↓ Lecture 18 [27.06.25]

6. Classification of Non-Interacting Fermionic Topological Phases

A good introduction to the classification of topological insulators and superconductors is given by Ludwig [92] (this section is partly based on his paper). A more technical description of the scheme with examples is given by Ryu *et al.* [122] (their detailed introduction is quite useful). A completely different angle on the classification is provided by Kitaev [57] (be warned: this paper looks "simple" as it is extremely high-level but the underlying mathematical framework is very deep).

Goal: By now we have seen various models of non-interacting fermions in one and two dimensions that are classified by different topological indices and protected by different symmetries (or none at all). Since all of these models are described by *band structures*, the question arises whether one can find a *unifying scheme* to classify the topological phases of non-interacting fermions.

The description of such an approach is the goal of this section.

6.1. Generic symmetries and the tenfold way

Our final goal is to fit all discussed topological models into a single classification scheme.

As a preliminary step, we must first decide on the *symmetries* to use for this classification:

1 | Goal: Classify TPs of non-interacting fermions

Approach: Use SP Hamiltonian H to describe & classify MB Hamiltonian $\hat{H}$

Here, H can either be a "standard" SP Hamiltonian or a Bogoliubov-de Gennes Hamiltonian if superconductivity is present.

→ We are interested in *constraints* on the matrix H that arise from the symmetries of $\hat{H}$.

2 | *Which symmetries of $\hat{H}$ to use?*

Remember: $\mathcal{X}$ symmetry of $\hat{H} :\Leftrightarrow [\mathcal{X},\hat{H}]=0$

← *Wigner's theorem* → $\mathcal{X}$ unitary or antiunitary (remember ➔ Problemset 1)

→ Four possibilities on Fock space:

In Chapters 2, 4 and 5, we encountered four distinct *classes* of symmetries that can act on Fock space:

$$\begin{aligned}
&\textbf{Unitary:} && \mathcal{U} i \mathcal{U}^{-1} = +i\,, && \mathcal{U} c_i \mathcal{U}^{-1} = U_{ij} c_j && \text{(6.1a)}\\
&\text{(unitary MB sym.)} && \left[\mathcal{U}, \hat{H}\right] = 0 \quad \Leftrightarrow && [U, H] = 0 \quad \text{(unitary SP sym.)} &&\\
&\textbf{Time-reversal:} && \mathcal{T} i \mathcal{T}^{-1} = -i\,, && \mathcal{T} c_i \mathcal{T}^{-1} = U_{ij} c_j && \text{(6.1b)}\\
&\text{(antiunitary MB sym.)} && \left[\mathcal{T}, \hat{H}\right] = 0 \quad \Leftrightarrow && [U\mathcal{K}, H] = 0 \quad \text{(antiunitary SP sym.)} &&\\
&\textbf{Particle-hole:} && \mathcal{C} i \mathcal{C}^{-1} = +i\,, && \mathcal{C} c_i \mathcal{C}^{-1} = U_{ij} c_j^\dagger && \text{(6.1c)}\\
&\text{(unitary MB sym.)} && \left[\mathcal{C}, \hat{H}\right] = 0 \quad \Leftrightarrow && \{U\mathcal{K}, H\} = 0 \quad \text{(antiunitary SP pseudosym.)} &&\\
&\textbf{Sublattice:} && \mathcal{S} i \mathcal{S}^{-1} = -i\,, && \mathcal{S} c_i \mathcal{S}^{-1} = U_{ij} c_j^\dagger && \text{(6.1d)}\\
&\text{(antiunitary MB sym.)} && \left[\mathcal{S}, \hat{H}\right] = 0 \quad \Leftrightarrow && \{U, H\} = 0 \quad \text{(unitary SP pseudosym.)} &&
\end{aligned}$$

Note that the unitary *mixing* of particles ($c_i^\dagger$) and holes (c_i) is not necessarily canonical, i.e., does not preserve the fermionic anticommutation relations in general (remember the ← *Bogoliubov transformation* in Chapter 5). By contrast, here we only mix annihilation operators among themselves or map them to creation operators only.

Using *unitary* symmetries of $\hat{H}$ (H) is possible but not universal!

In the sense that the classification would be "infinite" because there are infinitely many unitarily realized symmetries and the classification depends on the specific symmetry (representation); → *extended note below*.

$\rightarrow \sphericalangle$ TRS, PHS and SLS …

This is a conceptually important but subtle point: The decision to "factor out" all unitary symmetries is not so much physically motivated but more a decision based on systematics. One *can* classify fermionic SPTs with unitary symmetries, but this is a question that cannot really be conclusively answered because there are infinitely many possible symmetry groups. Thus the most systematic approach asks whether there is anything below that sprawling complexity that is simpler and more systematic. After all, one should first understand these basics before plunging into the never ending story that lies beyond. To put this into context: There *are* classifications for certain unitary symmetry groups for free fermions [176–179] (but only "certain" not "all"). Also for *bosonic* SPTs one considers unitary symmetries [47]. So there is nothing inherently "bad" about them. The difference becomes clear when one compares the classification table below (the "periodic table") with similar tables for bosonic SPTs [47]: The latter always have an exemplary character in that one must hope that the unitary symmetry one is interested in is listed; these lists are not exhaustive (they cannot be). However, once one throws all unitary symmetries away (= allows them to be explicitly broken), what is left is, quite unexpectedly, (1) *non-trivial* and (2) *finite* so that the classification introduced in the following *is* exhaustive (although in a more restricted sense).

…and only SP Hamiltonians <u>without</u> unitary symmetries:

$$H \text{ ⁂ } \textit{irreducible} \quad :\Leftrightarrow \quad \left([U, H] = 0 \;\Rightarrow\; U = e^{i\lambda}\mathbb{1} \right) \tag{6.2}$$

These irreducible Hamiltonians without unitary symmetries can be understood as the "atomic building blocks" of all Hamiltonians. To see this, consider an arbitrary Hamiltonian H with symmetry group G_0 that is unitarily realized on the SP Hilbert space $\mathcal{H}$. As always, we can decompose the Hilbert space into irreducible representations λ of G_0 (with possible multiplicities):

$$\mathcal{H} = \bigoplus_\lambda \mathcal{H}_\lambda\,. \tag{6.3}$$

Each subspace $\mathcal{H}_\lambda$ is composed of equivalent copies of the same irrep λ ("equivalent" in the sense of "isomorphic"):

$$\mathcal{H}_\lambda = \bigoplus_{\alpha=1}^{m_\lambda} \mathcal{H}_\lambda^{(\alpha)} \simeq \tilde{\mathcal{H}}_\lambda \otimes \mathcal{V}_\lambda \tag{6.4}$$

where $\mathcal{H}_\lambda^{(\alpha)} \simeq \mathcal{V}_\lambda$ for all α with the irrep $\mathcal{V}_\lambda$ and $\tilde{\mathcal{H}}_\lambda = \mathbb{C}^{m_\lambda}$. It is $d_\lambda = \dim \mathcal{V}_\lambda$ the dimension of the irrep λ and m_λ the multiplicity of the irrep λ in $\mathcal{H}$. The $\mathcal{H}_\lambda$ are known as G_0*-isotypic components* of $\mathcal{H}$ [180].

Since $[H, U_g] = 0$ for all $g \in G_0$ (with unitary representation U_g) and $\mathcal{V}_\lambda$ is irreducible, it is

$$H = \bigoplus_\lambda H_\lambda \otimes \mathbb{1}_{d_\lambda} \quad \text{and} \quad U_g = \bigoplus_\lambda \mathbb{1}_{m_\lambda} \otimes U_g^{(\lambda)}\,. \tag{6.5}$$

The Hamiltonian blocks H_λ act on $\tilde{\mathcal{H}}_\lambda$ and have no longer any unitary symmetry left, they are the "irreducible building blocks" of all Hamiltonians, just as the $U_g^{(\lambda)}$ are the irreducible building blocks of all representations of the symmetry group G_0. It is these irreducible Hamiltonians that we will focus on below (just like mathematicians study groups in terms of their irreducible representations $U_g^{(\lambda)}$).

3 | For a given irreducible SP Hamiltonian H check (henceforth we forget about $\hat{H}$) …

$$\exists\, U_T\,?:\ [U_T\mathcal{K}, H] = 0 \quad \text{and if so:} \quad U_T U_T^* \stackrel{?}{=} \pm\mathbb{1} \tag{6.6a}$$

$$\exists\, U_C\,?:\ \{U_C\mathcal{K}, H\} = 0 \quad \text{and if so:} \quad U_C U_C^* \stackrel{?}{=} \pm\mathbb{1} \tag{6.6b}$$

$$\exists\, U_S\,?:\ \{U_S, H\} = 0 \tag{6.6c}$$

To understand where the two possibilities for PHS come from, we can generalize the argument used for TRS in Section 2.1.2, but now exploit that we only consider irreducible Hamiltonians: Let $X = U\mathcal{K}$ and either $[X, H] = 0$ (TRS) or $\{X, H\} = 0$ (PHS). In *both* cases, it is $[X^2, H] = 0$, which implies $X^2 = \lambda\mathbb{1}$ because H is irreducible and $X^2 = UU^* = U(U^T)^{-1}$ is a unitary. Combined, we have $U = \lambda U^T$ and $U^T = \lambda U$ so that $U = \lambda^2 U$. This implies $\lambda = \pm 1$ and therefore $X^2 = \pm\mathbb{1}$. Note that this is true for both TRS and PHS, and we do not have to assume that these represent a $\mathbb{Z}_2$ symmetry! Note also that this sign cannot be transformed away by $U \mapsto e^{i\varphi}U$ due to the complex conjugation. This is different for SLS (which lacks the complex conjugation) where $U_S^2 = +\mathbb{1}$ can always be chosen with an appropriate transformation $U_S \mapsto e^{i\varphi}U_S$.

→ Define:

$$\text{TRS:} \quad T \equiv U_T\mathcal{K} \quad \text{(antiunitary symmetry)} \tag{6.7a}$$

$$\text{PHS:} \quad C \equiv U_C\mathcal{K} \quad \text{(antiunitary pseudosymmetry)} \tag{6.7b}$$

$$\text{SLS:} \quad S \equiv U_S \quad \text{(unitary pseudosymmetry)} \tag{6.7c}$$

¡! Here we switch from our previous notation $T_U = U\mathcal{K}$ to $T \equiv T_{U_T} = U_T\mathcal{K}$ (similarly for $C = C_U$ and $S = S_U$) because we will mix T, C and S below and then it is important to distinguish the unitaries U_T, U_C and U_S.

→ Labeling scheme:

$$[T,H]\neq 0 \quad :\Leftrightarrow: \quad T=0 \tag{6.8a}$$
$$[T,H]=0 \quad \text{with} \quad T^2=+\mathbb{1} \quad :\Leftrightarrow: \quad T=+1 \tag{6.8b}$$
$$[T,H]=0 \quad \text{with} \quad T^2=-\mathbb{1} \quad :\Leftrightarrow: \quad T=-1 \tag{6.8c}$$
$$\{C,H\}\neq 0 \quad :\Leftrightarrow: \quad C=0 \tag{6.8d}$$
$$\{C,H\}=0 \quad \text{with} \quad C^2=+\mathbb{1} \quad :\Leftrightarrow: \quad C=+1 \tag{6.8e}$$
$$\{C,H\}=0 \quad \text{with} \quad C^2=-\mathbb{1} \quad :\Leftrightarrow: \quad C=-1 \tag{6.8f}$$
$$\{S,H\}\neq 0 \quad :\Leftrightarrow: \quad S=0 \tag{6.8g}$$
$$\{S,H\}=0 \quad :\Leftrightarrow: \quad S=1 \tag{6.8h}$$

Note that this is an abuse of notation: In the left column, $T/C/S$ denote the *operators* of Eq. (6.7), whereas in the right column they are simply *variables* used to label the situation on the left. From the context it is always clear which use is intended.

→ Triple (T,C,S) encodes answers to classification in Eq. (6.6)

Note:

These constraints on the SP level can also be constructed quite systematically without deriving them from MB symmetries:

Imagine you are given a gapped SP Hamiltonian (= Hermitian matrix) H and a unitary U, and your job is to formulate a linear/antilinear constraint on H using only U and complex conjugation. The constraint can be written in the form

$$f(H,U)\overset{!}{=}H\,. \tag{6.9}$$

We want f to be linear/antilinear in H and its result must be Hermitian because H is; hence it should be $f(H,U)=\alpha UH^{(*)}U^\dagger$ with $\alpha\in\mathbb{R}$. Now note that $\det(H)=\det(\alpha UH^{(*)}U^\dagger)=\alpha^N\det(H)$; since H is gapped we can *w.l.o.g.* shift the Fermi energy (= zero energy) into the gap so that $\det(H)\neq 0$ and we have $\alpha^N=1$.

In general, this leaves only four possibilities:

$$f(H,U)=\begin{cases}+1\cdot UHU^\dagger & \text{(unitary symmetry)}\\ -1\cdot UHU^\dagger & \text{(unitary pseudosymmetry}\rightarrow\text{SLS)}\\ +1\cdot UH^*U^\dagger & \text{(antiunitary symmetry}\rightarrow\text{TRS)}\\ -1\cdot UH^*U^\dagger & \text{(antiunitary pseudosymmetry}\rightarrow\text{PHS)}\end{cases} \tag{6.10}$$

Since for an irreducible Hamiltonian (by construction) there is no unitary symmetry (except the trivial one), we are left with the latter three constraints that are nothing but the three symmetries (on the MB level) we have discussed before.

4 | Important:

For a given irreducible Hamiltonian, TRS T_U, PHS C_U and SLS S_U are *unique* (if present)

To see this, assume T_{U_1} and T_{U_2} were two *different* time-reversal symmetries:

$$\left[T_{U_1},H\right]=0 \quad \text{and} \quad \left[T_{U_2},H\right]=0 \tag{6.11}$$

Then $\tilde{U} := T_{U_1} T_{U_2} = U_1 U_2^*$ is a *unitary* symmetry of H:

$$[\tilde{U}, H] = 0 \xrightarrow{H \text{ irreducible}} \tilde{U} = e^{i\lambda}\mathbb{1}\,, \tag{6.12}$$

and therefore $T_{U_1} = U_1\mathcal{K} = e^{i\lambda} U_2^{*\dagger}\mathcal{K} \propto T_{U_2}^{-1}$. So we can replace T_{U_1} by T_{U_2} or vice versa. The same argument applies to PHS and similarly to SLS.

5 | Sublattice symmetry:

As already mentioned previously in Sections 4.1 and 5.3:

$$S = T \circ C = U_T U_C^* \quad \textit{unitary} \text{ operator with } (\textit{w.l.o.g.}) \quad S^2 = +\mathbb{1} \tag{6.13}$$

One the many-body level: $\mathcal{S} = \mathcal{T} \circ \mathcal{C}$.

In particular:

$$\left.\begin{array}{ll} \text{TRS:} & [T, H] = 0 \\ \text{PHS:} & \{C, H\} = 0 \end{array}\right\} \begin{array}{c} \Rightarrow \\ \not\Leftarrow \end{array} \quad \{S, H\} = 0 \tag{6.14}$$

→ C *cannot* be eliminated in favor of T since S is not a unitary *symmetry* (but a *pseudo*symmetry)

¡! This means that despite "factoring out" all unitary symmetries of the SP Hamiltonian H, there can still be a *unitary* PHS symmetry $\mathcal{C}$ of the MB Hamiltonian $\hat{H}$ left because (1) C is *anti*unitary on the SP level and (2) $\mathcal{S} = \mathcal{T} \circ \mathcal{C}$ is a *pseudo*symmetry on the SP level.

→ Keep T, C, and S

6 | The "Tenfold way":

Eq. (6.14) → (here T, C, S are used in their function as *labels*)

$$(T \neq 0 \vee C \neq 0) \quad \Rightarrow \quad S = |TC| \tag{6.15a}$$

$$\text{but:} \quad T = 0 = C \quad \Rightarrow \quad \begin{cases} \text{either} & S = 0 \\ \text{or} & S = 1 \end{cases} \tag{6.15b}$$

This is easy to understand: If T and/or C are present, the relation $S = T \circ C$ determines the absence/presence of S automatically. Only if both T *and* C are *absent*, the absence/presence of S is *not* determined. (Note that $\mathcal{T} \circ \mathcal{C}$ can be a symmetry even if $\mathcal{T}$ and $\mathcal{C}$ are not symmetries separately!)

$\to 3 \times 3 + 1 = 10$ symmetry classes:

Class	T	C	S
A	0	0	0
AIII	0	0	1
AI	+1	0	0
BDI	+1	+1	1
D	0	+1	0
DIII	−1	+1	1
AII	−1	0	0
CII	−1	−1	1
C	0	−1	0
CI	+1	−1	1

Remember: We encountered the classes **AI**, **D** and **BDI** before; the Kane-Mele model belonged to **AII** and the Chern insulator to **A**.

As mentioned before, the names of the classes go back to the mathematician Élie Cartan who assigned them to so called *(large) symmetric spaces (of compact type)*; in the present context, the labels are typically taken "as is" without assigning any deeper meaning to them. The order in the above table seems arbitrary but is actually not – this will become clear later.

6.2. The periodic table of topological insulators and superconductors

We are finally prepared to fit all our discussed topological models into a *single classification scheme*:

7 | $\sphericalangle$ Gapped Hamiltonians H of class **X** in dimension d

Question: How to label the topological phases that can be realized by these systems?

Note that a *specific* system H in **X** may have *additional* symmetries (both unitary and antiunitary). However, the classification below does not *rely* on these symmetries, so that they can be broken by perturbations without leaving the phase.

8 | Answer:

Periodic table of topological insulators & superconductors:

Symmetries | Dimensions →

	Class	T	C	S	0D	1D	2D	3D	...	8D	...
Complex (w/o T & C)	A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	2-periodic →	$\mathbb{Z}$	...
	AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	complex K-groups	0	...
Real (w/ T or C)	AI	+1	0	0	$\mathbb{Z}$	0	0	0		$\mathbb{Z}$	...
	BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0		$\mathbb{Z}_2$	...
	D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	"Bott periodicity"	$\mathbb{Z}_2$	...
	DIII	−1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		0	...
	AII	−1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$		$2\mathbb{Z}$	...
	CII	−1	−1	1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	real K-groups	0	...
	C	0	−1	0	0	0	$2\mathbb{Z}$	0	8-periodic →	0	...
	CI	+1	−1	1	0	0	0	$2\mathbb{Z}$		0	...

This course: Remember, remember...

- **Chern number** (A, 2D): • IQHE • QWZ • Haldane
- **Pfaffian Index** (AII, 2D): • Kane-Mele model
- **Winding number** (AIII, 1D): • SSH-chain
- **Endpoints = Poles** (D, 1D): • Majorana chain
- No Kramers pairs ⚡ (Kane-Mele model) (A, 2D / AI, 2D)
- All paths contractible ⚡ (Majorana chain) (AI, 1D)
- **Winding number** (BDI, 1D): • Majorana chain

The entries denote the classification of topological phases. 0 means "no TPs possible." $\mathbb{Z}$ means that there is an infinite number of different TPs labeled by an integer etc.

→ In every dimension, 5 out of 10 symmetry classes support TPs!

- The classification is referred to as "periodic table" because of its periodic structure for $d = 0, 1, \ldots$ where the period for the "complex" classes is 2 and for the "real" classes 8.
- There are several equivalent ways to derive this table (and its periodicity), none of which is trivial. We will *sketch* one of the approaches below.

  These methods were developed around 2008–2009 by different researchers [56, 57, 122].
- *In case you wonder about the column for $d = 0$:* One should think of these systems as "blobs" without spatial structure. Mathematically, this column follows naturally and is not really special (actually, it is simpler because the constraints on the Hamiltonians are easier to implement). The Brillouin zone is simply T^0 (which is a point). The origin of the $\mathbb{Z}$ index for symmetry class **A** is easy to understand: Let the gap of the system be at zero energy. Then $\mathbb{Z}$ corresponds to the *number* of negative energy states of the SP Hamiltonian (i.e., the ones filled in the many-body ground state). Note that one cannot change this number by continuously deforming the Hamiltonian (matrix) without crossing the gap with an eigenergy (which violates the gap constraint).

9 | Recipe:

If one studies a particular model (specified by a SP Hamiltonian H) and wants to find out whether any of its phases are topological, the standard procedure goes as follows:

i | Check whether the SP Hamiltonian H features TRS, PHS, and/or SLS and (if so) whether TRS/PHS square to ± 1.

$\rightarrow (T, C, S) \rightarrow$ Class **X**

ii | Use the periodic table above to check whether **X** supports topological phases in the spatial dimension d of the given system.

iii | Look up the associated topological invariant I for $(\mathbf{X}, d)$ in Ref. [122].

iv | Compute $I = I[H]$ as a function of the control parameters of the system and check whether it non-zero in (some of) the phases.

Without knowing about the periodic table and the systematic approach of Ref. [122] to construct topological invariants, we nevertheless succeeded for the Chern insulator (Class **A**) with the Chern number, the Kane-Mele model (Class **AII**) with the Pfaffian index, the SSH chain (Class **AIII**) with the winding number, and the Majorana chain (Class **D**) with the $\mathbb{Z}_2$-index constructed from the BdG-Hamiltonian ☺.

Note: The symmetry classes are not exclusive. E.g., every system in class **BDI** can also be considered a member of the classes **AI**, **D**, or **AIII**. We encountered this ambiguity for the Majorana chain which generically is considered a representative of **D** even if the "clean" Majorana chain Hamiltonian does not break TRS. In this situation, TRS is considered an "accidental" symmetry that one does not want to rely on. If, however, one considers the Majorana chain a representative of **BDI**, TRS becomes a crucial symmetry that must not be broken. This may seem arbitrary but is perfectly valid as the choice of a protecting symmetry essentially specifies which perturbations we consider allowed and which forbidden. This situation is typical for all SPT phases as they do not have intrinsic topological order (recall our discussion of SPTs in ← Section 0.5). In → Section 6.4 we discuss *stacks* of Majorana chains where this concept should become clear.

6.3. Frameworks for classification

There are different frameworks that can be used to derive the periodic table above. Unfortunately, none of them is straightforward and all of them make heavy use of highly non-trivial physical and/or mathematical facts. A deep study of any of these approaches would easily fill its own course, so we keep it simple and sketch only one of the approaches exemplarily:

- Anderson localization on the boundary (Details: ↑ Refs. [56, 122])

  Rationale: Study field theories (↑ *non-linear sigma models*) that describe the *boundary* of the system and determine when they retain delocalized states in the presence of disorder (i.e., whether they avoid ↑ *Anderson localization*). Mathematically, this happens if certain topological terms can be added to the action; the existence (and properties) of these terms depends on $\mathbf{X}$ and d and provides the periodic table.

- Quantum anomalies on the boundary (Details: ↑ Ref. [181])

  Rationale: Study ↑ *anomalous field theories* that can emerge as effective descriptions on the *boundaries* of the system (this approach relates to the one based on Anderson localization above). To cite Ludwig [92]:

  "[The approach] relies on the notion that the boundary of a topological insulator (superconductor) cannot exist as an isolated system in its own dimensionality. Rather it must always be attached to a higher dimensional bulk."

  We encountered such an anomaly before when we discussed the IQHE and realized that its chiral edge modes are in conflict with the ← *Nielsen-Ninomiya theorem*. These edge modes can only be consistently formulated on the boundary of a two-dimensional bulk.

- K-Theory: (Details: ↑ Ref. [57])

  In contrast to the other two frameworks which (1) do not require translational invariance, and (2) focus on the boundary of the system, the K-theory approach pioneered by Kitaev assumes *translational invariance* and describes the *bulk* of the system. Let us briefly sketch the rationale of this (very mathematical) approach to get a feeling how the classification problem can be tackled on a very high level:

↑ *(Topological) K-theory* is a very general mathematical framework that is used to study vector bundles over topological spaces. It goes back to the influential 20th-century mathematician ALEXANDER GROTHENDIECK. In its application to classify topological phases, the topological base space is essentially the Brillouin torus and the system/Hamiltonian is described by a (potentially non-trivial) vector bundle over this space. Before its application to topological phases, K-theory had already found applications in string theory.

1 | ⊲ Gapped (translation invariant) system with n filled (m empty) bands described by Bloch Hamiltonian $H(\boldsymbol{k})$

2 | Spectral flattening:

In this first step, we simplify the Hamiltonian without leaving the quantum phase to classify:

$$H(\boldsymbol{k}) \xrightarrow{\text{Continuous deformation}} \mathfrak{H}(\boldsymbol{k}) \tag{6.16a}$$

$$\text{with} \quad \sigma(\mathfrak{H}(\boldsymbol{k})) = (\underbrace{-1,\dots,-1}_{n\text{ filled bands}},\underbrace{+1,\dots,+1}_{m\text{ empty bands}}) \tag{6.16b}$$

$\sigma(A)$ denotes the spectrum (eigenvalues) of the operator A.

3 | ⊲ Simplest case: Class **A** →

(Hence we do not have to implement any symmetry constraint in the following.)

$$\mathfrak{H}(\boldsymbol{k}) = \mathcal{U}(\boldsymbol{k}) \underbrace{\begin{pmatrix} \mathbb{1}_m & 0 \\ 0 & -\mathbb{1}_n \end{pmatrix}}_{\mathbb{X}} \mathcal{U}^\dagger(\boldsymbol{k}) \quad \text{with} \quad \mathcal{U}(\boldsymbol{k}) \in U(m+n) \tag{6.17}$$

$U(m+n)$ is the matrix group of unitary $(m+n)\times(m+n)$-matrices.

4 | "Gauge symmetry":

The decomposition in Eq. (6.17) is not unique:

$$\mathcal{U} \sim \mathcal{U}' \quad :\Leftrightarrow \quad \mathcal{U} = \mathcal{U}' \cdot \begin{pmatrix} \mathcal{U}_1 & 0 \\ 0 & \mathcal{U}_2 \end{pmatrix} \quad \text{for} \quad \mathcal{U}_1 \in U(m),\ \mathcal{U}_2 \in U(n) \tag{6.18}$$

since then

$$\mathfrak{H}(\boldsymbol{k}) = \mathcal{U}(\boldsymbol{k})\,\mathbb{X}\,\mathcal{U}^\dagger(\boldsymbol{k}) = \mathcal{U}'(\boldsymbol{k})\,\mathbb{X}\,\mathcal{U}'^\dagger(\boldsymbol{k}) \tag{6.19}$$

That is, the $\mathfrak{H}$-encoding unitary $\mathcal{U}$ is only defined up to unitaries from $U(m)\times U(n)$.

→

$$\mathfrak{H}\,:\,T^d \ni \boldsymbol{k} \mapsto \mathfrak{H}(\boldsymbol{k}) \mathrel{\hat{=}} [\mathcal{U}(\boldsymbol{k})]_\sim \in \frac{U(m+n)}{U(m)\times U(n)} = G_{m,n+m}(\mathbb{C}) \tag{6.20}$$

$G_{m,n+m}(\mathbb{C})$: ↑ *complex Grassmannian*

In mathematics, Grassmannians are differentiable manifolds that parametrize the set of m-dimensional linear subspaces of an $n+m$-dimensional vector space. The concept was introduced by mathematician HERMANN GRASSMANN in the 19th century.

→ $G_{m,n+m}(\mathbb{C})$ is the ⁂ *classifying space* C_0 for symmetry class **A**
(and one of Cartan's *symmetric spaces*, which is where the label "**A**" comes from)

This statement is not completely correct, actually it is

$$C_0 = \bigcup_{k\in\mathbb{Z}} \lim_{s\to\infty} \frac{U(2s)}{U(s+k)\times U(s-k)} \simeq \lim_{n,m\to\infty} \frac{U(m+n)}{U(m)\times U(n)} \times \mathbb{Z}\,. \tag{6.21}$$

The idea behind this is that SP Hamiltonians of different sizes should be comparable (and the classification should not depend on system-specific parameters like m and n). In particular, for systems with $d > 0$ it should not matter whether one adds additional trivial bands to the system (like those from closed atomic shells). This leads to the concept of ↑ *stable equivalence* which has its counterpart in K-theory where one considers vector bundles *modulo trivial bundles*.

5 | Classifying spaces:

Similar arguments [taking the constraints (6.1) imposed by symmetries on the SP Hamiltonian into account] lead to the following table of classifying spaces:

Complex classes: $UHU^\dagger = -H$ (AIII)

Real classes: $UH^*U^\dagger = H$ (AI)

Class	T	C	S	Classifying Space	Name
A	0	0	0	$U(m+n)/U(n)\times U(m)$	C_0
AIII	0	0	1	$U(n)\times U(n)/U(n)$	C_1
AI	+1	0	0	$O(n+m)/O(n)\times O(m)$	R_0
BDI	+1	+1	1	$O(n)\times O(n)/O(n)$	R_1
D	0	+1	0	$O(2n)/U(n)$	R_2
DIII	−1	+1	1	$U(2n)/Sp(2n)$	R_3
AII	−1	0	0	$Sp(n+m)/Sp(n)\times Sp(m)$	R_4
CII	−1	−1	1	$Sp(n)\times Sp(n)/Sp(n)$	R_5
C	0	−1	0	$Sp(2n)/U(n)$	R_6
CI	+1	−1	1	$U(n)/O(n)$	R_7

- $\mathrm{Sp}(n)$ denotes the *compact symplectic group* which is the analog of the unitary group $\mathrm{U}(n)$ if one replaces the field $\mathbb{C}$ by quaternions $\mathbb{H}$.
- The distinction between the two *complex* classes **A** and **AIII** and the remaining eight *real* classes follows from the reality constraints (that is, the constraint on the SP Hamiltonian includes a complex conjugate) on the Hamiltonians for real classes, and the missing of such for complex classes. On the mathematical level, this leads to the distinction between complex and real vector bundles and henceforth complex and real K-theory with classifying spaces C_q (q mod 2) and R_q (q mod 8), respectively.