↓ **Lecture 4** [24.04.25]

1. The Integer Quantum Hall Effect

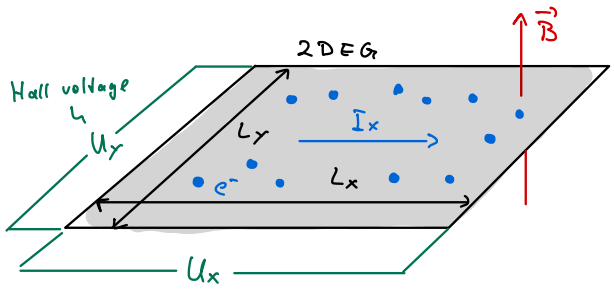
◊ **Topics**

- Review the classical & (integer) quantum Hall effect
- Derive Landau levels
- Motivate & define the Berry connection & holonomy
- Motivate & define the Berry curvature & phase
- Motivate & define the Chern number as a topological invariant
- Derive the Kubo formula and the TKNN formula for the Hall conductivity
- Comment on the role of disorder and edge states
- Locate the integer quantum Hall states in our classification of topological phases

- We start our discussion with topological phases that can be realized by *non-interacting fermions*. Such systems can be solved exactly in terms of single-particle Hamiltonians, the spectrum of which defines a ↓ *band structure*. The many-body ground states are then given by a ↓ *Fermi sea* of "filled bands" (in first-quantized language, the ground state is given by a ↓ *Slater determinant* of single-particle eigenstates). The resulting quantum phases will be symmetry-protected (SPT) phases and invertible topological orders. We will not encounter non-invertible topological orders (with anyons etc.) within this family of quantum many-body systems.
- Historically, the study of topological phases was kick-started by the experimental observation of the integer quantum Hall effect by Klaus von Klitzing in 1980 [13] who was awarded the 1985 Nobel Prize in Physics for his seminal discovery. The theoretical explanation of the effect by Thouless *et al.* in 1982 [17] highlighted the pivotal role that topological concepts can play in quantum many-body physics. For these theoretical contributions (among others) David J. Thouless (jointly with F. Duncan M. Haldane and J. Michael Kosterlitz) was awarded the 2016 Nobel Prize in Physics. We will use the integer quantum Hall effect and its theoretical description as motivation and starting point for the exploration of topological phases of non-interacting fermions in general.
- In the following, we have a quite detailed look at some aspects of the integer quantum Hall effect, especially the mathematics that underlies the quantization of the Hall conductivity. However, the integer quantum Hall effect is not the main focus of this course, and we will not cover the subject to its full extend (to do so would merit its own dedicated course!). If you are interested in more details, have a look at the textbook *Field Theories of Condensed Matter Systems* by Fradkin [63] (Chapter 12 and 13) or the *Lectures on the Quantum Hall Effect* by David Tong [64] on which parts of this chapter are based. You may also have a look at the collection [65] by Prange *et al.*.

1.1. From the classical to the quantum Hall effect

1 | ≺ 2D electron gas (2DEG) in perpendicular magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$:



Our sample is wired such that a current I_x can flow from a connection on the left boundary to a connection on the right boundary (there is no source/drain on the boundaries in y-direction, $I_y = 0$). There are voltage probes on all four boundaries to measure the voltages U_x and U_y.

2 | Drude model: (= Electrons as billard balls)

$$m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = \underbrace{-e\boldsymbol{E} - e\,\boldsymbol{v}\times\boldsymbol{B}}_{\text{Lorentz force}} \underbrace{-\frac{m}{\tau}\boldsymbol{v}}_{\text{Scattering}} \tag{1.1}$$

τ: scattering time (due to electrons bouncing of much heavier crystal ions)

3 | ≺ Equilibrium $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = \boldsymbol{0}$

Define the current density $\boldsymbol{J} = -ne\boldsymbol{v}$ (n: electron density) $\overset{\circ}{\rightarrow}$

Note that $I_x = L_y J_x$ and $U_y = L_y E_y$.

$$\underbrace{\boldsymbol{J} = \sigma\boldsymbol{E}}_{\text{Ohm's law}} \quad \text{with} \quad \underbrace{\sigma = \begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix} \overset{\circ}{=} \frac{\sigma_0}{1+\omega_B^2\tau^2}\begin{bmatrix}1 & -\omega_B\tau\\ \omega_B\tau & 1\end{bmatrix}}_{\text{Conductivity tensor}} \tag{1.2}$$

Note that $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$ is a consequence of the *rotation symmetry* of the system about the perpendicular z-axis.

with

$$\omega_B = \frac{eB}{m} \quad \text{⁂ cyclotron frequency} \tag{1.3}$$

and $\sigma_0 = ne^2\tau/m$ the ⁂ *DC conductivity* (conductivity w/o magnetic field).

4 | $\overset{\circ}{\rightarrow}$ ⁂ *Resistivity tensor:*

$$\rho \equiv \begin{bmatrix}\rho_{xx} & \rho_{xy}\\ \rho_{yx} & \rho_{yy}\end{bmatrix} := \sigma^{-1} = \frac{1}{\sigma_0}\begin{bmatrix}1 & \omega_B\tau\\ -\omega_B\tau & 1\end{bmatrix} \quad \text{with} \quad \boldsymbol{E} = \rho\boldsymbol{J} \tag{1.4}$$

Note that Hall *resistance* and Hall *resistivity* are (up to a sign depending on convention) the same:

$$R_{xy} := \frac{U_y}{I_x} = \frac{\cancel{L_y}E_y}{\cancel{L_y}J_x} = \frac{E_y}{J_x} = -\rho_{xy} \tag{1.5}$$

(Here we used $J_y = 0$ due to our experimental setup.)

This is *not* true for longitudinal resistance and resistivity:

$$R_{xx} := \frac{U_x}{I_x} = \frac{L_x E_x}{L_y J_x} = \frac{L_x}{L_y}\rho_{xx} \tag{1.6}$$

This already suggests that the Hall resistance is in some sense more robust than the longitudinal (ohmic) resistance as the former is independent of the sample geometry whereas the latter is not.

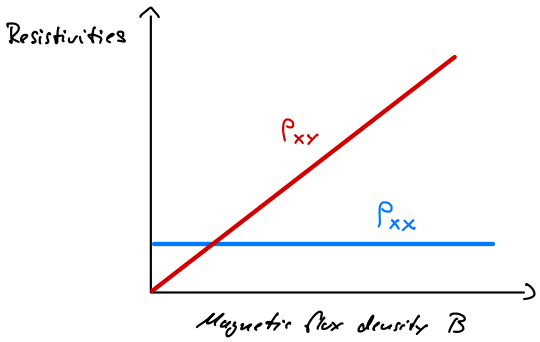
→ In particular:

$$\rho_{xy} = \frac{\omega_B \tau}{\sigma_0} = \frac{B}{ne} \quad \text{Independent of } \tau \text{ (= no dissipation) !} \tag{1.7a}$$

$$\rho_{xx} = \frac{m}{ne^2\tau} \tag{1.7b}$$

This implies that the Hall resistivity [and via Eq. (1.5) the Hall resistance] does not depend on the microscopic interactions of electrons with crystal ions and lattice defects (which determines the scattering time τ).

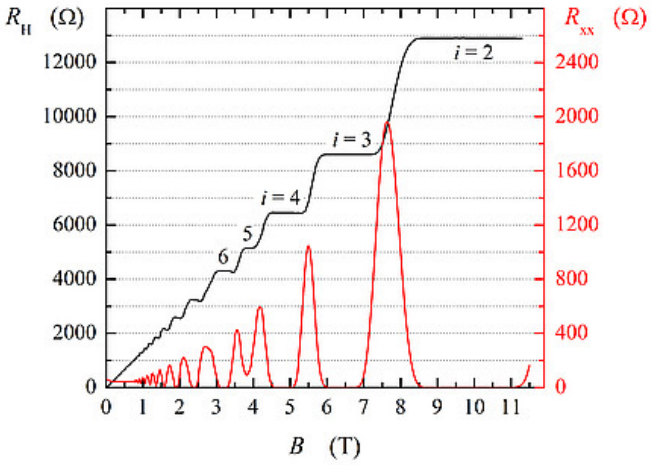
5 | → Classical prediction:



6 | Observation:

✓ Valid for high temperatures & weak magnetic fields ($\hbar\omega_B \ll k_B T$).

✗ *Not* valid for low temperatures & strong magnetic fields ($\hbar\omega_B \gg k_B T$):



– Note how the plot resembles the classical predictions in the lower-left corner (i.e., for weak magnetic fields).

- These results are from the electrical quantum metrology division of the PTB (the national metrology institute of Germany) and taken from this website; here $R_H = R_{xy} = L_y/L_y \cdot \rho_{xy} = \rho_{xy}$ and $R_{xx} = L_x/L_y \cdot \rho_{xx}$ and $i = \nu$ (see below). This phenomenon was first observed by KLAUS VON KLITZING in [13] for which he was awarded the 1985 Nobel Prize in Physics.
- The oscillations of the longitudinal resistance R_{xx} are called ↑ *Shubnikov-de Haas oscillations*. Here we are interested in the → *Hall plateaus* of the transversal resistance R_{xy} (→ *below*).

7 | → *Quantized* plateaus for Hall resistivity:

$$\rho_{xy} = \underbrace{\frac{\overbrace{2\pi\hbar}^{h/e^2}}{e^2}}_{R_K} \frac{1}{\nu} \quad \text{with} \quad \nu \in \{1, 2, 3, \dots\} \tag{1.8}$$

R_K: ⁂ *von Klitzing constant* or *quantum of resistivity* ($R_K \approx 25.8\,\mathrm{k\Omega}$)

At this point, Eq. (1.8) is an observational fact and a theoretical miracle!

Note: By the revision of the SI system of units in 2019, the numerical values of h and e are now fixed. Consequently, the value of the von Klitzing constant R_K is also fixed by definition and does not have to be measured. The integer quantum Hall effect can then be used as a universal (and defining) resistance measurement device (that's why the BTP is measuring the Hall resistance, see above). In particular, the quantization of the first → *Landau level* is perfect by definition: $\nu = 1.000\dots$. (Using your ohmmeter to measure this quantization would be as if using your balance to measure the weight of the primary kilogram in Paris before the revision of the SI. With one big difference: the primary kilogram was a unique artifact. By contrast, the integer quantum Hall effect is a universal phenomenon that can be reproduced everywhere with the right equipment. Thus "bootstrapping" universal units for measurements is much easier when artifacts are not involved. This was the motivation behind the 2019 revision of the SI system in the first place.)

8 | Historically, the miracle of the quantized Hall response and its "topological explanation" [17] (→ *below*) kick-started the study of topological phases in the first place:

> **¡! Important**
>
> The *exact quantization* of the (macroscopic) Hall resistivity in *disordered samples* of a 2DEG is a remarkable and unexpected feature that demands for an explanation!

With "exact quantization" one refers to the extraordinary precision to which the experimentally measured Hall resistivity of *different* samples coincides: the relative variations are of order 10^{-10}! A miracle indeed.

1.2. Landau levels

Up to now we used *classical* physics to describe the Hall effect – and we failed to explain the quantization of the Hall resistance. It is time for quantum mechanics to flex it's muscles …

> **¡! Important**
>
> The *integer* quantum Hall effect can be understood in the context of *non-interacting fermions*. Therefore we focus on *single-particle wavefunctions* in the following.
>
> This is *not* true for the ↑ *fractional quantum Hall effect*!

1 | ◁ Same setup as before, but now we qantize the system!

→ Single-particle Hamiltonian of an electron in a magnetic field:

$$H = \frac{1}{2m}\underbrace{(\boldsymbol{p} + e\boldsymbol{A})}_{\boldsymbol{\pi}}{}^2 \tag{1.9}$$

$\boldsymbol{\pi}$: *kinetic* momentum (gauge independent)
$\boldsymbol{p}$: *canonical* momentum (gauge dependent)
$\boldsymbol{A}(\boldsymbol{x})$: gauge potential with $\nabla \times \boldsymbol{A} = B\boldsymbol{e}_z$ (we do not yet fix a gauge!)

2 | Canonical quantization:

$$\left[x_i, p_j\right] = i\hbar\delta_{ij} \tag{1.10}$$

→ $[\pi_x, \pi_y] \mathrel{\hat{=}} -ie\hbar B$

Remember that the (static) gauge potential $\boldsymbol{A}(\boldsymbol{x})$ depends on the position (operator) $\boldsymbol{x}$, and that the canonical momentum (operator) that satisfies Eq. (1.10) is $p_i = -i\hbar\frac{\partial}{\partial x_i}$ (↑ *Stone-von Neumann theorem*).

→ The magnetic field couples the movement in x- and y-direction, so that the kinetic momenta form a pair of conjugate observables.

3 | This immediately suggests the introduction of ↓ *ladder operators*:

$$a := \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y) \tag{1.11}$$

$\overset{\circ}{\rightarrow}$ These satisfy as usual $[a, a^\dagger] = 1$ and we find with Eq. (1.9) $\overset{\circ}{\rightarrow}$

$$\boxed{H \mathrel{\hat{=}} \hbar\omega_B\left(a^\dagger a + \frac{1}{2}\right)} \tag{1.12}$$

→ Discrete spectrum $E_n = \hbar\omega_B\left(n + \frac{1}{2}\right)$ with $n = 0, 1, 2, \ldots$

→ ⁂ *Landau levels* (LL)

The term "Landau levels" refers to both the quantized *eigenenergies* E_n and the corresponding (degenerate) *eigenspaces* within the single-particle Hilbert space.

4 | Eigenstates? Degeneracy?

Note that we only used *one* degree of freedom (= one harmonic oscillator) although we started with *two* independent degrees of freedom (an electron moving in a 2D plane). The Landau levels must therefore be extensively degenerate to harbor all the needed states! So see this, we must first fix a gauge …

We stress that here the gauge field $\boldsymbol{A}$ is *not* a dynamical degree of freedom (like when you quantize the electromagnetic field). Thus gauge fixing is really just a classical inconvenience and does not lead to fundamental problems like negative norm states etc.

1.2.1. Landau gauge

Here we proceed with the particularly simple *Landau gauge* (which comes with a price: it breaks the rotational symmetry of the problem); the alternative *symmetric gauge* is discussed in Section 1.2.2 → *below*. Since these are gauges, their choice does not affect physical conclusions; however, they lead to different basis states in the Landau levels that paint different (but equivalent) pictures of the physics within them.

5 | ◁ Gauge choice $\boldsymbol{A} := xB\boldsymbol{e}_y$

This gauge breaks translation symmetry in x-direction and rotation symmetry in the plane. This is of course a mathematical artifact: the physics remains completely invariant under these transformations.

Eq. (1.9) → Hamiltonian:

$$H = \frac{1}{2m}\left[p_x^2 + (p_y + eBx)^2\right] \tag{1.13}$$

6 | ◁ Translation symmetry in y-direction

Here we assume either periodic boundaries in y-direction or take the limit $L_y \to \infty$.

→ Ansatz: $\Psi_k(x,y) = e^{iky} f_k(x)$

In Eq. (1.13) → *Shifted* harmonic oscillator:

$$H_k \mathrel{\overset{\circ}{=}} \frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x + kl_B^2)^2 \tag{1.14}$$

with

$$l_B = \sqrt{\frac{\hbar}{eB}} \quad ⁂ \textit{ magnetic length} \tag{1.15}$$

The magnetic length is the relevant length scale for electrons in a magnetic field (it is the length scale of their cyclotron orbits).

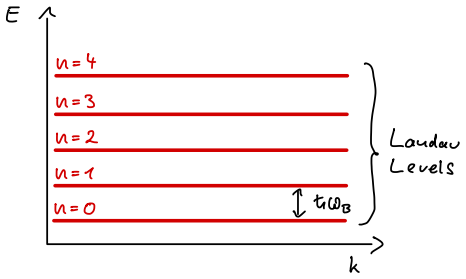
7 | $\overset{\circ}{\to}$ Eigenfunctions: (of H_k for each y-momentum k)

$$\Psi_{n,k}(x,y) = \mathcal{N} \underbrace{e^{iky}}_{\substack{\text{Plane wave}\\ \text{in } y\text{-direction}}} \underbrace{\underbrace{H_n\left(xl_B^{-1} + kl_B\right)}_{\text{Hermite polynomials}} e^{-\frac{1}{2}\left(xl_B^{-1}+kl_B\right)^2}}_{\text{Harmonic oscillator wavefunctions in } x\text{-direction}} \tag{1.16}$$

with $n = 0, 1, 2, \ldots$ the Landau level index and $k = \frac{2\pi}{L_y}\mathbb{Z}$ the y-momentum.

Note that the eigenspaces of H (and the eigenfunctions) are *physical* and therefore gauge independent. What is *unphysical* is the choice of a basis (and the labeling of the basis wavefunctions by "good" quantum numbers). Since the Landau gauge preserves translation symmetry in y-direction, the basis above can be labeled by momenta in y-direction. In other gauges (see below), this is not the case. However, the eigenspaces that are spanned by these wavefunctions are the same for all gauges (of course) and you can linearly combine basis functions of one gauge with basis functions of another.

8 | Spectrum: $E_n = \hbar\omega_B\left(n + \frac{1}{2}\right)$ (degenerate in the k quantum number!)



The Landau levels are prototypes for *perfectly flat bands*. If a LL is only partially filled, the many-body properties of the electrons that occupy this level are determined by their (Coulomb) interactions. This is crucial to understand the long-range entanglement (topological order) of ↑ *fractional quantum Hall states*.

9 | Degeneracy: $0 \le x \le L_x \to$ Restricted y-momenta k: $-L_x/l_B^2 \le k \le 0$

(Since the wavefunctions (1.16) are exponentially localized around $x_k = -kl_B^2$.)

→ Number of states in each Landau level:

$$N = \frac{L_x/l_B^2 - 0}{2\pi/L_y} = \frac{L_xL_y}{2\pi l_B^2} = \frac{AB}{\Phi_0} = \frac{\Phi}{\Phi_0} \tag{1.17}$$

$\Phi_0 = 2\pi\hbar/e$: ⁂ *quantum of flux* (cf. $R_K = 2\pi\hbar/e^2$ the *quantum of resistivity*)

$A = L_xL_y$: area of the sample

→ Extensive degeneracy of each Landau level (as expected)

In particular, the number of electrons N than can be crammed into each Landau level increases with the magnetic flux through the sample (one electron per quantum of flux). This implies that if we fix the electron density and increase the magnetic flux density, fewer and fewer Landau levels will be needed to distribute all electrons, until for very large B all electrons fit into the lowest Landau level (LLL). Conversely, at "every day" weak-field conditions, Landau levels up to very large indices n are occupied by fermions.

1.2.2. ‡ Symmetric gauge

You will do these calculations on ➲ Problemset 2.

In contrast to the ← *Landau gauge*, the *symmetric gauge* breaks translation invariance in *both* directions but retains the two-dimensional rotation invariance of the system. Thus, instead of k, we should expect a basis labeled by *angular momentum quantum numbers* m:

10 | ◁ Gauge choice

$$\boldsymbol{A} := -\frac{1}{2}\boldsymbol{r} \times \boldsymbol{B} = -\frac{yB}{2}\boldsymbol{e}_x + \frac{xB}{2}\boldsymbol{e}_y \tag{1.18}$$

11 | Hamiltonian: [recall Eq. (1.12)]

$$H = \frac{\boldsymbol{\pi}^2}{2m} = \hbar\omega_B \left(a^\dagger a + \frac{1}{2}\right) \tag{1.19}$$

with $a, a^\dagger$ defined via π_x and π_y [recall Eq. (1.11)]

So far, this procedure does not depend on the gauge choice since the kinetic momentum is a gauge independent quantity.

12 | Define additional "momentum": (which does not show up in the Hamiltonian!)

$$\tilde{\boldsymbol{\pi}} := \boldsymbol{p} - e\boldsymbol{A} \quad \Rightarrow \quad \left[\tilde{\pi}_x, \tilde{\pi}_y\right] \doteq ie\hbar B \tag{1.20}$$

(Recall that $\boldsymbol{\pi} = \boldsymbol{p} + e\boldsymbol{A}$.)

Important: In *symmetric gauge* (1.18) the two momenta are independent: $[\pi_i, \tilde{\pi}_j] \doteq 0$

This is not so in other gauges!

13 | → Define additional ladder operators:

$$b := \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x + i\tilde{\pi}_y) \quad \text{and} \quad b^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x - i\tilde{\pi}_y) \tag{1.21}$$

→ $[b, b^\dagger] = 1$ and $[a, b] = 0$ (The latter is only true in symmetric gauge!)

14 | → Eigenstates:

$$|n, m\rangle := \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n!m!}}|0, 0\rangle \quad \text{with} \quad a|0, 0\rangle = b|0, 0\rangle = 0 \tag{1.22}$$

$n = 0, 1, 2, \ldots$: Landau level index
$m = 0, 1, 2, \ldots$: Angular momentum index (→ *below*)
In symmetric gauge, m replaces the y-momentum k and generates the degeneracy of the LLs.

15 | ◁ Complex coordinates:

The unconventional sign makes the functions below holomorphic instead of antiholomorphic.

$$z := x - iy \quad \text{and} \quad \bar{z} := x + iy \tag{1.23}$$

and the corresponding ↓ *Wirtinger derivatives*

$$\partial := \frac{1}{2}(\partial_x + i\,\partial_y) \quad \text{and} \quad \bar{\partial} := \frac{1}{2}(\partial_x - i\,\partial_y) \tag{1.24}$$

Then $\partial z = \bar{\partial}\bar{z} = 1$ and $\partial\bar{z} = \bar{\partial}z = 0$. A function of complex variables is then holomorphic (= satisfies the Cauchy-Riemann equations) if and only if $\bar{\partial} f = 0$, i.e., $f = f(z)$.

16 | Use $p_i = -i\hbar\partial_i$ & Eqs. (1.18), (1.20), (1.21), (1.23) and (1.24) →

$$a = -i\sqrt{2}\left(l_B\bar{\partial} + \frac{z}{4l_B}\right) \quad \text{and} \quad a^\dagger = -i\sqrt{2}\left(l_B\partial - \frac{\bar{z}}{4l_B}\right) \tag{1.25a}$$

$$b = -i\sqrt{2}\left(l_B\partial + \frac{\bar{z}}{4l_B}\right) \quad \text{and} \quad b^\dagger = -i\sqrt{2}\left(l_B\bar{\partial} - \frac{z}{4l_B}\right) \tag{1.25b}$$

17 | ◁ *Lowest* Landau level wave functions $\Psi_0(z,\bar{z})$:

$$a\Psi_0 = 0 \quad \Leftrightarrow \quad \bar{\partial}\Psi_0 = -\frac{z}{4l_B^2}\Psi_0 \quad \Leftrightarrow \quad \Psi_0(z,\bar{z}) = f(z)e^{-z\bar{z}/4l_B^2} \tag{1.26}$$

$f(z)$: *arbitrary* holomorphic function

18 | ◁ Unique state with $m = 0$: (within the lowest Landau level)

$$b\Psi_0 = 0 \;\Leftrightarrow\; \partial\Psi_0 = -\frac{\bar{z}}{4l_B^2}\Psi_0 \quad \overset{(1.26)}{\Longleftrightarrow} \quad \partial f(z) = 0 \;\Leftrightarrow\; f(z) = \text{const} \tag{1.27}$$

→ $\Psi_{0,0}(z,\bar{z}) \propto e^{-|z|^2/4l_B^2}$ (Gaussian state)

19 | ◁ Other states in the LLL → Apply $b^\dagger$ to $\Psi_{0,0}$: (Remember that $\bar{\partial} z = 0$.)

$$\Psi_{0,m} \propto b^{\dagger m}\Psi_{0,0} \propto \left(l_B\bar{\partial} - \frac{z}{4l_B}\right)^m e^{-z\bar{z}/4l_B^2} \propto \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4l_B^2} \tag{1.28}$$

→ Holomorphic monomials × Gaussian

Since all wave functions $\Psi_{0,m}$ are degenerate, one can form arbitrary linear combinations of these holomorphic monomials (times a Gaussian) to form more general holomorphic polynomials.

20 | → In the LLL, m is an *angular momentum* quantum number:

$$J\Psi_{0,m} = \hbar m\Psi_{0,m} \quad \text{with} \quad \underbrace{J = i\hbar\left(x\partial_y - y\partial_x\right)}_{\text{Angular momentum operator}} = \hbar\left(z\partial - \bar{z}\bar{\partial}\right) \tag{1.29}$$

with $m = 0, 1, 2, \ldots$

Note: In 2D there is only one generator of angular momentum $J = J_z$ and the Lie algebra that generates the rotation group $\mathrm{SO}(2) \simeq U(1)$ (namely $\mathfrak{u}(1) \simeq \mathbb{R}$) is abelian. Consequently, there is no algebraic reason for spin to be quantized (as in 3D where spin can take only half-integer values) and all irreducible representations are one-dimensional. Thus there is only *one* spin quantum number needed (to label the irrep) but none to label distinct basis states of an irrep, i.e., $J = m$. So Eq. (1.29) is all there is to say about spin in this context. Note that the abelian "angular momentum algebra" in 2D has also consequences for particles with anyonic statistics which do not only feature "fractional charges" and "fractional statistic" but also "fractional spin" (→ Part III).

1.3. Berry connection and Berry holonomy

We now take a step back and discuss some rather abstract (and high-level) concepts of quantum mechanics. We return to the integer quantum Hall effect → *later*.

The following concepts are very generic and play a role in many areas of modern physics; they are also important throughout this course. Their application to the quantized Hall conductivity → *below* is only one of many examples.

The following derivation is quite common and leads to physically important (and valid) conclusions. However, it is mathematically not rigorous and uses hidden assumptions on the ↑ *connection* of the full ↑ *Hilbert bundle* on which the parametric family of Hamiltonians is defined, see Ref. [66] for a mathematical treatment of the problem geared towards physicists.

1 | ◁ Setting:

- Continuous family of gapped Hamiltonians $H(\mathbf{\Gamma})$ with k parameters $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \ldots, \Gamma_k)$ and n-fold degenerate ground state space $\mathcal{V}(\mathbf{\Gamma}) \equiv \mathcal{V}(H(\mathbf{\Gamma}))$

  Since $H(\mathbf{\Gamma})$ is continuous and gapped, the dimension of $\mathcal{V}(\mathbf{\Gamma})$ is constant.
  We set $H(\mathbf{\Gamma})|\Psi\rangle = 0$ for $|\Psi\rangle \in \mathcal{V}(\mathbf{\Gamma})$ and all $\mathbf{\Gamma}$, i.e., the ground state energy is zero.

- Slow "parameter path" $\mathbf{\Gamma}(t)$ for $0 \leq t \leq T$

  "Slow" compared to the (inverse) of the smallest energy gap along the path $\mathbf{\Gamma}(t)$.

- Initial ground state $|\Psi_0\rangle \in \mathcal{V}(\mathbf{\Gamma}_0)$

2 | Question: What happens with $|\Psi_0\rangle$ as $H(\mathbf{\Gamma}(0))$ evolves to $H(\mathbf{\Gamma}(T))$?

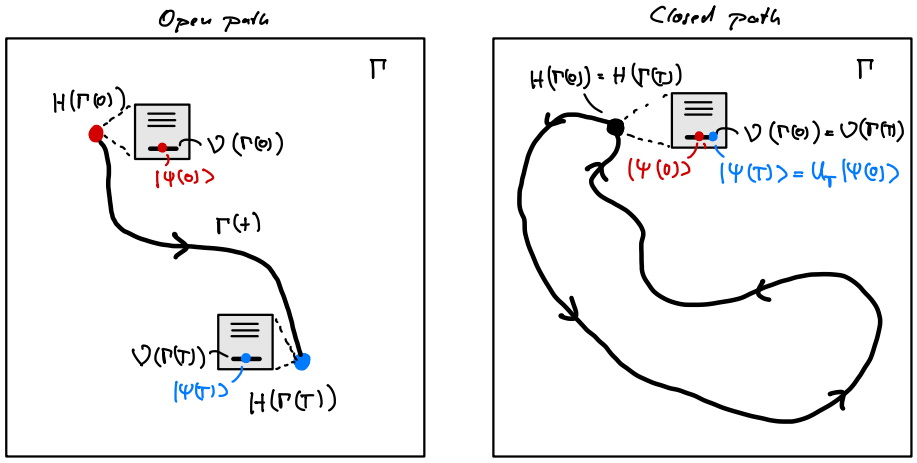
3 | To answer this question, we use the following well-known fact:

> **¡! Important: Adiabatic theorem**
>
> A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum.

This fundamental insight is due to MAX BORN and VLADIMIR FOCK [67].

4 | Solution:

Here is a sketch of the scenario/task that we want to solve:



We proceed step by step:

i | Pick a basis $\{|v_i(\mathbf{\Gamma})\rangle\}_{i=1,\ldots,n}$ of $\mathcal{V}(\mathbf{\Gamma})$ for every $\mathbf{\Gamma}$

This choice is not unique and leads to a $\mathrm{U}(n)$ gauge degree of freedom (→ *below*). Here we assume that the choice is differentiable (and therefore continuous) along the path $\mathbf{\Gamma}$. This makes it less arbitrary but leaves a lot arbitrariness to choose from. Note that a choice that is globally continuous is often impossible. Then one follows the arguments below on local patches in parameter space on which such a choice is possible.

ii | ◁ Time-dependent Schrödinger equation:

$$\underbrace{i\hbar\partial_t|\Psi(t)\rangle}_{\text{(L)}} = \underbrace{H(\mathbf{\Gamma}(t))|\Psi(t)\rangle}_{\text{(R)}} \tag{1.30}$$