↓ **Lecture 19** [03.07.25]

6 | ⊲ Simplification: $T^d \mapsto S^d$ (we did this before when discussing Skyrmions in Section 2.1.1)

¡! This simplification is done for pedagogic reasons; it is undone → *below* and not part of the full classification.

→

$$\{\text{Topological phases}\} \overset{\text{Physics}}{=} \left\{\begin{array}{l}\text{Equivalence classes of continuous maps}\\ \mathfrak{H}: S^d \to C_0 \text{ that can be continuously}\\ \text{deformed into each other}\end{array}\right\} \quad (6.22a)$$

$$\overset{\text{Math}}{=} \langle d\text{th} \uparrow \textit{Homotopy group} \text{ of } C_0\rangle \equiv \pi_d(C_0) \quad (6.22b)$$

The homotopy group $\pi_d(X)$ is the group of equivalence classes (= homotopy classes) of (base-point-preserving) homeomorphisms (= continuous maps) from the d-dimensional sphere S^d to the topological space X. The special group $\pi_1(X)$ is called ↑ *fundamental group* and describe the topologically different ways closed loops can be drawn on the space X, where two loops are equivalent if they can be continuously deformed into each other (this is the homotopy).

Example for $d = 2$: $\pi_2(C_0) \overset{*}{=} \mathbb{Z}$ → Chern number ☺

Remember that the IQHE (and relatives) belong to class **A** (classified by C_0) and we identified the $\mathbb{Z}$-valued Chern number as label for possible topological phases.

7 | Undo simplification ($S^d \mapsto T^d$) & Include symmetry constraints

(Here we focus on the eight real classes, i.e., **X** ≠ **A**,**AIII**.)

Kitaev & K-Theory [57] $\overset{*}{\to}$

$$\{\text{Topological phases of } (\mathbf{X}, d)\} =$$

$$\underbrace{\pi\left(\bar{T}^d, R_q\right)}_{K_{\mathbb{R}}^{-q}(\bar{T}^d)} \overset{K\text{-theory}}{=} \underbrace{\pi_0(R_{q-d})}_{\substack{\text{⁂ Strong topological index}\\ \to \text{Periodic table}}} \oplus \underbrace{\bigoplus_{s=0}^{d-1} \binom{d}{s} \pi_0(R_{q-s})}_{\substack{\text{⁂ Weak topological indices}\\ \text{(not part of the periodic table)}}} \quad (6.23)$$

- $\pi_0(X)$ is the 0th homotopy group of X; its elements label the *connected components* of X. Since the connectivity of the symmetric spaces R_q is known, the right-hand side of Eq. (6.23) can be looked up in the literature.

  In general, $\pi_0(X)$ is *not* a group but the *set* of path-connected components of X (only $\pi_d(X)$ has a natural group structure for $d \geq 1$). However, in the present case, the structure of the classifying spaces $X = R_q$ in the "stable limit" of many bands endows $\pi_0(R_q)$ with a group structure.

- $\pi\left(\bar{T}^d, R_q\right)$ describes the equivalence classes of all maps $\mathfrak{H}(\boldsymbol{k})$ from the BZ T^d into an appropriately restricted matrix space (which depends on the symmetry class **X**, ↑ *Table 1* in Ref. [92]; for $d = 0$ the target space is the classifying space R_q that belongs to **X**, for $d > 0$ this is only true at the TRIMs) that, in addition, satisfy the symmetry constraints on momenta demanded by the symmetry class **X** (the latter constraint is indicated by the bar of $\bar{T}^d$); this object is known in K-theory as the "real K-group $K_{\mathbb{R}}^{-q}(\bar{T}^d)$ of $\bar{T}^d$." Remember, for example, that TRS relates the Bloch Hamiltonian at momentum $\boldsymbol{k}$ to the Bloch Hamiltonian at momentum $-\boldsymbol{k}$, Eq. (2.31d). These constraints are hidden in the precise definition of $\pi\left(\bar{T}^d, R_q\right)$.

→ Computing $\pi_0(R_{q-d})$ (= strong topological indices) …

- for $q = 0, \ldots, 7$ (real symmetry classes = rows)
- and $d = 0, 1, \ldots$ (dimensions of space = columns)

…yields the periodic table (more precisely: the eight rows of the real symmetry classes)

Comments:

- There is an analog expression for the two *complex* classes **A** and **AIII** (first two rows of the periodic table).
- The contributions labeled "weak topological indices" are not part of the periodic table.

  These additional indices have physical consequences, e.g., for ↑ *weak topological insulators* [57, 96, 97].
- The indices of the classifying spaces R_{q-d} and R_{q-s} are defined modulo 8; for the complex classes, the periodicity is 2 (this is known in K-theory as ↑ *Bott periodicity*). This leads to the periodicity of the *periodic* table in the dimension d and finally explains its name.

  In contrast to the more famous periodic table in chemistry, this one is *really* periodic ☺.

8 | Example for $q = 4$ (**AII**) and $d = 2$ (e.g. ← *Kane-Mele model*):

$$\pi\left(\bar{T}^2, R_4\right) = \pi_0(R_2) \oplus 1 \times \pi_0(R_4) \oplus 2 \times \pi_0(R_3) \tag{6.24}$$

$$= \underbrace{\mathbb{Z}_2}_{\text{Pfaffian index}} \oplus \underbrace{\mathbb{Z}}_{\#\,\text{Valence bands}} \oplus \underbrace{2 \times 0}_{\text{No weak indices}} \tag{6.25}$$

The values for $\pi_0(R_q)$ are provided in Table 2 of Ref. [57] but can also be read off from the $d = 0$ column of the periodic table (replacing $2\mathbb{Z}$ by $\mathbb{Z}$).

6.4. Consequences of interactions

In this part, we focused on *non-interacting* fermions. The crucial feature of such theories is that their MB Hamiltonian $\hat{H}$ can be encoded by a SP Hamiltonian H so that their MB spectrum can be built from the SP spectrum; this makes them exactly (or efficiently) solvable. The periodic table is built on the SP Hamiltonians and is therefore only valid for systems that can be reasonably described by such theories.

The natural question is then of course:

> What happens to the periodic table if *interactions* are included?

It is clear that interactions allow for more "paths" to connect gapped Hamiltonians, so that the classification must become "coarser" (i.e., phases that are separated without interactions may no longer be if interactions are allowed).

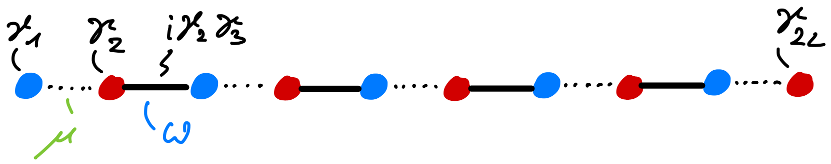
Quick answer:

- A full classification is known for *quartic* interactions (↑ Ref. [182]).
- In $d = 1$ dimensions, (interacting) fermions can be mapped to (interacting) *bosons* and fully classified via techniques that we discuss in Part II (↑ Refs. [29, 183, 184]).
- There is no complete classification known for arbitrary interactions and dimensions (as far as I know).
- This is a topic of ongoing research … (e.g. ↑ Refs. [185, 186])

However, there is an *example* that demonstrates that (and how) the periodic table is modified by interactions for a specific ($\mathbf{X}$, d):

This was worked out by Fidkowski and Kitaev in 2010 [187]. You study this example on ➲ Problemset 8.

1 | ⊲ Majorana chain for $w = \Delta > 0$ and $\mu = 0$: [Remember Eq. (5.39) in Section 5.5]



TRS & PHS → Symmetry class **BDI** in $d = 1$ → $\mathbb{Z}$-index

Remember the $\mathbb{Z}$-valued winding number defined in Section 5.3 which is quantized if TRS is present.

¡! Here we do *not* consider the Majorana chain as representative of class **D** *without* TRS; it turns out that the corresponding $\mathbb{Z}_2$-index is stable under interactions [29].

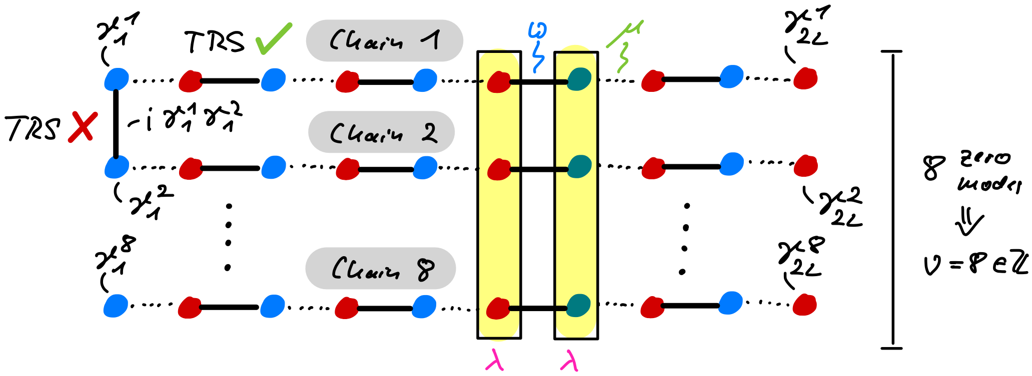
2 | Time-reversal symmetry:

$$\mathcal{T} i \mathcal{T}^{-1} = -i \quad \text{and} \quad \mathcal{T} \gamma_{2i-1} \mathcal{T}^{-1} = +\gamma_{2i-1}\,, \quad \mathcal{T} \gamma_{2i} \mathcal{T}^{-1} = -\gamma_{2i} \tag{6.26}$$

This follows from the "standard" TRS for *spinless* fermions: $\mathcal{T} c_i \mathcal{T}^{-1} = c_i$ and Eq. (5.33).

→ Only quadratic couplings between *even* (red) and *odd* (blue) Majorana modes allowed!

3 | ⊲ Stack of Majorana chains in the topological phase:



Note that one could gap out the edge modes with $i\gamma_1^\alpha \gamma_1^{\alpha+1}$ but these terms *break* TRS (the coupled modes are both either even or odd)!

→ $\mathbb{Z}$-index = # dangling Majorana modes γ_1^α (on one end of the stack)

The chains are *oriented* in that they start with an odd and end with an even mode (which transform differently under TRS). Reversing the orientation of a chain therefore gives a negative index and indeed, a pair of chains with opposite orientation can be gapped out without breaking TRS because the two Majorana modes on one end are even and odd.

⊲ 8 topological chains → **BDI**-index $\nu = 8$

Note that if there is an *odd* number of dangling Majorana modes on one end, you cannot gap them out completely even when breaking TRS because after gapping out all pairs a single mode will be left. This distinguishes the situations with an even and an odd number of Majorana zero modes and corresponds to the $\mathbb{Z}_2$-index of class **D** that does not require TRS.

4 | Idea: Connect topological to trivial phase via quartic interaction:

$$\hat{H}(\mu, w, \lambda) = \hat{H}_{\mathrm{MC}}(\mu, w) + \lambda \sum_{n=1}^{2L} W_n \tag{6.27}$$

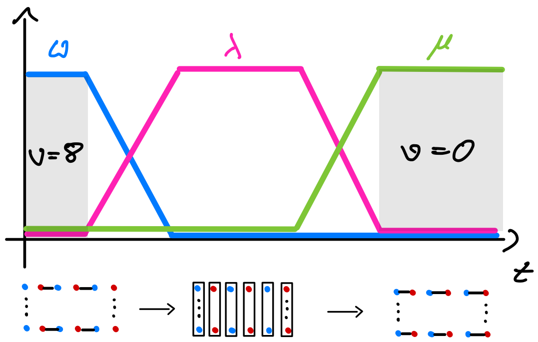
with quartic interaction between the 8 chains

$$W_n = \gamma_n^1 \gamma_n^2 \gamma_n^3 \gamma_n^4 \pm \dots \text{many more quartic terms} \tag{6.28}$$

See ↑ Ref. [187, Eq. (8)] for the full term and its derivation.

¡! The interaction terms W_n commute with the TRS in Eq. (6.26).

⊲ Protocol:



On this continuous path ...

- the bulk gap remains open ...
  This can be shown by exact diagonalization on a unit cell (which contains 8 fermion modes that span a $2^8 = 256$ dimensional Fock space).

  You show this numerically on ➔ Problemset 8.
- and TRS is not broken.
  This is easily checked by inspection.

5 | Conclusion:

$$\boxed{\text{With } \textit{interactions } \nu = 0 \text{ and } \nu = 8 \text{ are the } \textit{same} \text{ phase in } \mathbf{BDI}!} \tag{6.29}$$

→ $\mathbb{Z}$-index of **BDI** in $d = 1$ reduces to $\mathbb{Z}_8$-index

→ *With interactions* there are not infinitely many top. 1D superconductors in **BDI** but only 8!

For an overview how quartic interactions modify the periodic table in other dimensions and for other symmetry classes see Ref. [182].

7. ‡ Topological edge states in classical systems

At the beginning, in Section 0.1, I motivated the exploration of topological phases with a classical mechanics example (coupled classical pendulums) that features remarkably robust "edge modes." Later, in Chapter 4, we identified the underlying matrix structure as that of the quantum mechanical ← *SSH chain* which is a SPT phase protected by ← *sublattice symmetry*. However, we still miss an explicit example of how topological features of quantum systems can carry over to classical systems. Here we discuss an example in two-dimensions that has been experimentally realized and features robust edge modes.

More specifically, we discuss a classical mechanics realization of topological bands derived from the ← *quantum spin Hall effect* that has been described and implemented by Süsstrunk & Huber in 2015 [188]. There is also a review by Sebastian Huber on the broader field of "topological mechanics" [189]. For more references on classical systems with topological bands, see Section 7.3 below or my PhD thesis [126, Section 1.3.2].

7.1. Review: Effects of topological bands

In this course, we have studied various models that realize topological *quantum phases* at $T = 0$. The role of topology was to describe "twists" in the band structure that cannot be undone without closing the gap (or breaking a protecting symmetry). These topologially non-trivial band structures had observable, physical consequences:

- Quantized Hall response:

  Remember: [Eq. (1.98) in Section 1.4.2]

$$\underbrace{\sigma_{xy} = \frac{e^2}{2\pi\hbar}\nu}_{\substack{\text{Computed from} \\ \text{Many-Body Hamiltonian} \\ (\hbar\,\to\,\text{QM}^{\text{inside}})}} \quad \text{with} \quad \underbrace{\nu = \sum_{n:\varepsilon_n<E_F} C^{[n]} \in \mathbb{Z}}_{\substack{\text{Computed from} \\ \text{Single-Particle Hamiltonian} \\ (\text{no }\hbar\,\to\,\text{no QM}^{\text{inside}})}} \tag{7.1}$$

  → The quantization of σ_{xy} requires filled bands

  → Many-body phenomenon (Fermi statistics!)

  → Genuine *quantum effect*!

  Note that $\hbar$ appears in the expression for the quantized Hall conductance, so quantum mechanics must play a role. (You can trace the $\hbar$ back to the Kubo formula and therefore to the propagator/path integral; it is *not* related to the Chern number, that is, "topology.")

- Robust edge modes:

  Remember:

$$\text{Topological system in 2D \& OBC} \xrightarrow{\text{Sections 1.6 and 3.4}} \text{Gapless edge modes}$$

$$\text{Topological system in 1D \& OBC} \xrightarrow{\text{Sections 4.6 and 5.5}} \text{Zero-energy boundary modes}$$

→ Bulk-boundary correspondence
Formally, the ↑ *bulk-boundary correspondence* ensures the existence of robust, gapless edge modes on the boundaries of systems with topological bands (recall Problem 6.2 on ➔ Problemset 6)

→ Single-particle phenomenon (studying the band structure is sufficient)
We need neither the Fock space / Slater determinants (= Fermi statistics) nor time-evolution operators / path integrals to understand the appearance of edge modes.

→ *Not* a quantum effect!

We conclude:

> Topological features of the *band structure* (= single-particle features) are *not* quantum effects!

→ *Question:* Can we translate edge modes to *classical* systems?

¡! Since we will focus on *single-particle* features in the following, it does make no sense to talk about "topological *phases (of matter)*" – neither quantum nor classical. What we will study are topological *features* of classical systems that affect the behaviour of their finite-energy excitations.

7.2. Example: Topological mechanics and helical edge modes

This subsection is based on the paper by Süsstrunk and Huber [188].

1 | Goal:

Realization of the helical edge modes of the QSHE (← Kane-Mele model) in a classical mechanics setup governed by Newton's equation

Rationale: A "phononic topological insulator" would transmit energy (= vibrations/phonons) only along its surface but not through the bulk. The robustness of its topological edge modes would allow for wave guides of arbitrary shape (in contrast to ↑ *wispering gallery modes* that delicately depend on geometry).

2 | Quantum system (QS):

⊲ SP Schrödinger equation for a particle with internal states hopping on a lattice:

$$\underbrace{i\hbar\dot{\Psi}_i^\alpha}_{\text{QM}^{\text{inside}}} = H_{ij}^{\alpha\beta}\Psi_j^\beta \tag{7.2}$$

- i, j: sites
- α, β: spin (or other local degrees of freedom)
- H: SP Hamiltonian = *Hermitian matrix*

Note that *states* in the SP Hilbert space have the form $|\Psi\rangle = \sum_{i,\alpha}\Psi_i^\alpha|i,\alpha\rangle$ where $|i,\alpha\rangle$ describes a particle (fermion) on site i with spin α.

3 | Classical system (CS):

⊲ Newton's equation for N coupled 1D oscillators (e.g., pendulums coupled by springs)

$$\underbrace{\ddot{x}_i}_{\substack{\text{Classical}\\ \text{dynamics}}} = -D_{ij}x_j \qquad i = 1,\dots,N \tag{7.3}$$

The masses of the oscillators are set to one or, equivalently, absorbed into D.

- x_i: position of oscillator i (in 3D it would be $i = 1, \dots, 3N$)
- D: Dynamical coupling matrix = *real, symmetric, positive semi-definite matrix*

First, D only contains spring constants (and masses), thus it is real. Next, Newton's third law (action-reaction law) demands the symmetry of D because D_{ij} (D_{ji}) encodes the force by pendulum j (i) on pendulum i (j). And last, physically, the eigenvalues of D are the *frequencies squared* of the eigenmodes of the system and therefore must be nonnegative.

Remember that for a single 1D pendulum, $m\ddot{x} = -Dx$ leads to an eigenfrequency $\omega^2 = D/m$; to obtain periodic (bounded) solutions, D/m better be non-negative.

4 | Observation:

- QS characterized by *eigenvectors* of H (= Hermitian matrix)
- CS characterized by *eigenvectors* of D (= real, symmetric, positive semi-definite matrix)

In particular: *Edge modes* are simply special eigenvectors of H

→ *Idea:* Use H with edge modes (e.g., from QSHE) to construct D with edge modes!

Note that the *dynamics* of QS [Eq. (7.2)] and CS [Eq. (7.3)] are different (1st vs. 2nd order) but this has no effect on the existence of edge states/modes which is a property of the matrices H and D, respectively.

5 | Model construction:

We now construct a particular topological coupling matrix D that was implemented in Ref. [188].

i | ⊲ Two independent copies of the ← *Hofstadter model* (➲ Problemset 4) with spin-dependent flux $\alpha\hat{\Phi} = \pm p/q = \pm 1/3$:

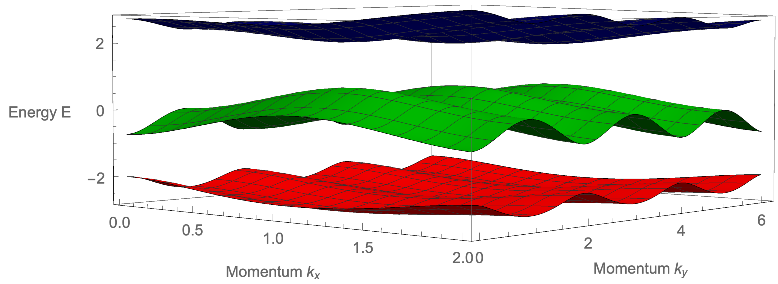
$$H = \begin{pmatrix} H_\uparrow & 0 \\ 0 & H_\downarrow \end{pmatrix} \tag{7.4a}$$

$$\text{with} \quad H_\alpha = \sum_{x,y} \begin{bmatrix} |x, y+1, \alpha\rangle\langle x, y, \alpha| \\ +e^{-2\pi i \alpha \hat{\Phi} y} \, |x+1, y, \alpha\rangle\langle x, y, \alpha| \end{bmatrix} + \text{h.c.} \tag{7.4b}$$

$\alpha \in \{\uparrow, \downarrow\} \equiv \{+1, -1\}$: pseudo-spin index

- H_α is given in Landau gauge.
- This construction is reminiscent to the construction of the ← *Kane-Mele model* in Chapter 3 where we combined two time-inverted copies of the ← *Chern insulator*. Here two copies of the ← *Hofstadter model* are combined to obtain a time-reversal invariant system. We must start from a time-reversal invariant system, since the goal is to realize its classical analogue with springs and pendulums alone (which only allows for time-reversal symmetric physics).
- The choice of the Hofstadter model on the square lattice is motivated by technical considerations (for example, constructing a square lattice of pendulums is easier than constructing a Honeycomb lattice). The flux $p/q = \pm 1/3$ is the one with (1) smallest magnetic unit cell ($q = 3$ sites) that has (2) fully gapped bands ($p/q = \pm 1/2$ is not gapped, recall Problem 4.2 on ➲ Problemset 4).

ii | $q = 3 \to$ Three gapped, spin-degenerate bands: [← Problem 4.2 on ➔ Problemset 4]



Chern numbers: [← Problem 5.1 on ➔ Problemset 5]

Red band: $1 = 3s_1 + t_1 \Rightarrow s_1 = 0,\ t_1 = 1 \quad \Rightarrow C_1 = t_1 - t_0 = +1$

Green band: $2 = 3s_2 + t_2 \Rightarrow s_2 = 1,\ t_2 = -1 \Rightarrow C_2 = t_2 - t_1 = -2$

Blue band: $3 = 3s_3 + t_3 \Rightarrow s_3 = 1,\ t_3 = 0 \quad \Rightarrow C_3 = t_3 - t_2 = +1$

$\to$ TKNN invariant Eq. (1.98):

- $\nu = C_1 = +1$ in red/green gap
- $\nu = C_1 + C_2 = -1$ in green/blue gap

$\lessdot$ Finite sample with open boundaries (like a square):

$\to$ Two helical edge modes (opposite spin/group velocity) *in each gap*

Note that the situation in each of the two gaps is comparable to the Kane-Mele model with upper/lower bands of Chern number ± 1. In each spin sector, a TKNN invariant of $|\nu| = 1$ demands for a single conducting edge channel. Because the signs of ν are different for the two gaps, the edge modes (for a fixed spin) have opposite chiralities (→ *experimental results below*).

iii | *Symmetries:*

- Time-reversal symmetry:

  $T_{\frac{1}{2}} = i\sigma^y \mathcal{K} \equiv U_T \mathcal{K}$ with $T_{\frac{1}{2}}^2 = -\mathbb{1}$ and

$$T_{\frac{1}{2}} H T_{\frac{1}{2}}^{-1} = H \tag{7.5}$$

  since $H_\uparrow = H_\downarrow^*$

  Section 6.2 $\to$ Class **AII** with $\mathbb{Z}_2$ Pfaffian index [← Section 3.3]

  - Here σ^y acts on the "spin space" in Eq. (7.4a), i.e., it is actually $T_{\frac{1}{2}} = i\sigma^y \otimes \mathbb{1}_{\text{Lattice}} \mathcal{K}$ where $\mathbb{1}_{\text{Lattice}}$ acts on the orbital space of H_α.
  - You might wonder: Would the TRS representation $T' = \sigma^x \mathcal{K}$ with $T'^2 = +\mathbb{1}$ not also commute with the Hamiltonian (7.4a)? The answer is of course "yes" since σ^z is also a (unitary) symmetry (→ *next point*) and $i\sigma^z\sigma^y = \sigma^x$. However, a representation with $T'^2 = +\mathbb{1}$ belongs to class **AI** which, according to the periodic table in Section 6.2, *cannot* protect a topological phase in $D = 2$ dimensions. So while the Hamiltonian is consistent with both representations, only $T_{\frac{1}{2}} = i\sigma^y\mathcal{K}$ is restrictive enough to separate a topological phase from the trivial phase (and thereby protect edge modes). [Note that the choice of the representation dictates which perturbations one can add to the Hamiltonian to try to connect it to the trivial phase!]

- Spin conservation:

$$\sigma^z\, H\, \sigma^z = H \tag{7.6}$$

This is the same unitary symmetry that we encountered for the Kane-Mele model in the absence of a Rashba term. It allows for the expression of the Pfaffian invariant in terms of the *Chern numbers* of the spin-polarized bands [← Eq. (3.21)] so that one can avoid the more complicated expression Eq. (3.35) we derived in Section 3.3.