↓ **Lecture 15** [06.06.25]

5. Topological superconductors in 1D: The Majorana Chain

5.1. Preliminaries: Particle-hole symmetry and mean-field superconductors

Before we can discuss the Majorana chain – the paradigmatic model of a *topological superconductor* – we first review a few important concepts needed for its description:

- Remember (← Section 4.1):

  Particle-hole symmetry (PHS) $\mathcal{C}_U$:

  The naming should be evident:
  $\mathcal{C}_U$ exchanges particles with holes ($c_i \leftrightarrow c_i^\dagger$) up to a unitary transformation U.

  $$\mathcal{C}_U i \mathcal{C}_U^{-1} = +i \quad \text{and} \quad \mathcal{C}_U c_i \mathcal{C}_U^{-1} = \sum_j U_{ij}^{*\dagger} c_j^\dagger \tag{5.1a}$$

  $$\left[\hat{H}, \mathcal{C}_U\right] = 0 \quad \Leftrightarrow \quad U H^* U^\dagger = -H \quad \Leftrightarrow \quad \{H, \underbrace{U\mathcal{K}}_{C_U}\} = 0 \tag{5.1b}$$

  (The complex conjugate at the U is convention and not crucial.)

  →

  - Unitary symmetry on MB Hamiltonian
  - Antiunitary pseudosymmetry on SP Hamiltonian

    As a pseudosymmetry, $C_U = U\mathcal{K}$ *anti*commutes with the SP Hamiltonian.

  Of course, this symmetry will be crucial to define a *new topological invariant*.

- Remember (↓ *your lecture on solid state physics*):

  BCS theory of superconductivity:

  Until now, we only considered (topological) *insulators*, i.e., quadratic fermion theories with *particle number conservation*. By contrast, the Majorana chain is a (topological) *superconductor*, where only *fermion parity* survives as symmetry. Let us briefly review how these particle-number violating terms emerge from a microscopic theory:

  1 | ◁ ⁂ *BCS Hamiltonian*: (BCS = Bardeen-Cooper-Schrieffer)

  $$\hat{H}_{\text{BCS}} = \underbrace{\sum_{\boldsymbol{k},\sigma} (\varepsilon_{\boldsymbol{k}} - \mu)\, c_{\boldsymbol{k}\sigma}^\dagger c_{\boldsymbol{k}\sigma}}_{\text{Free fermions}} + \underbrace{\sum_{\boldsymbol{k},\boldsymbol{k}'} V_{\boldsymbol{k}\boldsymbol{k}'}\, c_{\boldsymbol{k}\uparrow}^\dagger c_{-\boldsymbol{k}\downarrow}^\dagger c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow}}_{\text{Pairing term (interaction)}} \tag{5.2}$$

$\sigma \in \{\uparrow, \downarrow\}$: fermion spin

μ: chemical potential

$\varepsilon_{\boldsymbol{k}}$: free fermion dispersion

$V_{\boldsymbol{k}\boldsymbol{k}'}$: pairing potential

- *Rationale:* Superconductivity is a condensation mechanism that is triggered by attractive interactions $V_{\boldsymbol{k}\boldsymbol{k}'}$ (mediated by phonons) between fermions. The formation of bosonic ↓ *Cooper pairs* then lowers the energy, the Cooper pairs condense and form the superconducting condensate.
- Note that Eq. (5.2) is a theory of interacting fermions with particle-number conservation. The symmetry group U(1) is generated by the total particle number operator $N = \sum_{\boldsymbol{k},\sigma} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}$ with $[\hat{H}_{\text{BCS}}, N] = 0$. Due to the interactions, Eq. (5.2) cannot be diagonalized exactly.

2 | The BCS Hamiltonian is interacting (= not quadratic) and therefore hard to study.

→ ↓ *Mean-field theory:*

$$c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} = X^*_{\boldsymbol{k}} + (c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} - X^*_{\boldsymbol{k}}) \quad \text{with} \quad X^*_{\boldsymbol{k}} = \langle c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} \rangle \tag{5.3a}$$

$$c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} = \underbrace{X_{\boldsymbol{k}'}}_{\text{Mean}} + \underbrace{(c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} - X_{\boldsymbol{k}'})}_{\text{Small fluctations } \delta X_{\boldsymbol{k}'}} \quad \text{with} \quad X_{\boldsymbol{k}} = \langle c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} \rangle \tag{5.3b}$$

⁂ *Cooper pair condensation* ⇔ $X_{\boldsymbol{k}'} \neq 0$ and $\delta X_{\boldsymbol{k}'}$ small

[The approximation $c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} = X^*_{\boldsymbol{k}} \times \mathbb{1} + \delta X_{\boldsymbol{k}}$ means that we expect the ground state to be (approximately) invariant under the application of $c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow}$ (and similarly $c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow}$). This can only be true if the ground state is a superposition of states with all possible numbers of fermions (with the same parity). Such a superposition is usually called a ⁂ *condensate*.]

→ Drop terms of order $\mathcal{O}(\delta X^2_{\boldsymbol{k}'})$ (and a constant offset):

$$\hat{H}^{\text{mf}}_{\text{BCS}} \stackrel{\circ}{=} \underbrace{\sum_{\boldsymbol{k},\sigma} (\varepsilon_{\boldsymbol{k}} - \mu)\, c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma}}_{\text{Free fermions}} + \sum_{\boldsymbol{k}} \underbrace{\left[\Delta_{\boldsymbol{k}}\, c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} + \Delta^*_{\boldsymbol{k}}\, c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow}\right]}_{\text{Quadratic pairing terms}} \tag{5.4}$$

with order parameter

$$\Delta_{\boldsymbol{k}} = \sum_{\boldsymbol{k}'} V_{\boldsymbol{k}\boldsymbol{k}'} X_{\boldsymbol{k}'} \quad \in \mathbb{C} \tag{5.5}$$

Since here Cooper pairs are formed by fermions with total spin zero $[X_{\boldsymbol{k}} = \langle c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'\uparrow} \rangle]$ this is called ↓ *s-wave superconductivity*.

- ¡! $\hat{H}^{\text{mf}}_{\text{BCS}}$ is no longer particle-number conserving; only the fermion parity $\mathcal{P} = (-1)^N$ is conserved: $[\hat{H}^{\text{mf}}_{\text{BCS}}, \mathcal{P}] = 0$.
- The $\mathbb{Z}_2$ group generated by $\mathcal{P}$ is a subgroup of U(1) generated by N, hence this is an example of ← *spontaneous symmetry breaking* (Section 0.4), where the superconducting condensate breaks (global) particle-number conservation and only fermion parity survives. The Hamiltonian $\hat{H}^{\text{mf}}_{\text{BCS}}$ makes only sense as an *effective mean-field description* that excludes the superconducting condensate from/into which pairs of electrons can be transfered.

[As mentioned in Section 0.5, the correct classification of the superconducting phase is subtle [35]. When the fermions are charged and coupled to a *dynamical* gauge field, the transition is not described by SSB but a topological phase transition [39, 40]. When the fermions couple to a *static* (= background) gauge field, the transition is described by spontaneous breaking of the *global* U(1) symmetry. It is not correct (as one sometimes hears) that the (local) *gauge* symmetry is broken spontaneously [135–138].]

- The benefit of the mean-field description $\hat{H}^{\mathrm{mf}}_{\mathrm{BCS}}$ of superconductivity is that the Hamiltonian is *quadratic* in fermions and therefore fits our current class of models ("non-interacting fermions").

In this section, we consider quadratic fermion Hamiltonians of the form (5.4) (i.e., with superconducting pairing terms). We treat these models as *fundamental*, and ignore that they actually arise from microscopic, interacting, particle-number conserving theories via spontaneous symmetry breaking!

5.2. The Majorana chain

A detailed exposition of the Majorana chain is given in the textbook by Bernevig [1] but may also be found in almost any other textbook that covers topological superconductors. Furthermore, the original paper by KITAEV is worthwile to read [139]. There is also an introduction in my PhD thesis [126] (on which this section is based) and a more detailed account in my Master thesis [140].

1 | $\triangleleft$ 1D superconductor of spinless fermions c_i (= p-wave pairing):

$$\boxed{\hat{H}_{\mathrm{MC}} := -\sum_{i=1}^{L'} \left(w\, c_i^\dagger c_{i+1} - \Delta\, c_i c_{i+1} + \text{h.c.}\right) - \sum_{i=1}^{L} \mu \left(c_i^\dagger c_i - \frac{1}{2}\right)} \tag{5.6}$$

$w \in \mathbb{R}$: tunneling amplitude

$\Delta = e^{i\theta}|\Delta| \in \mathbb{C}$: superconducting gap ($\theta$ is the phase of the condensate)

$\mu \in \mathbb{R}$: chemical potential

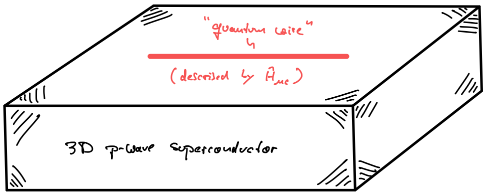
$L' = L$ (PBC) or $L' = L - 1$ (OBC)

- This is the mean-field theory (in real space!) of a "triplet superconductor" with p-wave pairing, i.e., Cooper pairs consist of *spin-polarized* (and therefore effectively *spinless*) electrons with total angular momentum of one.

  Of course there are no true "spinless fermions" because of the ↑ *spin-statistics theorem*. However, imagine you apply a strong magnetic field such that only fermions in spin-polarized modes $c_{i\uparrow}$ are relevant for the low-energy physics (in particular: ground state). If only operators like $c_{i\uparrow}$ show up in the (low-energy) Hamiltonian, one can drop the spin-index ↑ altogether: $c_{i\uparrow} \mapsto c_i$. This is what we mean by "spinless fermions."

- ¡! We are interested in topological phase transitions *between* different superconducting phases – and not in the superconducting phase transition itself (which is, as mentioned above, described by spontaneous symmetry breaking). Therefore we do not determine the gap Δ self-consistently (as done in BCS theory) but simply take it as a non-zero, translation invariant parameter of the theory.

- With a unitary transformation $c_i = e^{-i\theta/2} c_i'$ one can remove the superconducting phase, so that *w.l.o.g.* $\Delta = |\Delta|$ is real. Note that since the system is one-dimensional, there cannot be vortices in the superconducting condensate (= flux tubes).
- In one dimension, the ↑ *Mermin-Wagner theorem* forbids the spontaneous breaking of the continuous $U(1)$ symmetry (particle-number conservation) that is responsible for the superconducting phase. (Instead one finds a disordered phase known as a ↑ *Luttinger liquid* with correlations that decay algebraically.) Thus one should think of the superconducting terms in $\hat{H}_{\text{MC}}$ as being induced by the ↑ *proximity effect* of an attached *three dimensional* bulk superconductor:



This is also (roughly) the setting used to study the Majorana chain in experiments [141] (although there have been setbacks [142]).

2 | $\sphericalangle$ PBC → Fourier transform:

$$\tilde{c}_k = \frac{1}{\sqrt{L}} \sum_{n=1}^{L} e^{-ikn} c_n \quad \Leftrightarrow \quad c_n = \frac{1}{\sqrt{L}} \sum_{k\in\text{BZ}} e^{ikn} \tilde{c}_k \tag{5.7}$$

with $k = \frac{2\pi}{L} m$ for $m = 0, \dots, L-1$.

$\overset{\circ}{\to}$ (up to a constant)

$$\hat{H}_{\text{MC}} = -\sum_{k\in\text{BZ}} \left[(2w\cos k + \mu)\, \tilde{c}_k^\dagger \tilde{c}_k + i\Delta \sin(k)\, \tilde{c}_k \tilde{c}_{-k} - i\Delta \sin(k)\, \tilde{c}_{-k}^\dagger \tilde{c}_k^\dagger \right] \tag{5.8}$$

Note that because of the pairing terms, this Hamiltonian is not yet diagonal (despite there being only a single mode per unit cell and no spin involved). To diagonalize it, we can use a trick:

3 | <u>Bogoliubov-de Gennes Hamiltonian:</u>

We expand the cosine term artificially (using an index substitution $k \mapsto -k$ in the sum):

$$(2w\cos k + \mu)\, \tilde{c}_k^\dagger \tilde{c}_k \quad \mapsto \quad \frac{1}{2}[(2w\cos k + \mu)\, \tilde{c}_k^\dagger \tilde{c}_k + (2w\cos k + \mu)\, \tilde{c}_{-k}^\dagger \tilde{c}_{-k}] \tag{5.9}$$

→

$$\hat{H}_{\text{MC}} = -\frac{1}{2} \sum_{k\in\text{BZ}} \begin{bmatrix} (2w\cos k + \mu)\, \tilde{c}_k^\dagger \tilde{c}_k + (2w\cos k + \mu)\, \tilde{c}_{-k}^\dagger \tilde{c}_{-k} \\ + i2\Delta \sin(k)\, \tilde{c}_k \tilde{c}_{-k} - i2\Delta \sin(k)\, \tilde{c}_{-k}^\dagger \tilde{c}_k^\dagger \end{bmatrix} \tag{5.10}$$

Introduce ⁂ *Nambu spinors*

$$\boldsymbol{\Psi}_k := \begin{pmatrix} \tilde{c}_k \\ \tilde{c}_{-k}^\dagger \end{pmatrix} \tag{5.11}$$

Note that the degrees of freedom described by the components of the Nambu spinor are *not* independent but related by particle-hole symmetry. This is different from the introduction of other pseudo-spinors in the situation of multiple DOFs per unit cell (like sublattices or internal DOFs).

→ Rewrite the Hamiltonian (up to a constant)

$$\hat{H}_{\mathrm{MC}} = \frac{1}{2}\sum_{k\in\mathrm{BZ}} \begin{pmatrix}\tilde{c}_k^\dagger & \tilde{c}_{-k}\end{pmatrix} \cdot \underbrace{\begin{pmatrix}-2w\cos k-\mu & -2\Delta i\sin k\\ 2\Delta i\sin k & 2w\cos k+\mu\end{pmatrix}}_{H_{\mathrm{BdG}}(k)} \cdot \begin{pmatrix}\tilde{c}_k\\ \tilde{c}_{-k}^\dagger\end{pmatrix} \tag{5.12a}$$

$$= \frac{1}{2}\sum_{k\in\mathrm{BZ}} \mathbf{\Psi}_k^\dagger H_{\mathrm{BdG}}(k)\mathbf{\Psi}_k \tag{5.12b}$$

with ⁂ *Bogoliubov-de Gennes Hamiltonian*

$$H_{\mathrm{BdG}}(k) = -(2w\cos k+\mu)\,\sigma^z + 2\Delta\sin k\,\sigma^y = \vec{d}(k)\cdot\vec{\sigma} \tag{5.13}$$

and

$$\vec{d}(k) = \begin{pmatrix}0\\ 2\Delta\sin k\\ -2w\cos k-\mu\end{pmatrix} \tag{5.14}$$

The BdG Hamiltonian is a *redundant* matrix encoding of the MB Hamiltonian $\hat{H}_{\mathrm{MC}}$. It exists for all quadratic fermion Hamiltonians, but is non-trivial (= not diagonal) – and therefore useful – only for Hamiltonians with superconducting pairing terms. As the above construction demonstrates, its existence is rooted in the algebra of the fermion operators.

4 | Bogoliubov transformation:

To diagonalize Eq. (5.12), we must diagonalize the BdG Hamiltonian:

$$U_k^\dagger\, H_{\mathrm{BdG}}(k)\, U_k = \begin{pmatrix}E(k) & 0\\ 0 & -E(k)\end{pmatrix} \tag{5.15}$$

U_k: unitary rotation in Nambu space

The symmetry of the spectrum follows from PHS of the BdG Hamiltonian (→ *below*).

Define new fermion modes → ⁂ *Bogoliubov quasiparticles:*

$$\tilde{\mathbf{\Psi}}_k \equiv \begin{pmatrix}\tilde{a}_k\\ \tilde{a}_{-k}^\dagger\end{pmatrix} := U_k^\dagger\mathbf{\Psi}_k \mathrel{\hat=} \begin{pmatrix}u_k\tilde{c}_k + v_k\tilde{c}_{-k}^\dagger\\ v_{-k}^*\tilde{c}_k + u_{-k}^*\tilde{c}_{-k}^\dagger\end{pmatrix} \tag{5.16}$$

The coefficients u_k and v_k satisfy certain constraints to ensure that the new modes $\tilde{a}_k$ obey fermionic anticommutation relations:

$$\left\{\tilde{a}_k, \tilde{a}_k^\dagger\right\} \mathrel{\hat=} |u_k|^2 + |v_k|^2 \overset{!}{=} 1\,, \tag{5.17a}$$

$$\{\tilde{a}_k, \tilde{a}_{-k}\} \mathrel{\hat=} v_k u_{-k} + u_k v_{-k} \overset{!}{=} 0\,. \tag{5.17b}$$

That this structure for U_k is possible is again a consequence of the PHS of the BdG Hamiltonian (→ *below*). Note that this additional structure is necessary because the Bogoliubov transformation mixes particles and holes. By contrast, for the diagonalization of a non-superconducting Bloch Hamiltonian, any unitary U_k yields a canonical transformation (because there one does not mix annihilation with creation operators).

For Eq. (5.12) in the important special case $\Delta = w$ and $\mu = 0$ (→ *later*), one finds the explicit expressions

$$u_k \mathrel{\hat=} i\sin\frac{k}{2} \quad\text{and}\quad v_k \mathrel{\hat=} \cos\frac{k}{2}\,. \tag{5.18}$$

5 | Spectrum: Eqs. (2.9) and (5.14) →

$$E(k) = |\vec{d}(k)| = \sqrt{(2w\,\cos k + \mu)^2 + 4\Delta^2\,\sin^2 k} \tag{5.19}$$

Because of the redundancy of the BdG Hamiltonian, the second band (and therefore the second eigenenergy $-|\vec{d}(k)|$ of H_{BdG}) is "fake" …

…because

$$\hat{H}_{\text{MC}} = \frac{1}{2}\sum_{k\in\text{BZ}} \begin{pmatrix} \tilde{a}_k^\dagger & \tilde{a}_{-k} \end{pmatrix} \cdot \begin{pmatrix} +E(k) & 0 \\ 0 & -E(k) \end{pmatrix} \cdot \begin{pmatrix} \tilde{a}_k \\ \tilde{a}_{-k}^\dagger \end{pmatrix} \tag{5.20a}$$

$$= \frac{1}{2}\sum_{k\in\text{BZ}} \left[E(k)\,\tilde{a}_k^\dagger \tilde{a}_k - E(k)\,\tilde{a}_{-k}\tilde{a}_{-k}^\dagger \right] \tag{5.20b}$$

$$= \sum_{k\in\text{BZ}} E(k)\,\tilde{a}_k^\dagger \tilde{a}_k + \text{const} \tag{5.20c}$$

where we used $E(k) = E(-k)$ and $\{\tilde{a}_k, \tilde{a}_k^\dagger\} = 1$; i.e., for every k there is only *one* mode $\tilde{a}_k$ with energy $+E(k)$.

6 | Preliminary Phase diagram:

Let $\Delta \neq 0 \rightarrow E(k) = 0$ only possible for $k = 0, \pi \rightarrow$

$$E(0/\pi) = |\pm 2w + \mu| \overset{!}{=} 0 \quad \Rightarrow \quad 2|w| = |\mu| \tag{5.21}$$

→ Two gapped phases:

$$\boxed{\text{Phase A: } 2|w| > |\mu| \quad \text{and} \quad \text{Phase B: } 2|w| < |\mu|} \tag{5.22}$$

¡! In contrast to models with particle-number conservation, the gap here is not given by the separation of two bands (there is only one!), and the ground state is not obtained by "filling" the lower of two bands. Since $E(k) > 0$ for all $k \in$ BZ, Eq. (5.20c) implies that the many-body ground state is the state with all modes $\tilde{a}_k$ empty (→ *next*), and excited states are characterizied by occupied modes (↑ *Bogoliubov quasiparticles*) with a finite (system-size independent) energy. This is the gap of the system (induced by superconductivity); the quasiparticle excitations are "particle-hole excitations" (superpositions of a fermion above and a hole in the condensate) and can be thought of as "broken" Cooper pairs.

7 | Many-body ground state $|\Omega\rangle$ with

$$\tilde{a}_k|\Omega\rangle \overset{!}{=} 0 \quad \forall k \in \text{BZ} \tag{5.23}$$

→ *Unique* BCS ground state (unique in *both* phases, i.e., no symmetry breaking!)

$$|\Omega\rangle \propto \prod_{k:\,\tilde{a}_k|0\rangle\neq 0} \tilde{a}_k|0\rangle \overset{5.16}{\underset{\substack{w=\Delta\\ \mu=0}}{\overset{5.18}{\propto}}} \prod_{k\in(-\pi,\pi)} \tilde{a}_k|0\rangle \overset{\circ}{\propto} \tilde{a}_0 \prod_{k\in(0,\pi)} \left(u_k + v_k \tilde{c}_{-k}^\dagger \tilde{c}_k^\dagger\right)|0\rangle \tag{5.24a}$$

- This ground state is called *quasi*particle vacuum ($\tilde{a}_k|\Omega\rangle = 0$) and is different from the *physical* vacuum ($\tilde{c}_k|\Omega\rangle \neq 0$), i.e., $|\Omega\rangle$ contains superpositions of states with different particle numbers of c_i-fermions (this is true as long as $\Delta \neq 0$, i.e., in the presence of a superconducting condensate).

- As we will see → *below*, the parameter choice $w = \Delta$ and $\mu = 0$ corresponds to the fixpoint in Phase A (which is topological). That the model simplifies at this point is apparent from the spectrum (5.19) which becomes flat.
- Notice that in phase A (for $w = \Delta$ and $\mu = 0$) $|\Omega\rangle$ has *negative* fermion parity because of the zero-mode $\tilde{a}_0$ (the other TRIM mode $\tilde{a}_\pi$ annihilates $|0\rangle$ and must not be applied). This can be shown by deriving u_k and v_k explicitely for this case [Eq. (5.18)].

5.3. Symmetries and topological indices

Our next goal is to characterize (and distinguish) the two gapped phases A and B by topological features of the BdG Hamiltonian:

8 | $\sphericalangle$ Time-reversal symmetry:

i | *w.l.o.g.* Δ real $\to$ $\mathcal{T} := \mathbb{1}\mathcal{K}$ $\to$ $[\hat{H}_{\text{MC}}, \mathcal{T}] = 0$ $\to$ TRS ✓
More precisely: $\mathcal{T}\, i\, \mathcal{T}^{-1} = -i$ and $\mathcal{T}\, c_i^{(\dagger)} \mathcal{T}^{-1} = c_i^{(\dagger)}$.

$\overset{\circ}{\to}$ After a Fourier transform, TRS is represented as (acting on "Nambu space") [← Eq. (2.31d)]

$$\mathbb{1}\, H^*_{\text{BdG}}(k)\, \mathbb{1} = H_{\text{BdG}}(-k) \tag{5.25}$$

$\to \tilde{T} = \mathbb{1}\mathcal{K} \to$ TRS with $\tilde{T}^2 = +\mathbb{1}$

Systems with a TRS that squares to $+\mathbb{1}$ are combined into the …

→ ⁂ *Symmetry class* **AI** [→ ??]

The label has historical/mathematical meaning but is of no importance to us.
↑ *Cartan's classification of symmetric spaces*

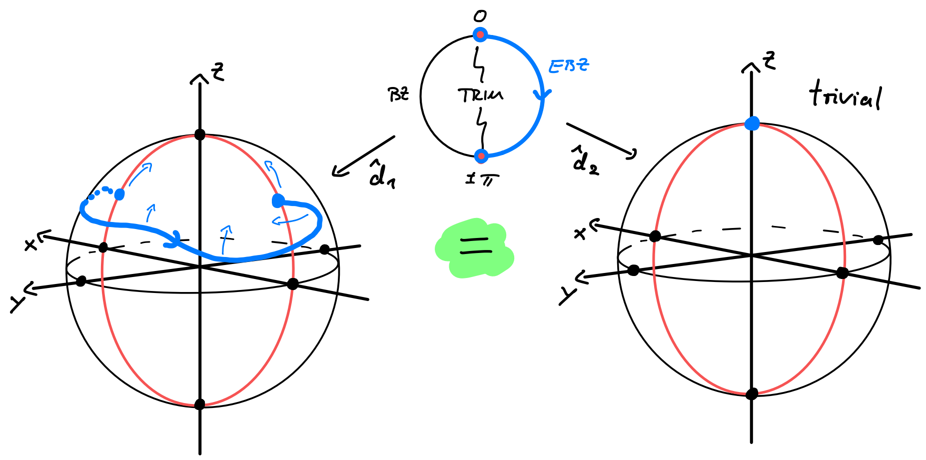
ii | Eqs. (5.13) and (5.25) $\to$ Constraints on the BdG vector:

$$d_x(-k) = d_x(k) \tag{5.26a}$$
$$d_y(-k) = -d_y(k) \tag{5.26b}$$
$$d_z(-k) = d_z(k) \tag{5.26c}$$

$\to \vec{d}(k)$ on EBZ $[0, \pi]$ determines $H_{\text{BdG}}(k)$ completely

iii | $\sphericalangle$ $K^* \in \{0, \pi\}$ TRIM $\to$ $d_y(K^*) = 0$

$\to$ Image $\hat{d}\,(\text{EBZ})$ $[\hat{d}\,(k) = \frac{\vec{d}\,(k)}{|\vec{d}\,(k)|}]$ on S^2 must *start & end* on great circle:



→ All paths (= gapped & symmetric Hamiltonians) can be continuously contracted

→ No topological phases ☹

Note that a BdG vector pointing in z-direction corresponds to the Hamiltonian (5.6) with $w = 0 = \Delta$ and only a chemical potential $\mu \neq 0$, which is obviously a trivial band insulator with all modes either empty or filled (depending on the sign of μ).

iv | Boldly generalizing these findings, we could hypothesize:

> One-dimensional systems of symmetry class **AI** do not allow for TPs.

This is true in general; → ?? on the classification of topological insulators & superconductors.

v | Conclusion for the Majorana chain:

TRS alone is not sufficient to characterize the phases of the Majorana chain.

→ We need something else ...

9 | $\sphericalangle$ Particle-hole "symmetry":

i | $\sphericalangle$ Eqs. (2.31d), (5.1b) and (5.13)

The BdG Hamiltonian (matrix) has an *intrinsic* PHS:

$$\sigma^x\, H^*_{\text{BdG}}(k)\, \sigma^x = -H_{\text{BdG}}(-k) \tag{5.27}$$

In real space this would read $UH^*_{\text{BdG}}U^\dagger = -H_{\text{BdG}}$, where U acts as σ^x on the Nambu subspace spanned by c_i and $c_i^\dagger$. Above we had no need to explicitly define H_{BdG} in real space.

→ $\tilde{C} := \sigma^x \mathcal{K}$ → PHS with $\tilde{C}^2 = +\mathbb{1}$

→ ⁂ *Symmetry class* **D** [→ ??]

- This "symmetry" is tautological in the sense that it derives solely from the fermion algebra. It does *not* correspond to a physical many body symmetry $\mathcal{C}$ of $\hat{H}_{\text{MC}}$, so that some authors do not call it a "symmetry" altogether. However, it is a valid antiunitary *pseudo*symmetry of the BdG Hamiltonian – and this is all that matters for the discussion that follows. Whether the algebraic constraint Eq. (5.27) on $H_{\text{BdG}}(k)$ derives from a *physical symmetry* or from the *algebraic structure* of the fermion algebra is irrelevant for the topological classification of $H_{\text{BdG}}(k)$.

  If this all seems a bit cryptic: ➔ Problemset 8

- This teaches us something important: The "symmetry classes" we started to introduce (like **AI** and **D**) should be thought of as classes/ensembles of *matrices* with certain constraints. If these matrices derive from a many-body Hamiltonian (like a Bloch-oder BdG Hamiltonian), these constraints *can* descend from real symmetries of the many-body Hamiltonian. However, this is not always the case (as for the PHS of superconductors). This explains the somewhat opaque statement that **D** describes the family of superconductors *without* symmetries – where "symmetries" refers to *physical* symmetries of the many-body Hamiltonian.

  Note that the proper concept of "particle-hole symmetry" has not yet been fully settled in the community [143], partially due to the tautological nature of the PHS above (which is then refered to as *charge conjugation* instead of *particle-hole transformation*).

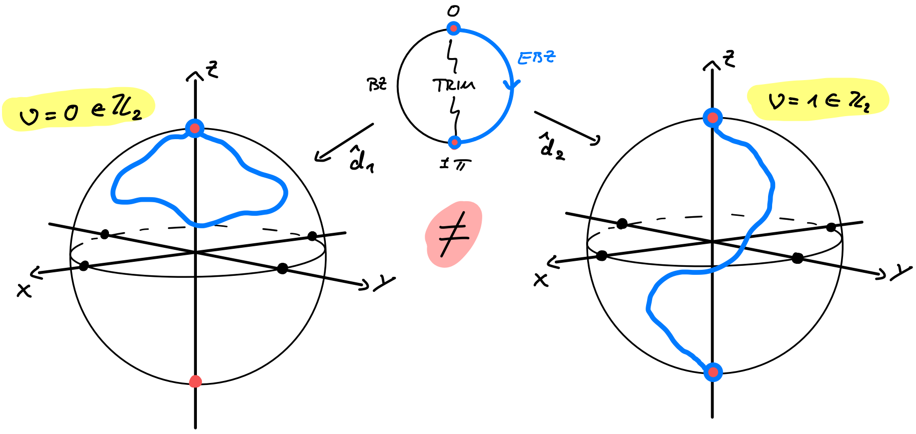
ii | Eqs. (5.13) and (5.27) → Constraints on the BdG vector:

$$d_x(-k) = -d_x(k) \tag{5.28a}$$
$$d_y(-k) = -d_y(k) \tag{5.28b}$$
$$d_z(-k) = d_z(k) \tag{5.28c}$$

→ Again $\vec{d}(k)$ on EBZ $[0,\pi]$ determines $H_{\text{BdG}}(k)$ completely

iii | $\sphericalangle$ $K^* \in \{0,\pi\}$ TRIM → $d_x(K^*) = 0 = d_y(K^*)$
Note that these are now *two* constraints as compared to TRS!

→ Image $\hat{d}$(EBZ) on S^2 must *start & end* either on "north" or "south pole":



→ Two topologicall distinct classes of paths
(Only one of which can be continuously contracted to a point.)

→ One topological phase possible ☺ → $\mathbb{Z}_2$-index

Note that the orientation of the Bloch sphere (and therefore the position of the poles) has no physical meaning as it can be changed continuously by SU(2) rotations in Nambu space (as we did with the Bogoliubov transformation). Consequently, a path attached to the *south* pole is unitarily equivalent to the shown path attached to the north pole.

iv | Boldly generalizing these findings, we could hypothesize:

> In 1D, systems of class **D** allow for a single TP labeled by a $\mathbb{Z}_2$-index.

Again, this is true in general; → ?? on the classification of topological insulators & superconductors.