↓ **Lecture 20** [04.07.25]

iv | Problem: H is complex ☹ (= we cannot interpret it as a classical coupling matrix)

Solution:

⊲ Unitary rotation in Kramers-degenerate subspace:

$$D := U^\dagger H U \hat{=} \underbrace{\begin{pmatrix} \mathrm{Re}\, H & \mathrm{Im}\, H \\ \mathrm{Im}\, H^T & \mathrm{Re}\, H \end{pmatrix}}_{\in \mathbb{R}^{2N\times 2N}} \quad \text{with} \quad U = \underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}}_{u} \otimes \mathbb{1}_{\text{Lattice}} \tag{7.7}$$

with $\mathrm{Re}\, H_\uparrow = \mathrm{Re}\, H_\downarrow \equiv \mathrm{Re}\, H$ and $\mathrm{Im}\, H_\downarrow = \mathrm{Im}\, H_\uparrow^* = \mathrm{Im}\, H_\uparrow^T \equiv \mathrm{Im}\, H^T$

Use that $\mathrm{Re}\, H = \frac{1}{2}(H + H^*)$ and $\mathrm{Im}\, H = -\frac{i}{2}(H - H^*)$.

¡! Note that U is a *local unitary* in pseudo-spin-space so that edge modes of H remain edge modes of D.

→ D real and symmetric ☺

Positive semi-definiteness can always be achieved through shifting by a constant offset: $D \mapsto D + \text{const} \times \mathbb{1}$; clearly this does not affect the existence of edge modes.

v | → Transformed basis:

$$\begin{pmatrix} X_{xy} \\ Y_{xy} \end{pmatrix} \equiv u^\dagger \begin{pmatrix} \psi_{xy\uparrow} \\ \psi_{xy\downarrow} \end{pmatrix} \qquad \text{with amplitudes} \quad \psi_{xy\{\uparrow,\downarrow\}} \in \mathbb{C} \tag{7.8}$$

$(X_{xy}, Y_{xy}) \in \mathbb{C}^2$: Position (and phase) of 2D harmonic oscillator on site (x, y)

¡! To accommodate the two "spin" degrees of freedom per lattice site, we will need a 2D harmonic oscillator or, equivalently, *two* 1D pendulums on every site (x, y), → *below*.

⊲ Spin-up mode on site (x, y):

$$\Psi_{xy\uparrow}(t) \equiv \overbrace{\Psi_0\, e^{-i\omega t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{xy}}^{\text{Eigen}\textit{state}\text{ “Spin UP”}} \in \mathbb{C}^2 \tag{7.9a}$$

$$\xleftrightarrow[\text{Translation}]{\text{QS-CS}} \quad u^\dagger \Psi_{xy\uparrow}(t) = \underbrace{\Psi_0 e^{-i\omega t} \overbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}_{xy}}^{\downarrow \textit{Jones vector}}}_{\text{Eigen}\textit{mode}\text{ “LEFT circular”}} \tag{7.9b}$$

As usual, the real-valued oscillation amplitude is given by the real part of the complex vector:

$$\Rightarrow \quad \mathrm{Re}\left[u^\dagger \Psi_{xy\uparrow}(t)\right] \propto \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}_{xy} \tag{7.10}$$

→

$$\boxed{\begin{array}{rcl} \text{Spin UP} & \leftrightarrow & \text{LEFT circular polarization} \\ \text{Spin DOWN} & \leftrightarrow & \text{RIGHT circular polarization} \end{array}} \qquad \begin{array}{r} (7.11a) \\ (7.11b) \end{array}$$

vi | Requirements on the …

Setup:

- Hofstadter model with $q = 3$ → 3 sites per magnetic unit cell
- Two-fold spin/polarization degeneracy per site
  → 2 harmonic oscillators (= 1D pendulums) per site: X_{xy} and Y_{xy}

→ 6 × 1D pendulums per unit cell coupled by springs according to D:

- Re H → XX- and YY-couplings
- Im H → XY-couplings

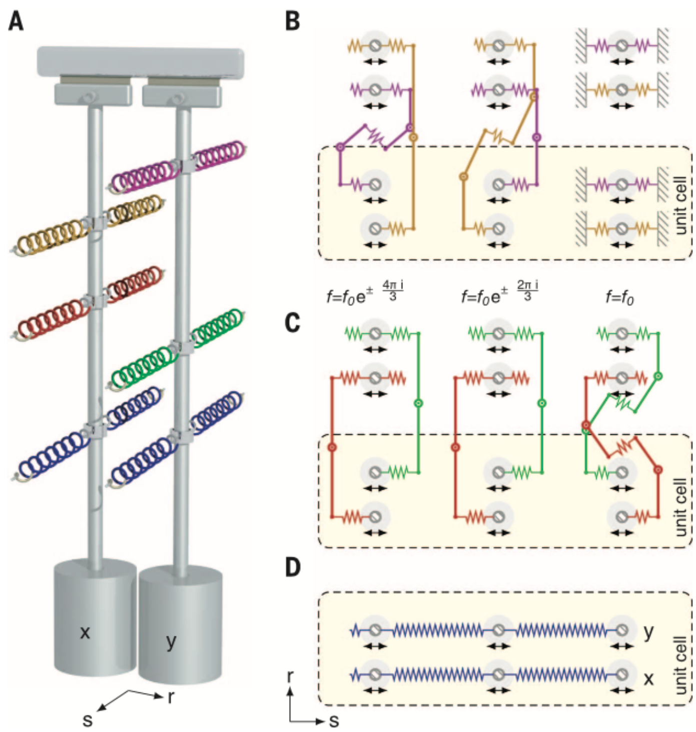
This follows from the coupling matrix in Eq. (7.7).

6 | Experiment & Results:

The following figures are taken from Ref. [188].

i | Construction:

Array of $9 \times 15 = 135$ sites with 270 pendula, each of 500 mm length and 500 g heavy, coupled by springs and lever arms; the whole setup is suspended from the ceiling:



Coupling in x/r-direction via lever arms:

- *One* lever arm → *Negative* coupling
- *Two* lever arms → *Positive* coupling

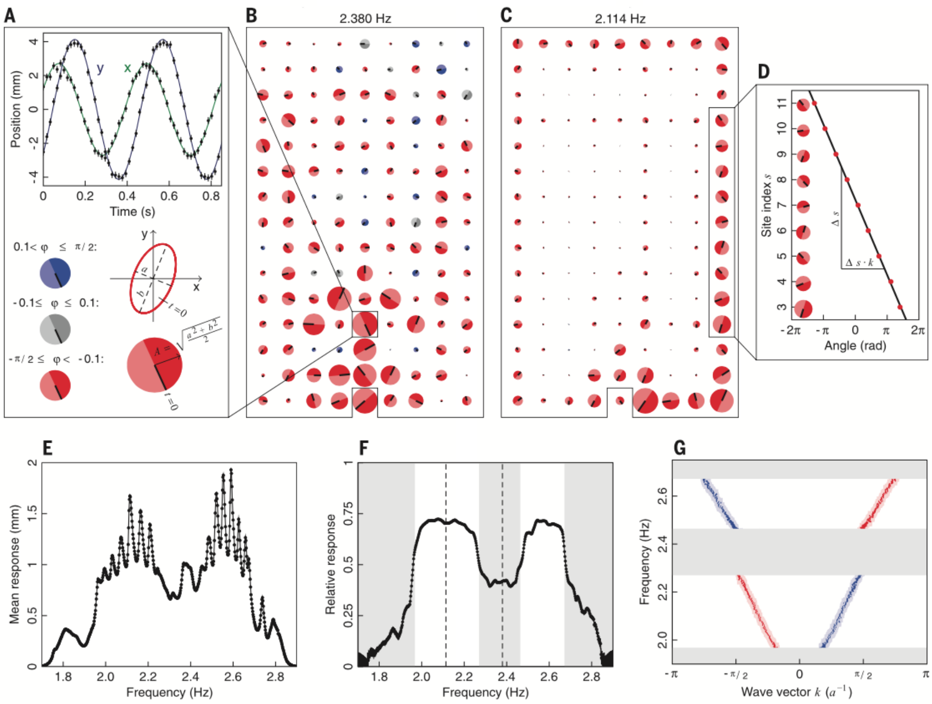
Note that due to the Landau gauge [Eqs. (7.4b) and (7.7)], there are only real, positive couplings in y/s-direction which can be implemented by direct XX- and YY-spring couplings without lever arms (panel D). In x/r-direction there are two classes of couplings (each with three subtypes depending on the y/s-position): Cross-couplings between pendulums of type X and Y given by the entries in Im H (panel B), and couplings between pendulums of the same type given by the entries in Re H (panel C).

Here is a video taken from the supplementary material of Ref. [190] where the lever arms coupling adjacent pendulums can be seen in action:

➲ **Topological Mechanics: Couplings**

ii | Edge modes:

The spectral/dynamical properties of the system can be probed by actively driving a pair of pendulums on the boundary and measuring the response of the other pendulums (by tracking their motion with a camera):



- **a** | Drive system with circular polarization on one site on the edge
- **b** | Track $(X_{xy}(t), Y_{xy}(t))$ for every site after steady state is reached (panel A)

  → Average amplitude A_{xy} (size of circle), polarization (color of circle)
- **c** | Strong response A_{xy} on edges for excitation frequency in band gap (panel F and C)

  → Edge modes ☺
- **d** | Measure orientation/phase in X-Y-plane at fixed time on boundary (black lines in circles)

  → Dispersion of edge modes

  → Two helical edge modes in each band gap (panel G)

Here are two videos taken from the supplementary material of Ref. [190]. They show the complete system from below with an overlay that indicates the oscillation amplitude of each pendulum:

➲ **Topological Mechanics: Edge modes**

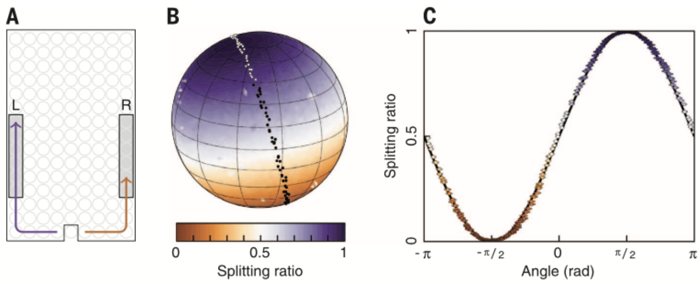
Here, one pair of pendulums on the boundary (center right) is actively driven with left circular polarization and a frequency within the band gap. This excites helical edge modes that propagate oscillation energy only in one direction along the boundary.

Here is another video where the active driving stops after a short time so that a wave packet of edge excitations propagates along the boundary:

➲ **Topological Mechanics: Wave packet**

iii | Application: Beam splitter

Classical systems tailored to feature topological bands (and edge modes) are referred to as ↑ *topological metamaterials*. They have various applications in engineering. For example, the construction discussed here can be used as a "beam splitter" that routes oscillations according to their (circular) polarization:



a | Drive single site on boundary with *linear* (or any other) polarization

→ Superposition of left- and right circular polarization

b | Helical edge modes

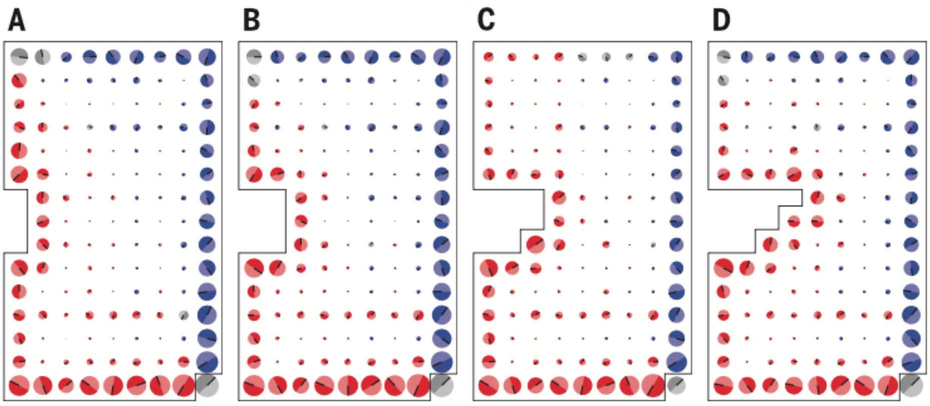
→ Right (left) circular polarized waves propagate to the left (right)

→ "Beam splitter" for phonons

For more applications of this setup see Ref. [190].

iv | Robustness:

The robustness of the topological band structure against weak, symmetry-preserving perturbations translates to the edge modes. Consequently, they should be robust (remain scattering-free) under any perturbation that keeps the bulk gap open and does not violate the time-reversal symmetry in Eq. (7.5). For example, removing some sites on the boundary should have no detrimental effect except deforming the edge modes to follow the new boundary:



- In the plots above, the site in the lower-right corner is driven with *linear* polarization. This triggers left- and right-polarized oscillations to propagate in different directions along the edge. Note how the oscillations stick to the edges even if the boundary is deformed by removing sites.
- Note that this behaviour is very different from ↑ *whispering gallery modes* which are bound to *concave* surfaces and would scatter into the bulk when hitting a *convex* obstacle on the boundary.

Let us derive the constraints needed for the presence & robustness of the edge modes:

a | Remember that the edge modes in class **AII** are protected by time-reversal symmetry:

$$H \overset{!}{\underset{7.5}{=}} T_{\frac{1}{2}} H T_{\frac{1}{2}}^{-1} = U_T H^* U_T^\dagger = U_T (U U^\dagger H U U^\dagger)^* U_T^\dagger \tag{7.12a}$$

$$\Leftrightarrow \quad D \overset{7.7}{=} U^\dagger H U \overset{!}{=} (U^\dagger U_T U^*) D (U^{*\dagger} U_T^\dagger U) \equiv S^\dagger D S \tag{7.12b}$$

Here we used that $D = U^\dagger H U$ is real-valued by construction.

with unitary symmetry

$$S := U^{*\dagger} U_T^\dagger U \overset{7.5}{\underset{7.7}{=}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{7.13}$$

b | → Constraints on the coupling matrix:

$$D = \begin{pmatrix} D_{XX} & D_{XY} \\ D_{YX} & D_{YY} \end{pmatrix} \xRightarrow[7.13]{7.12b} \begin{cases} D_{XX} \overset{!}{=} D_{YY} \quad \text{and} \\ D_{XY} \overset{!}{=} -D_{YX} \end{cases} \tag{7.14}$$

Note that $D_{XX} = D_{YY} = \operatorname{Re} H$, $D_{XY} = \operatorname{Im} H$ and $D_{YX} = \operatorname{Im} H^T = \operatorname{Im} H^* = -\operatorname{Im} H$ satisfy these constraints. In addition, Newton's third law always imposes the constraint $D^T = D$ so that $D_{XY}^T = D_{YX}$ and the constraint becomes $D_{XY}^T \overset{!}{=} -D_{XY}$ for the spring strength coupling different types of pendulums.

c | → In particular, Perturbations of the form

$$D' = D + \begin{pmatrix} \delta D & 0 \\ 0 & \delta D \end{pmatrix} \tag{7.15}$$

do not destroy the edge modes!

Note that the spring strength on different sites *can* vary as there are no additional constraints imposed by S on δD. Thus, to protect the symmetry, a *local* tuning of parameters on each site is sufficient.

→ *Local* symmetry constraint

So while the symmetry S that protects the edge modes is not an intrinsic symmetry of the system (governed by classical mechanics), it can be "fine-tuned" by *locally* adjusting the spring constants; global uniformity (= translational invariance) is not necessary!

7 | Final comment:

- In Section 4.6 we studied the SSH chain and identified two degenerate edge modes at the endpoints of the chain. These modes are protected by ← *sublattice symmetry* (symmetry class **AIII** in $D = 1$ dimensions).
- In this section, we have seen how such topological edge modes can be realized in classical mechanics systems. The bottom line is that topological band structures and edge modes are properties of certain classes of *matrices*. These matrices can play different physical roles [cf. Eq. (7.2) and Eq. (7.3)] – in quantum *and* classical systems.

→

This finally explains the construction in Section 0.1, where we used the topological edge modes of the SSH chain to transfer oscillations from one end of a chain of pendulums to the other end.

The two types of disorder we studied (imperfect springs vs. imperfect pendulums) correspond to SLS-symmetric and SLS-breaking disorder, respectively. This explains why imperfect springs could not remove the degeneracy of the edge modes (so that perfect transfer was still possible), whereas imperfect pendulums split the degeneracy of the edge modes (so that no perfect transfer could be achieved).

Note that the rather complicated construction needed in this section to translate the doubled Hofstadter model into a classical mechanics scenario is not necessary for the SSH chain, since the single particle Hamiltonian Eq. (4.10) of the SSH chain is already real and symmetric to begin with.

As a closing remark, let me stress again that …

> Classical systems with topological band structures ("topological metamaterials") are *not* topological phases (neither classical nor quantum)!

So, strictly speaking, topological mechanics has no place in a course on *topological quantum many-body physics*. However, to understand a domain in depth, it is mandatory to delineate its boundaries and learn about adjacent fields as well …

7.3. More classical systems with topological features

What follows is a brief (and incomplete) list of classical setups where topological edge modes have been proposed and/or experimentally realized. Many of these systems are studied with applications in mind (like efficient transport of signals or energy):

- ↑ *Topological mechanics* (Review: Ref. [189]):
  - The field was pioneered by Kane and Lubensky who studied mechanical constructs called *isostatic lattices* [191]. These systems can exhibit robust zero-frequency modes of topological origin at boundaries ("floppy modes").
  - While zero-frequency "floppy modes" can influence the low-energy response of a mechanical system, topologically protected, chiral edge modes with non-zero frequency allow for robust and scattering-free excitations.
  - Classical analogues of the time-reversal breaking ← *Chern insulator* can be realized as "gyroscopic phononic crystals" [192, 193].
  - Classical analogues of time-reversal protected quantum spin Hall insulators were studied in Refs. [188, 190] (as discussed in this section).
  - One can also realize ↑ *higher-order topological insulators* (which feature protected *corner* modes) in classical settings [194].
- ↑ *Topological acoustics*:
  - The rationale of topological acoustics is to transmit sound waves through topological (and therefore robust and scattering-free) edge modes [195].
  - An experimental demonstration of an "acoustic Chern insulator" was reported in Ref. [196].
- ↑ *Topological photonics* (Review: Ref. [197]):
  - The field of topological photonics was kick-started by Haldane and Raghu who proposed a classical analog of the Chern insulator for photonic crystals [198, 199].

  - Time-reversal breaking, chiral edge modes for microwaves can be realized in magneto-optical photonic crystals [200].
  - Chiral edge modes without magnetic fields can be realized in a Floquet setting in the optical regime [201].
  - Symmetry-protected analogs of topological insulators have been realized in lattices of photonic resonators [202].
  - A potential application of topological photonics is the construction of "topological insulator lasers" [203–205].
- ↑ *Topoelectrical circuits* (Review: Ref. [206]):
  - Electrical circuits (composed of "lumped elements" like inductors, capacitors, etc.) can exhibit symmetry-protected edge modes [207].
  - For example, ↑ *higher-order topological insulators* have been realized both in the microwave regime [208] and with "topoelectrical circuits" [209].
- ↑ *Topological fluid dynamics*:
  - Fluid dynamics on the surface of rotating spheres (time-reversal symmetry breaking!) exhibits robust, chiral modes that are trapped at the equator and protected by non-zero Chern numbers [210]. Remarkably, such modes have long been known to influence Earth's atmospheric and oceanic flow systems!

Part II.

Symmetry-Protected Topological Phases of Interacting Spin Systems

8. Preliminaries

Up to now we studied topological phases of *non-interacting fermions*.

What about *bosons*?

Remember: The many-body ground state of a system with non-interacting *bosons* is given by the symmetric product state where all bosons occupy the same (lowest energy) single-particle mode, i.e., a ↓ *Bose-Einstein condensate (BEC)* – which is not a gapped phase and in particular not a topological one.

A (physically realistic, i.e., weakly interacting) Bose-Einstein condensate is a (uncharged) superfluid that breaks the global U(1) symmetry (number conservation) spontaneously. Since U(1) is a continuous symmetry, there are necessarily gapless ↑ *Nambu-Goldstone modes* (↑ *Goldstone's theorem*), namely Bogoliubov quasi-particles with a linear dispersion (for low momenta). Thus, for us, a BEC is not a phase of interest for *two* reasons: it is described by ← *Landau symmetry breaking* and it is not a gapped phase to begin with.

→ Topological phases of bosons require *interactions*!

(In particular, there is no analog of Part I for bosons.)

Hence the machinery we developed in Part I to label topological phases of non-interacting fermions (based on topological indices computed from band structures) goes right out the window … ☹☹

This leads to the question:

> How to classify/characterize gapped phases of interacting bosons?

Answering this question in full generality is way too ambitious (this is a matter of ongoing research). To make things easier ("easier" does not mean "easy"), we focus on a particularly interesting subclass of interacting bosonic systems, namely:

⊲ *Interacting spins* in *one* spatial dimension:

> How to classify/characterize gapped phases of interacting spin systems in 1D?

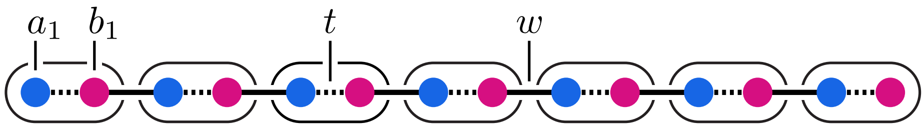
Remember (➔ Problemset 1) that spin systems can be mapped to bosonic systems with strong repulsive on-site interactions, e.g., $n_i(1-n_i)$ for $s=\frac{1}{2}$ (← *hard-core bosons*).

To answer even this simplified question, we need quite a bit of new machinery. As a preparation, we first introduce a toy model to exemplify the rather abstract constructions in this part, and then review some fundamental concepts that are useful later.

8.1. A toy model: Filling the SSH chain with hard-core bosons

The following simple model will be used throughout Chapters 8 to 10 to illustrate the rather abstract concepts that we will introduce:

1 | ⊲ 1D chain of length L with two ← *hard-core bosons* per unit cell:



a_i, b_i: Hard-core bosons $(i = 1, \ldots, L)$

Fermions satisfy the algebra

$$\left\{a_i, a_i^\dagger\right\} = 1 \quad \text{and} \quad \left\{b_i, b_i^\dagger\right\} = 1 \tag{8.1}$$

and all other *anticommutators* vanish. By contrast, hard-core bosons behave like fermions on-site,

$$\left\{x_i, x_i^\dagger\right\} = 1 \quad \text{and} \quad \left\{x_i, x_i\right\} = 0 \tag{8.2}$$

for $x \in \{a, b\}$, but as bosons otherwise:

$$\left[x_i, y_j^\dagger\right] = 0 \quad \text{and} \quad \left[x_i, y_j\right] = 0 \tag{8.3}$$

for $x \in \{a, b\}$ and $i \neq j$ and/or $x \neq y$.

→ ⁂ *Bosonic SSH chain Hamiltonian*:

$$\boxed{\hat{H}_{\text{bSSH}} = \underbrace{t \sum_{i=1}^{L} (a_i^\dagger b_i + b_i^\dagger a_i)}_{\textit{Intra-site}\text{ hopping}} + \underbrace{w \sum_{i=1}^{L'} (b_i^\dagger a_{i+1} + a_{i+1}^\dagger b_i)}_{\textit{Inter-site}\text{ hopping}}} \tag{8.4}$$

- $t, w \in \mathbb{R}$: alternating hopping amplitudes
- $L' = L - 1$ for OBC and $L' = L$ for PBC

- Eq. (8.4) is a strongly interacting bosonic system as there can be only one boson occupying each mode, i.e., there is an infinite repulsive on-site interaction between the bosons.
- ¡! Note that Eq. (8.4) *looks* like the fermionic SSH chain Hamiltonian (4.10). In general, two Hamiltonians that look the same are not necessarily the same (= unitarily equivalent) because the algebra of the operators determines the representation and therefore the spectrum. In this particular case, Eq. (8.4) indeed *is* unitarily equivalent to the fermionic SSH chain Eq. (4.10) (for open boundary conditions) via a ← *Jordan-Wigner transformation* (which is why one can diagonalize it exactly). However, keep in mind that interacting bosonic systems that map to non-interacting fermionic systems are the exception and not the norm.

2 | Rewrite in spin-$\frac{1}{2}$ language:

Remember (➔ Problemset 1) that hard-core bosons can be interpeted as spin-$\frac{1}{2}$ degrees of freedom via the mapping

$$a_i \equiv \sigma_{2i-1}^+, \quad a_i^\dagger \equiv \sigma_{2i-1}^- \quad \text{and} \quad b_i \equiv \sigma_{2i}^+, \quad b_i^\dagger \equiv \sigma_{2i}^- . \tag{8.5}$$

(That we count the spins with indices $j = 1, \dots, 2L$ instead of introducing a- and b-spins is not important and only used to save indices.) Remember that $\sigma_j^\pm = \frac{1}{2}(\sigma_j^x \pm \sigma_j^y)$ with Pauli matrices σ_j^α ($\alpha = x, y, z$). Convince yourself that the algebra of the Pauli matrices makes σ_j^+ and σ_j^- satisfy the algebra of hard-core bosons Eqs. (8.2) and (8.3).

→ Spin-$\frac{1}{2}$ Hamiltonian on $\mathcal{H} = \bigotimes_{j=1}^{2L} \mathbb{C}_j^2$:

$$\hat{H}_{\text{bSSH}} = t \sum_{i=1}^{L} (\sigma_{2i-1}^- \sigma_{2i}^+ + \sigma_{2i}^- \sigma_{2i-1}^+) + w \sum_{i=1}^{L'} (\sigma_{2i}^- \sigma_{2i+1}^+ + \sigma_{2i+1}^- \sigma_{2i}^+) \tag{8.6a}$$

$$= \frac{t}{2} \sum_{i=1}^{L} (\sigma_{2i-1}^x \sigma_{2i}^x + \sigma_{2i-1}^y \sigma_{2i}^y) + \frac{w}{2} \sum_{i=1}^{L'} (\sigma_{2i}^x \sigma_{2i+1}^x + \sigma_{2i}^y \sigma_{2i+1}^y) \tag{8.6b}$$

→ Spin-exchange interactions with alternating strength t and w

We will use the model (8.6) to illustrate various concepts throughout Chapters 8 to 10.

8.2. Reminder: Linear representations of groups

Linear representations of groups are ubiquitous in physics, in particular in quantum mechanics. In the present context, they are important to describe the action of *symmetries* of Hamiltonians on the Hilbert space. Let us briefly review a few important facts:

1 | $\sphericalangle$ Group G (this can be a finite group or a Lie group)

Remember that a group $(G, \bullet)$ is a set G of elements together with a binary map $\bullet : G \times G \to G$ that together satisfy the following axioms:

- *Associativity:* $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for all $a, b, c \in G$
- *Neutral element:* There is a element $e \in G$ such that $a \bullet e = e \bullet a = a$ for all $a \in G$.
- *Inverse elements:* For all $a \in G$ there exists $a^{-1} \in G$ such that $a \bullet a^{-1} = a^{-1} \bullet a = e$.

Note that the axioms imply that both the identity and the inverse elements are unique.

In physics, groups are used (inter alia) to describe *symmetries* of systems. Symmetries are operations that leave something (a Hamiltonian or a state) unchanged. Consequently, one can concatenate such operations to obtain new symmetries. The "do nothing" operation is trivially a symmetry, and for every operation that does not change a system there is naturally another operation that "undoes" it. Together, this makes the set of symmetries a group: a *symmetry group*.

2 | ¡! Note, however, that the *abstract* group G only encodes how the different symmetries relate to each other (for example: applying a 180° rotation twice does nothing). It does not specify how these operations *act* on states of the system …

$\sphericalangle$ Hilbert space $\mathcal{H}$ with ↓ *general linear group* $\mathrm{GL}(\mathcal{H})$

The general linear group of a vector space is the group of invertible linear maps ("matrices") from the space into itself, i.e., $\mathrm{GL}(\mathcal{H})$ is the set of invertible matrices with matrix multiplication $\cdot$ (concatenation of maps) as group multiplication.

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$\triangleleft$ Map $\rho : G \to \mathrm{GL}(\mathcal{H})$ with the property

$$\rho(\underbrace{a \bullet b}_{\substack{\text{Multiplication}\\ \text{in } G}}) = \underbrace{\rho(a) \cdot \rho(b)}_{\substack{\text{Multiplication}\\ \text{in } \mathrm{GL}(\mathcal{H})}} \tag{8.7}$$

$\to$ ρ is a ⁂ *(Linear) representation of G in $\mathcal{H}$*

In the following, we often omit the multiplication symbols $\bullet$ and $\cdot$ as the intended product usually follows from the context.

- Eq. (8.7) implies in particular for every representation:

$$\rho(a) = \rho(a \bullet e) = \rho(a) \cdot \rho(e) \quad \Rightarrow \quad \rho(e) = \mathbb{1} \tag{8.8}$$

  $\mathbb{1}$ is the identity matrix and plays the role of the identity element in $\mathrm{GL}(\mathcal{H})$.

$$\mathbb{1} = \rho(e) = \rho(a \bullet a^{-1}) = \rho(a) \cdot \rho(a^{-1}) \quad \Rightarrow \quad \rho(a^{-1}) = \rho(a)^{-1} \tag{8.9}$$

  Here, a^{-1} denotes the inverse in G and $\rho(a)^{-1}$ the inverse of the matrix $\rho(a)$ in $\mathrm{GL}(\mathcal{H})$.

- A representation "translates" *abstract group elements* to *operators* that act on states in a Hilbert space such that these operators satisfy the multiplicative structure of the group. Hence a representation *represents* the group on state vectors:

$$\mathcal{H} \ni |\Psi\rangle \xrightarrow[g \in G]{\text{Act with symmetry}} |\Psi'\rangle = \rho(g)|\Psi\rangle \in \mathcal{H} \tag{8.10}$$

- In quantum mechanics, we additionally want symmetry operators to map normalized quantum states $|\Psi\rangle$ onto normalized quantum states $|\Psi'\rangle$. Hence we restrict $\mathrm{GL}(\mathcal{H})$ (matrices with non-zero determinant) usually to the subgroup $\mathrm{U}(\mathcal{H})$ of *unitary* maps (matrices with unit-modulus determinants). I.e., we are interested in ⁂ *unitary representations* on the Hilbert space:

$$\rho(a^{-1}) = \rho(a)^{-1} \stackrel{\text{Unitary}}{=} \rho(a)^\dagger \tag{8.11}$$

  The most general form of quantum mechanical symmetries can also apply an additional complex conjugation $\mathcal{K}$, see the proof of ← *Wigner's theorem* on ➔ Problemset 1. Then one deals with ⁂ *antiunitary representations*.

- The map $\rho(g) := \mathbb{1}$ for all $g \in G$ is a representation of every group G, called the ⁂ *trivial representation*. Representations that are injective $[a \neq b \Rightarrow \rho(a) \neq \rho(b)]$ are called ⁂ *faithful*. Such representations carry the complete group structure over to $\mathrm{GL}(\mathcal{H})$. The trivial representation is never faithful (except for the trivial group $G = \{e\}$).

3 | Example:

$\triangleleft$ $G = D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, x, z, xz\}$ with multiplication table

(8.12)

$\bullet$	e	x	z	xz
e	e	x	z	xz
x	x	e	xz	z
z	z	xz	e	x
xz	xz	z	x	e

The defining properties of this group are $x^2 = e$, $z^2 = e$, and $xz = zx$. Since all elements commute, it is an ↓ *abelian group*.

D_2 is known as ↑ *Klein four-group* or ↑ *dihedral group of order 4*. Geometrically, it is the symmetry group of a (non-square) rectangle: x and y correspond to the two mirror symmetries and xy is the 180° rotation.

$\lessdot$ $\mathcal{H} = \mathbb{C}_1^2 \otimes \mathbb{C}_2^2$ (two spin-$\frac{1}{2}$) and define

$$\rho(x) := \sigma_1^x \sigma_2^x \quad \text{and} \quad \rho(z) := \sigma_1^z \sigma_2^z \tag{8.13}$$

$\overset{\circ}{\rightarrow}$ ρ is a 4-dimensional, faithful, linear representation of D_2

- To verify this, check that the defining relations of the group are satisfied on the operator level:

$$\mathbb{1} = \rho(e) \overset{G}{=} \rho(x^2) = \rho(x)\rho(x) = \sigma_1^x\sigma_2^x\sigma_1^x\sigma_2^x = \mathbb{1} \quad \checkmark \tag{8.14a}$$

$$\mathbb{1} = \rho(e) \overset{G}{=} \rho(z^2) = \rho(z)\rho(z) = \sigma_1^z\sigma_2^z\sigma_1^z\sigma_2^z = \mathbb{1} \quad \checkmark \tag{8.14b}$$

$$\sigma_1^x\sigma_2^x\sigma_1^z\sigma_2^z = \rho(x)\rho(z) = \rho(xz) \overset{G}{=} \rho(zx) = \rho(z)\rho(x) = \sigma_1^z\sigma_2^z\sigma_1^x\sigma_2^x \quad \checkmark \tag{8.14c}$$

  Here we used that Pauli matrices square to one and different Pauli matrices of the same spin anticommute with each other.

- Note that our choice to use *two* spin-$\frac{1}{2}$ is not arbitrary. On a *single* spin-$\frac{1}{2}$ ($\mathcal{H} = \mathbb{C}^2$) one could try

$$\rho^?(x) := \sigma^x \quad \text{and} \quad \rho^?(z) := \sigma^z \tag{8.15}$$

  but then one finds

$$\mathbb{1} = \rho^?(e) \overset{G}{=} \rho^?(x^2) = \rho^?(x)\rho^?(x) = \sigma^x\sigma^x = \mathbb{1} \quad \checkmark \tag{8.16a}$$

$$\mathbb{1} = \rho^?(e) \overset{G}{=} \rho(z^2) = \rho^?(z)\rho^?(z) = \sigma^z\sigma^z = \mathbb{1} \quad \checkmark \tag{8.16b}$$

$$\sigma^x\sigma^z = \rho^?(x)\rho^?(z) = \rho^?(xz) \overset{\lightning}{\neq} \rho^?(zx) = \rho^?(z)\rho^?(x) = \sigma^z\sigma^x = -\sigma^x\sigma^z \quad ✗ \tag{8.16c}$$

  The sign in the last equation violates Eq. (8.7) and will become important → *later*.

- The 4-dimensional representation (8.13) is not the "smallest" one. Since D_2 is an abelian group, all its ↑ *irreducible representations* are one-dimensional (a fact from group theory). These, however, are not faithful, and the smallest faithful representation is two-dimensional (can you find it?). We use the representation in (8.13) as example because it is relevant for the bosonic SSH chain → *below*.

4 | We say that a Hamiltonian H on the Hilbert space $\mathcal{H}$ has symmetry group G

$$:\Leftrightarrow \quad \exists \text{ Faithful representation } \rho \text{ of } G \text{ in } \mathcal{H} \text{ such that } [H, \rho(g)] = 0 \text{ for all } g \in G$$

Note that the concept "symmetry of a Hamiltonian" only makes sense when a representation is given since only *operators* on $\mathcal{H}$ can commute with H (and not abstract group elements). Furthermore, the qualifier "faithful" is needed because otherwise we could always choose the trivial representation and every Hamiltonian would have every group as symmetry group (which is a useless concept).

5 | Example:

$\lessdot$ Bosonic SSH chain (8.4) on $\mathcal{H} = \bigotimes_{i=1}^{L} \left[\mathbb{C}_{2i-1}^2 \otimes \mathbb{C}_{2i}^2\right]$

⋖ Faithful linear representation of D_2 on $\mathcal{H}$:

$$\rho(x) := \prod_{i=1}^{L} \sigma^x_{2i-1}\sigma^x_{2i} \equiv X \quad \text{and} \quad \rho(z) := \prod_{i=1}^{L} \sigma^z_{2i-1}\sigma^z_{2i} \equiv Z \tag{8.17}$$

This is the ↑ *product representation* obtained from Eq. (8.13) acting on every unit cell $i = 1, \dots, L$ of the chain. Hence the checks in Eq. (8.14) are sufficient to establish the validity of Eq. (8.7).

$[H_{\text{bSSH}}, X] = 0$ and $[H_{\text{bSSH}}, Z] = 0 \;\rightarrow\; H_{\text{bSSH}}$ has symmetry group D_2

¡! The bosonic SSH chain has many more symmetry groups (→ *later*), but D_2 is the important one for what follows.