↓ **Lecture 3** [17.04.25]

10 | Examples for topologically ordered systems that exist in nature (or in laboratories):

- Fractional quantum Hall states

Yes, this is all we actually know of (except for some special cases, see below)! There is a plethora of *theoretical* models, some of which are actively studied in labs; but none of them have been experimentally realized and characterized to the degree that fractional quantum Hall states have. Examples of promising models that are theoretically known to be topologically ordered and actively experimentally studied include ↑ *topological quantum spin liquids* like the → *toric code* (→ ??) [31, 32], ↑ *Kitaev materials* [33], and ↑ *fractional Chern insulators* [34].

- But actually, that's not quite correct:

  Integer quantum Hall states, first observed in 1980 by KLAUS VON KLITZING [13], are also long-range entangled, i.e., cannot be transformed into product states via constant-depth LU circuits [35]. However, their long-range entanglement is of a particularly simple type (so called *invertible* topological order, → *below*) that does *not* give rise to anyonic excitations and topological ground state degeneracies (→ *Part III*, see also Ref. [36]) which makes non-invertible topological orders like fractional quantum Hall states so interesting. This is why some use a different nomenclature where "topologically ordered" only refers to non-invertible topological order with anyonic excitations and non-vanishing → *topological entanglement entropy* [37].

- But *aaactually* ...that's also not quite correct:

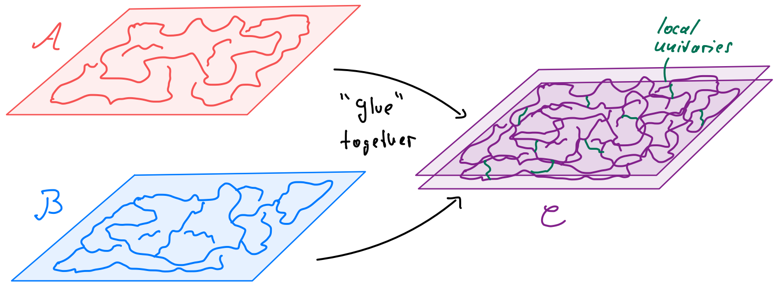
  Surprisingly, conventional s-wave superconductivity in 3D systems (discovered in 1911 by HEIKE KAMERLINGH ONNES) is also an example of intrinsic topological order [35, 38]. This is not true for simplistic models like the BCS-Hamiltonian where the electromagnetic gauge field is treated as a non-dynamical background. In the real world, however, the gauge field *is* dynamical and a superconductor is described by the interactions between charged particles (electrons) with themselves (which gives rise to pairing) and with the dynamical electromagnetic field (which gives rise to string-like excitations, namely quantized ↑ *flux tubes*, and massive photons). The combined system of charges and electromagnetic field turns out to be topologically ordered [39, 40] and is described by a ↑ *topological quantum field theory* called ↑ *BF-theory* [41]. The excitations of such systems are (1) Bogoliubov quasiparticles ("broken Cooper pairs"), (2) flux lines/loops, and (3) massive photons. The Bogoliubov quasiparticles have non-trivial braiding statistics with the flux lines/loops – which demonstrates the (non-invertible) topological order of such systems.

11 | ‡ Invertible topological orders (ITO):

With our definition of gapped quantum phases, one can define a "multiplication" of such phases:

⊲ Two topological phases $\mathcal{A}$ and $\mathcal{B}$ (in the same dimension):

$$\underbrace{\mathcal{A} \boxtimes \mathcal{B}}_{\text{Stacking two TOs}} = \underbrace{\mathcal{C}}_{\text{New TO}} \tag{0.23}$$



Observation: Trivial phase $\mathcal{E}$ (= product states) acts as identity:

$$\mathcal{A} \boxtimes \mathcal{E} = \mathcal{A} \tag{0.24}$$

→ Commutative monoid of quantum phases [42]

In mathematics, a ↓ *monoid* is a set with an associative binary operation and an identity. It is commutative if the binary operation is abelian. Elements are not required to have inverse elements (if all elements have inverse elements, the monoid becomes a group).

⊲ Class of topological phases that have an *inverse element*:

⁂ Definition: Invertible topological order (ITO)

$$\text{⁂ } \textit{Invertible TOs (ITO)} := \{\mathcal{A} \,|\, \exists \mathcal{A}^{-1} : \mathcal{A} \boxtimes \mathcal{A}^{-1} = \mathcal{E}\}$$

- The class of ITOs forms a *group* within the monoid of TOs.
- If they exist, the inverse phases are given by a time-reversal operation [43].
- In words: A topologically ordered ground state is invertible if and only if you can find another ground state (of a gapped, local Hamiltonian) such that the combination of both can be transformed into a product state by a constant-depth LU circuit.

$\overset{*}{\rightarrow}$ The entanglement patters of ITOs are of a particularly simple kind [42, 43]:

$$\text{ITO} \quad \Leftrightarrow \quad \begin{cases} \text{No} \rightarrow \textit{anyons} \\ \text{Vanishing} \rightarrow \textit{topological entanglement entropy} \end{cases}$$

In that sense, ITOs are a rather "boring" type of long-range entanglement, which is why some do not refer to ITOs as topologically ordered in the first place (in this course, we do).

Examples [35, 44]:

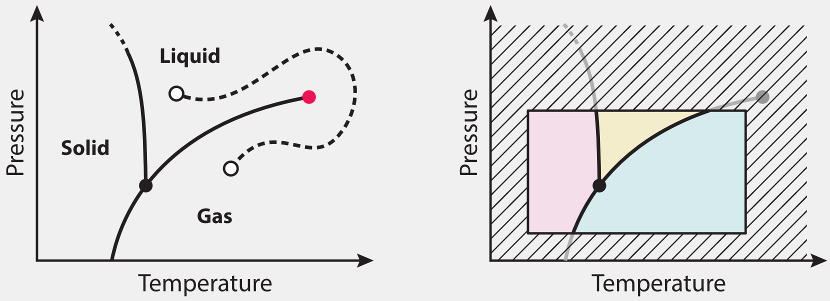
- → *Integer quantum Hall states* in 2D (Part I)
- → *Haldane model* in 2D (??)
- → *Kane-Mele model (Topological insulator)* in 2D (??)
  (This is the only example in this list that is truly short-range entangled.)
- → *Majorana chain* in 1D (??),

So despite their lack of fancy anyonic statistics, ITOs are not so boring after all and we will study them in detail (and discover much interesting physics).

12 | But wait! There is more ...

Adding patterns of long-range entanglement to our labeling scheme produces a more fine-grained classification of quantum phases. However, it can be useful to make this classification *even more fine-grained* by adding additional symmetry constraints.

We can motivate this rationale by a classical analog:

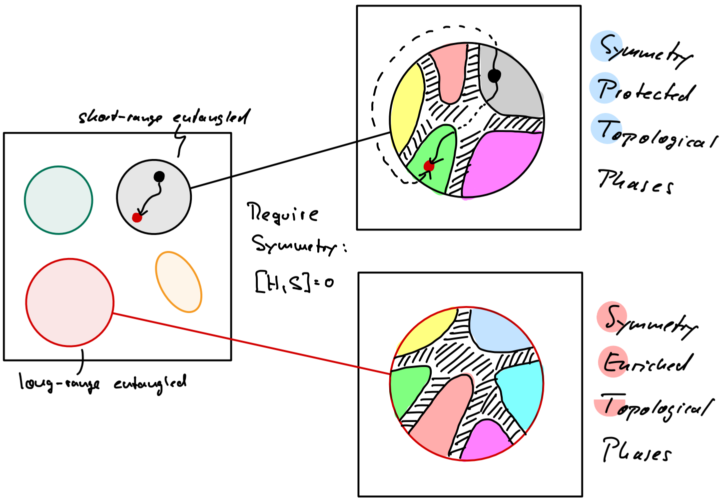
> ★ **Interlude: Restrictions on perturbations**
>
> ⊲ Phase diagram of water:
>
> 
>
>
> (This argument draws inspiration from classical thermodynamic phases, not quantum phases.)
>
> → Restrictions on allowed paths can be useful! (depending on the system ...)

13 | → Restrict Hamiltonians by *(protecting) symmetries* $G_P \subseteq G_S$:

> ⁂ **Definition: Symmetry-protected quantum phases**
>
> ⊲ Gapped, local Hamiltonians H_a and H_b with unique ground states $|\Omega_a\rangle$ and $|\Omega_b\rangle$, and a symmetry group G_P (represented by unitaries U_g on the Hilbert space) with $[H_x, U_g] = 0$ for all $g \in G_P$ and $x = a, b$.
>
> The ground states belong to the same *symmetry-protected quantum phase* if and only if there exists a family of *gapped* and *local* Hamiltonians $\hat{H}(\alpha)$ that depends continuously on $\alpha \in [0, 1]$ such that
>
> $$H_a = \hat{H}(0) \quad \text{and} \quad H_b = \hat{H}(1) \tag{0.25}$$
>
> and
>
> $$\left[\hat{H}(\alpha), U_g\right] = 0 \quad \text{for all} \quad g \in G_P \text{ and } \alpha \in [0, 1]\,. \tag{0.26}$$

14 | → Phases with the *same entanglement pattern* can *split* further:

In particular short-range entangled phases that belong to the trivial phase!



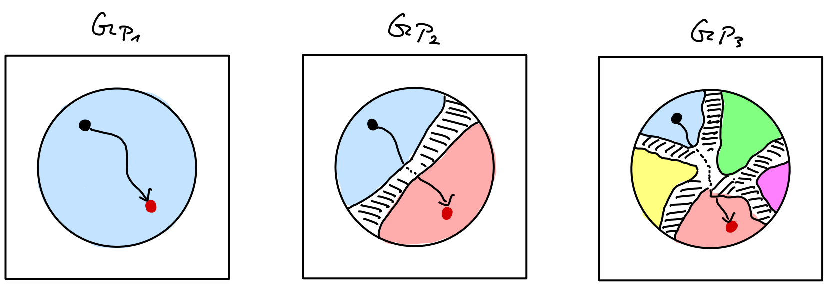
→ Conventional nomenclature:

> ⁂ **Definition: SPT and SET phases**
>
> - ⁂ *Symmetry-protected topological (SPT) phases* :=
>   Short-range entangled phases protected by symmetries
> - ⁂ *Symmetry-enriched topological (SET) phases* :=
>   Long-range entangled phases with additional symmetries
>
> We use the terms "SPT(s)" and "SET(s)" to mean "symmetry-protected/enriched topological quantum phase(s)" whenever the context requires it.

Comments:

- Typical examples for SPT phases are the → *SSH chain* in 1D (??) and the → *topological insulator* in 2D (??).
- Typical examples for SET phases are ↑ *factional quantum Hall states* with protected U(1) symmetry (= particle number conservation) [35].
- We will not study SET phases in this course!
- Since SPT phases are LU-equivalent to product states (when ignoring symmetries), they belong to the equivalence class $\mathcal{E}$ of "trivial" phases; in particular, they are ← *invertible topological orders* (of a special kind, namely without any long-range entanglement).

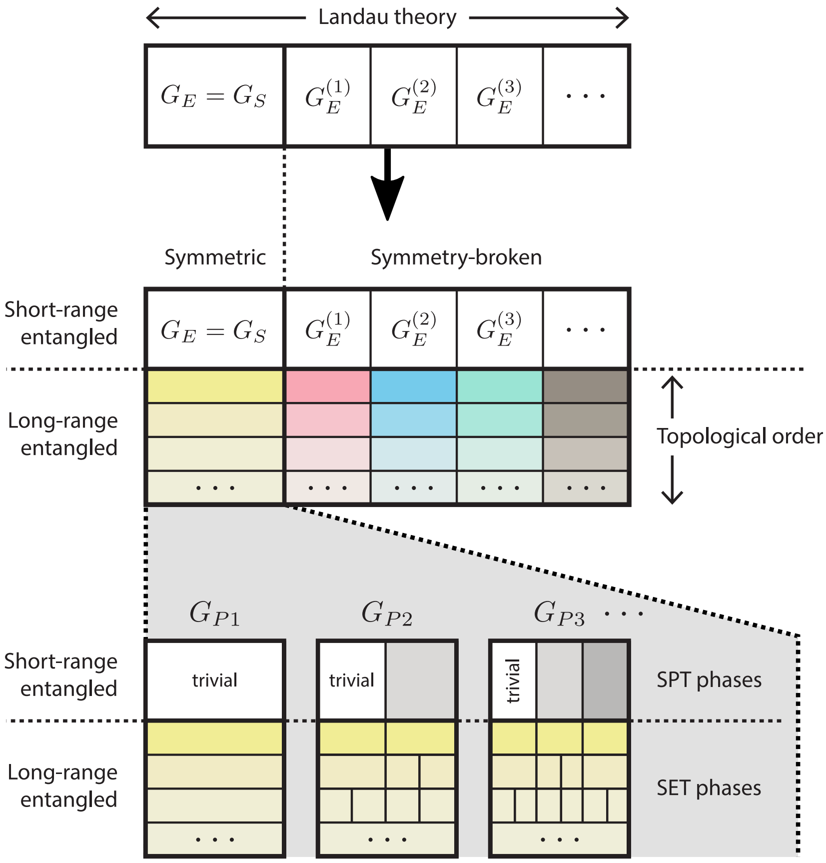
15 Important: Possible SPTs (and SETs) depend on the protecting symmetry G_P:



¡! Note that SPTs (and SETs) are not properties of ground states (like intrinsic topological order) but rather classifications of ground states with respect to a prescribed class of allowed Hamiltonian deformations (or a restricted class of constant-depth LU circuits). The relevance of this class typically derives from physical considerations (recall the motivation above).

Second extension of Landau's paradigm: Symmetry-protected topological phases

16 | These insights lead us to a second extension of our classification scheme:



¡! In this course we only study phases without SSB.

17 | *Question:* How to characterize SPT phases?

We cannot label them by patterns of long-range entanglement nor by the symmetries they break!

Answer: Complicated! (also: subject of ongoing research!)

→ Make simplifying assumptions: Consider restricted classes of models/Hamiltonians:

- ⊲ Non-interacting fermions → Part I

  The benefit of non-interacting fermions is that such models can be solved exactly. This provides us with powerful tools to classify them systematically. Note that some of these models will turn out to be invertible topological orders (we will not find *non*-invertible TOs with anyons etc. for this class of Hamiltonians).

- ⊲ Interacting bosons/spins (in 1D) → Part II

  Interacting systems of bosons/spins in 1D are usually not exactly solvable. However, since there is no topological order in these systems, such models realize true SPT phases with a powerful description in terms of matrix-product states (which again makes a systematic classification possible).

Let me comment on a few questions that might come up at this point:

- Why not *interacting* fermions?

  In 1D, this classification derives (via Jordan-Wigner transformation) from the classification of interacting spin systems (also in 1D) [29]. In higher dimensions, there are approaches to classify interacting fermions (this is ongoing research [45]), but this goes beyond the scope of this course.

- Why not *non-interacting* bosons?

  Because non-interacting bosons form a Bose-Einstein condensate (which is a well-understood non-topological quantum phase). Thus topological phases for bosons *require* interactions (in contrast to fermions).

- Why not interacting bosons in *higher dimensions*?

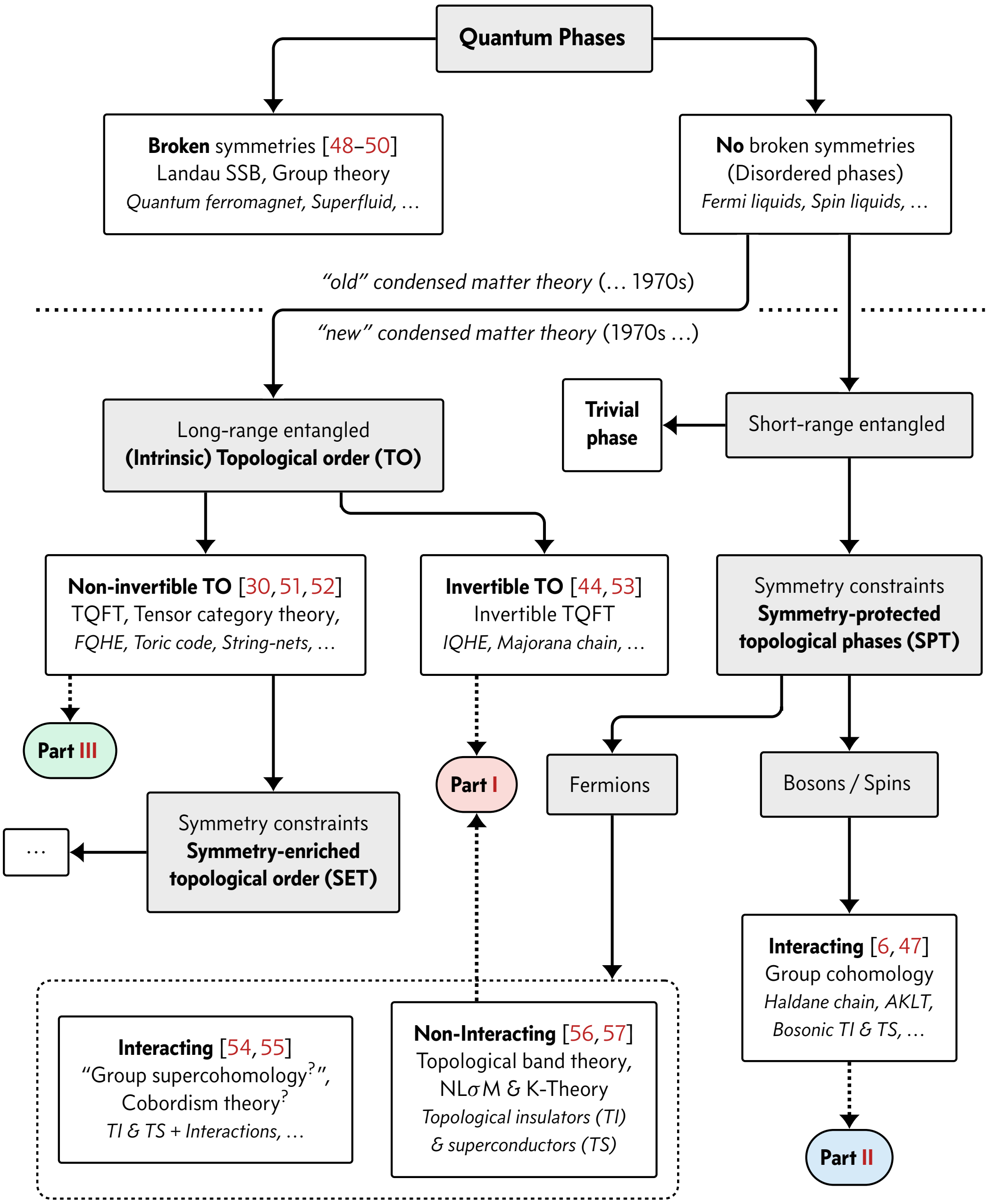
  This can be done with a generalization of the mathematical methods that we will discuss for the 1D case (Part II) [46, 47]. We will not discuss these generalizations in this course.

0.6. Overview and Outlook

We can combine all these concepts into a flowchart:

Highlight the three classes that we will discuss in this course.

(A few important original references are given that establish the various concepts.)



⁂ Definition: Nomenclature

In this course, gapped quantum phases that …

(1) do not break any symmetries

(2) and are either …

(2a) *topologically ordered (TO)* or
(invertible and non-invertible, with and without additional protecting symmetries)

(2b) *symmetry-protected (SPT)*

…will jointly be referred to as *topological phases (TP)*.

Note the difference between "topologically ordered phases" (which are long-range entangled) and "topological phases" (which can also refer to short-range entangled SPT phases)!

So far, we discussed topological phases on an abstract level. To further motivate these concepts, let us briefly list a few features of these intriguing phases of matter. Note that not every topological phase exhibits all these features! Some of them necessarily require long-range entanglement, some don't. Conversely, long-range entanglement does not necessarily imply all these features. It is quite a mess and we will study various models in this course to shed light on these features and their origins.

Some features of topological phases

- TPs cannot be characterized by local order parameters
(all correlations of local operators decay exponentially, cf. our discussion of the TIM).

  This is the defining property of topological phases and applies to intrinsic topological order and SPT phases alike.

- For some TPs, the ground state degeneracy on closed manifolds depends on their topology (whether it is a sphere, a torus, etc.) and is robust in the presence of perturbations [58].

  This is true for non-invertible topological orders (like fractional quantum Hall states), but not for SPT phases and invertible topological orders (topological insulators, integer quantum Hall states).

- Some TPs feature exponentially localized excitations (*quasiparticles*) that obey neither fermionic nor bosonic statistics – they are *anyons* and obey *fractional* or *anyonic statistics* [59, 60].

  The presence of anyons is closely related to the topological degeneracy mentioned above. For example, integer quantum Hall states and topological insulators do not have anyonic quasiparticles, but fractional quantum Hall states do.

- These quasiparticles can carry fractionalized charges (e.g. a fraction of the electron charge) [15].

  Fractional charges are a consequence of additional symmetries (for example, particle number conservation). Fractional charges are therefore a consequence of anyonic excitations in the presence of a conserved symmetry, i.e., SET order. Fractional quantum Hall states with $U(1)$ particle number conservation are an example.

- Some TPs have an effective low-energy description in terms of a *topological quantum field theory (TQFT)* [61] (a quantum field theory defined by an action that is a topological invariant).

This is closely related to features of intrinsic topological orders (topological degeneracy, anyonic excitations). Invertible topological orders are described by ↑ *invertible TQFTs*.

- In some TPs, (lattice) defects can exhibit anyonic statistics as well (under continuous deformations of the Hamiltonian).

  This can happen even for invertible topological phases like the Majorana chain in 1D and the $p_x + i p_y$ superconductor in 2D. Note that such defects are *not* intrinsic quasiparticle excitations but deformations of the Hamiltonian.

- Some TPs feature robust, gapless edge states on manifolds with boundaries that allow for scattering-free transport [62].

  This can happen for invertible topological phases and even SPT phases. Examples are the famous chiral edge states of integer quantum Hall systems.

- The linear response of TPs can be *quantized* to a remarkable degree (even in the presence of disorder!).

  This can happen for invertible and non-invertible topological orders (like integer and fractional quantum Hall states) and even SPT phases (like topological insulators). The quantization requires some additional symmetry (like particle number conservation), even if the phase itself does not require any symmetry protection.

† Note: Why topology?

After this introductory info-dump you might wonder:

Why are topological phases called "topological"?
Where does topology enter the picture?

The answer to these questions is, as usual, complicated and cannot be fully appreciated at this point (answering these questions is the purpose of this course). However, we can make a few high-level comments to set your expectations:

First, remember that ↓ *topology* is the field of mathematics that is concerned with the formalization of deformation-invariant properties of spaces. Topology is therefore considered with rather "robust" qualities of (potentially abstract) objects – in contrast to *geometry* that is concerned with concepts like distances and angles (for which one requires a metric). For instance, a donut (bagel) is topologically equivalent to a coffee mug because you can continuously deform these two shapes into each other (such a deformations is called a ↓ *homotopy*):

➔ Example of a homotopy (Source: Wikipedia)

We say that the donut and the coffee mug are *topologically equivalent* but *geometrically distinct*. Formalizing the concept of "topological equivalence" and studying its implications is the core subject of topology.

Generally speaking, topological phase are called "topological" because various (!) concepts from topology play a role in their description. It is important to appreciate that the term "topology" can refer to *different* topological concepts and their application in physics. Broadly speaking, there are two very distinct such applications in the realm of topological quantum many-body physics:

- *"Classical topology":*

  Our first encounter of topological concepts will concern so called → *topological invariants* that can be used to characterize certain manifolds (parametric paths, Brillouin zones, band structures, …) that describe *non-interacting* quantum systems (Part I). This is an application of topology to *single-particle* quantum mechanics (with many-body ramifications like the quantized Hall conductivity). As such, some (not all!) of these phenomena translate to classical systems (→ *topological edge modes*) with several applications in engineering (??). The SPT phases of non-interacting fermions (topological insulators and superconductors) are an example of "classical topology" at play.

- *"Quantum topology":*

  A completely different application of methods from topology concerns the description of long-range entangled quantum many-body phases, i.e., *intrinsic topological order*. The low-energy physics of quantum phases in general can be described by ↑ *quantum field theories* (QFT). It turns out that the quantum field theories of systems with topological order are of a particularly elegant type: they are so called ↑ *topological quantum field theories* (TQFT). These are QFTs that do not depend on the metric of the space on which the fields live; hence their degrees of freedom only depend on the *topology* of this space. The algebraic properties of TQFTs capture all the fascinating properties of topologically ordered systems (anyonic excitations, topological ground state degeneracies, …). This application of topology is a genuine feature of *quantum* systems and has no classical counterpart, hence "quantum topology".

Part I.

Topological Phases of Non-Interacting Fermions