

↓ Lecture 2 [11.04.25]

- **3** | Paradigmatic example:
 - $i \mid \triangleleft$ Periodic 1D chain of L spin- $\frac{1}{2}$ with Hamiltonian:

** Transverse-field Ising model (TIM):

$$H_{\text{TIM}} = -J \sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{L} \sigma_i^x \qquad (0.9)$$

where ...

- $J \ge 0$: ferromagnetic coupling strength
- $h \ge 0$: *transverse* magnetic field

"Transverse" since h points in x-direction, which is transverse to the z-direction of the ferromagnetic Ising interactions.

ii | Observation:

$$\left[\sigma_i^z \sigma_{i+1}^z, \sigma_i^x\right] \neq 0 \tag{0.10}$$

- \rightarrow The Ising interactions and the magnetic field terms *cannot* be diagonalized simultaneously!
- \rightarrow Quantum fluctuations

 \rightarrow Ground state(s) = (entangled) *superpositions* of product states $|\uparrow\downarrow\ldots\rangle$ for $h\neq 0$ Product states of the form $|\uparrow\downarrow\ldots\rangle$ are eigenstates of the classical Ising interaction $\sigma_i^z \sigma_{i+1}^z$.

- iii | Two qualitatively different parameter regimes:
 - $a \mid \underline{J \ll h}$:
 - $J \approx 0 \rightarrow Gapped$ phase with *unique* ground state:

$$\Omega_+\rangle \approx |++\dots+\rangle$$
(0.11)

< Spin-spin correlations:

$$\langle \Omega_+ | \sigma_i^z \sigma_j^z | \Omega_+ \rangle \xrightarrow{|i-j| \to \infty} 0$$
 (0.12)

 \rightarrow ** *Paramagnetic phase* (=*disordered* phase)

- Note that $\langle \Omega_+ | \sigma_i^z | \Omega_+ \rangle = 0$, i.e., measuring any spin yields ± 1 with equal probability. The vanishing of spin-spin correlations (0.12) means that there is no correlation between these random outcomes for distant spins. That is, there is *no order* in the ground state.
- For J = 0 and h > 0 the system has a stable bulk gap of △E = 2h, independent of L (the energy cost of flipping a single spin from |+⟩ to |-⟩).
- $\mathbf{b} \mid \underline{J \gg h}:$



 $h \approx 0 \rightarrow$ Gapped phase with two-fold degenerate ground state manifold:

$$|\Omega\rangle \approx \alpha \underbrace{|\uparrow\uparrow\dots\uparrow\rangle}_{|\Omega_{\uparrow}\rangle} + \beta \underbrace{|\downarrow\downarrow\dots\downarrow\rangle}_{|\Omega_{\downarrow}\rangle}$$
(0.13)

$$\langle \Omega | \sigma_i^z \sigma_j^z | \Omega \rangle \xrightarrow{|i-j| \to \infty} 1$$
 (0.14)

This is true for arbitrary amplitudes α and β !

\rightarrow *** Ferromagnetic phase (ordered phase)

- Note that now ⟨Ω|σ_i^z|Ω⟩ ≥ 0 depends on the particular values of α and β; for the "classical" product states it is ⟨Ω_{↑↓}|σ_i^z|Ω_{↑↓}⟩ = ±1. However, the non-vanishing correlations (0.14) imply in any case that z-measurements of distant spins are correlated. That is, there is *order* in the ground state.
- For J > 0 and h = 0 and periodic boundaries, the system has a stable bulk gap of ∆E = 4J, independent of L (the energy cost of flipping a contiguous domain of spins, e.g., |↑↑↑↑⟩ → |↑↓↓↑⟩).
- iv $| \rightarrow$ The z-magnetization σ_i^z is a * local order parameter for the ferromagnetic phase:

$$\lim_{|i-j| \to \infty} \langle \sigma_i^z \sigma_j^z \rangle = 0 \quad \text{in the paramagnetic (disordered) phase}$$
(0.15a)

$$\lim_{|i-j|\to\infty} \langle \sigma_i^z \sigma_j^z \rangle \neq 0 \quad \text{in the ferromagnetic (ordered) phase}$$
(0.15b)

- The very fact that there is a local order parameter that characterizes the ferromagnetic phase makes this particular kind of order locally testable, i.e., by looking at a finite patch of the system, you can decide whether you are in the ferromagnetic or the paramagnetic phase. This makes the ferromagnetic phase a *counterexample* of a *topological* phase (→ *later*).
- Note that [H, σ_i^z] ≠ 0, i.e. correlations of this observable at two distant points are a non-trivial phenomenon.
- v | <u>Comments:</u>
 - So far we only made heuristic arguments regarding the ground states of the TIM Hamiltonian (0.9). Fortunately, this model *can* be solved exactly! Despite the simplicity of the Hamiltonian, this calculation is not straightforward and requires quite a bit of machinery; you solve the model on
 Problemset 7 → *later*.
 - While the TIM Hamiltonian clearly has a stable bulk gap in the two extreme cases (J = 0 and h > 0 vs. J > 0 and h = 0), it is not clear what happens when one adds small perturbations. For example, whether the gap stays open for J > 0 and 0 < h ≪ J is not obvious. The problem is that the gap ΔE is of order unity, but the total operator norm of the magnetic field perturbation goes like h × L, which diverges in the thermodynamic limit L → ∞ for *arbitrarily small* perturbations! [For the TIM this does *not* happen, and the gap remains open for up to some critical value h_c of the magnetic field, but this must be proven (Problemset 7).]



0.4. Spontaneous symmetry breaking

vi | What happends between the two gapped phases for $J \ll h$ and $J \gg h$?

Since the ground state degeneracy of the two gapped phases is different, the gap must close at some critical ratio $g_c = h/J$.

As noted above, we cannot exclude $g_c = 0$ or $g_c = \infty$ with our current knowledge. Here we assume that $0 < g_c < \infty$ (which turns out to be correct).

 \rightarrow Schematic spectrum:



You compute this spectrum exactly later on *Problemset 7*.

vii | Tentative Phase diagram:



 \rightarrow Order parameter *continuous* at phase transition

Again, this is not obvious; but solving the model exactly shows that it is true.

viii | $\rightarrow **$ Continuous (second-order) phase transition:

This is the most typical situation (at least for the models studied in this course), with the following features at the phase transition:

- Bulk gap closes
- Long-range fluctuations and self-similarity (= quantum fluctuations on all length scales)
- Effective conformal field theory (CFT) description
- Algebraic decay of correlations (As compared to exponential decay in gapped phases.)



ix | What characterizes the phase transition?

LEV LANDAU: Spontaneous symmetry breaking!

LANDAU was awarded the Nobel Prize in Physics 1962 for his pioneering work on describing quantum phases of matter, especially the superfluid phase of liquid Helium.

(1) \triangleleft Symmetry group G_S of the TIM *Hamiltonian* (0.9):

$$G_S = \{\mathbb{1}, X\} \simeq \mathbb{Z}_2 \quad \text{with} \quad X := \prod_{i=1}^L \sigma_i^X$$
 (0.16)

X realizes a global flip of all spins: $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$.

Check that $[H_{\text{TIM}}, X] = 0$. Note that $X^2 = \mathbb{1}$ so that $G_S \simeq \mathbb{Z}_2$.

- (2) \triangleleft Symmetry groups G_E of the TIM ground states Eqs. (0.11) and (0.13):
 - Paramagnetic phase:

$$G_E^{\text{(para)}} = \{\mathbb{1}, X\} = G_S \text{ since } X|\Omega_+\rangle = |\Omega_+\rangle$$
 (0.17)

 $\rightarrow ** Symmetric phase$

• Ferromagnetic phase:

$$G_E^{(\text{ferro})} = \{1\} \subsetneq G_S \text{ since } X |\Omega_{\uparrow}\rangle = |\Omega_{\downarrow}\rangle \neq |\Omega_{\uparrow}\rangle$$
 (0.18)

 $\rightarrow *$ * Symmetry-broken phase

i! Important

In the ferromagnetic phase, the ground states $|\Omega_{\uparrow/\downarrow}\rangle$ *spontaneously break* the symmetry group G_S of the Hamiltonian H_{TIM} .

 \rightarrow ** Spontaneous symmetry breaking (SSB)

Landau's paradigm (Spontaneous symmetry breaking)

- 4 | This concept extends to many quantum phases and their phase transitions (e.g. superconductors/superfluids where the particle number symmetry $G_S = U(1)$ is spontaneously broken) and is also applicable to classical phases and phase transitions (e.g. the transition from liquid to solid where rotation and translation symmetry are broken down to crystallographic subgroups).
 - \rightarrow Phases are characterized by the symmetries they break & preserve:

Symmetric	Symmetry-broken			
$G_E = G_S$	$G_E^{(1)}$	$G_{E}^{(2)}$	$G_E^{(3)}$	

Labels of phases = Subgroups $G_E^{(i)}$ of symmetry group G_S

• This concept covers many (quantum and classical) phases and phase transitions, but in the realm of quantum mechanics there is more than just symmetry breaking—there is *entanglement*! This will become important → *below*...



• For the TIM the symmetry group G_S has only itself and the trivial group as subgroups. In general, G_S can be much larger so that many non-trivial subgroups exists (and therefore many different phases are possible). For example, if $G_S = E(3)$ is the Euclidean group of three-dimensional space (continuous rotations and translations), then G_S contains all possible space groups (symmetry groups of crystals) as subgroups.

5 | <u>Comments:</u>

- Note that the spontaneous symmetry breaking of the TIM in 1D is *not* forbidden by the
 ↑ Mermin-Wagner theorem because the broken symmetry is *discrete* (Z₂).
- In one dimension, the spontaneous symmetry breaking (and the ferromagnetic phase) does not survive at finite temperatures T > 0. (Recall that the *classical* Ising model does not have a thermodynamic phase transition in one dimension, i.e., there is no ferromagnetic phase in a classical 1D Ising chain since domain walls can move without energy penalty.) The quantum phase transition of the 1D TIM is therefore a genuine quantum phenomenon, without classical counterpart.
- By contrast, in *two* dimensions (and above) the spontaneous symmetry breaking (and the ferromagnetic phase) *does* survive at finite temperatures T > 0. (Recall that the classical 2D Ising model has a thermodynamic phase transition at a critical temperature T_c below which it enters a ferromagnetic phase that breaks ergodicity.)
- A note on "symmetry breaking" in the quantum case:

The ground state (for h = 0 and J > 0)

$$|\Omega_s\rangle := \frac{1}{\sqrt{2}}|\Omega_{\uparrow}\rangle + \frac{1}{\sqrt{2}}|\Omega_{\downarrow}\rangle = \frac{1}{\sqrt{2}}|\uparrow\uparrow\dots\uparrow\rangle + \frac{1}{\sqrt{2}}|\downarrow\downarrow\dots\downarrow\rangle \qquad (0.19)$$

is clearly *symmetric* under global spin-flips: $X|\Omega_s\rangle = |\Omega_s\rangle$. So what about the symmetry *breaking*? (Note that this is something without analog in a classical setting where you cannot superimpose arbitrary ground states to form new ground states.)

Mathematically, the two symmetry breaking states $|\Omega_{\uparrow}\rangle$ and $|\Omega_{\downarrow}\rangle$ belong to different \uparrow superselection sectors in the thermodynamic limit (they don't live in the same Hilbert space). As a consequence, the "symmetric state" $|\Omega_s\rangle$ is not a state in the Hilbert space of the *infinite* system (strictly speaking, this is the mathematical manifestation of SSB); \uparrow Refs. [24–26].

Physically, the symmetry-broken states $|\Omega_{\uparrow/\downarrow}\rangle$ behave very differently than the symmetryinvariant states $|\Omega_{\uparrow}\rangle \pm |\Omega_{\downarrow}\rangle$: Local measurements (of σ_i^z) immediately collapse the latter into a mixture of the former. I.e. the symmetric states are extremely *fragile* (in contrast to the symmetry-broken states). Thus, in an experiment, one would always observe the *symmetry-broken* states, so that the notion of "spontaneous symmetry breaking" effectively carries over to the quantum realm.

0.5. Extending Landau's paradigm: Topological phases

6 | To understand the deficits of Landau's paradigm, and the conceptual possibility of topological phases, we first need a mathematically more rigorous definition of quantum phases (without spontaneous symmetry breaking!):

* Definition: Gapped quantum phases (formal version)

 \triangleleft Gapped, local Hamiltonians H_a and H_b with unique ground states $|\Omega_a\rangle$ and $|\Omega_b\rangle$.

These two many-body ground states belong to the same quantum phase if and only if there is a family of *gapped* and *local* Hamiltonians $\hat{H}(\alpha)$ (which depends continuously on a parameter



 $\alpha \in [0, 1]$) such that

$$H_a = \hat{H}(0)$$
 and $H_b = \hat{H}(1)$. (0.20)

- The two constraints *"gapped"* and *"local"* ensure that the macroscopic properties of the ground states only change gradually along the path. (This precludes the traversal of phase boundaries where macroscopic properties change qualitatively.)
- i! Note that, strictly speaking, the two Hamiltonians H_a and H_b [and the family Ĥ(α)] are meant to be sequences of Hamiltonians for increasing system sized L → ∞. The condition that the gap remains open along the parameter path thus refers to the thermodynamic limit L → ∞, and not to any finite system. (Note that every finite system has a trivial gap that separates its ground state manifold from the first excited states!)
- The above definition can be extended in a straightforward way to systems with finite (but non-extensive) ground state degeneracies. This allows for an extension of the following concepts to symmetry-broken phases as well (→ *below*).
- 7 | \triangleleft Parameter-space of local Hamiltonians (without SSB, $G_E = G_S$):



* In $D \ge 2$ dimensions the parameter space decomposes into "islands" of gapped Hamiltonians that cannot be connected without closing the gap:

- *Trivial phase:* Ground state = disentangled product state (e.g. $|\Omega_+\rangle = |+\rangle \otimes |+\rangle \otimes \cdots$ or $|\Omega_{\uparrow}\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \otimes \cdots$)
- *Topological phase:* Ground state = long-range entangled state (different patterns of long-range entanglement = different topological phases)

Comments:

• The fact that one-dimensional systems cannot have intrinsic topological order is not obvious.

It follows because the ground states of gapped 1D Hamiltonians (without SSB) are shortrange correlated and feature an area law (the entanglement entropy between different parts of the sytem is constant) [27, 28]. One can therefore encode these states as (short-range correlated) \rightarrow matrix-product states (MPS) with finite \rightarrow bond dimension. It then follows that states of this form can always be mapped to a product state by a quantum circuit of finite depth (\rightarrow below) [6].

i! In this course, we often dinstinguish between *fermionic* systems and *bosonic* systems. Since in our context bosonic systems make only sense with interactions (→ *below*), we also count *spin* systems to this class and often use the terms interchangeably. What makes (interacting) systems *bosonic* is therefore not so much the existence of an infinite-dimensional bosonic Fock space, but rather that the operator algebras of local degrees of freedom commute. Note



that spin- $\frac{1}{2}$ (or \checkmark *qubits*) are equivalent to \checkmark *hard-core bosons* (\bigcirc Problemset 1), i.e., bosons with an infinite, repulsive on-site interaction.

- This splitting can also occur for Hamiltonians *with* SSB and a fixed subgroup G_E . We will not discuss this case in this course (\rightarrow *below*).
- Strictly speaking, the statement that there is no topological order in 1D is only true for *bosonic* systems (or spin systems). For 1D systems of *fermions*, there is a single non-trivial topological phase realized by the → *Majorana chain* (??) [29]. The subtle distinction between 1D bosonic and fermionic systems can be traced back to the non-locality of the → *Jordan-Wigner transformation* that translates between them, and the fact that parity is a locality constraint for fermionic systems.
- 8 | There is an alternative (but mathematically equivalent) definition of quantum phases in terms of local unitary circuits with constant depth:



 \triangleleft ** Local unitary (LU) circuit of depth D_L :

 $\stackrel{*}{\rightarrow}$ $|\Omega_a\rangle$ and $|\Omega_b\rangle$ belong to the same quantum phase, if and only if

$$|\Omega_a\rangle = U_{\rm QC}|\Omega_b\rangle \tag{0.21}$$

where U_{OC} is a local quantum circuit of constant depth $D_L = \text{const}$ for $L \to \infty$.

- This characterization clarifies that two states belong to the same quantum phase if they share the same "pattern of long-range entanglement" since this pattern can only be modified by long-range unitary gates (and not a LU-circuit of constant depth).
- With this characterization, it follows that a ground state |Ω_a⟩ is long-range entangled (= topologically ordered) iff it cannot be transformed into a trivial product state |↑↑↑ ...⟩ by a constant-depth quantum circuit that is local wrt. the geometry of the system.
- This definition can be shown to be equivalent to the one given in the definition above [30]. The unitary can be explicitly expressed as

$$U_{\rm QC} = \mathcal{P} \exp\left[-i \int_0^1 \mathrm{d}\alpha \,\tilde{H}(\alpha)\right] \tag{0.22}$$

where \mathcal{P} denotes the path-ordered exponential and $\tilde{H}(\alpha)$ is a sum of local Hermitian operators that is related (but generally not identical) to the gapped path $\hat{H}(\alpha)$.

• This makes the preparation of topologically ordered states experimentally challenging for quantum computers and quantum simulators with locality constraints: Quantum computers



must apply quantum circuits with a depth (= run time) that scales with the systems size. Similarly, quantum simulators that rely on adiabatic preparation schemes must cross a topological phase transition – which requires the duration of the preparation protocol to scale with the system size as well.

First extension of Landau's paradigm: (Intrinsic) Topological order

9 | The concept of long-range entanglement and equivalence via LU-circuits suggests the following extension of the classification of (gapped) quantum phase of matter:



This motivates the definition:



- We discuss this concept at the end of this course: \rightarrow *Part III*
- ¡! Sometimes the term "topological order" is used sloppily in the literature to refer to any phase of matter with some topological characteristics (e.g., → *symmetry-protected topological phases*). Then the modifier "intrinsic" is used to refer to states with non-trivial long-range entanglement. In this course "topological order" *always* refers to long-range entangled states; however, we still might add "intrinsic" to emphasize this point. By contrast, the term "topological phase" is used much broader and refers to any quantum phase with topological features.