↓ **Lecture 2** [11.04.25]

3 | Paradigmatic example:

i | $\sphericalangle$ Periodic 1D chain of L spin-$\frac{1}{2}$ with Hamiltonian:

⁂ *Transverse-field Ising model* (TIM):

$$H_{\text{TIM}} = -J \sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{L} \sigma_i^x \tag{0.9}$$

where ...

- $J \geq 0$: ferromagnetic coupling strength
- $h \geq 0$: *transverse* magnetic field

  "Transverse" since h points in x-direction, which is transverse to the z-direction of the ferromagnetic Ising interactions.

ii | Observation:

$$[\sigma_i^z \sigma_{i+1}^z, \sigma_i^x] \neq 0 \tag{0.10}$$

→ The Ising interactions and the magnetic field terms *cannot* be diagonalized simultaneously!

→ *Quantum fluctuations*

→ Ground state(s) = (entangled) *superpositions* of product states $|\uparrow\downarrow\dots\rangle$ for $h \neq 0$

Product states of the form $|\uparrow\downarrow\dots\rangle$ are eigenstates of the classical Ising interaction $\sigma_i^z \sigma_{i+1}^z$.

iii | Two qualitatively different parameter regimes:

a | $J \ll h$:

$J \approx 0 \rightarrow$ *Gapped* phase with *unique* ground state:

$$|\Omega_+\rangle \approx |+ + \cdots +\rangle \tag{0.11}$$

$\sphericalangle$ Spin-spin correlations:

$$\langle \Omega_+ | \sigma_i^z \sigma_j^z | \Omega_+ \rangle \xrightarrow{|i-j|\to\infty} 0 \tag{0.12}$$

→ ⁂ *Paramagnetic phase* (=*disordered* phase)

- Note that $\langle \Omega_+ | \sigma_i^z | \Omega_+ \rangle = 0$, i.e., measuring any spin yields ± 1 with equal probability. The vanishing of spin-spin correlations (0.12) means that there is no correlation between these random outcomes for distant spins. That is, there is *no order* in the ground state.
- For $J = 0$ and $h > 0$ the system has a stable bulk gap of $\Delta E = 2h$, independent of L (the energy cost of flipping a single spin from $|+\rangle$ to $|-\rangle$).

b | $J \gg h$:

$h \approx 0 \to$ *Gapped* phase with *two-fold degenerate* ground state manifold:

$$|\Omega\rangle \approx \alpha \underbrace{|\uparrow\uparrow\dots\uparrow\rangle}_{|\Omega_\uparrow\rangle} + \beta \underbrace{|\downarrow\downarrow\dots\downarrow\rangle}_{|\Omega_\downarrow\rangle} \tag{0.13}$$

$\sphericalangle$ Spin-spin correlations:

$$\langle\Omega|\sigma_i^z\sigma_j^z|\Omega\rangle \xrightarrow{|i-j|\to\infty} 1 \tag{0.14}$$

This is true for arbitrary amplitudes α and β!

$\to$ ⁂ *Ferromagnetic phase* (*ordered* phase)

- Note that now $\langle\Omega|\sigma_i^z|\Omega\rangle \gtreqless 0$ depends on the particular values of α and β; for the "classical" product states it is $\langle\Omega_{\uparrow\downarrow}|\sigma_i^z|\Omega_{\uparrow\downarrow}\rangle = \pm 1$. However, the non-vanishing correlations (0.14) imply in any case that z-measurements of distant spins are correlated. That is, there is *order* in the ground state.
- For $J > 0$ and $h = 0$ and periodic boundaries, the system has a stable bulk gap of $\Delta E = 4J$, independent of L (the energy cost of flipping a contiguous domain of spins, e.g., $|\uparrow\uparrow\uparrow\uparrow\rangle \mapsto |\uparrow\downarrow\downarrow\uparrow\rangle$).

iv | $\to$ The z-magnetization σ_i^z is a ⁂ *local order parameter* for the ferromagnetic phase:

$$\lim_{|i-j|\to\infty} \langle\sigma_i^z\sigma_j^z\rangle = 0 \quad \text{in the paramagnetic (disordered) phase} \tag{0.15a}$$

$$\lim_{|i-j|\to\infty} \langle\sigma_i^z\sigma_j^z\rangle \neq 0 \quad \text{in the ferromagnetic (ordered) phase} \tag{0.15b}$$

- The very fact that there is a local order parameter that characterizes the ferromagnetic phase makes this particular kind of order locally testable, i.e., by looking at a finite patch of the system, you can decide whether you are in the ferromagnetic or the paramagnetic phase. This makes the ferromagnetic phase a *counterexample* of a *topological* phase (→ *later*).
- Note that $[H, \sigma_i^z] \neq 0$, i.e. correlations of this observable at two distant points are a non-trivial phenomenon.

v | Comments:

- So far we only made heuristic arguments regarding the ground states of the TIM Hamiltonian (0.9). Fortunately, this model *can* be solved exactly! Despite the simplicity of the Hamiltonian, this calculation is not straightforward and requires quite a bit of machinery; you solve the model on ➔ Problemset 7 → *later*.
- While the TIM Hamiltonian clearly has a stable bulk gap in the two extreme cases ($J = 0$ and $h > 0$ vs. $J > 0$ and $h = 0$), it is not clear what happens when one adds small perturbations. For example, whether the gap stays open for $J > 0$ and $0 < h \ll J$ is not obvious. The problem is that the gap ΔE is of order unity, but the total operator norm of the magnetic field perturbation goes like $h \times L$, which diverges in the thermodynamic limit $L \to \infty$ for *arbitrarily small* perturbations $h > 0$. In general, bulk gaps can therefore vanish under infinitesimally small perturbations! [For the TIM this does *not* happen, and the gap remains open for up to some critical value h_c of the magnetic field, but this must be proven (➔ Problemset 7).]

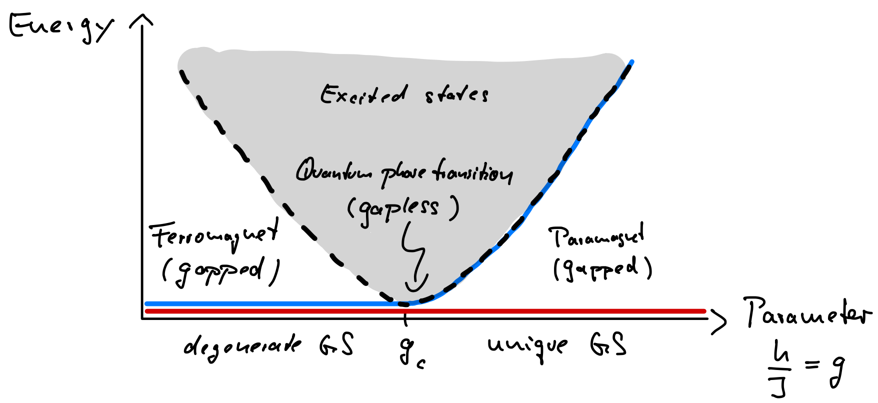
0.4. Spontaneous symmetry breaking

vi | What happends between the two gapped phases for $J \ll h$ and $J \gg h$?

Since the ground state degeneracy of the two gapped phases is different, the gap must close at some critical ratio $g_c = h/J$.

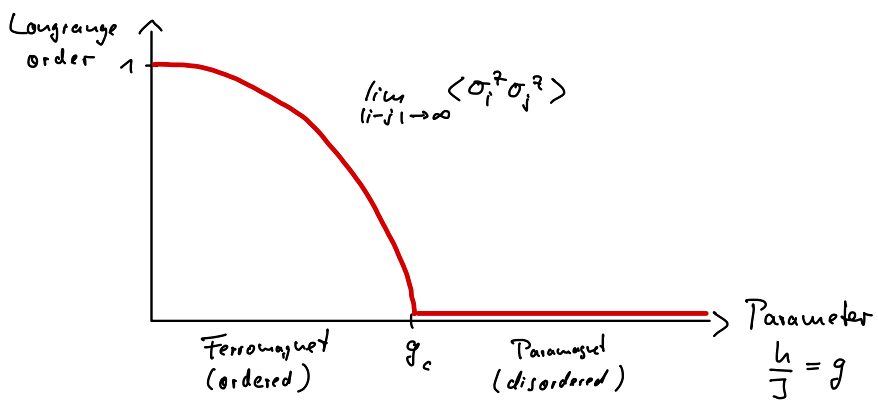
As noted above, we cannot exclude $g_c = 0$ or $g_c = \infty$ with our current knowledge. Here we assume that $0 < g_c < \infty$ (which turns out to be correct).

→ Schematic spectrum:



You compute this spectrum exactly later on ➔ Problemset 7.

vii | Tentative Phase diagram:



→ Order parameter *continuous* at phase transition

Again, this is not obvious; but solving the model exactly shows that it is true.

viii | → ⁂ *Continuous (second-order) phase transition*:

This is the most typical situation (at least for the models studied in this course), with the following features at the phase transition:

- Bulk gap closes
- Long-range fluctuations and self-similarity
  (= quantum fluctuations on all length scales)
- Effective conformal field theory (CFT) description
- Algebraic decay of correlations
  (As compared to exponential decay in gapped phases.)

ix | What characterizes the phase transition?

LEV LANDAU: *Spontaneous symmetry breaking*!

LANDAU was awarded the Nobel Prize in Physics 1962 for his pioneering work on describing quantum phases of matter, especially the superfluid phase of liquid Helium.

(1) ◁ Symmetry group G_S of the TIM *Hamiltonian* (0.9):

$$G_S = \{\mathbb{1}, X\} \simeq \mathbb{Z}_2 \quad \text{with} \quad X := \prod_{i=1}^{L} \sigma_i^x \tag{0.16}$$

X realizes a global flip of all spins: $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$.

Check that $[H_{\text{TIM}}, X] = 0$. Note that $X^2 = \mathbb{1}$ so that $G_S \simeq \mathbb{Z}_2$.

(2) ◁ Symmetry groups G_E of the TIM *ground states* Eqs. (0.11) and (0.13):

- Paramagnetic phase:

$$G_E^{(\text{para})} = \{\mathbb{1}, X\} = G_S \quad \text{since} \quad X|\Omega_+\rangle = |\Omega_+\rangle \tag{0.17}$$

→ ⁂ *Symmetric phase*

- Ferromagnetic phase:

$$G_E^{(\text{ferro})} = \{\mathbb{1}\} \subsetneq G_S \quad \text{since} \quad X|\Omega_\uparrow\rangle = |\Omega_\downarrow\rangle \neq |\Omega_\uparrow\rangle \tag{0.18}$$

→ ⁂ *Symmetry-broken phase*

¡! Important

In the ferromagnetic phase, the ground states $|\Omega_{\uparrow/\downarrow}\rangle$ *spontaneously break* the symmetry group G_S of the Hamiltonian H_{TIM}.

→ ⁂ *Spontaneous symmetry breaking (SSB)*

Landau's paradigm (Spontaneous symmetry breaking)

4 | This concept extends to many quantum phases and their phase transitions (e.g. superconductors/-superfluids where the particle number symmetry $G_S = U(1)$ is spontaneously broken) and is also applicable to classical phases and phase transitions (e.g. the transition from liquid to solid where rotation and translation symmetry are broken down to crystallographic subgroups).

→ Phases are characterized by the symmetries they break & preserve:

Symmetric	Symmetry-broken			
$G_E = G_S$	$G_E^{(1)}$	$G_E^{(2)}$	$G_E^{(3)}$	$\cdots$

Labels of phases = Subgroups $G_E^{(i)}$ of symmetry group G_S

- This concept covers many (quantum and classical) phases and phase transitions, but in the realm of quantum mechanics there is more than just symmetry breaking—there is *entanglement*! This will become important → *below* …

- For the TIM the symmetry group G_S has only itself and the trivial group as subgroups. In general, G_S can be much larger so that many non-trivial subgroups exists (and therefore many different phases are possible). For example, if $G_S = E(3)$ is the Euclidean group of three-dimensional space (continuous rotations and translations), then G_S contains all possible space groups (symmetry groups of crystals) as subgroups.

5 | Comments:

- Note that the spontaneous symmetry breaking of the TIM in 1D is *not* forbidden by the ↑ *Mermin-Wagner theorem* because the broken symmetry is *discrete* ($\mathbb{Z}_2$).
- In *one* dimension, the spontaneous symmetry breaking (and the ferromagnetic phase) does *not* survive at finite temperatures $T > 0$. (Recall that the *classical* Ising model does not have a thermodynamic phase transition in one dimension, i.e., there is no ferromagnetic phase in a classical 1D Ising chain since domain walls can move without energy penalty.) The quantum phase transition of the 1D TIM is therefore a genuine quantum phenomenon, without classical counterpart.
- By contrast, in *two* dimensions (and above) the spontaneous symmetry breaking (and the ferromagnetic phase) *does* survive at finite temperatures $T > 0$. (Recall that the classical 2D Ising model has a thermodynamic phase transition at a critical temperature T_c below which it enters a ferromagnetic phase that breaks ergodicity.)
- A note on "symmetry breaking" in the quantum case:

  The ground state (for $h = 0$ and $J > 0$)

  $$|\Omega_s\rangle := \tfrac{1}{\sqrt{2}}|\Omega_\uparrow\rangle + \tfrac{1}{\sqrt{2}}|\Omega_\downarrow\rangle = \tfrac{1}{\sqrt{2}}|\uparrow\uparrow\dots\uparrow\rangle + \tfrac{1}{\sqrt{2}}|\downarrow\downarrow\dots\downarrow\rangle \tag{0.19}$$

  is clearly *symmetric* under global spin-flips: $X|\Omega_s\rangle = |\Omega_s\rangle$. So what about the symmetry *breaking*? (Note that this is something without analog in a classical setting where you cannot superimpose arbitrary ground states to form new ground states.)

  Mathematically, the two symmetry breaking states $|\Omega_\uparrow\rangle$ and $|\Omega_\downarrow\rangle$ belong to different ↑ *superselection sectors* in the thermodynamic limit (they don't live in the same Hilbert space). As a consequence, the "symmetric state" $|\Omega_s\rangle$ is not a state in the Hilbert space of the *infinite* system (strictly speaking, this is the mathematical manifestation of SSB); ↑ Refs. [24–26].

  Physically, the symmetry-broken states $|\Omega_{\uparrow/\downarrow}\rangle$ behave very differently than the symmetry-invariant states $|\Omega_\uparrow\rangle \pm |\Omega_\downarrow\rangle$: Local measurements (of σ_i^z) immediately collapse the latter into a mixture of the former. I.e. the symmetric states are extremely *fragile* (in contrast to the symmetry-broken states). Thus, in an experiment, one would always observe the *symmetry-broken* states, so that the notion of "spontaneous symmetry breaking" effectively carries over to the quantum realm.

0.5. Extending Landau's paradigm: Topological phases

6 | To understand the deficits of Landau's paradigm, and the conceptual possibility of topological phases, we first need a mathematically more rigorous definition of quantum phases (without spontaneous symmetry breaking!):

⁂ Definition: Gapped quantum phases (formal version)

⊲ Gapped, local Hamiltonians H_a and H_b with unique ground states $|\Omega_\mathrm{a}\rangle$ and $|\Omega_\mathrm{b}\rangle$.

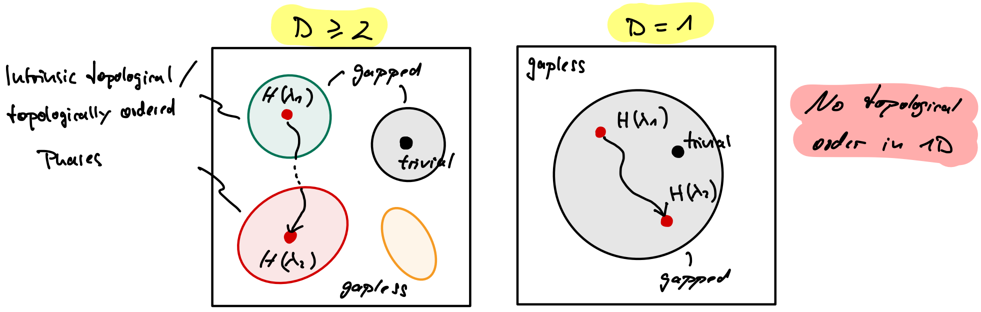
These two many-body ground states belong to the same quantum phase if and only if there is a family of *gapped* and *local* Hamiltonians $\hat{H}(\alpha)$ (which depends continuously on a parameter

$\alpha \in [0,1]$) such that

$$H_a = \hat{H}(0) \quad \text{and} \quad H_b = \hat{H}(1)\,. \tag{0.20}$$

- The two constraints *"gapped"* and *"local"* ensure that the macroscopic properties of the ground states only change gradually along the path. (This precludes the traversal of phase boundaries where macroscopic properties change qualitatively.)
- ¡! Note that, strictly speaking, the two Hamiltonians H_a and H_b [and the family $\hat{H}(\alpha)$] are meant to be *sequences* of Hamiltonians for increasing system sized $L \to \infty$. The condition that the gap remains open along the parameter path thus refers to the *thermodynamic limit* $L \to \infty$, and not to any finite system. (Note that every finite system has a trivial gap that separates its ground state manifold from the first excited states!)
- The above definition can be extended in a straightforward way to systems with finite (but non-extensive) ground state degeneracies. This allows for an extension of the following concepts to symmetry-broken phases as well (→ *below*).

7 | ⊲ Parameter-space of local Hamiltonians (without SSB, $G_E = G_S$):



$\overset{*}{\to}$ In $D \geq 2$ dimensions the parameter space decomposes into "islands" of gapped Hamiltonians that cannot be connected without closing the gap:

- *Trivial phase:* Ground state = disentangled product state
  (e.g. $|\Omega_+\rangle = |+\rangle \otimes |+\rangle \otimes \cdots$ or $|\Omega_\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \otimes \cdots$)
- *Topological phase:* Ground state = long-range entangled state
  (different patterns of long-range entanglement = different topological phases)

Comments:

- The fact that one-dimensional systems cannot have intrinsic topological order is not obvious.

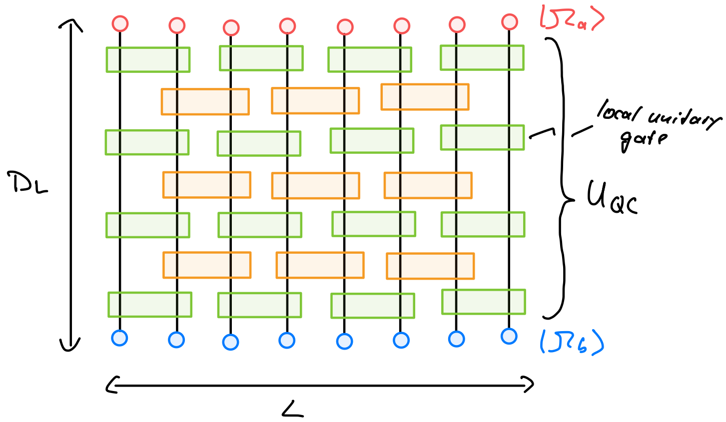
  It follows because the ground states of gapped 1D Hamiltonians (without SSB) are short-range correlated and feature an area law (the entanglement entropy between different parts of the sytem is constant) [27, 28]. One can therefore encode these states as (short-range correlated) → *matrix-product states* (MPS) with finite → *bond dimension*. It then follows that states of this form can always be mapped to a product state by a quantum circuit of finite depth (→ *below*) [6].
- ¡! In this course, we often dinstinguish between *fermionic* systems and *bosonic* systems. Since in our context bosonic systems make only sense with interactions (→ *below*), we also count *spin* systems to this class and often use the terms interchangeably. What makes (interacting) systems *bosonic* is therefore not so much the existence of an infinite-dimensional bosonic Fock space, but rather that the operator algebras of local degrees of freedom commute. Note

that spin-$\frac{1}{2}$ (or ↓ *qubits*) are equivalent to ↓ *hard-core bosons* (➲ Problemset 1), i.e., bosons with an infinite, repulsive on-site interaction.

- This splitting can also occur for Hamiltonians *with* SSB and a fixed subgroup G_E. We will not discuss this case in this course (→ *below*).
- Strictly speaking, the statement that there is no topological order in 1D is only true for *bosonic* systems (or spin systems). For 1D systems of *fermions*, there is a single non-trivial topological phase realized by the → *Majorana chain* (??) [29]. The subtle distinction between 1D bosonic and fermionic systems can be traced back to the non-locality of the → *Jordan-Wigner transformation* that translates between them, and the fact that parity is a locality constraint for fermionic systems.

8 | There is an alternative (but mathematically equivalent) definition of quantum phases in terms of local unitary circuits with constant depth:

⊲ ⁂ *Local unitary (LU) circuit of depth D_L:*



$\xrightarrow{*}$ $|\Omega_a\rangle$ and $|\Omega_b\rangle$ belong to the same quantum phase, if and only if

$$|\Omega_a\rangle = U_{\mathrm{QC}}|\Omega_b\rangle \tag{0.21}$$

where U_{QC} is a local quantum circuit of constant depth $D_L = \text{const}$ for $L \to \infty$.

- This characterization clarifies that two states belong to the same quantum phase if they share the same "pattern of long-range entanglement" since this pattern can only be modified by long-range unitary gates (and not a LU-circuit of constant depth).
- With this characterization, it follows that a ground state $|\Omega_a\rangle$ is long-range entangled (= topologically ordered) iff it cannot be transformed into a trivial product state $|\uparrow\uparrow\uparrow\dots\rangle$ by a constant-depth quantum circuit that is local wrt. the geometry of the system.
- This definition can be shown to be equivalent to the one given in the definition above [30]. The unitary can be explicitly expressed as

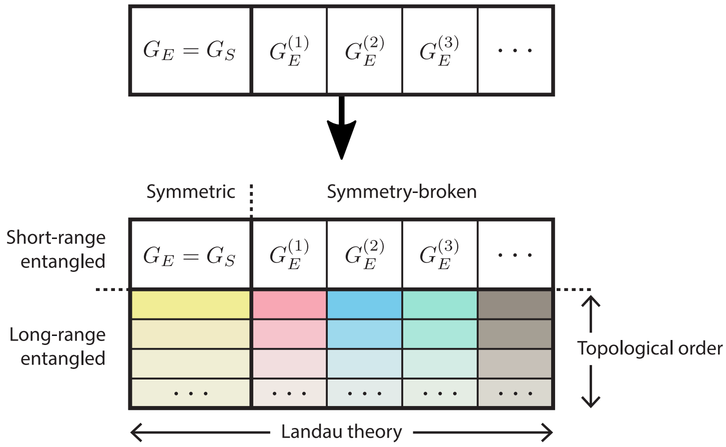
$$U_{\mathrm{QC}} = \mathcal{P}\exp\left[-i\int_0^1 \mathrm{d}\alpha\, \tilde{H}(\alpha)\right] \tag{0.22}$$

where $\mathcal{P}$ denotes the path-ordered exponential and $\tilde{H}(\alpha)$ is a sum of local Hermitian operators that is related (but generally not identical) to the gapped path $\hat{H}(\alpha)$.

- This makes the preparation of topologically ordered states experimentally challenging for quantum computers and quantum simulators with locality constraints: Quantum computers must apply quantum circuits with a depth (= run time) that scales with the systems size. Similarly, quantum simulators that rely on adiabatic preparation schemes must cross a topological phase transition – which requires the duration of the preparation protocol to scale with the system size as well.

First extension of Landau's paradigm: (Intrinsic) Topological order

9 | The concept of long-range entanglement and equivalence via LU-circuits suggests the following extension of the classification of (gapped) quantum phase of matter:



This motivates the definition:

> ⁂ **Definition: Topological order (TO)**
>
> ⁂ *(Intrinsic) Topological order* := Patterns of long-range entanglement

- We discuss this concept at the end of this course: → *Part III*
- ¡! Sometimes the term "topological order" is used sloppily in the literature to refer to any phase of matter with some topological characteristics (e.g., → *symmetry-protected topological phases*). Then the modifier "intrinsic" is used to refer to states with non-trivial long-range entanglement. In this course "topological order" *always* refers to long-range entangled states; however, we still might add "intrinsic" to emphasize this point. By contrast, the term "topological phase" is used much broader and refers to any quantum phase with topological features.