↓ **Lecture 24** [18.07.25]

10.4. Symmetry fractionalization and edge modes

Here we consider one dimensional systems with *open boundary conditions*. We show how the projective representations on virtual indices, identified in Section 10.2, necessitate physical degrees of freedom on the edges. These *edge modes* transform under projective representations of the symmetry and lead to robust ground state degeneracies. We exemplify this phenomenon with the bosonic SSH chain.

18 | ◁ Bosonic SSH chain with open boundary conditions (OBC):

$$\hat{H}^{\text{OBC}}_{\text{bSSH}} \stackrel{8.6}{=} t \sum_{n=1}^{L} (\sigma^-_{2n-1}\sigma^+_{2n} + \sigma^-_{2n}\sigma^+_{2n-1}) + w \sum_{n=1}^{L-1} (\sigma^-_{2n}\sigma^+_{2n+1} + \sigma^-_{2n+1}\sigma^+_{2n}) \tag{10.59}$$

$\xrightarrow{\circ}$ Fixpoint ground states as matrix product states with open boundaries: [← Eq. (9.10)]

- Trivial Phase B ($t < 0$ and $w = 0$):

  The OBC do not cut entangled pairs, so that the ground state is the same as for PBC:

$$|B\rangle \stackrel{9.25}{=} \bigotimes_{n=1}^{L} (|0\rangle_{2n-1}|1\rangle_{2n} + |1\rangle_{2n-1}|0\rangle_{2n}) \tag{10.60a}$$

$$\stackrel{!}{=} \sum_{i} B_{i_1 i_2} B_{i_3 i_4} \dots B_{i_{2L-1} i_{2L}} \, |i_1 i_2, \dots\rangle \tag{10.60b}$$

  → Same (site-independent) matrices:

$$B^{ii'} \equiv B_{ii'} \stackrel{9.26}{=} \sigma^x_{ii'} \tag{10.61}$$

  Note that these are numbers (1×1-matrices), so that this matrix product state can be interpreted both in OBC form [Eq. (9.10)] and in PBC form [Eq. (9.29)].

  Here the bond dimension is $D = 1$ so that we can omit the virtual indices k and k'.

- Topological Phase A ($t = 0$ and $w < 0$):

  One coupling term less due to OBC
  → One triplet pair less
  → One dangling spin-$\frac{1}{2}$ on each boundary:

$$|A(\alpha,\beta)\rangle = |\alpha\rangle_0 \otimes \bigotimes_{n=1}^{L-1} (|0\rangle_{2n}|1\rangle_{2n+1} + |1\rangle_{2n}|0\rangle_{2n+1}) \otimes |\beta\rangle_{2L} \tag{10.62a}$$

$$= \sum_{i} \underbrace{\boldsymbol{L}^{\alpha}_{i_1 i_2}}_{1\times 2} \cdot \underbrace{A_{i_3 i_4} \cdots A_{i_{2L-3} i_{2L-2}}}_{2\times 2} \cdot \underbrace{\boldsymbol{R}^{\beta}_{i_{2L-1} i_{2L}}}_{2\times 1} \, |i_1 i_2, \dots\rangle \tag{10.62b}$$

  → 4-fold degenerate ground state: $\alpha, \beta \in \{\uparrow, \downarrow\}$

  - Same 2×2 matrices In the bulk:

$$\left(A^{ii'}\right)_{kk'} \equiv (A_{ii'})_{kk'} \stackrel{9.23}{=} \delta_{ik}\sigma^x_{i'k'} \qquad k, k' \in \{1, 2 = D\} \tag{10.63}$$

    For explicit matrices see ← Eq. (9.24).

– New 1×2 and 2×1 matrices On the boundaries:

$$\left(\boldsymbol{L}^{\alpha}_{ii'}\right)_{k'} = \delta_{i\alpha}\sigma^{x}_{i'k'} \quad \text{and} \quad \left(\boldsymbol{R}^{\beta}_{ii'}\right)_{k} = \delta_{ik}\delta_{i'\beta} \tag{10.64}$$

Since these are row and column vectors, there is only *one* virtual index k and k' with bond dimension $D = 2$ for each "matrix."

Explicitly, the left-boundary vectors read ($\uparrow = 0$ and $\downarrow = 1$):

$$\boldsymbol{L}^{\uparrow}_{00} = \begin{bmatrix}0 & 1\end{bmatrix} \quad \boldsymbol{L}^{\uparrow}_{01} = \begin{bmatrix}1 & 0\end{bmatrix} \quad \boldsymbol{L}^{\uparrow}_{10} = \begin{bmatrix}0 & 0\end{bmatrix} \quad \boldsymbol{L}^{\uparrow}_{11} = \begin{bmatrix}0 & 0\end{bmatrix} \tag{10.65a}$$

$$\boldsymbol{L}^{\downarrow}_{00} = \begin{bmatrix}0 & 0\end{bmatrix} \quad \boldsymbol{L}^{\downarrow}_{01} = \begin{bmatrix}0 & 0\end{bmatrix} \quad \boldsymbol{L}^{\downarrow}_{10} = \begin{bmatrix}0 & 1\end{bmatrix} \quad \boldsymbol{L}^{\downarrow}_{11} = \begin{bmatrix}1 & 0\end{bmatrix} \tag{10.65b}$$

Similarly, the right-boundary vectors are:

$$\boldsymbol{R}^{\uparrow}_{00} = \begin{bmatrix}1 \\ 0\end{bmatrix} \quad \boldsymbol{R}^{\uparrow}_{01} = \begin{bmatrix}0 \\ 0\end{bmatrix} \quad \boldsymbol{R}^{\uparrow}_{10} = \begin{bmatrix}0 \\ 1\end{bmatrix} \quad \boldsymbol{R}^{\uparrow}_{11} = \begin{bmatrix}0 \\ 0\end{bmatrix} \tag{10.66a}$$

$$\boldsymbol{R}^{\downarrow}_{00} = \begin{bmatrix}0 \\ 0\end{bmatrix} \quad \boldsymbol{R}^{\downarrow}_{01} = \begin{bmatrix}1 \\ 0\end{bmatrix} \quad \boldsymbol{R}^{\downarrow}_{10} = \begin{bmatrix}0 \\ 0\end{bmatrix} \quad \boldsymbol{R}^{\downarrow}_{11} = \begin{bmatrix}0 \\ 1\end{bmatrix} \tag{10.66b}$$

19 | Action of symmetries:

¡! Here we change several symbols for indices to match our current nomenclature.

Let us again study how the symmetry operators act these matrices:

i | ⊲ Trivial Phase B:

→ Same MPS matrices
→ Same transformation

$$\sum_{i,j} \sigma^{x}_{i'i}\sigma^{x}_{j'j}\, B^{ij} \stackrel{10.49a}{=} \gamma_B(x)\,[1 \cdot B^{i'j'} \cdot 1] \tag{10.67a}$$

$$\sum_{i,j} \sigma^{z}_{i'i}\sigma^{z}_{j'j}\, B^{ij} \stackrel{10.49b}{=} \gamma_B(z)\,[1 \cdot B^{i'j'} \cdot 1] \tag{10.67b}$$

In particular, the state is still invariant under the global symmetry operators $\rho(g)$ for $g \in D_2$, and there is no *ground state degeneracy*.

Mathematically speaking, the ground state space forms a *one-dimensional* representation $\gamma^{L}_{B}(g)$ of the symmetry group D_2.

ii | ⊲ Topological Phase A:

→ Same MPS matrices *in the bulk*
→ Same transformation on the bulk

$$\sum_{i,j} \sigma^{x}_{i'i}\sigma^{x}_{j'j}\, A^{ij}_{kk'} \stackrel{10.9}{=} \gamma_A(x)\left[\hat{\sigma}^{x} \cdot A^{i'j'} \cdot \hat{\sigma}^{x}\right]_{kk'} \tag{10.68a}$$

$$\sum_{i,j} \sigma^{z}_{i'i}\sigma^{z}_{j'j}\, A^{ij}_{kk'} \stackrel{10.11}{=} \gamma_A(z)\left[\hat{\sigma}^{z} \cdot A^{i'j'} \cdot \hat{\sigma}^{z}\right]_{kk'} \tag{10.68b}$$

iii | Transformation on the boundary?

The matrices (10.64) on the boundaries are new, so we have to evaluate the action of the symmetry on them explicitly:

$\triangleleft$ Left boundary & Action of $\rho(x) = X$:

$$\sum_{i,j} \sigma^x_{i'i}\sigma^x_{j'j} \left(L^\alpha_{ij}\right)_{k'} = \sum_i \sigma^x_{i'i}\delta_{i\alpha} \sum_j \sigma^x_{j'j}\sigma^x_{jk'} \tag{10.69a}$$

$$\stackrel{\circ}{=} \sum_i \sigma^x_{i'i}\delta_{i\alpha} \sum_k \sigma^x_{kk'}\sigma^x_{j'k} \tag{10.69b}$$

$$= \sum_i \sigma^x_{i'i} \left[\sum_k \left(\delta_{i\alpha}\sigma^x_{j'k}\right)\sigma^x_{kk'}\right] \tag{10.69c}$$

$$\stackrel{10.64}{\equiv} \underbrace{\gamma_L(x)}_{1} \sum_i \underbrace{[U_L(x)]}_{\sigma^x}{}_{i'i} \left[1 \cdot L^\alpha_{ij'} \cdot \hat\sigma^x\right]_{k'} \tag{10.69d}$$

Here we used the definition Eq. (10.64) in the last line and the same reordering as in Eq. (10.9) in the second line (for the second summand).

¡! Note that the left sum $\sum_i$ cannot be rewritten as a sum $\sum_k$ over virtual bond indices because $i = 0, 1$ whereas the left bond dimension is $D = 1$, i.e., $k = 0$ (which is why we omit this index completely), *cf.* Eq. (10.9). Therefore the unitary action $U_L(x) = \sigma^x$ has to act on the *physical index* and cannot descent to a "gauge transformation" $V_A^{-1}(x)$ on the virtual bonds of the row vector $L^\alpha_{ij'}$.

- $\triangleleft$ Left boundary & Action of $\rho(z) = Z \rightarrow$ Analogous result

  $\stackrel{\circ}{\rightarrow}$ Transformation valid for all $g \in D_2$:

$$\sum_{i,j} [\rho_1(g)]_{i'i;j'j} \left(L^\alpha_{ij}\right)_{k'} = \gamma_L(g) \sum_i [U_L(g)]_{i'i} \left[L^\alpha_{ij'} \cdot V_A(g)\right]_{k'} \tag{10.70}$$

  Here we omit the trivial multiplication $1 \cdot \ldots$ by a 1×1-identity matrix on the left of the row vector.

  with …

$$\text{“Half” physical symmetry:} \quad \begin{cases} U_L(x) = \sigma^x \\ U_L(z) = \sigma^z \end{cases} \tag{10.71a}$$

$$\text{Virtual (gauge) transformation [Eq. (10.45)]:} \quad \begin{cases} V_A(x) = \hat\sigma^x \\ V_A(z) = \hat\sigma^z \end{cases} \tag{10.71b}$$

- $\triangleleft$ Right boundary & Eq. (10.64) $\stackrel{\circ}{\rightarrow}$ Transformation valid for all $g \in D_2$:

$$\sum_{i,j} [\rho_L(g)]_{i'i;j'j} \left(R^\beta_{ij}\right)_k = \gamma_R(g) \sum_j [U_R(g)]_{j'j} \left[V_A^\dagger(g) \cdot R^\beta_{i'j}\right]_k \tag{10.72}$$

  Note that here the gauge transformation on the *right* virtual bond is necessarily trivial and the physical "half" symmetry acts on the rightmost spin (index by j and j').

  with …

$$\text{“Half” physical symmetry:} \quad \begin{cases} U_R(x) = \sigma^x \\ U_R(z) = \sigma^z \end{cases} \tag{10.73a}$$

$$\text{Virtual (gauge) transformation:} \quad \begin{cases} V_A^\dagger(x) = \hat\sigma^x \\ V_A^\dagger(z) = \hat\sigma^z \end{cases} \tag{10.73b}$$

iv | Observation:

$$U_L(x)U_L(z) = -U_L(z)U_L(x) \quad \text{and} \quad U_R(x)U_R(z) = -U_R(z)U_R(x) \tag{10.74}$$

→ Projective representations as *physical* symmetry operators on boundaries!

¡! Note that the site-local physical symmetries ρ_k are *not* projective but linear.

That this must happen can also be seen by very general arguments: Consider either the left boundary with transformation (10.70) or the right boundary with Eq. (10.72). Now apply two consecutive symmetry transformations $g = g_1 \bullet g_2 \in D_2$. Then the left-hand side transforms according to the *linear* representation $\rho_k(g)$ [Eq. (8.13)] whereas the *single* bond representation on the right-hand side transforms *projectively* according to Eq. (10.45). But the latter *violates* the multiplication law of linear representations by phases (here by signs) given by the cocycle $\chi_A(g_1, g_2)$ on the left and $\chi_A^*(g_1, g_2)$ on the right boundary [Eq. (10.46)].

The relations (10.70) and (10.72) can therefore be only valid if these additional phases are canceled by conjugate phases that arise from the multiplication law of the *physical* representations $U_{L/R}$ on the respective boundary:

$$U_L(g_1)U_L(g_2) = \chi_A^*(g_1, g_2)\, U_L(g_1 g_2)\,, \tag{10.75a}$$

$$U_R(g_1)U_R(g_2) = \chi_A(g_1, g_2)\, U_R(g_1 g_2)\,. \tag{10.75b}$$

That is, the boundary spin must also transform *projectively* to ensure that the the combined action of boundary symmetry and virtual "gauge" symmetry transforms *linearly*:

$$\begin{aligned}[U_L(g_1) \otimes V_A(g_1)]\,[U_L(g_2) \otimes V_A(g_2)] &= \chi_A^*(g_1, g_2)\chi_A(g_1, g_2)\, U_L(g_1 g_2)V_A(g_1 g_2) \\ &= U_L(g_1 g_2)V_A(g_1 g_2)\,,\end{aligned} \tag{10.76}$$

in accordance with the the on-site symmetry on the left-hand side of Eq. (10.70).

v | Conclusion:

- For *periodic* boundaries, the emergence of projective representations is "hidden," as they affect only the *virtual* bonds of the MPS (where they obstruct the deformation into trivial product state).
- By contrast, for *open* boundaries, the projective class of the topological phase is "revealed" by the – now projective – action of the *physical* symmetry on the boundaries.

  This is the mathematical reason for the emergence of degenerate edge states at the boundaries (→ *below*).

20 | Symmetry fractionalization:

Upshot of our findings for topological Phase A of the bosonic SSH chain:

- Eqs. (10.70) and (10.72): Projective symmetry representation on the left/right boundaries
- Eq. (10.10): Invariant bulk (= trivial representation)

→ Generally true for 1D SPT phases:

⊲ Ground state $|G[\chi]\rangle$ of 1D OBC SPT with $[\chi] \in H^2(G, \mathrm{U}(1))$

$\overset{\circ}{\to}$ Action of global symmetry:

$$\boxed{\underbrace{\rho(g)}_{\text{Linear}}\, |G[\chi]\rangle = \overbrace{\underbrace{U_L(g)}_{\substack{\text{Projective}\\ \in[\chi]^{-1}=[\chi^*]}}}^{\text{Left boundary}} \otimes \overbrace{\mathbb{1}}^{\text{Bulk}} \otimes \overbrace{\underbrace{U_R(g)}_{\substack{\text{Projective}\\ \in[\chi]}}}^{\text{Right boundary}} |G[\chi]\rangle} \tag{10.77}$$

- ¡! Note that the ground states are only invariant in the bulk (this is the condition of "no symmetry breaking"). However, on the boundaries the projective symmetries act non-trivially, so that the ground state is *no longer an invariant state* and the ground state manifold is degenerate (→ *below*).
- For our example of the bosonic SSH chain (in the topological Phase A), it is

$$|G[\chi_A]\rangle \in \text{span}\{|A(\uparrow,\uparrow)\rangle, |A(\uparrow,\downarrow)\rangle, |A(\downarrow,\uparrow)\rangle, |A(\downarrow,\downarrow)\rangle\} \tag{10.78}$$

  and $U_L(g)$ ($U_R(g)$) act only on the left (right) boundary spin-$\frac{1}{2}$.
- Notice how the *product* of both boundary representations is again linear:

$$\begin{aligned}&[U_L(g_1) \otimes U_R(g_1)]\,[U_L(g_2) \otimes U_R(g_2)] \\ &\overset{10.75}{=} \chi^*(g_1,g_2)\chi(g_1,g_2)\,U_L(g_1g_2) \otimes U_R(g_1g_2) &(10.79a)\\ &= U_L(g_1g_2) \otimes U_R(g_1g_2)\,, &(10.79b)\end{aligned}$$

  consistent with $\rho(g)$ being a linear representation of G.

This spatial "separation" of a linearly realized symmetry into two (or more) projective "parts" is known as …

$$\boxed{⁂\ \textit{Symmetry fractionalization}} \tag{10.80}$$

Note that this is an emergent phenomenon *on the ground state manifold*. There is nothing intrinsically "fractionalized" about the symmetry representation ρ on the full Hilbert space.

- This boundary-based approach was applied by TURNER and collaborators in Ref. [184] to systematically derive the breakdown of the $\mathbb{Z}$ topological index to $\mathbb{Z}_8$ in symmetry class **BDI** for $d = 1$-dimensional interacting fermion systems, recall ← Section 6.4.
- Symmetry fractionalization can become physically relevant whenever a global (linear) symmetry acts trivially on an extended system, except for a few spatially separate regions (here: boundaries). In systems with ← *intrinsic topological order* with → *anyonic excitations*, global symmetries act trivially on the vacuum (no anyons = excitations), but can act non-trivially on the anyons. Hence the symmetry can fractionalize, so that anyons transform under projective representations of the symmetry. Since charges are mathematically speaking "the generators of (continuous) symmetry groups," these anyons can carry ↑ *fractional charges*. This interplay between long-range entanglement and symmetry fractionalization is crucial to understand ← *symmetry-enriched topological order (SET)* [245, 246].

21 | Ground state degeneracy:

Let us again focus on the bosonic SSH chain to understand how projective symmetries on boundaries and robust ground state degeneracy are related:

i | ⊲ Fixpoint in Phase A ($t = 0$ and $w < 0$):

→ Projective boundary symmetries Eqs. (10.71a) and (10.73a):

$$U_L(x) = \sigma_1^x\,,\quad U_L(z) = \sigma_1^z\,,\quad U_L(xz) = \sigma_1^x\sigma_1^z \tag{10.81a}$$

$$U_R(x) = \sigma_{2L}^x\,,\quad U_R(z) = \sigma_{2L}^z\,,\quad U_R(xz) = \sigma_{2L}^x\sigma_{2L}^z \tag{10.81b}$$

Here we interpret the physical symmetries as operators on the full Hilbert space and indicate the subsystems (spin-$\frac{1}{2}$) on which they act by spin indices $i = 1$ (left boundary) and $i = 2L$ (right boundary).

$\rightarrow [\chi_L] = [\chi_R] \overset{10.45}{=} [\chi_A] \overset{10.52}{\neq} [1] \in H^2(D_2, \mathrm{U}(1))$

Strictly speaking $[\chi_L] = [\chi_A]^{-1}$ but since $H^2(D_2, \mathrm{U}(1)) = \mathbb{Z}_2$ these are identical.

ii | Observation:

These boundary operators are symmetries *independently* of each other:

Eqs. (10.59) and (10.81) & $t = 0 \rightarrow$

$$\boxed{\left[H_{\text{bSSH}}^{\text{OBC}}, U_L(g)\right] = 0 \quad \text{and} \quad \left[H_{\text{bSSH}}^{\text{OBC}}, U_R(g)\right] = 0} \tag{10.82}$$

- It is a feature of the fixpoint where $w = 0$ that these boundary representations are *exact symmetries* of the Hamiltonian H_{bSSH}.
- For $t \neq 0$ and finite chain length $L < \infty$, this holds only approximately as long as $|t| < |w|$, i.e., in the topological phase. Then, the operators U_L and U_R are no longer localized on the left- and rightmost spins but "leak" into the bulk. For $L \rightarrow \infty$, the symmetry (10.82) is restored for $t \neq 0$ where the ground state degeneracy is lifted for $L < \infty$ and restored for $L \rightarrow \infty$. Only at $t = 0$ the degeneracy is exact even for finite chains.

  This is similar to the fermionic edge modes of the SSH chain on ➔ Problemset 7.

iii | Eq. (10.81) $\rightarrow$

$$\begin{aligned} U_L(x)U_L(z) &= -U_L(z)U_L(x) \\ U_R(x)U_R(z) &= -U_R(z)U_R(x) \end{aligned} \quad \text{and} \quad [U_L(g), U_R(g)] = 0 \tag{10.83}$$

Note that the anti-commutation relations are a manifestation of $[\chi_L] = [\chi_R] \neq [1]$ being non-trivial cohomology classes.

Eqs. (10.82) and (10.83) $\overset{\circ}{\rightarrow}$ (4-fold) Degenerate ground state space

- Let $|A(\uparrow,\uparrow)\rangle$ be one ground state of $H_{\text{bSSH}}^{\text{OBC}}$. Then the following three states are also ground states [Eq. (10.82)] and linearly independent [Eq. (10.83)]:

$$U_L(x)|A(\uparrow,\uparrow)\rangle = |A(\downarrow,\uparrow)\rangle \tag{10.84a}$$
$$U_R(x)|A(\uparrow,\uparrow)\rangle = |A(\uparrow,\downarrow)\rangle \tag{10.84b}$$
$$U_L(x)U_R(x)|A(\uparrow,\uparrow)\rangle = |A(\downarrow,\downarrow)\rangle \tag{10.84c}$$

  Note that this is a consequence of the non-abelian algebra of the boundary symmetry operators – and this algebra is a consequence of the projective representation of D_2.
- Again, these conclusions can be generalized to 1D SPT phases in general:

  To this end, assume that Eq. (10.82) is valid (at least in the limit $L \rightarrow \infty$) for some boundary operators $U_L(g)$ and $U_R(g)$, and that $[\chi_L] = [\chi_R^*] \neq [1] \in H^2(G, \mathrm{U}(1))$. Since $U_X(g)$ is a symmetry of the Hamiltonian ($X = L, R$), we can consider the representation of these operators on the ground state manifold. The claim is that this manifold must be at least two-dimensional, i.e., there must be a robust (→ *below*) *ground state degeneracy*.

  To see why, assume that the ground state is unique instead. Then $U_X : G \rightarrow \mathrm{U}(1)$ are one-dimensional *projective* representations:

$$U_X(g_1)U_X(g_2) = \chi_X(g_1, g_2)\, U_X(g_1 g_2)\,. \tag{10.85}$$

Since the $U_X(g)$ are U(1) phases, we can express the cocycle as

$$\chi_X(g_1, g_2) = 1 \cdot \frac{U_X(g_1)U_X(g_2)}{U_X(g_1 g_2)} \tag{10.86}$$

which shows that $\chi_X \sim 1$ is the trivial cocycle [using $U_X : G \to \mathrm{U}(1)$ as the function f in Eq. (10.32)]. This contradicts our assumption that the boundary symmetries transform in a *non-trivial* projective representation!

This argument shows that non-trivial projective representations must be at least *two-dimensional*, which implies a degeneracy of the ground state manifold.

iv | Conclusion:

Let us summarize the line of arguments we have built throughout this section:

(i) Sections 9.1 and 10.1: Ground states of gapped 1D spin systems can be represented as matrix product states.

(ii) Sections 10.2 and 10.3: Symmetric matrix product states are characterized by a projective representation V on their *virtual bonds* which, in turn, is characterized by a cohomology class $[\chi] \in H^2(G, \mathrm{U}(1))$.

(iii) Eq. (10.43): This cohomology class cannot change under arbitrary perturbations that do not close the gap or violate the protecting symmetry.

(iv) Eq. (10.77): For *open boundaries*, the global symmetry acts on ground state(s) via boundary representations $U_{L,R}$ in the same (or inverse) non-trivial cohomology class $[\chi]$ (← *symmetry fractionalization*)– so this class is robust to perturbations as well.

(Even though the specific *form and domain* of the operators $U_{L,R}$ may change.)

(v) Eq. (10.82): In the thermodynamic limit (or at the fixpoint), the ground state space carries projective representations of both U_L and U_R separately.

(vi) Eq. (10.86): Non-trivial projective representations cannot be one-dimensional.

→ Hence the ground state space must be degenerate.

This line of arguments remains valid as long as the gap does not close and the protecting symmetry is not violated; i.e., the ground state degeneracy is robust.

→

$$\boxed{\begin{array}{lcl} \text{1D SPT Phase} & \leftrightarrow & \text{Cohomology class } [\chi] \in H^2(G, \mathrm{U}(1)) \\ & \leftrightarrow & [\chi]\text{-projective boundary symmetries} \\ & \rightarrow & \text{Robust ground state degeneracy} \end{array}}$$

- This is an explicit example of a ↑ *bulk-boundary correspondence*: A topological bulk necessitates the presence of robust zero-energy degrees of freedom on the boundary.
- In *finite* systems, Eq. (10.82) is only *approximately* valid for generic parameters in the topological phase. Therefore the conclusion that the ground state space carries a representation of U_L and U_R *separately* is no longer exactly valid; only their product $\rho(g) = U_L(g)U_R(g)$ is a true symmetry of the system. Since this symmetry is represented *linearly*, we cannot exclude the possibility that $\rho(g) \propto 1$ acts with the trivial representation on a one-dimensional ground state space. This is what happens in finite systems where the ground state degeneracy is lifted.

10.5. ‡ Antiunitary symmetries and twisted group cohomology

In this section, we show that a single unitary $\mathbb{Z}_2$-symmetry cannot protect a SPT phase (in contrast to the $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry discussed previously). We then show how *antiunitary* symmetries fit into the cohomology framework by introducing a "twisted" cocycle condition. With this new concept, we then show that a antiunitary $\mathbb{Z}_2$-symmetry protects a non-trivial SPT phase. We demonstrate this with the bosonic SSH chain.

You study the following derivations in detail on ➔ Problemset 11.

The following discussion is loosely based on Refs. [247, 248].

1 | So far:

- Only symmetries represented by local unitary operators: $\rho = \pi \otimes \cdots \otimes \pi$
- Only example of non-trivial cohomology: $H^2(D_2, \mathrm{U}(1)) = \mathbb{Z}_2$

→ Goal of this section:

- How to incorporate *antiunitary* symmetries into the cohomology framework?
- Determine more cohomology groups to identify non-trivial SPT phases.

2 | $\sphericalangle$ Symmetry group G

← *Wigner's theorem*: (➔ Problemset 1)

In principle, each $g \in G$ can be represented by a *unitary* or *antiunitary* operator.

→ Label group elements as follows:

$$\sigma : G \to \mathbb{Z}_2, \quad g \mapsto \sigma(g) = \begin{cases} +1 & \Leftrightarrow \; g \text{ represented unitarily} \\ -1 & \Leftrightarrow \; g \text{ represented antiunitarily} \end{cases} \tag{10.87}$$

$\overset{\circ}{\to}$ σ: Group homomorphism

¡! Since the product of two antiunitary operators is unitary, the map σ must be a group homomorphism: $\sigma(g_1 \bullet g_2) = \sigma(g_1) \cdot \sigma(g_2)$. Note that this restricts the possibilities for antiunitary representations, e.g., the cyclic group $\mathbb{Z}_3$ does not allow for any antiunitary representation as it is of odd order.

Example: $\sphericalangle$ $G = \mathbb{Z}_2 = \{1, T\}$ → Two possible homomorphisms:

$$\begin{Bmatrix} \sigma_0(1) = 1 \\ \sigma_0(T) = 1 \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} \sigma_1(1) = 1 \\ \sigma_1(T) = -1 \end{Bmatrix} \tag{10.88}$$

σ_0: T represented unitarily
σ_1: T represented antiunitarily

3 | Henceforth we are interested in a group G *augmented* by such a group homomorphisms: $\sphericalangle$ (G, σ)

→ ⁂ *Twisted cocycle condition*: [cf. Eq. (10.24)]

$$\boxed{\chi(g_1, g_2)\chi(g_1 g_2, g_3) \overset{\circ}{=} \chi^{\sigma(g_1)}(g_2, g_3)\chi(g_1, g_2 g_3)} \tag{10.89}$$

Here, χ^{-1} is just the complex conjugate since $\chi(g_1, g_2) \in \mathrm{U}(1)$.

This is easy to show by applying the group associativity and taking into account complex conjugations due to antiunitary operators. Indeed, let ρ be a projective representation of G with $\rho(g_1)$ antiunitary (i.e., $\sigma(g_1) = -1$) and $\rho(g_2)$, $\rho(g_3)$ arbitrary for $g_1, g_2, g_3 \in G$. Then associativity implies the relation

$$\begin{aligned}[\rho(g_1)\rho(g_2)]\rho(g_3) &= \chi(g_1, g_2)\rho(g_1g_2)\rho(g_3) = \chi(g_1, g_2)\chi(g_1g_2, g_3)\rho(g_1g_2g_3) \\ \overset{!}{=} \rho(g_1)[\rho(g_2)\rho(g_3)] &= \rho(g_1)\chi(g_2, g_3)\rho(g_2g_3) = \chi^*(g_2, g_3)\chi(g_1, g_2g_3)\rho(g_1g_2g_3)\,, \end{aligned} \tag{10.90}$$

where we used that $\rho(g_1)\chi(g_2, g_3) = \chi^*(g_2, g_3)\rho(g_1)$. Thus the cocycle condition Eq. (10.89) is "twisted" whenever g_1 is represented antiunitarily.

→ Set of ⁂ *σ-twisted 2-cocycles:*

$$Z^2_\sigma(G, \mathrm{U}(1)) := \{\, \chi : G \times G \to \mathrm{U}(1) \mid \chi \text{ satisfies Eq. (10.89)} \,\} \tag{10.91}$$

This is the generalization of Eq. (10.33).

4 | A similar "twist" occurs when we multiply the operators by g-dependent phases (remember that we are ultimately interested in projective representations):

$\overset{\circ}{\to}$ Twisted equivalence R_σ of cocycles: [cf. Eq. (10.32)]

$$\boxed{\begin{aligned}&\chi \overset{R_\sigma}{\sim} \tilde\chi \quad :\Leftrightarrow \\ &\exists f : G \to \mathrm{U}(1) : \quad \frac{f(g_1) f^{\sigma(g_1)}(g_2)}{f(g_1g_2)}\chi(g_1, g_2) = \tilde\chi(g_1, g_2)\end{aligned}} \tag{10.92}$$

This follows along Eq. (10.30) by taking the antiunitarity of operators into account. Let $\tilde\rho(g) := f(g)\rho(g)$ with phase factor $f : G \to \mathrm{U}(1)$; then

$$\tilde\rho(g_1)\tilde\rho(g_2) = f(g_1)\rho(g_1)f(g_2)\rho(g_2) \tag{10.93a}$$
$$= f(g_1)f^{\sigma(g_1)}(g_2)\,\rho_1(g_1)\rho_1(g_2) \tag{10.93b}$$
$$= f(g_1)f^{\sigma(g_1)}(g_2)\chi(g_1, g_2)\,\rho(g_1g_2) \tag{10.93c}$$
$$= \frac{f(g_1)f^{\sigma(g_1)}(g_2)}{f(g_1g_2)}\chi(g_1, g_2)\,\tilde\rho(g_1g_2) \tag{10.93d}$$
$$\equiv \tilde\chi(g_1, g_2)\,\tilde\rho(g_1g_2)\,. \tag{10.93e}$$

5 | This relation allows us to define the …

⁂ *σ-twisted 2nd cohomology group* of G in U(1):

$$\boxed{H^2_\sigma(G, \mathrm{U}(1)) := Z^2_\sigma(G, \mathrm{U}(1)) \,/\, R_\sigma} \tag{10.94}$$

This is the generalization of Eq. (10.34) (which is recovered for the trivial homomorphism $\sigma \equiv 1$).

The elements of Eq. (10.94) correspond to physically distinct projective representations of G where the elements with $\sigma(g) = -1$ are represented by antiunitary operators.

Examples:

In the following, we consider the simplest example of $G = \mathbb{Z}_2$ with the two homomorphisms Eq. (10.88):

$$H^2(\mathbb{Z}_2, \mathrm{U}(1)) \equiv H^2_{\sigma_0}(\mathbb{Z}_2, \mathrm{U}(1)) \tag{10.95a}$$

$$H^2(\mathbb{Z}_2^T, \mathrm{U}(1)) \equiv H^2_{\sigma_1}(\mathbb{Z}_2, \mathrm{U}(1)) \tag{10.95b}$$

The left-hand notation is often used in the literature, e.g. in Refs. [46, 47, 54]; the "T" in the superscript stands for "time reversal" and indicates the antiunitary representation of T in $\mathbb{Z}_2 = \{1, T\}$.

6 | Evaluation of $H^2(\mathbb{Z}_2, \mathrm{U}(1))$:

i | Cocycle condition (10.89) $\xrightarrow{\sigma_0}$

$$\begin{aligned}(g_1, g_2, g_2) \equiv (1, T, T) \;&\Rightarrow\; \chi(1, T)\chi(T, T) = \chi(T, T)\chi(1, 1) \\ &\Rightarrow\; \chi(1, T) = \chi(1, 1)\end{aligned} \tag{10.96a}$$

$$\begin{aligned}(T, T, T) \;&\Rightarrow\; \chi(T, T)\chi(1, T) = \chi(T, T)\chi(T, 1) \\ &\Rightarrow\; \chi(1, T) = \chi(T, 1)\end{aligned} \tag{10.96b}$$

→ Only $\chi(1, 1)$ and $\chi(T, T)$ are independent

ii | ⊲ Trivialization: [Eq. (10.51)]

$$\chi(1, 1) \overset{!}{=} \frac{f(1)f(1)}{f(1)} = f(1) \quad \text{and} \quad \chi(T, T) \overset{!}{=} \frac{f(T)f(T)}{f(1)} \tag{10.97}$$

→ Can be solved for $f(1)$ and $f(T)$ for arbitrary cocycles $\chi(g_1, g_2)$

→ All cocycles are trivial →

$$\boxed{H^2(\mathbb{Z}_2, \mathrm{U}(1)) = H^2_{\sigma_0}(\mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_1 = \{1\}} \tag{10.98}$$

→ No SPT protected by *unitary* $\mathbb{Z}_2$ symmetry

7 | Evaluation of $H^2(\mathbb{Z}_2^T, \mathrm{U}(1))$:

i | Cocycle condition (10.89) $\xrightarrow{\sigma_1}$

$$\begin{aligned}(1, T, T) \;&\Rightarrow\; \chi(1, T)\chi(T, T) = \chi(T, T)\chi(1, 1) \\ &\Rightarrow\; \chi(1, T) = \chi(1, 1)\end{aligned} \tag{10.99}$$

This is the same as for σ_0.

By contrast:

$$\begin{aligned}(T, T, T) \;&\Rightarrow\; \chi(T, T)\chi(1, T) = \chi^{-1}(T, T)\chi(T, 1) \\ &\Rightarrow\; \chi^2(T, T) = \frac{\chi(T, 1)}{\chi(1, T)},\end{aligned} \tag{10.100a}$$

$$\begin{aligned}(T, 1, 1) \;&\Rightarrow\; \chi(T, 1)\chi(T, 1) = \chi^{-1}(1, 1)\chi(T, 1) \\ &\Rightarrow\; \chi(T, 1) = \frac{1}{\chi(1, 1)},\end{aligned} \tag{10.100b}$$

→ Again only $\chi(1,1)$ and $\chi(T,T)$ are independent.

However: Eqs. (10.99), (10.100a) and (10.100b) →

$$\chi^2(T,T) = \frac{1}{\chi^2(1,1)} \quad \Rightarrow \quad \underbrace{\chi(T,T) = \frac{\pm 1}{\chi(1,1)}}_{\text{2 consistent solutions}} \tag{10.101}$$

Are both solutions trivial?

ii | ◁ Trivialization: [Eqs. (10.51) and (10.92)]

$$\chi(1,1) \overset{!}{=} \frac{f(1)f(1)}{f(1)} = f(1) \tag{10.102a}$$

$$\text{and} \quad \chi(T,T) \overset{!}{=} \frac{f(T)f^{-1}(T)}{f(1)} = \frac{1}{f(1)} = \frac{1}{\chi(1,1)} \tag{10.102b}$$

→ Only solution $\chi(T,T) = +\chi^{-1}(1,1)$ can be trivialized

→ Solution $\chi(T,T) = -\chi^{-1}(1,1)$ is non-trivial →

$$\boxed{H^2(\mathbb{Z}_2^T, \mathrm{U}(1)) = H^2_{\sigma_1}(\mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_2} \tag{10.103}$$

→ <u>One</u> non-trivial SPT protected by *antiunitary* $\mathbb{Z}_2$ symmetry

8 | <u>Summary:</u>

We have now identified the following cohomology groups & allowed SPTs:

$$H^2(D_2, \mathrm{U}(1)) = \mathbb{Z}_2 \quad \rightarrow \quad \mathbf{1}\ \text{Topological phase} \tag{10.104a}$$

$$H^2(\mathbb{Z}_2, \mathrm{U}(1)) = \mathbb{Z}_1 \quad \rightarrow \quad \textbf{No}\ \text{topological phase} \tag{10.104b}$$

$$H^2(\mathbb{Z}_2^T, \mathrm{U}(1)) = \mathbb{Z}_2 \quad \rightarrow \quad \mathbf{1}\ \text{Topological phase} \tag{10.104c}$$

- You can check that these results match the literature [47, Table I] (for $d = 1$).
- We can illustrate these abstract results with the bosonic SSH chain for open boundary conditions at the fixpoint $|A(\alpha,\beta)\rangle$ [$t = 0$ and $w < 0$, Eq. (10.62)]. To this end, we consider global symmetries that induce edge symmetries in the respective cohomology class, and probe whether these allow for perturbations (= magnetic fields) that lift the ground state degeneracy by polarizing the edge spins:
  - <u>$H^2(D_2, \mathrm{U}(1)) = \{[1], [-1]\}$</u> with $D_2 = \{1, x, z, xz\}$ and $[-1] \overset{10.46}{\equiv} [\chi_A]$:
    - ◁ Global symmetry $\rho(x) := X$ and $\rho(z) := X$ of H_{bSSH}:

      → Boundary representation on $|A(\alpha,\beta)\rangle$:

$$\left.\begin{aligned} U_L(1) &:= \mathbb{1} \\ U_L(x) &:= \sigma_1^x \\ U_L(z) &:= \sigma_1^x \\ U_L(xz) &:= \mathbb{1} \end{aligned}\right\} \Rightarrow \chi(g_1, g_2) = 1 \Rightarrow \chi \in [1] \tag{10.105}$$

      ◁ Perturbation $H_{\text{pert}} = h_x \sigma_1^x$

      → Allowed since $[H_{\text{pert}}, U_L(g)] = 0$ for all $g \in D_2$ & Lifts ground state degeneracy

      → No topological phase ✗ (consistent with $\chi \in [1]$)

* $\sphericalangle$ Global symmetry $\rho(x) := X$ and $\rho(z) := Z$ of H_{bSSH}:
  $\to$ Boundary representation on $|A(\alpha,\beta)\rangle$:

$$\left.\begin{aligned} U_L(1) &:= \mathbb{1} \\ U_L(x) &:= \sigma_1^x \\ U_L(z) &:= \sigma_1^z \\ U_L(xz) &:= \sigma_1^x \sigma_1^z \end{aligned}\right\} \Rightarrow \chi(z,x) = -1 = -\chi(x,z) \Rightarrow \chi \in [-1] \tag{10.106}$$

  $\sphericalangle$ *Any* perturbation $H_{\text{pert}} = h_a \sigma_1^a$ for $a = x, y, z$
  $\to$ *Not* allowed since $[H_{\text{pert}}, U_L(g)] \neq 0$ for all $a = x, y, z$ and some $g \in D_2$
  $\to$ Symmetry-protected topological phase ✓ (consistent with $\chi \in [-1] \neq [1]$)

- $H^2(\mathbb{Z}_2, \mathrm{U}(1)) = \{[1]\}$ with $\mathbb{Z}_2 = \{1, T\}$:
  * $\sphericalangle$ Global symmetry $\rho(T) := X$ of H_{bSSH}:
    $\to$ Boundary representation on $|A(\alpha,\beta)\rangle$:

$$\left.\begin{aligned} U_L(1) &:= \mathbb{1} \\ U_L(T) &:= \sigma_1^x \end{aligned}\right\} \Rightarrow \chi(g_1, g_2) = 1 \Rightarrow \chi \in [1] \tag{10.107}$$

    $\sphericalangle$ Perturbation $H_{\text{pert}} = h_x \sigma_1^x$
    $\to$ Allowed since $[H_{\text{pert}}, U_L(g)] = 0$ for all $g \in \mathbb{Z}_2$ & Lifts ground state degeneracy
    $\to$ No topological phase ✗ (consistent with $\chi \in [1]$)

- $H^2(\mathbb{Z}_2^T, \mathrm{U}(1)) = \{[1], [-1]\}$ with $\mathbb{Z}_2 = \{1, T\}$ and T represented antiunitarily:
  * $\sphericalangle$ Global symmetry $\rho(T) := X\mathcal{K}$ of H_{bSSH}:
    $\to$ Boundary representation on $A(\alpha,\beta)$:

$$\left.\begin{aligned} U_L(1) &:= \mathbb{1} \\ U_L(T) &:= \sigma_1^x \mathcal{K} \end{aligned}\right\} \Rightarrow \chi(T,T) = +1 = \chi(1,1) \Rightarrow \chi \in [1] \tag{10.108}$$

    $\sphericalangle$ Perturbation $H_{\text{pert}} = h_x \sigma_1^x$
    $\to$ Allowed since $[H_{\text{pert}}, U_L(g)] = 0$ for all $g \in \mathbb{Z}_2$ & Lifts ground state degeneracy
    $\to$ No topological phase ✗ (consistent with $\chi \in [1]$)
  * $\sphericalangle$ Global symmetry $\rho(T) := Y\mathcal{K}$ of H_{bSSH}:
    $\to$ Boundary representation on $A(\alpha,\beta)$:

$$\left.\begin{aligned} U_L(1) &:= \mathbb{1} \\ U_L(T) &:= \sigma_1^y \mathcal{K} \end{aligned}\right\} \Rightarrow \chi(T,T) = -1 = -\chi(1,1) \Rightarrow \chi \in [-1] \tag{10.109}$$

    $\sphericalangle$ *Any* perturbation $H_{\text{pert}} = h_a \sigma_1^a$ for $a = x, y, z$
    $\to$ *Not* allowed since $[H_{\text{pert}}, U_L(T)] \neq 0$ for all $a = x, y, z$
    $\to$ Symmetry-protected topological phase ✓ (consistent with $\chi \in [-1] \neq [1]$)

These considerations allow us to conclude:

> The topological Phase A of the bosonic SSH chain can be protected by …
> … either D_2 (180° spin rotations)
> … or $\mathbb{Z}_2^T$ (time reversal).