbf{n}\rangle\langle\mathbf{n}|\mathfrak{J}_x|\mathbf{0}\rangle - \langle\mathbf{0}|\mathfrak{J}_x|\mathbf{n}\rangle\langle\mathbf{n}|\mathfrak{J}_y|\mathbf{0}\rangle}{(E_{\mathbf{n}} - E_{\mathbf{0}})^2} \tag{1.75}$$

Note that the sum goes over all possible excited many-body states (which are all states except the Fermi sea ground state). However, below we will see that only states with a single particle-hole excitation contribute.

5 | Current operator = Single-particle operator:

$$\mathfrak{J}_i = \sum_{n\boldsymbol{k},m\boldsymbol{q}} \langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{q}}\rangle\, c^\dagger_{n\boldsymbol{k}} c_{m\boldsymbol{q}} \tag{1.76}$$

$c^\dagger_{n\boldsymbol{k}}$: Creation operator for fermion in Bloch state $|\Psi_{n\boldsymbol{k}}\rangle$

Remember that this recipe produces an operator on Fock space that acts like the single-particle operator J_i within the one-fermion subspace.

6 | Eq. (1.75) → [Here $n\boldsymbol{k}'$ is short for $(n\boldsymbol{k})' = n'\boldsymbol{k}'$.]

$$\sum_{\boldsymbol{n}\neq\boldsymbol{0}} \frac{\langle \boldsymbol{0}|\mathfrak{J}_y|\boldsymbol{n}\rangle\langle \boldsymbol{n}|\mathfrak{J}_x|\boldsymbol{0}\rangle}{(E_{\boldsymbol{n}}-E_{\boldsymbol{0}})^2} = \sum_{n\boldsymbol{k}',m\boldsymbol{q}'}\sum_{n\boldsymbol{k},m\boldsymbol{q}} \langle\Psi_{n\boldsymbol{k}}|J_y|\Psi_{m\boldsymbol{q}}\rangle\langle\Psi_{n\boldsymbol{k}'}|J_x|\Psi_{m\boldsymbol{q}'}\rangle \underbrace{\sum_{\boldsymbol{n}\neq\boldsymbol{0}} \frac{\langle \boldsymbol{0}|c^\dagger_{n\boldsymbol{k}}c_{m\boldsymbol{q}}|\boldsymbol{n}\rangle\langle \boldsymbol{n}|c^\dagger_{n\boldsymbol{k}'}c_{m\boldsymbol{q}'}|\boldsymbol{0}\rangle}{(E_{\boldsymbol{n}}-E_{\boldsymbol{0}})^2}}_{\frac{\delta_{n\boldsymbol{k}=m\boldsymbol{q}'}\delta_{m\boldsymbol{q}=n\boldsymbol{k}'}\delta_{\varepsilon_m(\boldsymbol{q})>E_F}\delta_{\varepsilon_n(\boldsymbol{k})<E_F}}{[\varepsilon_m(\boldsymbol{q})-\varepsilon_n(\boldsymbol{k})]^2}} \tag{1.77}$$

$$\mathrel{\hat{=}} \sum_{\substack{n\boldsymbol{k},m\boldsymbol{q}\\ \varepsilon_n(\boldsymbol{k})<E_F<\varepsilon_m(\boldsymbol{q})}} \frac{\langle\Psi_{n\boldsymbol{k}}|J_y|\Psi_{m\boldsymbol{q}}\rangle\langle\Psi_{m\boldsymbol{q}}|J_x|\Psi_{n\boldsymbol{k}}\rangle}{[\varepsilon_m(\boldsymbol{q})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.78}$$

To evaluate the sum $\sum_{\boldsymbol{n}\neq\boldsymbol{0}}$ over all excited many-body states, convince yourself that you can *w.l.o.g.* replace the denominator by $[\varepsilon_m(\boldsymbol{q})-\varepsilon_n(\boldsymbol{k})]^2$ (which is independent of $\boldsymbol{n}$!). Then $\sum_{\boldsymbol{n}\neq\boldsymbol{0}}|\boldsymbol{n}\rangle\langle\boldsymbol{n}|$ can be written as $\mathbb{1}-|\boldsymbol{0}\rangle\langle\boldsymbol{0}|$ and the rest follows.

7 | Assume $\varepsilon_n(\boldsymbol{k}) \lessgtr E_F$ for all $\boldsymbol{k}\in T^2$

¡! This means that the Fermi energy falls into a *band gap*. This is absolutely crucial for what follows.

(Note that statements like "$\varepsilon_n < E_F$" are now well-defined since $\varepsilon_n(\boldsymbol{k}) < E_F$ is true for all momenta and only depends on the band index n.)

→

$$\sigma_{xy} \mathrel{\hat{=}} \frac{i\hbar}{A} \sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}} \sum_{\boldsymbol{k},\boldsymbol{q}\in T^2} \frac{\left\{\begin{array}{l}\langle\Psi_{n\boldsymbol{k}}|J_y|\Psi_{m\boldsymbol{q}}\rangle\langle\Psi_{m\boldsymbol{q}}|J_x|\Psi_{n\boldsymbol{k}}\rangle\\ -\langle\Psi_{n\boldsymbol{k}}|J_x|\Psi_{m\boldsymbol{q}}\rangle\langle\Psi_{m\boldsymbol{q}}|J_y|\Psi_{n\boldsymbol{k}}\rangle\end{array}\right\}}{[\varepsilon_m(\boldsymbol{q})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.79}$$

8 | As a first simplification, we want to get rid of one of the two momentum summations. To do so, we must show that the current operator cannot change the momentum of a state:

i | Define the *single-particle current operator*

$$\boldsymbol{J} := e\dot{\boldsymbol{x}} = i\frac{e}{\hbar}[H_0,\boldsymbol{x}] \tag{1.80}$$

Here we use the ↓ *Heisenberg equation of motion* to express the velocity operator in terms of a commutator. Remember that we are in the *interaction picture*, i.e., operators evolve in time under the unperturbed Hamiltonian H_0.

ii | ◁ Translation operator $T_{\boldsymbol{R}}$ with lattice vector $\boldsymbol{R}$:

$$T_{\boldsymbol{R}}\boldsymbol{x}T_{\boldsymbol{R}}^{-1} = \boldsymbol{x}+\boldsymbol{R} \tag{1.81a}$$

$$T_{\boldsymbol{R}}H_0T_{\boldsymbol{R}}^{-1} = H_0 \tag{1.81b}$$

$$T_{\boldsymbol{R}}|\Psi_{n\boldsymbol{k}}\rangle = e^{i\boldsymbol{k}\boldsymbol{R}}|\Psi_{n\boldsymbol{k}}\rangle \tag{1.81c}$$

- The first equation follows from the definition of the translation operator.
- The commutativity with the Hamiltonian follows from our assumption that the system features a discrete translation invariance ("periodic potential").
- The energy eigenstates of such a Hamiltonian are Bloch states $|\Psi_{n\boldsymbol{k}}\rangle$ which are also eigenstates of these lattice translations (this is just the statement of ← *Bloch's theorem*).

iii | Consequently

$$T_{\boldsymbol{R}}\boldsymbol{J}T_{\boldsymbol{R}}^{-1} = i\frac{e}{\hbar}[H_0, \boldsymbol{x}+\boldsymbol{R}] = i\frac{e}{\hbar}[H_0, \boldsymbol{x}] = \boldsymbol{J} \tag{1.82}$$

→ $\boldsymbol{J}$ cannot change lattice momenta

Formally: $\langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{q}}\rangle = \langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{k}}\rangle\delta_{\boldsymbol{k},\boldsymbol{q}}$

iv | Thus Eq. (1.79) →

$$\sigma_{xy} \stackrel{\circ}{=} \frac{i\hbar}{A}\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\sum_{\boldsymbol{k}\in T^2}\frac{\left\{\begin{matrix}\langle\Psi_{n\boldsymbol{k}}|J_y|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_x|\Psi_{n\boldsymbol{k}}\rangle\\ -\langle\Psi_{n\boldsymbol{k}}|J_x|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_y|\Psi_{n\boldsymbol{k}}\rangle\end{matrix}\right\}}{[\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.83}$$

9 | ⊲ Continuum limit: $L_x, L_y \to \infty$

In the thermodynamic limit, the sum over momenta turns into an integral over the Brillouin zone T^2:

$$\sigma_{xy} \stackrel{\circ}{=} i\hbar\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\left\{\begin{matrix}\langle\Psi_{n\boldsymbol{k}}|J_y|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_x|\Psi_{n\boldsymbol{k}}\rangle\\ -\langle\Psi_{n\boldsymbol{k}}|J_x|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_y|\Psi_{n\boldsymbol{k}}\rangle\end{matrix}\right\}}{[\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.84}$$

- The continuum limit is convenient because we can now use tools from calculus to simplify this expression further.
- Here we used the usual approximation of a Riemann sum:

$$\frac{1}{L_i}\sum_{k_i} = \frac{1}{2\pi}\sum_{k_i}\frac{2\pi}{L_i} \xrightarrow{L_i\to\infty} \int\frac{\mathrm{d}k_i}{2\pi} \tag{1.85}$$

Remember that $A = L_xL_y$.

10 | Our next goal is to get rid of the current operators:

i | Use $|\Psi_{n\boldsymbol{k}}\rangle = e^{i\boldsymbol{k}\boldsymbol{x}}|u_{n\boldsymbol{k}}\rangle$ (← *Bloch theorem*) and define $\tilde{\boldsymbol{J}}(\boldsymbol{k}) := e^{-i\boldsymbol{k}\boldsymbol{x}}\boldsymbol{J}e^{i\boldsymbol{k}\boldsymbol{x}}$ so that

$$\langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{k}}\rangle = \langle u_{n\boldsymbol{k}}|\tilde{J}_i(\boldsymbol{k})|u_{m\boldsymbol{k}}\rangle \tag{1.86}$$

¡! Note that in $e^{i\boldsymbol{k}\boldsymbol{x}}$, $\boldsymbol{x}$ is the position *operator*.

ii | Define $\tilde{H}_0(\boldsymbol{k}) := e^{-i\boldsymbol{k}\boldsymbol{x}}H_0e^{i\boldsymbol{k}\boldsymbol{x}}$ so that

$$H_0|\Psi_{n\boldsymbol{k}}\rangle = \varepsilon_n(\boldsymbol{k})|\Psi_{n\boldsymbol{k}}\rangle \;\Leftrightarrow\; \tilde{H}_0(\boldsymbol{k})|u_{n\boldsymbol{k}}\rangle = \varepsilon_n(\boldsymbol{k})|u_{n\boldsymbol{k}}\rangle \tag{1.87}$$

iii | With these preliminaries, we can write:

$$\tilde{J}_i \stackrel{\circ}{=} \frac{e}{\hbar}\tilde{\partial}_i\tilde{H}_0 \quad\text{with}\quad \tilde{\partial}_i := \frac{\partial}{\partial k_i} \tag{1.88}$$

To show this use the definition of $\tilde{H}_0(\boldsymbol{k})$ and show that $\tilde{\partial}_i\tilde{H}_0 = i[\tilde{H}_0, x]$.

iv | Eqs. (1.84), (1.86) and (1.88) →

$$\sigma_{xy} \stackrel{\circ}{=} i\frac{e^2}{\hbar}\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\left\{\begin{matrix}\langle u_{n\boldsymbol{k}}|\tilde{\partial}_y\tilde{H}_0|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde{\partial}_x\tilde{H}_0|u_{n\boldsymbol{k}}\rangle\\ -\langle u_{n\boldsymbol{k}}|\tilde{\partial}_x\tilde{H}_0|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde{\partial}_y\tilde{H}_0|u_{n\boldsymbol{k}}\rangle\end{matrix}\right\}}{[\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.89}$$

11 | Use

$$\langle u_{nk}|\tilde\partial_y \tilde H_0|u_{mk}\rangle = \langle u_{nk}|\tilde\partial_y \left(\tilde H_0|u_{mk}\rangle\right) - \langle u_{nk}|\tilde H_0|\tilde\partial_y u_{mk}\rangle \tag{1.90a}$$

$$= [\varepsilon_m(\boldsymbol{k}) - \varepsilon_n(\boldsymbol{k})]\langle u_{nk}|\tilde\partial_y u_{mk}\rangle \tag{1.90b}$$

$$= [\varepsilon_n(\boldsymbol{k}) - \varepsilon_m(\boldsymbol{k})]\langle \tilde\partial_y u_{nk}|u_{mk}\rangle \tag{1.90c}$$

The first line is just the product rule, in the second line we used that $\tilde H_0 = \tilde H_0^\dagger$ and that $\langle u_{nk}|u_{mk}\rangle = 0$ for $n \neq m$ (which is the case in our expression for the Hall conductivity). The last line follows if in the first line the derivative acts on the bra to the left instead on the ket to the right.

→

$$\sigma_{xy} \hat{=} i\frac{e^2}{\hbar} \sum_{\substack{n,m \\ \varepsilon_n < E_F < \varepsilon_m}} \int_{T^2} \frac{\mathrm{d}^2k}{(2\pi)^2} \begin{Bmatrix} \langle \tilde\partial_y u_{nk}|u_{mk}\rangle\langle u_{mk}|\tilde\partial_x u_{nk}\rangle \\ -\langle \tilde\partial_x u_{nk}|u_{mk}\rangle\langle u_{mk}|\tilde\partial_y u_{nk}\rangle \end{Bmatrix} \tag{1.91}$$

Yay! The denominator is gone …☺

12 | Use

$$\sum_m |u_{mk}\rangle\langle u_{mk}| = \mathbb{1} \tag{1.92a}$$

$$\Rightarrow \quad \sum_{m:\varepsilon_m > E_F} |u_{mk}\rangle\langle u_{mk}| = \mathbb{1} - \sum_{m:\varepsilon_m < E_F} |u_{mk}\rangle\langle u_{mk}| \tag{1.92b}$$

These statements are true on the subspace spanned by the Bloch functions $|u_{nk}\rangle$ for fixed $\boldsymbol{k}$.

More rigorously, one should replace $\mathbb{1}$ by the projector $P_{\boldsymbol{k}}$ onto states with lattice momentum $\boldsymbol{k}$ and do the derivatives in the expression for σ_{xy} properly; the result will be the same, though.

→

$$\sigma_{xy} \hat{=} i\frac{e^2}{\hbar} \sum_{n:\varepsilon_n < E_F} \int_{T^2} \frac{\mathrm{d}^2k}{(2\pi)^2} \left\{\langle \tilde\partial_y u_{nk}|\tilde\partial_x u_{nk}\rangle - \langle \tilde\partial_x u_{nk}|\tilde\partial_y u_{nk}\rangle\right\} \tag{1.93}$$

Only the term with $\mathbb{1}$ survives. The second term vanishes as it replaces the sum over empty bands by a sum over filled bands. But then the sum in the expression for the Hall conductance vanishes identically if one shifts the derivatives to the states with $m\boldsymbol{k}$ in the first term [using Eq. (1.90)] and substitutes $n \leftrightarrow m$ in the sums (the last step only works because m and n now run over the same range of filled bands).

13 | Finally, we can relate our findings to the geometrical quantities introduced in Section 1.3:

i | Define the Berry connection of band n:

$$\mathcal{A}_i^{[n]}(\boldsymbol{k}) := -i\,\langle u_{nk}|\tilde\partial_i u_{nk}\rangle \tag{1.94}$$

This is a U(1) connection on the Brillouin zone which is the compact 2D manifold T^2. The parameters are the momenta ($\boldsymbol{\Gamma} = \boldsymbol{k}$) and the local Hilbert spaces are one dimensional: $\mathcal{V}^{[n]}(\boldsymbol{k}) = \mathrm{span}\,\{|u_{nk}\rangle\}$; these are the non-degenerate eigenspaces (no band crossings!) of the Hamiltonian family $\tilde H_0(\boldsymbol{k})$ with discrete spectrum $\varepsilon_n(\boldsymbol{k})$ (fix $\boldsymbol{k}$ as a parameter!). Thus $n = 1$ and $k = 2$ in the context of our general discussion in Section 1.3; in the present context, n denotes the band index.

ii | → Berry curvature of band n:

$$\mathcal{F}^{[n]}_{ij}(\boldsymbol{k}) = \tilde{\partial}_j \mathcal{A}^{[n]}_i - \tilde{\partial}_i \mathcal{A}^{[n]}_j = -i\,\langle \tilde{\partial}_j u_{n\boldsymbol{k}} | \tilde{\partial}_i u_{n\boldsymbol{k}} \rangle + i\,\langle \tilde{\partial}_i u_{n\boldsymbol{k}} | \tilde{\partial}_j u_{n\boldsymbol{k}} \rangle \tag{1.95}$$

The cross terms cancel.

iii | → Chern number of band n:

$$C^{[n]} = \frac{1}{2\pi}\int_{T^2} \mathcal{F}_{ij}\,\mathrm{d}\sigma^{ij} = -\frac{1}{2\pi}\int_{T^2} \mathcal{F}_{xy}\,\mathrm{d}^2k = \frac{i}{2\pi}\int_{T^2} \left\{ \langle \tilde{\partial}_y u_{n\boldsymbol{k}} | \tilde{\partial}_x u_{n\boldsymbol{k}} \rangle - \langle \tilde{\partial}_x u_{n\boldsymbol{k}} | \tilde{\partial}_y u_{n\boldsymbol{k}} \rangle \right\} \mathrm{d}^2k \tag{1.96}$$

The integral is best evaluated with differential forms where $\mathcal{F} = \mathrm{d}\mathcal{A}$ is a 2-form and $\mathcal{A} = A_x \mathrm{d}k_x + A_y \mathrm{d}k_y$ is a 1-form. Then

$$C = \frac{1}{2\pi}\int_{T^2} \mathcal{F} = \frac{1}{2\pi}\int_{T^2} \left(\tilde{\partial}_y A_x\,\mathrm{d}k_y \wedge \mathrm{d}k_x + \tilde{\partial}_x A_y\,\mathrm{d}k_x \wedge \mathrm{d}k_y \right) \tag{1.97a}$$

$$= -\frac{1}{2\pi}\int_{T^2} \underbrace{\left(\tilde{\partial}_y A_x - \tilde{\partial}_x A_y \right)}_{\mathcal{F}_{xy}} \underbrace{\mathrm{d}k_x \wedge \mathrm{d}k_y}_{\mathrm{d}^2k} \tag{1.97b}$$

where we used $\mathrm{d}k_i \wedge \mathrm{d}k_j = -\mathrm{d}k_j \wedge \mathrm{d}k_i$.

14 | Compare Eq. (1.93) with Eq. (1.96) →

¡! Important: TKNN formula

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar} \sum_{n:\varepsilon_n < E_F} C^{[n]} = \frac{e^2}{h}\nu \quad \text{with} \quad \nu := \sum_{n:\varepsilon_n < E_F} C^{[n]} \in \mathbb{Z} \tag{1.98}$$

- In summary: The Hall conductivity of a system with non-degenerate bands that are either completely filled or completely empty is an integer multiple ν of $e^2/2\pi\hbar = e^2/h$, where ν is the sum of the Chern numbers of the filled bands. This quantization is robust and independent of microscopic details because the Chern numbers are topological invariants that are necessarily integer, as long as they are well-defined (= no gaps close).
- ¡! If the Fermi energy lies *within* a (then partially filled) band, our proof of the quantization of the Hall conductivity breaks down (where?). In this situation, we cannot make any statements about the value of σ_{xy}.
- ¡! You might wonder: Where is the magnetic field? In our derivation of the TKNN formula we didn't use it. But in experiments, the quantized Hall plateaus arise when tuning the magnetic flux through the sample. The answer is that the quantization of the Hall conductivity itself has nothing to do with a magnetic field. The statement is very clear: Whenever the Fermi energy lies within a gap, the Hall conductivity is quantized and given by the sum of Chern numbers of the filled bands. Note that our result is perfectly consistent with these Chern

numbers (and thereby the Hall conductivity) being *zero*! In that sense we didn't prove the exact "staircase" shape of the Hall resistance observed in 2DEGs penetrated by a magnetic field. We only showed that *if* the Hall conductivity happens to be non-zero, then it must come in steps. The role of the magnetic field is twofold: First, it opens gaps $\hbar\omega_B$ between the Landau levels, so that the conditions for a quantization of σ_{xy} are met (namely when all Landau levels are either full or empty). Second, and this is both crucial and not obvious, it makes the Landau levels "topological" in that their Chern number is $C^{[n]} = \pm 1$ (the same for all n, the sign depends on conventions and the direction of the perpendicular magnetic field). This then explains the exact structure of the famous Hall resistance plots. One can study the emergence of Landau levels and their Chern numbers in the ↑ *Hofstadter model* [17, 78] (➔ Problemset 4). Two different approaches to explicitly compute the Chern numbers of Landau levels are discussed by Fradkin [63, Chapter 12].)

- In our proof, we explicitly used that the many-body ground state is given by a Fermi sea. This description is invalidated by interactions between the fermions (e.g. Coulomb interactions). Similarly, our use of Bloch wave functions is invalidated by disorder in the system. Remarkably, it can be shown that the quantization Eq. (1.98) remains robust under general perturbations (that break translation invariance and/or add interactions) if these perturbations are not too strong [76, 79].
- Another subtlety is that all our calculations refer to *bulk properties* (namely the linear response of the bulk to a homogeneous electric field). This is *not* what one measures in experiments where one attaches point contacts to the *boundary* of a "Hall bar" (which hosts the 2DEG). The conductivity (both longitudinal and transversal) is then determined by the properties of the system boundary and not the bulk. However, due to the → *bulk-boundary correspondence*, the topological nature of the bulk directly influences the property of the edge (→ *below*); in particular, the total Chern number of the bulk (= filled Landau levels) correlates one-to-one with gapless chiral edge modes on the boundary. It is the scattering-free transport in these edge modes that one measures in actual experiments, and the quantized Hall resistance is due to the number of edge modes that contribute (= are partially filled). Formally, this is described by the ↑ *Landauer-Büttiger formalism* [80].
- This formula was first derived by **T**houless, **K**ohmoto, **N**ightingale, and **N**ijs in Ref. [17]; hence the name. It is one of the achievements that earned D. J. Thouless the 2016 Nobel Prize in Physics. Since Thouless got a half-share of the prize, and the Nobel Committee cited both his description of the KT phase transition and the TKNN result as motivation, one can put a Prize tag on Eq. (1.98): 1/4 of a Nobel Prize. I hope you are duly impressed (you can also be a bit proud of having followed the derivation to this point ☺).
- One can show that, without adding additional symmetry constraints, the TKNN invariant (Chern number) is the *only* quantized topological invariant that can be used to distinguish gapped bands [81].
- Historically, the first convincing (but more heuristic) argument for the quantization of the Hall plateaus was already given by Robert Laughlin in 1981 [82]. However, from this derivation one cannot establish a connection to the Chern number as a topological invariant.

15 | Closing remarks:

The salient feature of the integer quantum Hall effect is that a quantity that describes a macroscopic response of system (the Hall conductivity) is exactly quantized and hence impervious to microscopic disorder. This magic turns into comprehension when we go back [to Eq. (1.70)] and realize that we only showed that the *antisymmetric* part of the conductivity tensor has a topological character (remember that we argued the symmetric part away to evade a divergence in the DC limit). Note that in a conventional conductor (w/o magnetic field) the conductivity tensor is *not* antisymmetric but symmetric. So in general we should start with the decomposition

$$\sigma = \sigma_s + \sigma_a \tag{1.99}$$

with $\sigma_s^T = \sigma_s$ and $\sigma_a^T = -\sigma_a$. W/o magnetic field σ_a vanishes (this is an example of an *Onsager relation* [83]). Strictly speaking, we have only shown that the contribution of this antisymmetric part is topologically quantized. But this contribution is also special in another way. The current $\boldsymbol{J}$ is the response due to an external electric field: $\boldsymbol{J} = \sigma\boldsymbol{E}$. The power that is dissipated in an equilibrium setting (through bumps of the charge carriers with the crystal structure) is then $P = \boldsymbol{J}\cdot\boldsymbol{E}$ (if $\boldsymbol{J}$ is the current *density* this is of course the power *density*); this is known as *Joule's law*. Putting everything together, we find

$$P = \boldsymbol{E}^T\sigma\boldsymbol{E} = \boldsymbol{E}^T\sigma_s\boldsymbol{E} \tag{1.100}$$

since $\boldsymbol{E}^T\sigma_a\boldsymbol{E} = (\boldsymbol{E}^T\sigma_a\boldsymbol{E})^T = \boldsymbol{E}^T\sigma_a^T\boldsymbol{E} = -\boldsymbol{E}^T\sigma_a\boldsymbol{E} = 0$. Thus only the *symmetric* part of the conductivity tensor plays a role for dissipation! But we didn't show that this part is quantized, only the "non-dissipative" contribution σ_a is. So our intuition that a *dissipative* quantity should depend on microscopic details and hence *not* be quantized was right, after all. What we missed is that not everything about the conductivity *tensor* is dissipative; there is also a topological (or geometric) contribution that has nothing to do with microscopic physics. It is this contribution that gives rise to the integer quantum Hall effect.

There is much more to be said about the physics of the integer quantum Hall effect. Since this a course on the broader topic of topological phases, we should not linger too long, though. However, there are three last topics that must be mentioned to prevent misconceptions and embed the IQHE into the Big Picture. For students who want to dig deeper into quantum Hall physics, I can highly recommend the lecture notes by David Tong [64].