↓ **Lecture 23** [17.07.25]

10.3. Projective symmetries and group cohomology

Here we introduce the concept of *projective representations* of symmetry groups and their relation to the *second cohomology group*. The latter is the mathematical framework to classify SPT phases of interacting spin systems in one-dimension. We illustrate these rather abstract concepts with the bosonic SSH chain and prove that its two phases belong to different phases protected by the D_2 symmetry.

7 | Remember: $\pi : G \to \mathbb{C}^{d\times d}$ satisfies [Eq. (8.7)]

$$\pi(g_1)\pi(g_2) = \pi(g_1 g_2) \quad \text{for} \quad g_1, g_2 \in G \tag{10.16}$$

→ *Linear* representation of G

8 | ⊲ Eq. (10.15): $V : G \to \mathbb{C}^{D\times D}$ satisfies *almost* the same relation:

$$V(g_1)V(g_2) \overset{*}{=} \chi(g_1, g_2)V(g_1 g_2) \quad \text{for} \quad g_1, g_2 \in G \tag{10.17}$$

$\chi : G \times G \to \mathrm{U}(1)$: ⁂ *(2-)Cocycle* or ⁂ *factor system*
$V : G \to \mathbb{C}^{D\times D}$: ⁂ *Projective representation* of G

- The reason for this "relaxed" multiplication law is that in Eq. (10.15) all phases $\chi(g_1, g_2)$ that violate the multiplication rules of the abstract group cancel because $V(g)$ and $V^\dagger(g)$ always pair up.
- Projective representations are like "normal" (linear) representations "up to a phase." Linear representations are trivial projective representations with $\chi(g_1, g_2) \equiv 1$ (up to a "gauge freedom", → *below*).
- To derive Eq. (10.17) from the transformation (10.15) requires one additional input, namely that the MPS $|M\rangle$ is ↑ *injective*. This means that the set of d matrices $\{M^1, \dots, M^d\}$ spans the full space of $D \times D$-matrices $\mathbb{C}^{D\times D}$. You can check this explicitly for both states $|A\rangle$ and $|B\rangle$ of the bosonic SSH chain by inspecting Eqs. (9.24) and (9.27).

  One can show that injectivity follows for MPS with exponentially decaying correlations [243]. [More precisely: Exponentially decaying correlations imply ↑ *normality*, which leads to ↑ *injectivity* after combining finitely many tensors/sites to new sites (blocking).] This is always true for non-degenerate ground states of gapped Hamiltonians [27], so that we can assume injectivity as given (potentially after blocking). Note that *non-injective* MPS are typically associated to systems with ← *symmetry breaking* [7].

  So let us assume injectivity. First, without loss of generality, we can absorb the 1D representation $\gamma(g)$ into the representation $\pi(g)$ so that Eq. (10.15) reads

$$\sum_j [\tilde\pi(g)]_{ij} M^j = V^\dagger(g) \cdot M^i \cdot V(g) \tag{10.18}$$

  where $\tilde\pi(g) := \pi(g)/\gamma(g)$ is again a linear unitary representation of G. For $g = g_1 g_2$ the left hand side reads:

$$\sum_j [\tilde\pi(g_1g_2)]_{ij} M^j \overset{10.16}{=} \sum_{j,k} [\tilde\pi(g_1)]_{ik} [\tilde\pi(g_2)]_{kj} M^j \tag{10.19a}$$

$$\overset{10.18}{=} V^\dagger(g_2)\cdot \left[\sum_k [\tilde\pi(g_1)]_{ik}\, M^k\right]\cdot V(g_2) \tag{10.19b}$$

$$\overset{10.18}{=} V^\dagger(g_2)\cdot V^\dagger(g_1)\cdot M^i\cdot V(g_1)\cdot V(g_2)\,. \tag{10.19c}$$

On the other hand, we have

$$\sum_j [\tilde\pi(g_1g_2)]_{ij} M^j \overset{10.18}{=} V^\dagger(g_1g_2)\cdot M^i\cdot V(g_1g_2) \tag{10.20}$$

and therefore

$$V^\dagger(g_1g_2)\cdot M^i\cdot V(g_1g_2) = V^\dagger(g_2)\cdot V^\dagger(g_1)\cdot M^i\cdot V(g_1)\cdot V(g_2)\,. \tag{10.21}$$

Both sides of the equation are linear maps from $\mathbb{C}^{D\times D}\simeq\mathbb{C}^D\otimes\mathbb{C}^D$ to itself. Since this relation is true for all $i=1,\dots,d$ and $\{M^1,\dots,M^d\}$ forms a basis of $\mathbb{C}^{D\times D}$, it follows

$$V^\dagger(g_1g_2)|k\rangle\langle l|V(g_1g_2) = \left[V^\dagger(g_2)\cdot V^\dagger(g_1)\right]|k\rangle\langle l|\,[V(g_1)\cdot V(g_2)] \tag{10.22}$$

with $|k\rangle\langle l|$ a basis of the virtual space $\mathbb{C}^{D\times D}$ for $k,l=1,\dots,D$.

This condition is satisfied if and only if

$$V(g_1)\cdot V(g_2) = \chi(g_1,g_2)\, V(g_1g_2) \tag{10.23a}$$

$$V^\dagger(g_2)\cdot V^\dagger(g_1) = \chi^*(g_1,g_2)\, V^\dagger(g_1g_2) \tag{10.23b}$$

for an arbitrary g_1,g_2-dependent phase $\chi(g_1,g_2)$. This shows Eq. (10.17).

- Projective representations also play important roles in other places: Since quantum states are only defined modulo phases (the state space of quantum mechanics is a ↑ *projective Hilbert space*, ➔ Problemset 1), physical symmetries can be represented on the Hilbert space "up to phases" as well. In quantum mechanics, we are therefore interested in *projective* representations (10.17) of physical symmetry groups. For example, rotations in three-dimensional space are described by the symmetry group SO(3). A neat fact from mathematics tells us that (under some technical assumptions) the projective representations of a Lie group correspond to the *linear* representation of its ↑ *universal covering group* – which for SO(3) happens to be SU(2). Hence rotations on quantum states are described by linear representations of SU(2). The *half-integer* spin representations (spin-$\frac{1}{2}$, spin-$\frac{3}{2}$, …) then correspond to *projective* representations of SO(3). (The integer spin representations correspond to linear representations.)

9 | $\chi(g_1,g_2)$ is not arbitrary:

⊲ Associativity $(g_1g_2)g_3 = g_1(g_2g_3) \overset{\circ}{\rightarrow}$ ⁂ *Cocycle condition*

$$\boxed{\chi(g_1,g_2)\chi(g_1g_2,g_3) = \chi(g_2,g_3)\chi(g_1,g_2g_3)} \tag{10.24}$$

$\chi(g_1,g_2)$ must satisfy this constraint to make Eq. (10.17) well-defined on the entire group G.

10 | Example:

Let us again illustrate these concepts with the bosonic SSH chain in the (topological) phase A:

i | Recall:

$$\underbrace{\left.\begin{matrix}\pi(x)=\sigma^x_{2k-1}\sigma^x_{2k}\\ \pi(z)=\sigma^z_{2k-1}\sigma^z_{2k}\end{matrix}\right\}}_{\substack{\text{Linear representation (on sites)}\\ \text{[Eq. (10.5)]}}} \xrightarrow{\text{Eqs. (10.9) and (10.11)}} \underbrace{\left\{\begin{matrix}V(x)=\hat\sigma^x\\ V(z)=\hat\sigma^z\end{matrix}\right.}_{\substack{\text{Projective representation}\\ \text{(on bonds)}}} \tag{10.25}$$

ii | Observation: $xz = zx$ in D_2 $\overset{\lightning}{\leftrightarrow}$ $V(x)V(z) = -V(z)V(x)$

→ V cannot satisfy Eq. (10.16) but satisfies Eq. (10.17)

→ Projective representation

Note that we have two choices to *define* $V(xz) = V(zx)$, namely:

$$\text{For } i=1,2:\quad \left.\begin{matrix}V_i(1)=\mathbb{1}\\ V_i(x)=\hat\sigma^x\\ V_i(z)=\hat\sigma^z\end{matrix}\right\} \quad\text{and}\quad \left\{\begin{matrix}V_1(xz)=V_1(zx):=\hat\sigma^x\hat\sigma^z\\ \text{or}\\ V_2(xz)=V_2(zx):=\hat\sigma^z\hat\sigma^x\end{matrix}\right. \tag{10.26}$$

Both definitions satisfy Eq. (10.17) with different cocycles χ (→ *below*).

iii | → Multiplication rules:

$$\begin{aligned} V_1(x)V_1(z) &= \hat\sigma^x\hat\sigma^z = +1\cdot V_1(xz)\,,\\ V_1(z)V_1(x) &= \hat\sigma^z\hat\sigma^x = -1\cdot V_1(zx) \end{aligned} \tag{10.27a}$$

$$\text{or}\quad \begin{aligned} V_2(x)V_2(z) &= \hat\sigma^x\hat\sigma^z = -1\cdot V_2(xz)\,,\\ V_2(z)V_2(x) &= \hat\sigma^z\hat\sigma^x = +1\cdot V_2(zx)\,. \end{aligned} \tag{10.27b}$$

→ Cocycles:

$$\chi_1(x,z) = +1\,,\quad \chi_1(z,x) = -1 \tag{10.28a}$$

$$\text{or}\quad \chi_2(x,z) = -1\,,\quad \chi_2(z,x) = +1\,. \tag{10.28b}$$

There are also other non-trivial elements, e.g., $\chi_1(xz,x) = -1$ and $\chi_2(xz,z) = -1$.

→ Why this ambiguity ($i = 1,2$)? Has this physical consequences?

¡! Note that this choice is *not* fixed by Eq. (10.15) since the sign cancels.

11 | Consider again the general transformation Eq. (10.15):

$$\sum_j [\pi(g)]_{ij} A^j = \gamma(g)\, V^\dagger(g)\cdot A^i\cdot V(g) \tag{10.29a}$$

$$= \gamma(g)\,[f(g)V(g)]^\dagger\cdot A^i\cdot[f(g)V(g)] \tag{10.29b}$$

$$\equiv \gamma(g)\,\tilde V(g)^\dagger\cdot A^i\cdot\tilde V(g) \tag{10.29c}$$

Here we introduced the new projective representation $\tilde V(g) := f(g)V(g)$ (→ *below*).

$f: G \to \mathrm{U}(1)$: *Arbitrary* g-dependent phase

→ Projective representations V and $\tilde V$ are equivalent!

Put differently: A symmetric MPS does not uniquely determine the projective representation on its virtual bonds.

12 | Is $\tilde{V}$ a valid projective representation?

Check Eq. (10.17) → Compute cocycle $\tilde{\chi}$:

$$\tilde{V}(g_1)\tilde{V}(g_2) = f(g_1)f(g_2)\,V(g_1)V(g_2) \tag{10.30a}$$

$$\overset{10.17}{=} f(g_1)f(g_2)\chi(g_1,g_2)\,V(g_1g_2) \tag{10.30b}$$

$$= f(g_1)f(g_2)[f(g_1g_2)]^*\chi(g_1,g_2)\,\tilde{V}(g_1g_2) \tag{10.30c}$$

$$\equiv \tilde{\chi}(g_1,g_2)\,\tilde{V}(g_1g_2) \quad ✓ \tag{10.30d}$$

with new cocycle

$$\tilde{\chi}(g_1,g_2) := \frac{f(g_1)f(g_2)}{f(g_1g_2)}\chi(g_1,g_2) \tag{10.31}$$

Recall that f is a phase so that $f(g_1g_2)^* = f(g_1g_2)^{-1}$.

Check that $\tilde{\chi}$ satisfies Eq. (10.24) if χ does!

V and $\tilde{V}$ physically equivalent → Equivalence relation between cocycles:

$$\boxed{\chi \sim \tilde{\chi} \quad :\Leftrightarrow \quad \exists f : G \to \mathrm{U}(1) : \quad \frac{f(g_1)f(g_2)}{f(g_1g_2)}\chi(g_1,g_2) = \tilde{\chi}(g_1,g_2)} \tag{10.32}$$

If $\chi \sim \tilde{\chi}$ one calls the two cocycles are ⁂ *cohomologous*.

13 | Define the Set of all U(1)-valued 2-cocycles of G:

$$Z^2(G,\mathrm{U}(1)) := \{\, \chi : G \times G \to \mathrm{U}(1) \,|\, \chi \text{ satisfies Eq. (10.24)} \,\} \tag{10.33}$$

→ Set of all equivalence classes wrt. (10.32):

⁂ *(2nd) Cohomology group* of G in U(1):

$$\boxed{H^2(G,\mathrm{U}(1)) := Z^2(G,\mathrm{U}(1))\,/_\sim} \tag{10.34}$$

- $H^2(G,\mathrm{U}(1))$ is an abelian group with the natural multiplication on cocycles. This means that for $[\chi_1],[\chi_2] \in H^2(G,\mathrm{U}(1))$ it is

$$[\chi_1]\circ[\chi_2] := [\chi_1\cdot\chi_2] = [\chi_2\cdot\chi_1] = [\chi_2]\circ[\chi_1] \in H^2(G,\mathrm{U}(1)) \tag{10.35}$$

  where $\chi_1 \cdot \chi_2$ denotes the product of the two functions $\chi_i : G \times G \to \mathrm{U}(1)$.
- Let $[1] \in H^2(G,\mathrm{U}(1))$ be the identity element. Then every cocycle $\chi \in [1]$ can be written in the form

$$\chi(g_1,g_2) = \frac{f(g_1)f(g_2)}{f(g_1g_2)} \tag{10.36}$$

  for some function $f : G \to \mathrm{U}(1)$. This follows from Eq. (10.32).

14 | Intermediate status:

$$\underbrace{\rho(g) = \pi(g) \otimes \ldots \otimes \pi(g)}_{\text{Linear (physical) representation}} \xrightarrow[\text{Eqs. (10.15), (10.17) and (10.32)}]{\text{Symmetric MPS } |M\rangle} \underbrace{[\chi_M] \in H^2(G, \mathrm{U}(1))}_{\text{Depends on } \rho \text{ and } |M\rangle} \tag{10.37}$$

This means that the action of a linear representation $\rho(g)$ on an invariant MPS is *not* characterized by a *particular* projective representation $V(g)$ (with cocycle χ) on its (virtual) bonds but by the *cohomology class* $[\chi] \in H^2(G, \mathrm{U}(1))$ that its cocycle belongs to.

15 | Example:

With this concept of equivalence of cocycles, we can revisit phase A of the bosonic SSH chain:

i | $\sphericalangle$ Transformation $f : G \to \mathrm{U}(1)$ to relate V_1 and V_2:

Eqs. (10.26) and (10.27) $\to$

$$V_1(1) = \mathbb{1} \quad = +1 \cdot V_2(1) \equiv f(1) \cdot V_2(1) \tag{10.38a}$$
$$V_1(x) = \hat{\sigma}^x \quad = +1 \cdot V_2(x) \equiv f(x) \cdot V_2(x) \tag{10.38b}$$
$$V_1(z) = \hat{\sigma}^z \quad = +1 \cdot V_2(z) \equiv f(z) \cdot V_2(z) \tag{10.38c}$$
$$V_1(xz) = \hat{\sigma}^x \hat{\sigma}^z = -1 \cdot V_2(xz) \equiv f(xz) \cdot V_2(xz) \tag{10.38d}$$

$\to$ V_1 and V_2 are physically equivalent projective representations

ii | Eqs. (10.28) and (10.32) $\to$

$$\chi_1(z, x) = -1 = \frac{1 \cdot 1}{-1} \cdot 1 \overset{10.38}{=} \frac{f(z) f(x)}{f(zx)} \chi_2(z, x)\,. \tag{10.39}$$

$\to$ χ_1 and χ_2 are cohomologous

$\to [\chi_1] = [\chi_2] \in H^2(D_2, \mathrm{U}(1))$

16 | Classification of SPT phases:

We now have all tools to formulate the classification of SPT phases in 1D spin systems.

i | $\sphericalangle$ Algorithm (10.40):

$$\overbrace{\lambda}^{\text{System parameters}} \mapsto \underbrace{H(\lambda)}_{\substack{\text{Symmetric}\\ \text{gapped 1D}\\ \text{Hamiltonian}}} \mapsto \underbrace{\overbrace{|M(\lambda)\rangle}^{\substack{\text{Symmetric}\\ \text{ground state}}}}_{\substack{\downarrow \\ \text{Injective}\\ \text{MPS}}} \mapsto \underbrace{M^i(\lambda)}_{D \times D\text{-matrices}} \mapsto \overbrace{V_\lambda}^{\substack{\text{Projective}\\ \text{representation}}} \mapsto \underbrace{\chi_\lambda}_{\text{Cocycle}} \mapsto \underbrace{\overbrace{[\chi_\lambda]}^{\substack{\text{Cohomology}\\ \text{class}}}}_{\in H^2(G,\mathrm{U}(1))}$$

Assumptions:

- $\lambda \mapsto [\chi_\lambda]$ continuous
- $H^2(G, \mathrm{U}(1))$ discrete set (this is true for all relevant examples)

$\to [\chi_\lambda] = \text{const}$ (as long as the map $\lambda \mapsto [\chi_\lambda]$ is well-defined in the thermodynamic limit)

$\to$ This suggests:

$$\text{Same phase} \quad \Rightarrow \quad \text{Same cohomology class} \tag{10.41}$$

→ Cohomology classes are "labels" of 1D SPT phases ☺

It turns out that our intuition is correct. One can show that for any finite system size, a variation of parameters in the Hamiltonian changes the cocycle in linear order only by functions that belong to the trivial class [1], i.e., the equivalence class of the cocycle is "rigid" and cannot change due to continuous deformations. See Ref. [7, Section II.F.3] for the technical details.

Let us boldly hypothesize (though this is not obvious) that the inverse is also true:

$$\text{Same phase} \quad \overset{?}{\Leftarrow} \quad \text{Same cohomology class} \tag{10.42}$$

It turns out that this hypothesis is correct. The proof is rather technical as one has to construct an explicit path connecting two Hamiltonians. To do so, one makes use of ↑ *entanglement renormalization* [244] to first connect the ↑ *parent Hamiltonians* of MPS ground states to certain fixpoint Hamiltonians with particularly simple MPS ground states (which are provably in the same phase). One can then construct explicitly a gapped path connecting any such fixpoint MPS that transforms under the same projective representation. See Ref. [7, Section II.F.2] for the technical details.

ii | This suggests the following classification scheme:

$$\boxed{\left\{\begin{matrix}\text{Topological phases}\\ \text{in 1D protected by } G\end{matrix}\right\} \hat{=} \left\{\begin{matrix}\text{2nd Cohomology classes}\\ [\chi] \text{ of } G \text{ over } \mathrm{U}(1)\end{matrix}\right\} = H^2(G,\mathrm{U}(1))} \tag{10.43}$$

- In words:

  Let H_A and H_B be two one-dimensional, gapped Hamiltonians on a common Hilbert space $\mathcal{H}$ with symmetry $\rho(g)$ for $g \in G$ and unique ground states $|A\rangle$ and $|B\rangle$, respectively. The latter are invariant under the action of ρ and can be described by MPS with matrices A^i and B^i of bond dimensions $D_A = \text{const}$ and $D_B = \text{const}$ for $L \to \infty$. The action of the linear representation $\rho(g)$ on these states induces projective representations V_A and V_B on their bond spaces with cocycles χ_A and χ_B and matrix dimensions D_A and D_B.

  Then there exists a path $H(\lambda)$ of gapped, ρ-symmetric Hamiltonians on $\mathcal{H}$ with $H(0) = H_A$ and $H(1) = H_B$ if and only if $\chi_A \sim \chi_B$, i.e., iff V_A and V_B are projective representations of the same cohomology class $[\chi_A] = [\chi_B] \in H^2(G, \mathrm{U}(1))$.

  This implies that two symmetric states $|A\rangle$ and $|B\rangle$ belong to the same quantum phase if and only if their corresponding cocycles (defined via their MPS representation) are representatives of the same cohomology class. This fact leads to the somewhat cryptic statement that the one-dimensional symmetry-protected topological phases of interacting spin systems (with symmetry group G), are in one-to-one correspondence with elements of the second cohomology group $H^2(G, \mathrm{U}(1))$.

- It is important to stress that this concept of equivalence allows for the comparison of projective representations V_A and V_B *even if they do not have the same (bond) dimension* $D_A \neq D_B$ as the equivalence relation (10.32) only relies on their cocycles χ_A and χ_B. In this more general case, the equivalence $\chi_A \sim \chi_B$ does not imply $V_A(g) = f(g)V_B(g)$ [this equation does not make sense because $V_A(g)$ and $V_B(g)$ have different dimensions]. However, one can show that there is still a path of an extended Hamiltonian that connects the two phases [6, 7].

iii | $\lessdot H(\lambda)$ that connects *different* phases:

I.e., $|A\rangle$ and $|B\rangle$ belong to different cohomology classes: $[\chi_A] \neq [\chi_B]$

→ Algorithm (10.40) must be ill-defined

→ Two possibilities:

- $H(\lambda)$ is gapless for some λ_c

  The construction of a well-defined projective representation via injective MPS ground states fails because the bond dimension D_{λ_c} is unbounded in the thermodynamic limit (the ground states of gapless Hamiltonians are weakly long-range entangled in one dimension).

- $H(\lambda)$ violates the symmetry for some λ^*

  The ground state $|M(\lambda^*)\rangle$ is no longer invariant under $\rho(g)$ and the construction of a well-defined projective representation fails even though D_{λ^*} remains bounded.

17 | Example:

We can finally piece everything together to classify the two phases of the bosonic SSH chain:

i | *Summary for Phase A:*

$$\left.\begin{matrix} t=0 \\ w<0 \end{matrix}\right\} \overset{8.6}{\mapsto} \hat{H}^A_{\text{bSSH}} \overset{8.29}{\mapsto} |A\rangle \overset{9.23}{\mapsto} \{A^{ij}\} \overset{10.9}{\underset{10.11}{\mapsto}} V_A \overset{10.28}{\mapsto} \chi_A \tag{10.44}$$

with projective representation ... $(V_A \equiv V_1)$

$$V_A(1) = \mathbb{1}\,,\quad V_A(x) = \hat{\sigma}^x\,,\quad V_A(z) = \hat{\sigma}^z\,,\quad V_A(xz) = \hat{\sigma}^x\hat{\sigma}^z \tag{10.45}$$

¡! The whole point of our previous discussion was that it does not matter which cocycle/projective representation from the cohomology class we choose.

...and representative cocycle $(\chi_A \equiv \chi_1)$

$$\boxed{\chi_A(x,z) = +1 \quad \text{and} \quad \chi_A(z,x) = -1} \tag{10.46}$$

There are of course more combinations of group elements.
However, these two are sufficient for the arguments → *below*.

ii | *Summary for Phase B:*

$$\left.\begin{matrix} t<0 \\ w=0 \end{matrix}\right\} \overset{8.6}{\mapsto} \hat{H}^B_{\text{bSSH}} \overset{8.31}{\mapsto} |B\rangle \overset{9.26}{\mapsto} \{B^{ij}\} \overset{\circ}{\mapsto} V_B \overset{\circ}{\mapsto} \chi_B \tag{10.47}$$

with projective representation ...

$$V_B(1) = 1\,,\quad V_B(x) \mathrel{\hat{=}} 1\,,\quad V_B(z) \mathrel{\hat{=}} 1\,,\quad V_B(xz) = 1 \tag{10.48}$$

This is straightforward to show:

$$\tilde{B}^{i'j'}_{\alpha\beta} \overset{10.8}{\underset{9.26}{=}} \sum_{i,j} \sigma^x_{i'i}\sigma^x_{j'j}\sigma^x_{ij} = \sigma^x_{i'j'} \quad \equiv \gamma(x)\,[1\cdot B^{i'j'}\cdot 1]_{\alpha\beta} \tag{10.49a}$$

$$\tilde{B}^{i'j'}_{\alpha\beta} = \sum_{i,j} \sigma^z_{i'i}\sigma^z_{j'j}\sigma^x_{ij} = -\sigma^x_{i'j'} \equiv \gamma(z)\,[1\cdot B^{i'j'}\cdot 1]_{\alpha\beta} \tag{10.49b}$$

with the (non-trivial) one-dimensional representation $\gamma(g)$ of D_2.

...and representative *trivial* cocycle

$$\chi_B(g_1,g_2)=1 \quad \text{for all} \quad g_1,g_2\in D_2=\{1,x,z,xz\} \tag{10.50}$$

This cocycle belongs to the cohomology class [1] which is the identity element of $H^2(D_2,\mathrm{U}(1))$.

Note that this also follows directly from the fact that $|B\rangle$ is a *product state* and therefore has bond dimension $D=1$. Consequently, $V_B(g)$ must be a phase (a 1×1 unitary matrix) and the representation (10.48) can always be achieved by introducing appropriate factors $f(g)$.

iii | *Observation:* $|B\rangle$ product state → Phase B = Trivial phase

Hypothesis: Phase A is *non-trivial* SPT phase protected by D_2 ...

To show: $[\chi_A]\neq[\chi_B]=[1]$

To prove this, we must show that the equivalence relation (10.32)

$$\chi_A(g_1,g_2)\overset{?}{=}\frac{f(g_1)f(g_2)}{f(g_1g_2)}\chi_B(g_1,g_2)\overset{10.50}{=}\frac{f(g_1)f(g_2)}{f(g_1g_2)} \tag{10.51}$$

has *no solution* $f(g)$.

⊲ Reductio ad absurdum:

Assume $f:G\to\mathrm{U}(1)$ exists →

$$+1\overset{10.46}{=}\chi_A(x,z)\overset{10.51}{=}\frac{f(x)f(z)}{f(xz)}\underset{\text{abelian}}{\overset{D_2}{=}}\frac{f(z)f(x)}{f(zx)}\overset{10.51}{=}\chi_A(z,x)\overset{10.46}{=}-1 \quad ↯ \tag{10.52}$$

→ $[\chi_A]\neq[1]$ is non-trivial element in $H^2(D_2,\mathrm{U}(1))$

→ Phase A = Non-trivial SPT phase protected by D_2 ☺

- With this we showed that Phase A (represented by $|A\rangle$) is a non-trivial topological phase protected by D_2 with the on-site representation $\rho_k(a)=\sigma^a_{2k-1}\sigma^a_{2k}$ $(a=x,z)$. Without either breaking this symmetry or closing the gap, $|A\rangle$ cannot be adiabatically connected to a product state in Phase B!
- In addition, we showed that $H^2(D_2,\mathrm{U}(1))\neq\{1\}$ is non-trivial as it contains at least two elements: [1] and $[\chi_A]$. One can show that there are no more inequivalent classes and therefore $H^2(D_2,\mathrm{U}(1))=\mathbb{Z}_2$ (which is a well-known fact in mathematics). Physically, this means that there are only *two* phases possible in one-dimensional bosonic systems that are protected by $D_2=\mathbb{Z}_2\times\mathbb{Z}_2$: the trivial phase [1] (a representative of which is $|B\rangle$) and a topological phase $[\chi_A]$ (a representative of which is $|A\rangle$). This can be read off the respective classification tables for SPT phases, see e.g. Ref. [47, Table I].

iv | Comparison to ← *fermionic classification*: (Chapter 6)

a | ⊲ Jordan-Wigner transformation $\mathfrak{J}$: (➔ Problemset 8)

$$\mathfrak{J}[a_i]=\prod_{k=1}^{2i-2}\sigma^z_k\cdot\sigma^+_{2i-1} \quad \text{and} \quad \mathfrak{J}[b_i]=\prod_{k=1}^{2i-1}\sigma^z_k\cdot\sigma^+_{2i} \tag{10.53}$$

a_i, b_i ($i = 1, \dots, L$): Fermionic annihilation operators

$\triangleleft$ Fermionic SSH chain for OBC $\overset{\circ}{\to}$

$$\mathfrak{J}\left[\hat{H}_{\mathrm{SSH}}\right] \overset{8.6\,10.53\,4.10}{=} \hat{H}_{\mathrm{bSSH}} \qquad (L' = L-1) \tag{10.54}$$

→ $\hat{H}_{\mathrm{bSSH}}$ and $\hat{H}_{\mathrm{SSH}}$ are unitarily equivalent!

This implies that $\hat{H}_{\mathrm{bSSH}}$ can be solved exactly (via the Jordan-Wigner transformation).

→ Did we gain anything? Why not use the fermion classification in Chapter 6?

b | $\triangleleft$ Perturbations $\sigma^z_{2i-1}\sigma^z_{2i}$:

$$\hat{H}_{\mathrm{bSSH}} \quad\mapsto\quad \hat{H}_{\mathrm{bSSH}} + \delta_t \sum_{i=1}^{L} \sigma^z_{2i-1}\sigma^z_{2i} + \delta_w \sum_{i=1}^{L-1} \sigma^z_{2i}\sigma^z_{2i+1} \tag{10.55}$$

$\delta_t, \delta_w \in \mathbb{R}$: Additional parameters

Question: Can δ_t, δ_w (together with t, w) be used to adiabatically connect $|A\rangle$ and $|B\rangle$?

Observation:

$$\left[\sigma^z_k\sigma^z_{k+1}, X\right] \overset{8.17}{=} 0 \quad \text{and} \quad \left[\sigma^z_k\sigma^z_{k+1}, Z\right] \overset{8.17}{=} 0 \tag{10.56}$$

→ Ground state of Eq. (10.55) remains D_2-symmetric

→ Cohomology classification:

We cannot use the new terms to connect $|A\rangle$ and $|B\rangle$ unless the gap closes!

c | Could we have concluded the same from the fermion classification?

→ $\triangleleft$ Jordan-Wigner transformation of perturbation:

$$\mathfrak{J}\Big[\underbrace{(1-2a_i^\dagger a_i)(1-2b_i^\dagger b_i)}_{\text{↯ Interactions!}}\Big] \overset{\circ}{=} \sigma^z_{2i-1}\sigma^z_{2i} \tag{10.57}$$

→ The classification of *non-interacting* fermions has nothing to say about this ☹

Note that the Jordan-Wigner transformed fermionic Hamiltonian of Eq. (10.55) for $\delta_t, \delta_w \neq 0$ *can no longer be encoded by a single-particle Hamiltonian* so that our complete fermionic classification toolbox (bands, SP matrices, …) becomes useless!

→ Cohomology classification is much more versatile ☺

- There is of course another reason why the periodic table in Section 6.2 cannot answer our question: Our D_2 symmetry is *unitarily* realized – and we factored out all unitary symmetries and focussed only on the "generic symmetries" T, C, and S. This, however, is only a technical inconvenience and not a fundamental problem. Within the framework of non-interacting fermions, one *can* also study SPTs protected by unitary symmetries. Conversely, we could have considered an antiunitary time-reversal symmetry on the spin system (instead of D_2, → Section 10.5). In any case, the fundamental difference between the cohomology classification of this chapter and the classification developed Chapter 6 is that the former captures interacting systems and the latter does not.

- Remember that we already know that the periodic table is changed if interactions are allowed (← Section 6.4 and ➲ Problemset 8). However, we only studied one example and lack a systematic classification of *interacting* fermions (→ *below*).

d | Corollary:

This example suggests that the cohomology classification of interacting spin systems in one dimension can be used to classify *interacting fermionic* SPT phases as well:

$$\left\{ \begin{matrix} \text{Interacting 1D} \\ \textit{fermionic}\text{ SPTs} \end{matrix} \right\} \xleftrightarrow[\text{transformation}]{\text{Jordan-Wigner}} \left\{ \begin{matrix} \text{Interacting 1D} \\ \textit{spin}\text{ SPTs} \end{matrix} \right\} \tag{10.58}$$

- This is a special feature of *one-dimensional* systems because there *local* spin systems (with parity symmetry) map to *local* fermion systems and vice versa.
- There are a few subtleties regarding this mapping (related to fermion parity, recall Section 5.5). For more details see Ref. [29, Section V].