11. The Haldane chain and the AKLT model

Historically, the first symmetry-protected topological phase discovered was the *Haldane phase* realized by the spin-1 antiferromagnetic Heisenberg model (AFHM) in one dimension. In the same phase is the Affleck-Kennedy-Lieb-Tasaki (AKLT) model. In contrast to the AFHM, the ground state of the AKLT Hamiltonian can be derived exactly, which provides important insight into its entanglement structure. In this chapter we briefly discuss both models and their relation to the formal framework introduced in Chapter 10.

11.1. The Haldane conjecture and the Haldane phase

Here we briefly review the insights published by Duncan Haldane around 1983 concerning the spin-1 AFHM. We postpone a classification of its SPT phase to the more tractable AKLT model in Section 11.2 → *below*.

Haldane's early studies of antiferromagnetic spin chains are one of the seeds from which the flourishing field of topological phases originated. His insights regarding the relevance of topological terms in effective field theories of condensed matter systems are one of the contributions cited by the Nobel committee for the 2016 Nobel Prize in physics; see the scientific summary [250]. For a historical account see also Haldane's Nobel Lecture [104].

1 | ⊲ ⁂ *(Isotropic) Heisenberg chain (IHC)* for spin $S \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$:

$$\boxed{H_S := J \sum_{i=1}^{L'} \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} = J \sum_{i=1}^{L'} \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z \right]} \tag{11.1}$$

- $\boldsymbol{S}_i = (S_i^x, S_i^y, S_i^z)$: Spin-$S$ operator on site $i = 1, \dots, L$
- Hilbert space: $\mathcal{H} = \bigotimes_{i=1}^{L} \mathbb{C}_i^{2S+1} \equiv S \otimes S \ldots \otimes S$:

  Here S denotes the irreducible representation of SU(2) with dimension $2S + 1$.
- As usual:

  $L' = L$: periodic boundaries (PBC)
  $L' = L - 1$: open boundaries (OBC)
- Spin-spin coupling $J \in \mathbb{R}$:

  $J > 0$: ⁂ *antiferromangetic IHC*
  $J < 0$: ⁂ *ferromangetic IHC*

The Hamiltonian (11.1) is also referred to as ⁂ *XXX-model* since the couplings of the three terms are equal (isotropic). When there is an anisotropy $\Delta \in \mathbb{R}$ in z-direction,

$$\tilde{H}_S = J \sum_{i=1}^{L'} \left[S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right] , \tag{11.2}$$

the model is referred to ⁂ *XXZ-model*. The regime $|\Delta| < 1$ is called ⁂ *easy-plane anisotropy*, whereas for $|\Delta| > 1$ one speaks of an ⁂ *easy-axis anisotropy*. A model with anisotropy in all directions is called ⁂ *XYZ-model*.

2 | Symmetries:

The following symmetries of the Hamiltonian (11.1) are important → *later*:

- *Global* SU(2) *symmetry:*

$$\text{Total spin:} \quad \boldsymbol{S}_{\text{tot}} = \sum_{i=1}^{L} \boldsymbol{S}_i \quad \overset{\circ}{\Rightarrow} \quad [H_S, \boldsymbol{S}_{\text{tot}}] = 0 \tag{11.3}$$

  The total spin $\boldsymbol{S}_{\text{tot}}$ generates a *reducible* representation of SU(2) on the full Hilbert space $\mathcal{H}$.

  To derive Eq. (11.3), note that all spin operators $\boldsymbol{S}_i$ transform as ↓ *vector operators* under the total spin $\boldsymbol{S}_{\text{tot}}$. Hence the scalar products $\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}$ transform as ↓ *scalar operators* under $\boldsymbol{S}_{\text{tot}}$, which immediately implies $[H_S, \boldsymbol{S}_{\text{tot}}] = 0$.

- *Global D_2 symmetry:*

  An important *subgroup* of SU(2) is given by 180° rotations about the x-, y- and z-axes:

$$\left.\begin{aligned} U_x &:= e^{i\pi S^x_{\text{tot}}} \\ U_y &:= e^{i\pi S^y_{\text{tot}}} \\ U_z &:= e^{i\pi S^z_{\text{tot}}} \end{aligned}\right\} \quad \xRightarrow{\alpha\in\{x,y,z\}} \quad [H_S, U_\alpha] = 0 \tag{11.4}$$

  It turns out (→ *below*) that this subgroup is sufficient to protect the SPT phase of the → *AFHM* / → *AKLT model*. This matches our findings in Section 10.3, where we showed that the cohomology group of D_2 is non-trivial.

  For *integer* total spin (which is the situation we are interested in → *below*), it is

$$U_\alpha^2 \overset{\circ}{=} \mathbb{1} \quad \text{and} \quad U_x U_y \overset{\circ}{=} U_z \overset{\circ}{=} U_y U_x\,, \tag{11.5}$$

  so that $\{\mathbb{1}, U_x, U_y, U_z\}$ form a (linear) representation of $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$.

- *Time reversal symmetry:*

$$\mathcal{T} \boldsymbol{S}_i \mathcal{T}^{-1} = -\boldsymbol{S}_i \quad (\text{for all } i = 1, \dots, L) \quad \overset{\circ}{\Rightarrow} \quad [H_S, \mathcal{T}] = 0 \tag{11.6}$$

  As usual, $\mathcal{T}$ is an *anti*unitary operator (we omit its explicit construction). This symmetry can also protect the SPT phase of the → *AFHM* / → *AKLT model*, consistent with the results in Section 10.5 where we showed that the cohomology group of an antiunitary $\mathbb{Z}_2$ symmetry is non-trivial.

  Remember our discussion of time-reversal in Section 2.1.2. To construct $\mathcal{T}$ explicitly, recall ➔ Problemset 6 for the representation of time-reversal on arbitrary spins.

3 | Facts:

- There is no known method to diagonalize H_S analytically for general S and J.
- For $S = \frac{1}{2}$ and $J \in \mathbb{R}$, the *anisotropic* ("easy-plane") version of $\tilde{H}_{\frac{1}{2}}$ (where the $S_i^z S_{i+1}^z$ term is missing: $\Delta = 0$) can be mapped to a chain of free fermions via a ← *Jordan-Wigner transformation* (➔ Problemset 8 and Eq. (10.53) ff.). This yields access to the full spectrum, eigenstates and correlation functions. For $0 < |\Delta| < 1$ the model maps to *interacting*

fermions and can no longer be solved via a unitary transformation of the fermion algebra; however, it can be effectively described as a ↑ *Luttinger liquid* and studied via ↑ *bosonization*. It also can be solved exactly via the Bethe ansatz (→ *next*).

- For $S = \frac{1}{2}$ and $J \in \mathbb{R}$, the *isotropic* Heisenberg model $H_{\frac{1}{2}}$ ($\Delta = 1$) can be solved exactly via the ↑ *Bethe ansatz* [251]. This solution shows that the model is *gapless* in the thermodynamic limit. Note that Bethe ansatz solutions are notoriously difficult to evaluate. It took until the 1970s until Bethe's methods were generalized to the fully anisotropic $S = \frac{1}{2}$ Heisenberg chain (↑ *XYZ-model*) [252, 253]. Computing correlations functions from Bethe ansatz solutions was not achieved until 1975 [254–256] (they decay algebraically).
- For $S \in \{\frac{1}{2}, \frac{3}{2}, \dots\}$ and $J > 0$ (antiferromagnetic isotropic Heisenberg chain), it has been shown rigorously by Affleck and Lieb that H_S is *gapless* [257]. This result generalizes previous findings by Lieb, Schultz and Mattis for the case $S = \frac{1}{2}$ [258]. For this reason, the rigorous result that half-integer antiferromagnetic Heisenberg chains cannot have a gapped ground state is known as ↑ *Lieb-Schultz-Mattis theorem*. Over the year, this theorem has been generalized to other (non-spin) systems and higher dimensions [259, 260]. For an introduction see Ref. [6, Chapter 6.2].
- For $S \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ and $J < 0$ (ferromagnetic isotropic Heisenberg chain), one can write down the ground states exactly: They are extensively ($2SL + 1$-fold) degenerate, fully spin-polarized states (states with maximum total spin) and have ferromagnetic correlations. Low-lying excitations can also be explicitly constructed (↑ *magnons* or ↑ *spin waves*) and show that the spectrum is *gapless* for all spins S (integer and half-integer). This is very different from the *anti*ferromagnetic case (→ *below*). See Ref. [6, Chapter 2.4] for an introduction.

  [Note that this does not contradict the ↑ *Mermin-Wagner theorem*, according to which there is no spontaneous breaking of continuous symmetries in two spatial dimensions (or less) *at finite temperature*. Via a quantum-to-classical mapping (using transfer matrices), one can translate 1D quantum systems at zero temperature to two-dimensional classical systems at (typically) finite temperature. The ground state entanglement of the quantum system is then reflected by thermal fluctuations in the classical counterpart. At this point, Mermin-Wagner kicks in and forbids spontaneous symmetry breaking. This is the case for the 1D Heisenberg *antiferromagnet*. The Heisenberg *ferromagnet* is special in this regard, as its ground states at $T = 0$ are *product states* without entanglement, *i.e.*, there are no quantum fluctuations (or thermal fluctuations after the mapping to 2D) that could destroy long-range order. This is a consequence of the total spin conservation symmetry of the system. Since the Mermin-Wagner theorem does not apply to (classical) systems without fluctuations ($T = 0$), it does not preclude the correlations in the ferromagnetic ground states of H_S.]

4 | Haldane (1983, Refs. [19, 261]):

$\sphericalangle$ $J > 0$ and $S \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$: Isotropic *antiferromagnetic* Heisenberg chain

Recall that the ground states of *ferromagnetic* chains are fully polarized product states and therefore "boring." Furthermore, these phases are gapless for all S due to spin wave excitations – so there is nothing interesting for us in this direction.

The following is a *sketch* of the arguments that led Haldane to its → *conjecture*:

For more details see Ref. [262, Chapters 7,10,12,15] (on which the following steps are based).

i | Goal: Effective field theory for (generic) Spin Hamiltonian $\hat{H} = \hat{H}(\boldsymbol{S}_1, \dots, \boldsymbol{S}_L)$

→ ∢ ↑ *Spin coherent states (SCS)*:

$$|\hat{\Omega}\rangle := e^{iS^z\phi}e^{iS^y\theta}e^{iS^z\chi}|S,S\rangle \quad \text{with} \quad \hat{\Omega} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \tag{11.7}$$

- ϕ, θ, χ: ↓ *Euler angles* (χ is a global phase = gauge freedom)
- As usual, spin coherent states are the "most classical" spin states (they saturate the Heisenberg uncertainty relation for the two spin components transversal to $\hat{\Omega}$). They approximate classical angular momentum states in the limit $S \to \infty$.
- Spin coherent states form an ↓ *overcomplete basis* of the $2S+1$-dimensional spin representation.

→ Spin coherent state of L spins: $|\hat{\mathbf{\Omega}}\rangle := \bigotimes_{i=1}^{L} |\hat{\Omega}_i\rangle$
Here $\hat{\mathbf{\Omega}} = (\hat{\Omega}_1, \dots, \hat{\Omega}_L)$ encodes the direction of all spins.

→ Matrix elements in SCS basis:

$$H[\hat{\mathbf{\Omega}}] \equiv \langle\hat{\mathbf{\Omega}}|\hat{H}|\hat{\mathbf{\Omega}}\rangle \qquad \in \mathbb{R} \tag{11.8}$$

ii | ↓ *Time slicing* & Eq. (11.8) & Resolution of the identity $\mathbb{1} = \frac{2S+1}{4\pi}\int \mathrm{d}\hat{\Omega}\,|\hat{\Omega}\rangle\langle\hat{\Omega}|$

$\xrightarrow{*}$ ↑ *Spin coherent state path integral*:

$$Z \stackrel{*}{=} \oint \mathcal{D}\hat{\mathbf{\Omega}}\ \underbrace{e^{i\Upsilon[\hat{\mathbf{\Omega}}(t)]}}_{\text{Berry phase}}\ \underbrace{\exp\left(-i\int_0^T H[\hat{\mathbf{\Omega}}(t)]\mathrm{d}t\right)}_{\text{Dynamical phase}} \tag{11.9}$$

$\hat{\mathbf{\Omega}}(t) : [0,T] \to S^2 \times \cdots \times S^2$: Time-evolution of all L spin polarizations

Here we consider periodic boundaries and evolutions, $\hat{\mathbf{\Omega}}(0) = \hat{\mathbf{\Omega}}(T)$, so that the $1+1$-dimensional spacetime has the topology of a torus; this periodicity is indicated by $\oint$.

...with ← *Berry phase*

$$\Upsilon[\hat{\mathbf{\Omega}}(t)] = S\sum_{i=1}^{L}\omega[\hat{\Omega}_i(t)] \tag{11.10}$$

¡! The Berry phase shows up naturally in the derivation; it is not put in by hand.

...given by

$$\omega[\hat{\Omega}(t)] = -\int_0^T \mathrm{d}t\,\dot{\phi}\cos\theta \stackrel{\circ}{=} \left\{\begin{matrix}\text{Area on } S^2 \\ \text{traced out by } \hat{\Omega}(t)\end{matrix}\right\} \tag{11.11}$$

This quantity is geometric and independent of the specific parametrization by t.

iii | Now specialize to the antiferromagnetic ($J > 0$) Heisenberg chain for $S \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$:

$$\text{Eq. (11.1)} \quad \xrightarrow[\text{Eq. (11.8)}]{\circ} \quad H[\hat{\mathbf{\Omega}}] = JS^2\sum_{i=1}^{L}\hat{\Omega}_i\cdot\hat{\Omega}_{i+1} \tag{11.12}$$

Here we set $L' = L$ for periodic boundary conditions.

iv | Continuum limit:

Ansatz by Haldane:

$$\hat{\Omega}_i \equiv \overbrace{(-1)^i}^{\text{Antiferromagnetic ordering}} \underbrace{\hat{\boldsymbol{n}}(x_i)}_{⁂\ \textit{Néel field}} \sqrt{1-\left|\frac{\boldsymbol{L}(x_i)}{S}\right|^2} + \overbrace{\frac{\boldsymbol{L}(x_i)}{S}}^{⁂\ \textit{Canting field}} \tag{11.13}$$

- $x_i = a \cdot i$: Lattice sites with lattice spacing a
- Néel field is unimodular: $|\hat{\boldsymbol{n}}(x_i)| = 1$
- Canting field is transversal to the Néel field: $\boldsymbol{L}(x_i) \cdot \hat{\boldsymbol{n}}(x_i) = 0$

$\xrightarrow{*}$ Effective field theory: (here specified by its action functional)

$$\boxed{S_{\text{eff}}[\hat{\boldsymbol{n}}] = \underbrace{\frac{1}{2g}\int_{\text{Spacetime}} \mathrm{d}t\,\mathrm{d}x\,\left[\tfrac{1}{c}(\partial_t \hat{\boldsymbol{n}})^2 - c(\partial_x \hat{\boldsymbol{n}})^2\right]}_{\mathrm{O}(3)\ \uparrow\ \textit{Nonlinear Sigma Model}\ (\mathrm{N}\sigma\mathrm{ML})} + \overbrace{\Upsilon[\hat{\boldsymbol{n}}]}^{\substack{\text{Topological term}\\ \text{from Berry phase}}}} \tag{11.14}$$

- $g = \frac{2}{S}$: Coupling constant
- c: Spin wave velocity

Note: Semiclassical limit ($S \to \infty$) $\leftrightarrow$ Weak coupling limit ($g \to 0$)

But from the Berry phase we also get an additional …

Topological term:

$$\Upsilon[\hat{\boldsymbol{n}}] \stackrel{*}{=} 2\pi S\,\Theta[\hat{\boldsymbol{n}}] \tag{11.15a}$$

with ⁂ *Pontryagin index* (= winding number)

$$\Theta[\hat{\boldsymbol{n}}] := \frac{1}{4\pi}\int_{\text{Spacetime}} \mathrm{d}t\,\mathrm{d}x\,\left[\hat{\boldsymbol{n}} \cdot (\partial_t \hat{\boldsymbol{n}} \times \partial_x \hat{\boldsymbol{n}})\right] \quad \in \mathbb{Z} \tag{11.15b}$$

Due to the periodic boundaries of spacetime and the continuity of the unimodular field $\hat{\boldsymbol{n}}$, the winding number $\Theta[\hat{\boldsymbol{n}}]$ must be an integer.

- Mathematically, this is the same winding number introduced in Eq. (2.13) to compute the Chern number of two-band models. There, the domain was not spacetime but the 2D Brillouin zone and the unimodular field was given by the (normalized) Bloch vector. The geometrical interpretation from Section 2.1.1 still applies, though: $\Theta[\hat{\boldsymbol{n}}]$ counts the numbers of "skyrmions" of the Néel field in spacetime.
- Topological terms like (11.15) get "lost" when one transitions from the quantum theory to its classical limit in which the equations of motion derive from the variational principle $\delta S_{\text{eff}}[\hat{\boldsymbol{n}}] \stackrel{!}{=} 0$. This is so because topological terms are *constant* under infinitesimal

(smooth) variations of the fields: $\delta\Theta[\hat{\boldsymbol{n}}] = 0$ (this is what makes them topological after all). Adding topological terms to an action therefore does *not* affect the classical equations of motion. As the discussion below demonstrates, they *can* have significant impact on the *quantum* theory, though.

- Topological terms also play a role in fundamental physics, namely the Standard Model. Starting from a non-abelian gauge field A^μ, one easily arrives at the gauge- and Lorentz invariant Yang-Mills Lagrangian $\mathcal{L}_{\text{YM}} = -\frac{1}{2}\operatorname{Tr}[F_{\mu\nu}F^{\mu\nu}]$, where $F^{\mu\nu}$ is the field strength tensor and related to $F_{\mu\nu}$ via the metric of spacetime. (See my script on *Quantum Field Theory* [102, Chapter 9.2] for details.) However, there is another allowed term one can construct: $\mathcal{L}_\theta \propto \theta\operatorname{Tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}]$ where $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is the ↑ *dual field strength tensor*. This so called ↑ *θ-term* does not depend on the spacetime geometry (= metric) and can be written as a total derivative. (See my script on *Relativity* [69, Chapter 6.2] for details.) Hence it only contributes as a surface term in the action and does not affect the classical equations of motion. Just as our topological Θ-term above, this one has physical consequences on the quantum level, for instance in ↑ *quantum chromodynamics* where it leads to the ↑ *strong CP-problem*.

v | Known facts:

- The O(3) NLσM (without the topological Θ-term) has a (dynamically generated) *mass gap* and therefore exponentially decaying vacuum correlations.

  This gap is *dynamically generated* because there is no explicit mass term in the action. (This is different from the ↑ *Dirac mass terms* in the Standard Model that are responsible for fermion masses.) The occurrence of a mass gap is due to strong (non-perturbative) quantum fluctuations at small momenta where the coupling constant renormalizes to strong coupling (↑ *asymptotic freedom*) [263–266]. This mechanism is analogous to the (conjectured) mass gap of non-abelian ↑ *Yang-Mills gauge theories* like quantum chromodynamics (this is one of the ↑ *Millennium Prize Problems*).

- From the Bethe ansatz (and the original Lieb-Schultz-Mattis theorem), we known that the $S = \frac{1}{2}$ Heisenberg chain is *gapless*.

  Today (thanks to Affleck and Lieb) we know that this is true for *all* half-integer antiferromagnetic Heisenberg chains [257]; but these results came after Haldane's 1983 papers.

vi | How can this be consistent with our findings above?

→ Gaplessness for $S \in \{\frac{1}{2}, \frac{3}{2}, \dots\}$ must be due to the *topological Θ-term*!

Note that the mass gap of NLσMs has only been shown *without* the topological Θ-term.

→ Relevance of Θ-term depends on whether S is *integer* or *half-integer*:

$$S \in \{1, 2, \dots\} \quad\Rightarrow\quad e^{i\Upsilon[\hat{\boldsymbol{n}}]} = e^{i2\pi S\Theta[\hat{\boldsymbol{n}}]} = 1 \quad\Rightarrow\quad \text{irrelevant} \tag{11.16a}$$

$$S \in \{\tfrac{1}{2}, \tfrac{3}{2}, \dots\} \quad\Rightarrow\quad e^{i\Upsilon[\hat{\boldsymbol{n}}]} = e^{i2\pi S\Theta[\hat{\boldsymbol{n}}]} = (-1)^{\Theta[\hat{\boldsymbol{n}}]} \quad\Rightarrow\quad \text{relevant} \tag{11.16b}$$

Remember that the effective action (11.14) determines the amplitude $e^{iS_{\text{eff}}}$ in the path integral.

→ *Cancellations* (interference) of amplitudes due to the term $(-1)^{\Theta[\hat{\boldsymbol{n}}]}$ must be responsible for the vanishing of the NLσM mass gap …

5 | ⁂ *Haldane conjecture*:

These arguments motivated Haldane to conjecture that …

$$\boxed{\begin{aligned} S \in \{1,2,\dots\} \quad &\rightarrow \quad H_S \text{ for } J>0 \text{ is } \textit{gapped} \\ S \in \{\tfrac{1}{2},\tfrac{3}{2},\dots\} \quad &\rightarrow \quad H_S \text{ for } J>0 \text{ is } \textit{gapless} \end{aligned}} \qquad \begin{aligned}(11.17\text{a})\\(11.17\text{b})\end{aligned}$$

The (at the time unexpected) gap for integer spins S is called ⁂ *Haldane gap*

Recall that there is no such gap for *ferromagnetic* Heisenberg chains!

- You verify this conjecture numerically on ➲ Problemset 11 using DMRG.
- Haldane's prediction was confirmed numerically in 1983 [267], and even experimental evidence from excitation spectra of systems that are well-described by a spin-1 antiferromagnetic Heisenberg chain was reported not long after [268]. Today, the existence of the Haldane gap is easily demonstrated with off-the-shelf numerical methods like DMRG (➲ Problemset 11).
- While the evidence for Haldane's claim is incontrovertible, there is no rigorous mathematical proof of the full conjecture (as far as I am aware). At this point, only half of the conjecture stands on mathematically rigorous grounds (namely that half-integer antiferromagnetic spin chains are gapless, recall the proof by Affleck and Lieb [257]).
- Note that the presence of a gap for integer spin was well-known in the field-theoretic setting of nonlinear sigma models, whereas it was surprising in the context of spin chains (where one only knew about gapless phases). By contrast, the gaplesseness of antiferromagnetic half-integer spin chains was rigorously established, whereas it was surprising that (and how) a nonlinear sigma-model could become gapless. Haldane's groundbreaking contribution was to tie these two fields together.
- Only later it was shown explicitly that the topological Θ-term $(-1)^{\Theta[\hat{n}]}$ makes the O(3) nonlinear sigma model indeed gapless [269].

6 | Now that we identified a class of 1D spin models with *gapped* ground states, we can ask whether these are interesting phases from our point of view (namely: SPT phases). → Simplest choice:

∢ Spin-1 Heisenberg antiferromagnet $H_{\text{AFHM}} \equiv H_{S=1}$ $(J>0)$:

$$\boxed{⁂\ \textit{Haldane phase} \quad \equiv \quad \text{Gapped phase of } H_{\text{AFHM}}} \tag{11.18}$$

The ground state of this gapped phase …

- does not break any symmetry ("spin liquid state")
- and is short-range correlated [has no (quasi-)long range order].

→ How to characterize this state?

7 | *Problem:* The exact ground state of H_{AFHM} is unknown ☹

This is different from the *ferromagnetic* case (where the ground states are known).

Attack vectors:

- Use numerics like DMRG (➲ Problemset 11) …
- Find another Hamiltonian *in the same phase* with exactly known ground state … → *next*

11.2. The AKLT model

The Affleck-Kennedy-Lieb-Tasaki (AKLT) model can be connected to the antiferromagnetic spin-1 Heisenberg model without closing the gap or breaking any symmetry. Remarkably, its ground state can be derived exactly, so that this so called AKLT-state can be used to study the entanglement structure that characterizes the Haldane phase.

The AKLT model was introduced by Affleck, Kennedy, Tasaki, and Lieb in Ref. [270].

The following discussion is loosely based on the textbook by Tasaki [6] to which I refer the reader for more details.

8 | $\triangleleft$ Symmetric perturbation of H_{AFHM}:

$$\tilde{H}_\beta := \sum_{i=1}^{L'} \left[J(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) + \beta(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 \right] \tag{11.19}$$

$\rightarrow H_{\text{AFHM}} = \tilde{H}_0$

Note that the additional term does not violate any of the symmetries Eqs. (11.3), (11.4) and (11.6).

Expectation: $|\beta| \ll J \rightarrow \tilde{H}_\beta$ adiabatically connected to H_{AFHM}

Mathematically, this statement rests on the stability of the Haldane gap of the AFHM; I am not aware of a rigorous proof of this claim.

9 | Leap of faith: Gap does not close for $0 \xrightarrow{\beta} \frac{1}{3}J$

Numerics shows that this is true (➔ Problemset 11).

To the best of my knowledge, there is no mathematically rigorous proof of this assumption.

$\rightarrow$ (Henceforth we set $J = 1$)

$$\tilde{H}_{\text{AKLT}} \equiv \tilde{H}_{\frac{1}{3}} = \sum_{i=1}^{L'} \left[(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) + \tfrac{1}{3}(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 \right] \tag{11.20}$$

$\rightarrow \tilde{H}_{\text{AKLT}}$ and H_{AFHM} belong to the same phase (namely, the Haldane phase)

10 | Why $\beta = \frac{1}{3}$?

$\triangleleft$ Scaled & shifted Hamiltonian:

(This does not change the eigenvectors and only scales & shifts the spectrum.)

$$\boxed{\begin{aligned} H_{\text{AKLT}} &:= \tfrac{1}{2}\tilde{H}_{\text{AKLT}} + \tfrac{L'}{3}\mathbb{1} && (11.21\text{a}) \\ &= \sum_{i=1}^{L'} \left[\tfrac{1}{2}(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) + \tfrac{1}{6}(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 + \tfrac{1}{3} \right] && (11.21\text{b}) \\ &\stackrel{\circ}{=} \sum_{i=1}^{L'} P^{S=2}_{i,i+1} && (11.21\text{c}) \end{aligned}}$$

$P^{S=2}_{i,i+1}$: Projector onto $S = 2$ subspace of two adjacent spin-1: $1 \otimes 1 = 0 \oplus 1 \oplus 2$

¡! Adjacent projectors do not commute so that they cannot be diagonalized simultaneously.

This is straightforward to show: Note that the projector onto $S = 2$ can be written as

$$P_{i,i+1}^{S=2} \stackrel{\text{def}}{=} \tfrac{1}{4}(\boldsymbol{S}^2 - 2) \cdot \tfrac{1}{6}\boldsymbol{S}^2 \tag{11.22}$$

with total spin operator $\boldsymbol{S} = \boldsymbol{S}_i + \boldsymbol{S}_{i+1}$ acting on $1 \otimes 1 = 0 \oplus 1 \oplus 2$. Here we used $\boldsymbol{S}^2|S,m\rangle = S(S+1)|S,m\rangle$ for $S = 0, 1, 2$ to obtain the constants. With $\boldsymbol{S}_i^2 = 1(1+1) = 2$, we find

$$\begin{aligned} P_{i,i+1}^{S=2} &= \tfrac{1}{4}(\boldsymbol{S}_i^2 + \boldsymbol{S}_{i+1}^2 + 2\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} - 2) \cdot \tfrac{1}{6}(\boldsymbol{S}_i^2 + \boldsymbol{S}_{i+1}^2 + 2\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) && \text{(11.23a)} \\ &= \tfrac{1}{2}(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + 1) \cdot \tfrac{1}{3}(2 + \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}) && \text{(11.23b)} \\ &= \tfrac{1}{2}\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \tfrac{1}{6}(\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1})^2 + \tfrac{1}{3} && \text{(11.23c)} \end{aligned}$$

and we are done.

11 | Ground state:

The fact that H_{AKLT} is the sum of local projectors suggests that we might be able to construct the ground state exactly:

i | Goal: Construct state $|\Omega\rangle \in \mathcal{H}_{\text{AKLT}} = \bigotimes_{i=1}^{L} \mathbb{C}_i^3 \equiv 1 \otimes \ldots \otimes 1$ with

$$P_{i,i+1}^{S=2}|\Omega\rangle \stackrel{!}{=} 0 \qquad \text{for all } i = 1, \ldots, L' \tag{11.24}$$

If such a state exists, it is necessarily a ground state of H_{AKLT} (if not, we are stuck ...).

Note that

$$\langle\Psi|H_{\text{AKLT}}|\Psi\rangle = \sum_{i=1}^{L'} \underbrace{\langle\Psi|P_{i,i+1}^{S=2}|\Psi\rangle}_{\geq 0} \geq 0 \tag{11.25}$$

since projectors have only non-negative eigenvalues (= are positive-semidefinite operators). Hence H_{AKLT} is a positive-semidefinite operator as well and the smallest possible eigenvalue is zero. If we find an eigenstate with this eigenvalue, it must be the ground state. Note that the projectors do not have to commute for this argument, *i.e.*, it is not necessary that they can be simultaneously diagonalized. Note also that this argument does *not guarantee* the existence of a zero-energy ground state!

ii | $\sphericalangle$ Trick:

Let us consider an *extended* Hilbert space:

$\mathcal{H}_{\text{ext}} = \bigotimes_{i=1}^{L}[\mathbb{C}^2 \otimes \mathbb{C}^2]_i \equiv \bigotimes_{i=1}^{L}[\tfrac{1}{2} \otimes \tfrac{1}{2}]_i$

This space contains the Hilbert space of the AKLT model as a subspace:

$$\begin{aligned} \mathcal{H}_{\text{ext}} &= [\tfrac{1}{2} \otimes \tfrac{1}{2}] \otimes [\tfrac{1}{2} \otimes \tfrac{1}{2}] \otimes \ldots \otimes [\tfrac{1}{2} \otimes \tfrac{1}{2}] && \text{(11.26a)} \\ &= [0 \oplus 1] \otimes [0 \oplus 1] \otimes \ldots \otimes [0 \oplus 1] && \text{(11.26b)} \\ &= \underbrace{[1 \otimes 1 \otimes \ldots \otimes 1]}_{\mathcal{H}_{\text{AKLT}} \;☺} \oplus [\ldots] && \text{(11.26c)} \end{aligned}$$

→ $\sphericalangle$ Projector onto spin-1 subspace of two spin-$\tfrac{1}{2}$:

$$P^{S=1} := |+\rangle\langle\uparrow\uparrow| + |0\rangle\frac{\langle\uparrow\downarrow| + \langle\downarrow\uparrow|}{\sqrt{2}} + |-\rangle\langle\downarrow\downarrow| \tag{11.27}$$

→ Projector from $\mathcal{H}_{\text{ext}}$ onto $\mathcal{H}_{\text{AKLT}}$:

$$\mathcal{P}_{\text{AKLT}} := \bigotimes_{i=1}^{L} P_i^{S=1} \tag{11.28}$$

Idea: Construct "ansatz ground state" $|\Omega_{\text{ext}}\rangle$ in $\mathcal{H}_{\text{ext}}$ and project into $\mathcal{H}_{\text{AKLT}}$ afterwards …

iii | Eq. (11.24) →

We seek a state that has *No spin-2 components* between two adjacent sites i and $i+1$.

Observation:

$$\mathcal{H}_{\text{ext}} = \dots[\tfrac{1}{2}\otimes\tfrac{1}{2}]\otimes\overbrace{\underbrace{[\tfrac{1}{2}\otimes\tfrac{1}{2}]\otimes[\tfrac{1}{2}\otimes\tfrac{1}{2}]}_{2\times 0\,\oplus\,3\times 1\,\oplus\,2}}^{P^{S=2}_{i,i+1}|\Omega\rangle\overset{!}{=}0}\otimes[\tfrac{1}{2}\otimes\tfrac{1}{2}]\dots \tag{11.29a}$$

$$= \dots[\tfrac{1}{2}\otimes\tfrac{1}{2}]\otimes\underbrace{[\tfrac{1}{2}\otimes\overbrace{(\tfrac{1}{2}]\otimes[\tfrac{1}{2})}^{0\oplus\cancel{1}}\otimes\tfrac{1}{2}]}_{\cancel{2\times}0\,\oplus\,\cancel{3\times}1\,\oplus\,\cancel{2}}\otimes[\tfrac{1}{2}\otimes\tfrac{1}{2}]\dots \tag{11.29b}$$

→ Start with a product of *Singlets* between adjacent spin-$\frac{1}{2}$ on neighboring sites:

$$|\Omega\rangle_{\text{ext}} := \dots\otimes\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)\otimes\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle)\otimes\dots \tag{11.30}$$

¡! Here we consider *periodic boundaries*. For open boundaries, there is one disentangled spin-$\frac{1}{2}$ left on each boundary, so that one can construct *four* orthogonal ground states.

Note that the entangled spin-$\frac{1}{2}$ come from *different* sites:

$$\mathcal{H}_{\text{ext}} = \dots\underbrace{[\tfrac{1}{2}\otimes\tfrac{1}{2}]}_{\text{Site } i-1}\otimes\underbrace{[\tfrac{1}{2}\otimes\tfrac{1}{2}]}_{\text{Site } i}\otimes\underbrace{[\tfrac{1}{2}\otimes\tfrac{1}{2}]}_{\text{Site } i+1}\dots \tag{11.31}$$

→ $P^{S=2}_{i,i+1}|\Omega_{\text{ext}}\rangle = 0$ by construction

iv | Finally: Projection →

$$\boxed{|\Omega\rangle := \frac{1}{\mathcal{N}}\mathcal{P}_{\text{AKLT}}|\Omega_{\text{ext}}\rangle \quad \in \mathcal{H}_{\text{AKLT}}} \tag{11.32}$$

$\mathcal{N}$: Normalization

$|\Omega\rangle$ is know as ⁂ *Valence-Bond Solid (VBS)* state

Spin singlets are called *valence bonds* due to the role they play in chemistry. The term "solid" is motivated by other models where translation and/or rotation symmetries are broken by the pattern of valence bonds. For the AKLT ground state this is not the case, and a better name would be "valence bond spin liquid."

$|\Omega\rangle$ ground state of Eq. (11.21)?

One must prove two things:

- $|\Omega\rangle \neq 0\,?\ \overset{\circ}{\rightarrow} ✓$

  Note that we constructed $|\Omega\rangle$ by applying a projector to a given state. In principle, this can yield the zero vector and therefore not a normalizable quantum state.

  First, note that $\mathcal{P}_{\text{AKLT}}$ is the product of *commuting* projectors $P_i^{S=1}$ [Eq. (11.28)]. This implies

$$\mathcal{P}_{\text{AKLT}}|\Omega_{\text{ext}}\rangle = 0 \quad \Leftrightarrow \quad \exists i \in \{1,\dots,L\}: \ P_i^{S=1}|\Omega_{\text{ext}}\rangle = 0\,. \tag{11.33}$$

  Due to translation invariance, it suffices to check that $P_i^{S=1}|\Omega_{\text{ext}}\rangle \neq 0$ for one site. The unprojected state reads in the vicinity of site i

$$|\Omega_{\text{ext}}\rangle = \dots \otimes \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \otimes \tfrac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) \otimes \dots \tag{11.34a}$$

$$= \dots \otimes \tfrac{1}{2}(|\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle) \otimes \dots \tag{11.34b}$$

  so that it is sufficient to show that

$$P_i^{S=1}\tfrac{1}{2}\,(|\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle) \tag{11.35a}$$

$$\overset{11.27}{=} \tfrac{1}{2}\left(\tfrac{1}{\sqrt{2}}|\uparrow\mathbf{0}\downarrow\rangle - |\downarrow\mathbf{+}\downarrow\rangle - |\uparrow\mathbf{-}\uparrow\rangle + \tfrac{1}{\sqrt{2}}|\downarrow\mathbf{0}\uparrow\rangle\right) \neq 0\,. \tag{11.35b}$$

- $P_{i,i+1}^{S=2}|\Omega\rangle = 0\,?\ \overset{\circ}{\rightarrow} ✓$

  Our singlet construction was heuristically motivated. We must show that our construction survives the projection into $\mathcal{H}_{\text{AKLT}}$.

  To show this, we must make the spin representations explicit. Let $\boldsymbol{S}$ denote the spin-1 matrices acting in $\mathcal{H}_{\text{AKLT}}$ on every site,

$$S^x = \tfrac{1}{\sqrt{2}}\,(|0\rangle\langle+| + |+\rangle\langle 0| + |-\rangle\langle 0| + |0\rangle\langle -|) \tag{11.36a}$$

$$S^y = \tfrac{i}{\sqrt{2}}\,(|0\rangle\langle+| - |+\rangle\langle 0| + |-\rangle\langle 0| - |0\rangle\langle -|) \tag{11.36b}$$

$$S^z = |+\rangle\langle+| - |-\rangle\langle-|\,, \tag{11.36c}$$

  and let $\boldsymbol{S}_L$ and $\boldsymbol{S}_R$ denote the two spin-$\frac{1}{2}$ matrices acting on the two spins that make up one site in the extended Hilbert space $\mathcal{H}_{\text{ext}}$,

$$S_X^x = \tfrac{1}{2}\,(|\downarrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow|) \tag{11.37a}$$

$$S_X^y = \tfrac{i}{2}\,(|\downarrow\rangle\langle\uparrow| - |\uparrow\rangle\langle\downarrow|) \tag{11.37b}$$

$$S_X^z = \tfrac{1}{2}\,(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)\,, \tag{11.37c}$$

  where $X = L, R$. It is now an easy task to show that

$$\boldsymbol{S}\,P^{S=1} \underset{11.27}{\overset{\circ}{=}} P^{S=1}\,(\boldsymbol{S}_L \otimes \mathbb{1} + \mathbb{1} \otimes \boldsymbol{S}_R)\,. \tag{11.38}$$

  [Show this by applying the left- and right-hand side to the basis $\{|\uparrow\uparrow\rangle,\dots\}$; it suffices to show this for one component (say S^z) due to SU(2) invariance.]

  With this it follows immediately for two adjacent sites

$$(\boldsymbol{S}_i + \boldsymbol{S}_{i+1})\,\mathcal{P}_{\text{AKLT}} = \mathcal{P}_{\text{AKLT}}\,(\boldsymbol{S}_{i,L} + \boldsymbol{S}_{i,R} + \boldsymbol{S}_{i+1,L} + \boldsymbol{S}_{i+1,R})\,, \tag{11.39}$$

  where we now label the tensor factor on which an operator acts by site indices. This implies

$$P_{i,i+1}^{S=2}\,\mathcal{P}_{\text{AKLT}} = \mathcal{P}_{\text{AKLT}}\,\tilde{P}_{i,i+1}^{S=2} \tag{11.40}$$

where $\tilde{P}^{S=2}_{i,i+1}$ is given by Eq. (11.22) where the total spin of two spin-1 ($\boldsymbol{S} = \boldsymbol{S}_i + \boldsymbol{S}_{i+1}$) has been replaced by the total spin of four spin-$\frac{1}{2}$: $\boldsymbol{S} = \boldsymbol{S}_{i,L} + \boldsymbol{S}_{i,R} + \boldsymbol{S}_{i+1,L} + \boldsymbol{S}_{i+1,R}$. Hence we find

$$P^{S=2}_{i,i+1}|\Omega\rangle \propto \mathcal{P}_{\text{AKLT}}\, \underbrace{\tilde{P}^{S=2}_{i,i+1}|\Omega_{\text{ext}}\rangle}_{=0} = 0\,. \tag{11.41}$$

The projector vanishes because $|\Omega_{\text{ext}}\rangle$ was constructed such that it belongs to the subspace $\frac{1}{2} \otimes 0 \otimes \frac{1}{2} = 0 \oplus 1$ on sites i and $i+1$, *i.e.*, there is no spin-2 component on adjacent sites (this was the original motivation for the singlet pairs, after all).

12 | MPS Representation of AKLT ground state:

$$|\Omega\rangle \equiv \frac{1}{\mathcal{N}} \sum_{\boldsymbol{\sigma}} \mathrm{Tr}\left[M^{\sigma_1} M^{\sigma_2} \cdots M^{\sigma_L}\right] |\sigma_1, \sigma_2, \dots, \sigma_L\rangle \tag{11.42}$$

with $\sigma_i \in \{-, 0, +\}$

$\overset{*}{\rightarrow}$

$$M^- = -\tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad M^0 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M^+ = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{11.43}$$

→ AKLT ground state is Exact MPS with bond dimension $D = 2$

- You derive these matrices on ➔ Problemset 10.
- For a derivation of Eq. (11.43) see [6, Section 7.2.2]. Note that I use here the convention from ➔ Problemset 10 which relates via $|+\rangle \leftrightarrow |-\rangle$ to Tasaki's convention. Remember that the ground state is SO(3)-symmetric which allows for this transformation.

→ Normalization:

$$\langle\Omega|\Omega\rangle \overset{!}{=} 1 \quad \Leftrightarrow \quad \mathcal{N}^2 \overset{*}{=} \left(\tfrac{3}{4}\right)^L + 3\left(-\tfrac{1}{4}\right)^L \tag{11.44}$$

Asymptotically for $L \to \infty$ we can drop the second summand and the normalization constant is $\mathcal{N} = (\sqrt{3}/2)^L$. Distributing the L factors onto the L matrices yields the *asymptotically normalized* MPS matrices

$$\tilde{M}^- = -\sqrt{\tfrac{2}{3}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{M}^0 = \sqrt{\tfrac{1}{3}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{M}^+ = \sqrt{\tfrac{2}{3}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{11.45}$$

which you use on ➔ Problemset 10.

Eq. (11.45) $\xrightarrow{\text{Eq. (9.39)}}$ Spin-spin-correlations:

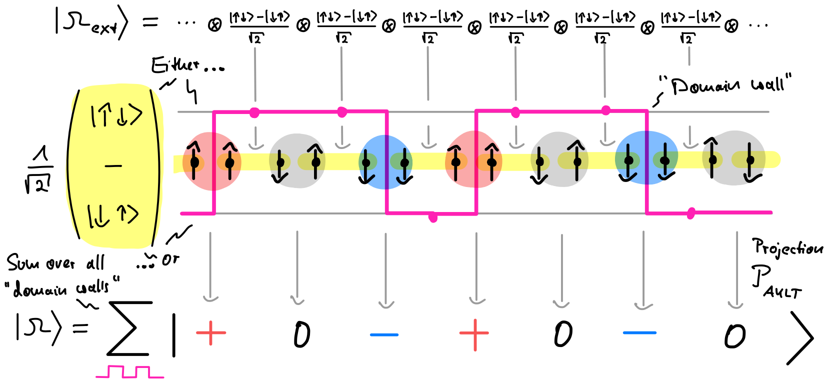
$$\boxed{\begin{aligned} &\lim_{L\to\infty} \langle\Omega|\boldsymbol{S}_i \cdot \boldsymbol{S}_j|\Omega\rangle \overset{*}{=} 4(-1)^{|i-j|} e^{-\frac{|i-j|}{\xi}} \\ &\text{with correlation length} \quad \xi = \frac{1}{\ln 3} \end{aligned}} \tag{11.46}$$

- The alternating term $(-1)^{|i-j|}$ shows that the spin-spin correlations are *antiferromagnetic*, as expected for a Hamiltonian in the AFHM (Haldane) phase. Note, however, that there is *no* antiferromagnetic long-range order due to the $e^{-\frac{|i-j|}{\xi}}$ term, consistent with Haldane's prediction of a (disordered) spin liquid ground state.

- You derive the correlation length on ➔ Problemset 10 via the transfer matrix derived from the MPS representation (11.45).
- A step-by-step derivation is also given in [6, Section 7.2.2].

13 | Hidden antiferromagnetic order:

$\sphericalangle$ $|\Omega_{\text{ext}}\rangle$ and expand in superposition of product basis states:



Expanding all singlets yields 2^L spin-$\frac{1}{2}$ configurations. Each configuration is determined by a binary choice on each *bond* (pink bullets). These choices determine a pattern of "domain walls" between adjacent choices. After projecting into the spin-1 subspace via $\mathcal{P}_{\text{AKLT}}$, sites without domain wall are projected onto $|0\rangle$, whereas sites with domain wall to either $|+\rangle$ or $|-\rangle$, depending on whether the domain wall goes "up" or "down." The AKLT ground state $|\Omega\rangle$ is then the superposition of all possible domain wall configurations. Since after each "up" wall there must be a "down" wall (while the length of the domain is arbitrary), this yields a (hidden) antiferromagnetic pattern in $|+\rangle$ and $|-\rangle$ states. Note that this immediately shows that for *periodic boundaries* the total spin must vanish since for every "up" wall ($|+\rangle$) there is a matching "down" wall ($|-\rangle$).

→ Superposition of all antiferromagnetic spin-1 patterns immersed in a "sea" of $|0\rangle$.

→ No antiferromagnetic long-range order (due to randomly distributed $|0\rangle$)

→ ⁂ *Hidden antiferromagnetic order*

- For a detailed discussion see [6, Section 7.2.1].
- Hidden order of this type can be detected by so called ↑ *string order parameters*, non-local modifications of spin-spin correlation functions. You evaluate the string order parameter that detects the hidden antiferromagnetic order of the AKLT state analytically on ➔ Problemset 10.

  String order parameters were originally introduced in Ref. [271]. More details can be found in [6, Section 7.2.1].

14 | Cohomology classification:

i | $\sphericalangle$ Spin-1 representation of D_2: (π-rotations about the x, y, z-axes)

$$\pi_i(x) := e^{i\pi S_i^x}\,, \quad \pi_i(y) := e^{i\pi S_i^y}\,, \quad \pi_i(z) := e^{i\pi S_i^z}\,, \tag{11.47}$$

with $y \equiv xz = zx \in D_2$ and S_i^α spin-1 operators (on site $i = 1, \dots, L$)

It is a straightforward exercise to show that these operators satisfy the multiplication table Eq. (8.12) of D_2 without phase factors,

$$\pi_i(\alpha)^2 \stackrel{\circ}{=} \mathbb{1}\,, \quad [\pi_i(\alpha), \pi_i(\beta)] \stackrel{\circ}{=} 0 \quad \text{and} \quad \pi_i(x)\pi_i(z) \stackrel{\circ}{=} \pi_i(y) \tag{11.48}$$

$(\alpha, \beta \in \{x, y, z\})$; *i.e.*, they form a *linear* representation of D_2.

Eq. (11.4) $\rightarrow U_\alpha \equiv \rho(\alpha) := \bigotimes_{i=1}^{L} \pi_i(\alpha)$ is *unitary symmetry* of H_{AKLT}

ii | Injectivity:

In contrast to the bosonic SSH chain [*cf.* Eq. (9.24)], the *three* AKLT matrices (11.43) cannot span the *four*-dimensional space of 2×2-matrices. However, they become ← *injective* after combining (blocking) two adjacent sites.

This is easy to check:

$$M^0 M^+ = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad M^0 M^- = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{11.49a}$$

$$M^+ M^- = -\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M^- M^+ = -\frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{11.49b}$$

$\overset{*}{\rightarrow}$ Eqs. (10.15) and (10.17) remain valid ☺

A proof of this statement is given in [6, Section 8.3.4].

iii | $\overset{\circ}{\rightarrow}$ Projective representation on bonds:

$$V(x) = \sigma^x, \quad V(z) = \sigma^z \quad \text{and} \quad \begin{aligned} V(xz) &= V(y) = -i\sigma^y \\ &= \sigma^x \sigma^z = V(x)V(z) \end{aligned} \tag{11.50}$$

→ In particular:

$$\begin{aligned} V(z)V(x) &= -V(x)V(z) \\ &= -V(xz) \\ &= -V(zx) \quad \Rightarrow \quad \chi_{\text{AKLT}}(z, x) = -1 \end{aligned} \tag{11.51}$$

Eq. (10.52) $\rightarrow [1] \neq [\chi_{\text{AKLT}}] \in H^2(D_2, \text{U}(1)) = \mathbb{Z}_2 \quad \rightarrow$

$$\boxed{H_{\text{AKLT}} \text{ is in an } \underline{\text{SPT phase}} \text{ protected by } D_2 \text{ representation (11.47)}} \tag{11.52}$$

- Since the antiferromagnetic Heisenberg Hamiltonian H_{AFHM} is adiabatically connected to H_{AKLT}, this shows that the former is in the *same* SPT phase. In conclusion, we found that the *Haldane phase* is an SPT phase protected by the D_2 symmetry realized by π-rotations in spin-1 space.
- Comparing this result with our findings in Section 10.3 for the bosonic SSH chain (introduced in Section 8.1) shows that Phase A of $\hat{H}_{\text{bSSH}}$ and the Haldane phase are actually *the same SPT phase*.

  This follows since they are characterized by the same non-trivial cohomology class in $H^2(D_2, \text{U}(1))$, and we claimed in Eq. (10.42) that this is sufficient to find a gapped path between two Hamiltonians. This path can be explicitly constructed, see [127, Section 5.1.4] and the supplement of Ref. [135].
- Consistent with our findings in Section 10.5 for the bosonic SSH chain, the Haldane phase can alternatively be protected by ← *time-reversal symmetry* $\mathbb{Z}_2^T$ with $\mathcal{T} \boldsymbol{S}_i \mathcal{T}^{-1} = -\boldsymbol{S}_i$. Furthermore, one can show that ↑ *bond-centered inversion symmetry* is also sufficient to protect the Haldane phase [214, 242], although this type of "non-local" symmetry is not covered by our formalism (it can be generalized to such symmetries as well, see [6, Section 8.3.2]).

15 | Comments:

- Above we assumed that by switching on β, the Haldane gap of the AFHM does not close. As already mentioned, this is easily numerically verified but has so far not been proven. However, it can be rigorously shown that the AKLT Hamiltonian (11.21) has a gap in the thermodynamic limit. This gap can even be lower bounded by $\Delta \geq 0.248$ for $L > 10$; see [6, Section 7.1.4] for a proof and Ref. [272] for the original results.
- Above we showed rigorously that Eq. (11.32) is a ground state of the AKLT Hamiltonian (11.21) (for periodic boundaries!). We did not show that this ground state is *unique*, though. This indeed is the case because one can show that the singlet structure we started from is the only way to construct a state which satisfies Eq. (11.24) on every site, see [6, Section 7.1.3] for details.
- As for the bosonic SSH chain, the ground state manifold of the AKLT model is one dimensional for periodic boundaries but *four-fold degenerate* for *open boundaries* (← Section 10.4). To show this, one starts again with a dimer state of singlets like $|\Omega_{\text{ext}}\rangle$, but now with one free spin-$\frac{1}{2}$ on each boundary; applying $\mathcal{P}_{\text{AKLT}}$ then yields exact ground states of H_{AKLT}. The two boundary spin-$\frac{1}{2}$ allow for the construction of *four* linearly independent, exact ground states; see [6, Section 7.2.3] for the explicit construction. You show the existence of four ground states numerically on ➔ Problemset 11.

  [Note that the four-fold degeneracy is *exact* for chains of finite length – despite their non-zero correlation length. This is an atypical feature of the fine-tuned AKLT Hamiltonian.]
- Although we were able to construct the *ground state(s)* of the AKLT Hamiltonian (11.21), there is no known method to diagonalize it exactly, *i.e.*, to access all excited states (and the spectrum) analytically. In this sense, the AKLT model is *not* "exactly solvable." Note that *some* exact excited states are known, see *e.g.* Ref. [273].
- In many cases, ground states with a closed analytical form are also renormalization fixpoints, *i.e.*, states with zero correlation length. This was certainly the case for the states $|A\rangle$ and $|B\rangle$ of the bosonic SSH chain discussed in Chapters 8 to 10. By contrast, Eq. (11.46) shows that the AKLT ground state has a built in length scale (the correlation length $0 < \xi < \infty$) and therefore is not a fixpoint wave function of the Haldane phase (at least not in the usual "Wilsonian" sense). Under ↑ *entanglement renormalization*, the AKLT ground state $|\Omega\rangle$ flows towards the ground state $|A\rangle$ [Eq. (8.29)] of the ← *bosonic SSH chain* (a dimer state of maximally entangled spin-$\frac{1}{2}$) [245].

16 | Fun fact:

Here is a picture of Haldane's Nobel Diploma [274]:

I hope you can see now where the artist's (Ingela Berntsson) inspiration came from.

↓ **Lecture 26** [??.??.??]