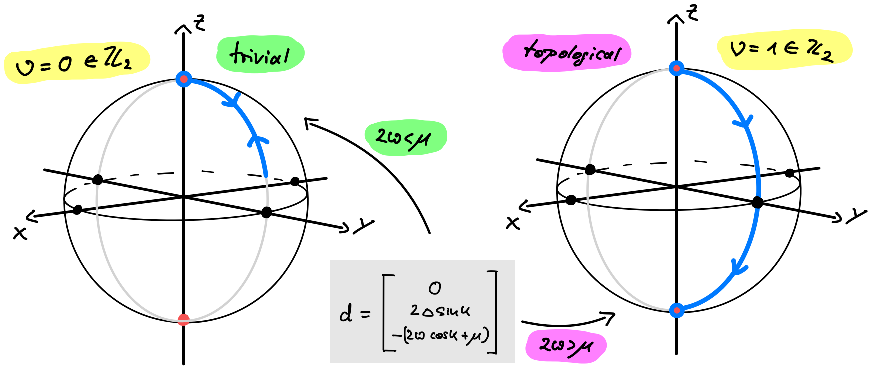
↓ **Lecture 16** [20.06.25]

v | Conclusion for the Majorana chain:

The $\mathbb{Z}_2$-index classifies the phase for $2|w| < |\mu|$ as *trivial* and $2|w| > |\mu|$ as *topological*:



→

$$\boxed{\begin{array}{lll} \text{Phase A:} & 2|w| > |\mu| & \to \text{topological} \\ \text{Phase B:} & 2|w| < |\mu| & \to \text{trivial} \end{array}} \tag{5.29}$$

In his original paper [139], Kitaev classified the two phases differently (using the Pfaffian to distinguish two classes of quadratic fermion Hamiltonians). The classification presented here, based on the BdG Hamiltonian, is conceptually very different. However, it can be shown that the two approaches lead to the same notion of trivial and topological phases [144].

We could be satisfied at this point, but there is actually more to be learned if we *combine* both PHS and TRS …

10 | $\sphericalangle$ PHS & TRS:

i | As argued above, PHS is intrinsic to the form of the BdG Hamiltonian (it cannot be broken). Furthermore, for an open chain we can always find a TRS representation by gauging away complex phases. Hence it is reasonable to consider the situation where both symmetries are preserved.

$\sphericalangle$ TRS with $\tilde{T}^2 = +\mathbb{1}$ and PHS with $\tilde{C}^2 = +\mathbb{1}$

→ ⁂ *Symmetry class* **BDI** [→ ??]

ii | Eqs. (5.26) and (5.28) → Constraints on the BdG vector:

$$d_x(-k) = 0 \tag{5.30a}$$

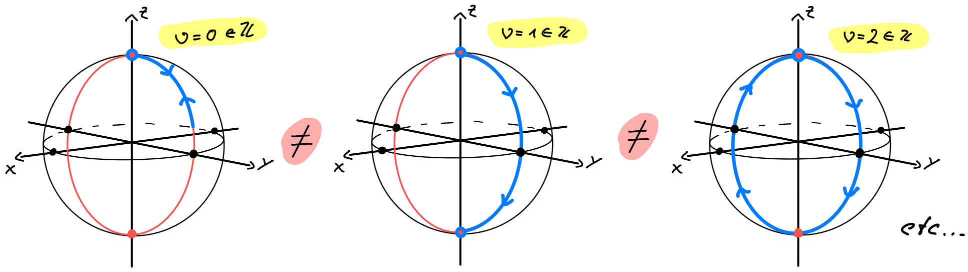
$$d_y(-k) = -d_y(k) \tag{5.30b}$$

$$d_z(-k) = d_z(k) \tag{5.30c}$$

→ Still $\vec{d}(k)$ on EBZ $[0, \pi]$ determines $H_{\text{BdG}}(k)$ completely

iii | Image $\hat{d}$(EBZ) on S^2 …

- … is constrained to the great circle with $d_x = 0$
- … *and* must *start & end* either on "north" or "south pole":



→ Infinitely many topologically distinct classes of paths
(distinguished by their winding number)

→ Infinitely many topological phases possible → $\mathbb{Z}$-index

iv | Boldly generalizing these findings, we could hypothesize:

> In 1D, systems of class **BDI** allow for many TPs labeled by a $\mathbb{Z}$-index.

Again, this is true in general; → ?? on the classification of topological insulators & superconductors.

v | Conclusion for the Majorana chain:

Although TRS is not useful on its own, in combination with PHS it boosts the $\mathbb{Z}_2$-index of **D** to a $\mathbb{Z}$-index of **BDI**. For a single Majorana chain, this has the only benefit that we can user either the topological index of **D** or the winding number of **BDI** to characterize the topological phase; in this situation, they are equivalent. This is different if one considers *stacks* of multiple parallel Majorana chains, where one can create infinitely many different SPT phases when TRS is present (**BDI**) but only one if it is broken (**D**).

On ➔ Problemset 8 you study stacks of time-reversal symmetric Majorana chains in class **BDI**. There you show that *interactions* modify the $\mathbb{Z}$-index constructed here to a $\mathbb{Z}_8$-index (see also → ??).

vi | Final note: For the SP Hamiltonian having PHS *and* TRS means:

$$\text{PHS:}\quad U_C H^* U_C^\dagger \overset{5.1b}{=} -H \tag{5.31a}$$

$$\text{TRS:}\quad U_T H^* U_T^\dagger \overset{2.31b}{=} +H \tag{5.31b}$$

which implies

$$U_S H U_S^\dagger = -H \quad \text{with} \quad U_S = U_T U_C^* \tag{5.32}$$

→ Sublattice symmetry [Eq. (4.4)]

This is true in general and will be important → *later* (??).

5.4. Majorana fermions

Why do we call the Majorana chain “Majorana chain” in the first place?

To answer this, we need a bit of algebra. As a bonus, we will find an unexpected relation between the Majorana chain and the SSH chain discussed in Chapter 4:

11 | $\triangleleft$ Set of fermion L operators $\{c_1, c_2, \dots, c_L\}$ and define $2L$ ⁂ *Majorana operators*

$$\boxed{\gamma_{2i-1} = c_i + c_i^\dagger \quad \text{and} \quad \gamma_{2i} = i(c_i^\dagger - c_i)} \tag{5.33}$$

→ There are *two* Majorana operators per fermion mode.

12 | $\overset{\circ}{\rightarrow}$ Properties:

$$\boxed{\gamma_n^\dagger = \gamma_n \quad \text{and} \quad \{\gamma_n, \gamma_m\} = 2\delta_{nm} \quad \text{for} \quad n, m \in \{1, \dots, 2L\}} \tag{5.34}$$

- Up to a normalization, Majorana fermions behave like *self-adjoint* or *real* fermions. The name originates from a similar concept in high-energy physics due to Ettore Majorana (namely, real-valued solutions of the Dirac equation in Majorana representation). In condensed matter physics, however, the properties Eq. (5.34) should be seen as the defining relations of *Majorana operators*.
- While the γ_n describe "real" (Majorana) fermions, the c_i describe "complex" (Dirac) fermions. Eq. (5.33) demonstrates that the two Majoranas γ_{2i-1} and γ_{2i} can be thought of as the "real" and "imaginary part" of the complex fermion c_i.
- We stress that Majorana fermions are *not* → *anyons*, they are fermionic quasiparticles (as the name clearly states); only → *Majorana zero modes* can make their hosts (like vortices in 2D $p_x + i p_y$-superconductors) behave like anyons under adiabatic deformations of the Hamiltonian.

13 | Pairs of Majoranas can be recombined to form a complex fermion:

$$\boxed{c_i \overset{5.33}{=} \tfrac{1}{2}(\gamma_{2i-1} + i\gamma_{2i}) \quad \text{and} \quad c_i^\dagger \overset{5.33}{=} \tfrac{1}{2}(\gamma_{2i-1} - i\gamma_{2i})} \tag{5.35}$$

Observation: We do not have to combine the original pairs of Majoranas! Actually, it is possible to combine *any* pair of Majoranas to form a new fermion mode (→ *below*). This follows from Eq. (5.34) which shows that all $2L$ Majorana modes "are made equal."

14 | We can now rewrite the Majorana chain Hamiltonian in terms of Majorana operators:

Eqs. (5.6) and (5.35) →

$$\hat{H}_{\text{MC}} \overset{\circ}{=} \frac{i}{2}\sum_{i=1}^{L'} [(\Delta + w)\,\gamma_{2i}\gamma_{2i+1} + (\Delta - w)\,\gamma_{2i-1}\gamma_{2i+2}] - \frac{i}{2}\sum_{i=1}^{L} \mu\,\gamma_{2i-1}\gamma_{2i} \tag{5.36}$$

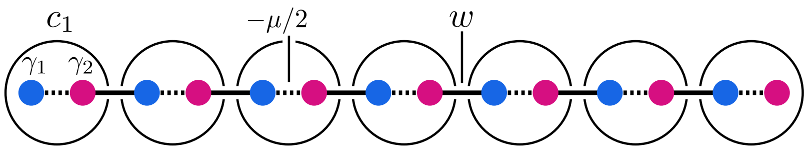
¡! Note that the factors of i are needed for Hermiticity.

15 | $\triangleleft$ Special case $\Delta = w$ (this simplifies expressions but still allows us to access both phases)

$$\boxed{\rightarrow \qquad \hat{H}_{\text{MC}} = -\frac{\mu}{2}\sum_{i=1}^{L}(i\gamma_{2i-1}\gamma_{2i}) + w\sum_{i=1}^{L'}(i\gamma_{2i}\gamma_{2i+1})} \tag{5.37}$$

Remember that the choice $\Delta = w$ also simplified the Bogoliubov transformation [e.g. Eq. (5.18)].

→ SSH-like dimerization:



16 | Comparison to the SSH chain:

The connection to the SSH chain is more than superficial. If one identifies $-\mu/2 \leftrightarrow t$ and $w \leftrightarrow w$ [where t and w are the alternating hopping amplitudes of the SSH chain, Eq. (4.10)], then the gapless points coincide: $|\mu| = 2|w|$ for the Majorana chain [Eq. (5.21)] and $|t| = |w|$ for the SSH chain [Section 4.3].

One can consider a hybrid model of Majorana and SSH chain and study their competing phases on the same footing [145]. This approach is also didactically valuable as it contrasts the different symmetries of the two models quite nicely.

5.5. Edge modes

Due to the SSH-like dimerization, we should again expect topologically protected *zero-energy modes* on the boundary of an open Majorana chain (in the topological phase). As usual, it is most instructive to focus on the fixpoints of the two phases with zero correlation length:

17 | $\sphericalangle$ Trivial phase (Phase B):

Let $w = \Delta = 0$ and $\mu > 0 \xrightarrow{5.37}$

$$\hat{H}_{\mathrm{MC}} = -\frac{\mu}{2}\sum_{i=1}^{L}(i\gamma_{2i-1}\gamma_{2i}) \overset{5.33}{=} -\mu\sum_{i=1}^{L}\left(c_i^\dagger c_i - \frac{1}{2}\right) \tag{5.38}$$

→ Pairing of Majorana modes *on* each site

→ Unique ground state (with all physical fermion modes c_i filled)

18 | Topological phase (Phase A):

Let $w = \Delta > 0$ and $\mu = 0 \xrightarrow{5.37}$

$$\hat{H}_{\mathrm{MC}} = w\sum_{i=1}^{L'}(i\gamma_{2i}\gamma_{2i+1}) \overset{\mathrm{OBC}}{=} w\sum_{i=1}^{L-1}(i\gamma_{2i}\gamma_{2i+1}) \tag{5.39}$$

→ Pairing of Majorana modes *between* adjacent sites

→ Unique ground state for PBC but 2-fold degenerate ground state space for OBC

Let us try to understand the (claimed) degeneracy for OBC in more detail:

i | Define new fermion modes ($i = 1, \dots, L-1$):

$$a_i := \tfrac{1}{2}(\gamma_{2i} + i\gamma_{2i+1}) \qquad \text{and} \qquad a_i^\dagger = \tfrac{1}{2}(\gamma_{2i} - i\gamma_{2i+1}) \tag{5.40}$$

¡! Compare this pairing of Majorana modes with Eq. (5.35).

Check that these are indeed fermions: $\{a_i, a_j^\dagger\} = \delta_{ij}$.

Eq. (5.39) $\overset{\circ}{\rightarrow}$

$$\hat{H}_{\mathrm{MC}} = 2w\sum_{i=1}^{L-1}\left(a_i^\dagger a_i - \frac{1}{2}\right) \tag{5.41}$$

ii | Observation: There is One fermion mode missing!

Note that γ_1 and γ_{2L} do not show up in Eq. (5.39), so we can use them to construct another fermion mode:

$$e := \tfrac{1}{2}(\gamma_{2L} + i\gamma_1) \qquad \text{and} \qquad e^\dagger = \tfrac{1}{2}(\gamma_{2L} - i\gamma_1) \tag{5.42}$$

Note that the $L-1$ modes a_i together with e obey the algebra of L fermionic modes, e.g., $\{e, e^\dagger\} = 1$ and $\{e, a_i\} = 0$.

→ One fermionic edge mode

Indeed, e describes a single fermion *delocalized* between the two endpoints of the chain:

$$e \overset{5.33}{=} \frac{i}{2}\Big(\underbrace{c_L^\dagger - c_L}_{\text{Right edge}} + \underbrace{c_1^\dagger + c_1}_{\text{Left edge}}\Big) \tag{5.43}$$

iii | Ground states for OBC:

$$|\Omega_n\rangle \;\text{ GS of } \hat{H}_{\text{MC}} \quad \Leftrightarrow \quad a_i|\Omega_n\rangle \overset{!}{=} 0 \quad \forall\, i = 1 \dots L-1 \tag{5.44}$$

$[\hat{H}_{\text{MC}}, e] = 0 \rightarrow$ *Two* ground states:

$$\underbrace{e^\dagger e\,|\Omega_0\rangle = 0\,|\Omega_0\rangle}_{\text{Edge mode empty}} \quad \text{and} \quad \underbrace{e^\dagger e\,|\Omega_1\rangle = 1\,|\Omega_1\rangle}_{\text{Edge mode occupied}} \tag{5.45}$$

with $|\Omega_1\rangle = e^\dagger|\Omega_0\rangle$

$e^\dagger e$ measures the occupancy of the edge mode.

19 | Comments:

- Comparison to the SSH chain:

  Remember that the SSH chain also has edge modes (Section 4.6). However, these are *fermionic*, i.e., the SSH chain (in the topological phase) has one independent (complex) fermion on *each edge*. Consequently, the ground state degeneracy for an open chain is *four-fold*. By contrast, the Majorana chain as a *Majorana* fermion per edge (and a Majorana fermion can be thought of as "half" a fermion because it is the real/imaginary part of a complex fermion). Both edges *combined* form a single (complex) fermion, so that the ground state degeneracy is only *two-fold*.

- Many-body ground states (in detail):

  As for the SSH chain, the two-fold degeneracy survives beyond the fixpoint for $\mu = 0$ as long as $|\mu| < 2|w|$ (up to finite-size effects). However, at the fixpoint, the two states $|\Omega_0\rangle$ and $|\Omega_1\rangle$ have a particularly simple description that makes their condensate nature clear and also explains the robustness of their degeneracy (↑ [126] for details):

$$|\Omega_0\rangle \propto \sum_{\boldsymbol{n}:\,|\boldsymbol{n}|\text{ odd}} |\boldsymbol{n}\rangle \quad \text{and} \quad |\Omega_1\rangle \propto \sum_{\boldsymbol{n}:\,|\boldsymbol{n}|\text{ even}} |\boldsymbol{n}\rangle \tag{5.46}$$

  with

$$|\boldsymbol{n}\rangle \equiv (c_1^\dagger)^{n_1}(c_2^\dagger)^{n_2}\dots(c_L^\dagger)^{n_L}\,|0\rangle \tag{5.47}$$

  and $|\boldsymbol{n}|$ the number of fermions in configuration $\boldsymbol{n}$.

- The ground states are the *equal-weight superposition* of all fermion configurations with a fixed parity, in particular, of fermion configurations with different particle number. This is the man-body manifestation of the superconducting condensate (note that $\langle c_i c_{i+1}\rangle \neq 0$ for $|\Omega_n\rangle$).
- Locally, the states $|\Omega_0\rangle$ and $|\Omega_1\rangle$ "look" the same. They can only be distinguished by a *global* measurement of the total fermion parity. To lift their degeneracy, one has to add the term $e^\dagger e$ to the Hamiltonian $\hat{H}_{\text{MC}}$. But for an open chain, this operator is highly *non local* [as can be seen from Eq. (5.43)].

  This scenario, namely multiple orthogonal ground states that are *indistinguishable by local operators*, is actually the hallmark of ← *topological order* (→ Part III).
- There is actually another way to lift the degeneracy. Note that $\gamma_1 e = -e^\dagger\gamma_1$ so that $|\Omega_1\rangle = \gamma_1|\Omega_0\rangle$, i.e., $\langle\Omega_1|\gamma_1|\Omega_0\rangle \neq 0$ so that the Hamiltonian $\hat{H}_{\text{MC}} + \gamma_1$ lifts the degeneracy (recall that $\gamma_1^\dagger = \gamma_1$). In contrast to $e^\dagger e$, γ_1 is localized on the left endpoint of the chain. However, γ_1 violates fermion parity and it is believed that in nature only Hamiltonians that commute with the parity operator are realizable (this is known as ↑ *parity superselection*), so this modification is mathematically sound but physically impossible (→ *comment below*).

- Classification and the role of symmetries:

  The above arguments have shown that the degeneracy of $|\Omega_0\rangle$ and $|\Omega_1\rangle$ is actually very robust and does not rely on any symmetry (note that this does not contradict the topological classification of $\hat{H}_{\text{MC}}$ as part of symmetry class **D** because of the discussed tautological nature of the PHS). Consequently, the topological phase of the Majorana chain is *not* an SPT phase but a topologically ordered phase (of the invertible kind) [29, 35, 44]. This is in stark contrast to the SSH chain which *is* an SPT phase protected by sublattice symmetry.

  [Remember (Section 4.5) that we had no trouble connecting the two phases of the SSH chain with a chemical potential that breaks SLS. You cannot do the same thing with a single Majorana chain! (Try it!) However, you *can* connect the two phases with *two* parallel chains, which demonstrates the invertibility of the topological order.]

- A note on fermion parity:

  The statement that the Majorana chain does not require any symmetry is subtle. To see this, one can check that the Majorana edge modes $\gamma_l = \gamma_1$ and $\gamma_r = \gamma_{2L}$ act on the ground states as follows:

  $$\gamma_l|\Omega_0\rangle = |\Omega_1\rangle \qquad \text{and} \qquad \gamma_r|\Omega_0\rangle = -i\,|\Omega_1\rangle\,. \tag{5.48}$$

  Since these operators are Hermitian and can be constructed from local fermion modes, we could add them to the Hamiltonian as a perturbation, e.g., $\tilde{H}_{\text{MC}} = \hat{H}_{\text{MC}} + \gamma_l$. This perturbation lifts the degeneracy such that the ground state of $\tilde{H}_{\text{MC}}$ is unique, namely $|\Omega_1\rangle - |\Omega_0\rangle$. This is not surprising as γ_l *violates* the fermion parity symmetry $\mathbb{Z}_2^f = \{\mathbb{1}, \mathcal{P}\}$.

  So the Majorana chain *is* protected by a symmetry after all: fermion parity. However, this "symmetry" should not be counted as a real symmetry but as an implicit feature of fermionic Hamiltonians (for instance, quadratic Hamiltonians automatically commute with $\mathcal{P}$) due to the following reason:

  Assume that the Hermitian (and unitary) operators γ_l and γ_r were admissible observables of the theory. Make the length L of the chain large and assume that Alice can measure $\gamma_l = c_1 + c_1^\dagger$ on the left endpoint while Bob can apply the unitary gate $\gamma_r = i(c_L^\dagger - c_L)$ on the right endpoint. Define the basis $|x\rangle \equiv |\Omega_1\rangle + (-1)^x\,|\Omega_0\rangle$ and let the system be initialized in the symmetric state $|x = 0\rangle$ so that Alice measures $+1$ with certainty. Now Bob can send Alice a classical bit $x \in \mathbb{Z}_2$ of information by flipping or not flipping this state with γ_r:

$$(\gamma_r)^x\,|0\rangle \propto \begin{cases}|0\rangle & \text{for} \quad x=0\\ |1\rangle & \text{for} \quad x=1\,.\end{cases} \tag{5.49}$$

This clearly violates causality since the bit x can be transmitted instantaneously over arbitrary distances L; this really is a "spooky action at a distance" and should not be possible with *local* measurements and operations. Therefore γ_l and γ_r are actually *non-local* operators, despite their local appearance in terms of fermion modes!

The reason is that fermions are intrinsically non-local objects due to their statistics, and this non-locality becomes relevant for operators that violate fermion parity. The upshot is that the parity symmetry required for the Majorana chain (or any other fermion Hamiltonian) is a logical consequence of *locality* – and not an additional symmetry constraint.

5.6. ‡ Application as topological quantum memory

Here we focused on the "condensed-matter side" of the Majorana chain (since this is a course on topological quantum phases). However, the topological robustness of the ground state degeneracy suggests the use of this system for quantum information storage (and processing):

1 | ⊲ Topological phase @ $\mu = 0$ and $\Delta = w = 1$ & Open boundary conditions:

$$\hat{H}_{\text{MC}} = \sum_{j=1}^{L-1}(i\gamma_{2j}\gamma_{2j+1}) \equiv -\sum_{j=1}^{L-1} S_j \tag{5.50}$$

with ⁂ *stabilizer generators* S_j that satisfy

$$[S_i, S_j] = 0\,, \quad S_j^\dagger = S_j\,, \quad S_j^2 = \mathbb{1} \tag{5.51}$$

→ ⁂ *stabilizer group* $\mathcal{S} := \langle\{S_1,\dots,S_{L-1}\}\rangle$

- Here $\langle\bullet\rangle$ denotes the (abelian) group generated by $\bullet$.
- You study the stabilizer formalism on ➔ Problemset ?.
- The stabilizer generators $S_j = -i\gamma_{2j}\gamma_{2j+1} \triangleq (-1)^{a_j^\dagger a_j}$ measure the parity of the quasiparticle modes a_i defined in Eq. (5.40).

2 | Ground state space of Eq. (5.50):

$$\mathcal{C} = \{\,|\Psi\rangle \in \mathcal{H} \mid \forall S \in \mathcal{S} \,:\, S|\Psi\rangle = |\Psi\rangle\,\} = \text{span}\,\{|\Omega_0\rangle, |\Omega_1\rangle\} \tag{5.52}$$

Here $|\Omega_0\rangle$ and $|\Omega_1\rangle$ denote the two degenerate many-body ground states introduced in Eq. (5.44) and explicitly written in Eq. (5.46).

→ $\dim\mathcal{C} = 2$ → Use ground state space to store a qubit:

$$|0\rangle \equiv |\Omega_0\rangle \quad \text{and} \quad |1\rangle \equiv |\Omega_1\rangle \tag{5.53}$$

→ Ground state space $\mathcal{C}$ = ⁂ *Code space*

In quantum information theory, a code space is a linear subspace of a larger Hilbert space that is used to encode quantum information.

3 | Qubit = Representation of U(2)

A qubit is a two-dimensional representation of U(2) which is generated by three Pauli matrices $\Sigma^x, \Sigma^y, \Sigma^z$ (and the identity Σ^0) which satisfy $\Sigma^a\Sigma^b = \delta_{ab}\mathbb{1} + i\varepsilon_{abc}\Sigma^c$ and therefore $[\Sigma^a, \Sigma^b] = 2i\varepsilon_{abc}\Sigma^c$.

$\overset{\circ}{\rightarrow}$ Pauli matrices acting on $\mathcal{C}$:

$$\Sigma^z \equiv -i\gamma_{2L}\gamma_1 \stackrel{\circ}{=} (-1)^{e^\dagger e} \quad \text{with} \quad \begin{cases} \Sigma^z|0\rangle = +|0\rangle \\ \Sigma^z|1\rangle = -|1\rangle \end{cases} \tag{5.54a}$$

$$\Sigma^x \equiv \gamma_{2L} \quad = e^\dagger + e \quad = i(c_L^\dagger - c_L) \quad \text{with} \quad \begin{cases} \Sigma^x|0\rangle = |1\rangle \\ \Sigma^x|1\rangle = |0\rangle \end{cases} \tag{5.54b}$$

$$\Sigma^y \equiv \gamma_1 \quad = i(e^\dagger - e) \quad = c_1^\dagger + c_1 \quad \text{with} \quad \begin{cases} \Sigma^y|0\rangle = +i|1\rangle \\ \Sigma^y|1\rangle = -i|0\rangle \end{cases} \tag{5.54c}$$

→ Satisfy all properties of Pauli matrices ✓

Indeed, it is easy to check that $(\Sigma^a)^\dagger = \Sigma^a$ and $\Sigma^a\Sigma^b = \delta_{ab}\mathbb{1} + i\varepsilon_{abc}\Sigma^c$ using the properties Eq. (5.34) of Majorana operators.

The operators Σ^a are called ⁂ *logical operators* as they operate on the encoded (= logical) qubit. To emphasize this, we denote them by Σ^a and not σ^a.

4 | Observation:

$$\left[\Sigma^a, S_j\right] = 0 \quad \forall_{a,j} \tag{5.55}$$

→ Measuring S_j does not destroy the qubit encoded in $\mathcal{C}$ ☺

This feature is crucial to combat errors (→ *below*).

5 | Error model:

We assume that random errors on the Majorana chain can be described by unitary operators with the following properties:

- *Local*
  This is a basic assumption of most error models: the environment acts *locally* on the system that encodes quantum information (here the Majorana chain). Note that essentially all Hamiltonians we study in physics have a locality structure.
- *Parity-symmetric*
  In superconductors, fermionic parity is considered a natural symmetry that can be enforced to high precision because fermions are created by breaking Cooper pairs (which costs energy).

  This is not a fundamental symmetry and it can be violated by ↑ *quasiparticle poisioning* [146, 147].
- *Rare & Uncorrelated*
  We assume that local errors happen independently of each other with a low probability $p \ll 1$ per site $j = 1, \dots, L$ and timestep (iid = independent and identically distributed). This is often (but not always) a good approximation.

→ Elementary (= physical) errors: $E_j = -i\gamma_{2j-1}\gamma_{2j}$ $(j = 1, \dots, L)$

¡! Note that pairs shifted by a single site ($-i\gamma_{2j}\gamma_{2j+1}$) are *stabilizer operators* that act trivially on the code space [Eq. (5.52)]. If such an error occurs on a state $|\Psi\rangle \in \mathcal{C}$ it doesn't do anything and we can ignore it.

With Eq. (5.38) we can write elementary errors as $E_j = 1 - 2c_j^\dagger c_j = (-1)^{n_j}$. Measuring this Hermitian operator therefore means that one observes whether a physical fermion site (mode) c_i is occupied or not. Unitarily applying this operator imprints phases on the many-body wave function depending on the occupancy of the fermion modes.

6 | Logical errors induced by combinations of physical errors?

Logical errors are errors that affect the state of the logical qubit encoded in the codes space $\mathcal{C}$.

- $\triangleleft$ Bit-flip errors: $\Sigma^x = \gamma_{2L}$ or $\Sigma^y = \gamma_1$

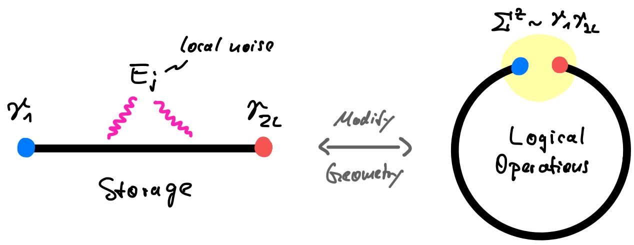
  Not parity-symmetric → *Cannot* occur ☺
- $\triangleleft$ Phase errors: $\Sigma^z = -i\gamma_{2L}\gamma_1$

  Parity-symmetric but Non-local → *Cannot* occur ☺…

  …except elementary errors accumulate:

$$\underbrace{\prod_{j=1}^{L} E_j}_{\text{All errors}} \stackrel{5.50}{=} -i\gamma_1 \underbrace{\left[\prod_{i=1}^{L-1} S_i\right]}_{=\mathbb{1} \text{ on } \mathcal{C}} \gamma_{2L} = -\Sigma^z \quad ☹ \tag{5.56}$$

To prevent a logical phase error Σ^z due to a *single* elementary (physical) error, it is crucial that the two endpoints of the chain are far apart from each other (otherwise Σ^z is not non-local and therefore a permissible error!). However, sometimes one might need to measure (or apply) the logical operator Σ^z (after all, we want to do quantum computing with the encoded qubit). This means that the endpoints of the Majorana chain must be moved close to operate on the encoded qubit, but must remain far apart when storing the qubit for future use:



Modifying the geometry to apply controlled unitary operations while suppressing unwanted perturbations is a characteristic feature of → *topological quantum memories* and → *topological quantum computing*.

→ How can we prevent elementary errors from accumulating?

→ Solution: