

# 0. Setting the Stage

## ◇ Topics

- Motivation: A classical system with topological edge modes
- Localization within physics: Where we are on the energy ladder
- Introduce our objects of interest: Quantum phases and phase transitions
- Sketch the Landau paradigm: Spontaneous symmetry breaking
- Concepts beyond the Landau paradigm: Topological phases
- Sketch different types of topological phases

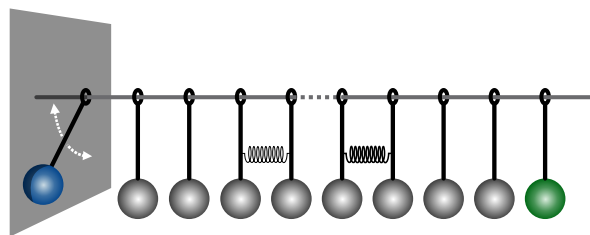
## 0.1. Motivation: Transferring energy with pendulums

To get you hooked (hopefully!), we start with a series of simple *classical mechanics* “experiments” (= computer simulations). The point of this adventure is to highlight some of the surprising effects *topological* features can have (where exactly topology enters is not obvious and will be discussed in due time):

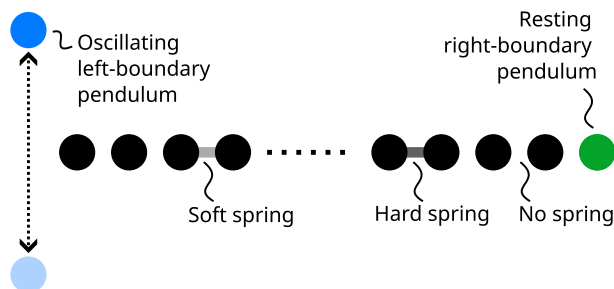
The following is inspired by on one of my papers [21].

Beamer and internet connection required!

1 | < 1D chain of  $N$  identical pendulums, coupled by *tunable* springs:



→ Schematic view from bottom:



We encode the strength of springs by their color:  
White: no spring / Light: soft spring / Dark: stiff spring

2 | **Goal:** Transfer oscillation energy from one boundary to the other:

$$\underbrace{\vec{x}(t < 0) = (1, 0, \dots, 0) \cdot e^{i\omega t}}_{\text{Left pendulum excited}} \xrightarrow[\text{How??}]{\text{Time evolution}} \underbrace{\vec{x}(t > T) = (0, \dots, 0, 1) \cdot e^{i\omega t}}_{\text{Right pendulum excited}} \quad (0.1)$$

Here,  $x_i(t)$  denotes the displacement of pendulum  $i$  at time  $t$ ; our protocol starts at  $t = 0$  and ends at  $t = T$ . The eigenfrequency of the (identical) pendulums is  $\omega$ .

3 | **Time evolution** → Classical equation of motion:

$$\ddot{\vec{x}} + \mathbb{D}(t)\vec{x} = 0 \quad (0.2)$$

This is the Newtonian equation of motion for  $N$  coupled harmonic oscillators.

$\mathbb{D}(t) \in \mathbb{R}^{N \times N}$ : Time dependent coupling matrix

4 | **Rules:**

- We can choose the stiffness for each spring independently.
- We can modify the stiffness of an arbitrary subset by *a single* time dependent factor.
- We can choose the time dependence of this factor freely.

→ Allowed form of the coupling matrix:

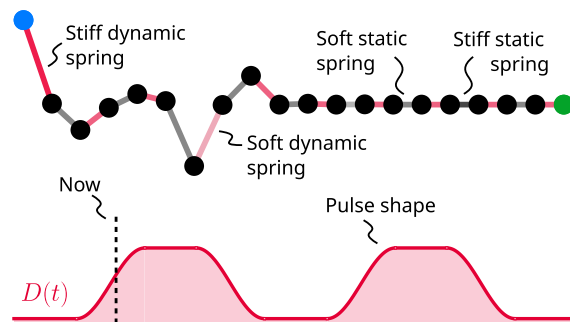
$$\mathbb{D}(t) = \underbrace{\begin{pmatrix} \omega_1^2 & s_1 & & & \\ s_1 & \omega_2^2 & s_2 & & \\ & s_2 & \omega_3^2 & & \\ & & & \ddots & \end{pmatrix}}_{\text{Static springs \& pendulums}} + D(t) \underbrace{\begin{pmatrix} 0 & d_1 & & & \\ d_1 & 0 & d_2 & & \\ & d_2 & 0 & & \\ & & & \ddots & \end{pmatrix}}_{\text{Time dependent springs}} \quad (0.3)$$

With ...

- $\omega_i = \sqrt{g/l_i} \equiv \omega$ : Frequency of pendulums (uniform and fixed)
- $s_j$ : Static stiffness of spring coupling pendulums  $i$  and  $i + 1$
- $D(t) \cdot d_j$ : Time dependent stiffness of spring coupling pendulums  $i$  and  $i + 1$
- Global time dependence of spring stiffness:

$$D(t) = \begin{cases} 0 & t < T \\ P(t) & 0 \leq t \leq T \\ 0 & t > T \end{cases} \quad \text{with pulse shape } P(t) : [0, T] \rightarrow [0, 1] \quad (0.4)$$

→ Schematic view:



We color static (tunable) springs in shades of black (red). The shape of  $D(t)$  is plotted below the pendulum chain; the current point in time is marked by a vertical line in this plot.

5 | Questions:

- How to choose the spring couplings  $s_i$  and  $d_i$ ?
- How to choose the pulse shape  $P(t)$ ?

6 | Experiments:

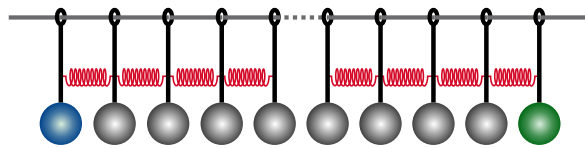
The simulations below are based on numerical integration of Eq. (0.2) with initial configuration (0.1):

[↻ Download Mathematica notebook](#)

i | < Variant 1:

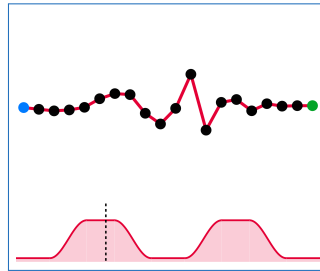
$$s_i = 0 \quad \text{and} \quad d_i = d > 0 \quad \text{for all} \quad i = 1, \dots, N - 1 \quad (0.5)$$

In this approach, we couple all pendulums *uniformly* by springs of *time dependent* stiffness:



As pulse  $P(t)$  we choose a smoothed-out rectangular double pulse to transfer the excitation from left to right and back. The latter is of course not necessary; it allows us to amplify the effects of a single transfer. We normalize the pulse such that  $\max_t P(t) \approx 1$ .

→ **Simulation:** [Click on figure \(internet required\).](#)



<https://itp3.info/pendulumv1>

→ **Result:** No perfect transfer possible! ☹

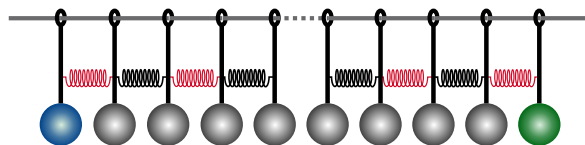
The reason is quite obvious: The boundary excitation is transferred via an elastic wave that travels through the bulk. Because of  $\downarrow$  *dispersion*, this excitation cannot be relocalized on the other boundary; we loose inevitably energy to bulk excitations.

ii | < Variant 2: (We assume  $N$  to be even!)

$$s_i = 0 \quad \text{and} \quad d_i = d > 0 \quad \text{for odd} \quad i = 1, 3, \dots, N - 1 \quad (0.6a)$$

$$s_i \approx 2 \times d \quad \text{and} \quad d_i = 0 \quad \text{for even} \quad i = 2, 4, \dots, N - 2 \quad (0.6b)$$

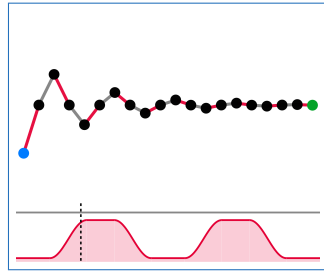
Now we couple pendulums *alternating* with weak & dynamic and strong & static springs:



- We use the same pulse  $P(t)$  as for **Variant 1** above. Now it affects only every other spring, of course!

- If you wonder *how* one might come up with this contraption: This is why you should attend this course ☺.

→ Simulation:



<https://itp3.info/pendulumv2>

→ Result: (Almost) perfect transfer possible! ☺☺

- The video above is “stroboscopic”, *i.e.*, the pendulums are oscillating with a much higher frequency; the visible oscillations are therefore determined by the actual frequency and the chosen frame times (↓ *beat frequency*). The transfer also works with lower frequencies (as in the **Variante 1** video above), but would then take much longer.
- The reason why this approach works perfectly is not obvious. In particular, its robustness to certain types of disorder (→ *next*) are not trivial to understand. We need to introduce quite a bit of machinery to tackle this problem (→ *much later*).

What happens to this method if the constituents of our contraption have *Imperfections*?

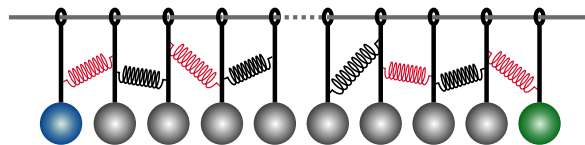
a | < Imperfect springs:

$$s_i = 0 \quad \text{and} \quad d_i \in \mathcal{N}(d, \sigma_d) \quad \text{for odd } i \quad (0.7a)$$

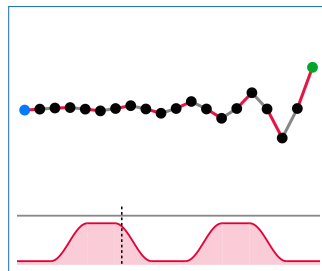
$$s_i \in \mathcal{N}(2d, \sigma_s) \quad \text{and} \quad d_i = 0 \quad \text{for even } i \quad (0.7b)$$

- $\mathcal{N}(\mu, \sigma)$  denotes the ↓ *normal distribution* with mean  $\mu$  and standard deviation  $\sigma$ .
- We choose  $\sigma_d \approx 0.1 \times d$  and  $\sigma_s \approx 0.1 \times 2d$ , *i.e.*, tolerances of about 10%.

→ We modify all non-zero spring couplings randomly by a small amount:



→ Simulation:



<https://itp3.info/pendulumv2a>

→ Result: Still perfect transfer possible! ☺

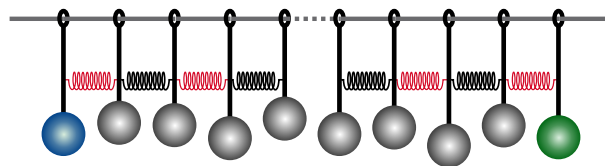
- ¡! This is not what one typically expects for an imperfect system. In particular for rather large imperfections of about 10%.
- To achieve perfect transfer, one has to tune the pulse slightly (either its height or its duration). However, one always finds an appropriately tuned pulse that achieves (almost) perfect transfer.
- Note that even if the pulse is not tuned, there is (almost) no energy loss to bulk modes. A non-optimal pulse therefore leads to an incomplete transfer but not to losses.
- If you look closely, there actually are weak excitations of the pendulum pairs in the bulk after the double transfer. This is a consequence of weak adiabaticity breaking; an ideal transfer would take infinitely long.

**b** | < Imperfect pendulums:

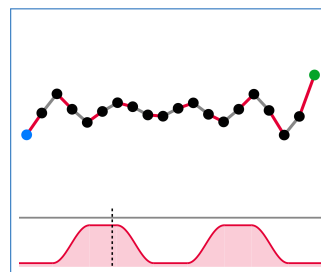
$$\text{Eq. (0.6) together with } \omega_i \equiv \omega \mapsto \omega_i \in \mathcal{N}(\omega, \sigma_\omega) \quad (0.8)$$

We choose  $\sigma_\omega \approx 0.1 \times \omega$ , i.e., tolerances of about 10%.

→ We modify all frequencies (= lengths of pendulums)  $\omega_i$  randomly by a small amount:



→ Simulation:



<https://itp3.info/pendulumv2b>

→ Result: No perfect transfer possible! ☹️

- This is the typical effect one might expect for an imperfect system.
- If one optimizes over the pulse length (or height), one typically does *not* find a pulse that achieves perfect transfer.
- Note that there is still no energy loss to bulk modes. This means that for small-enough imperfections, the time-evolution remains almost adiabatic.
- That the two boundary pendulums oscillate with drastically different periods is a consequence of the frequency imperfections (which of course also affect the boundary pendulums) in combination with the “stroboscopic” visualization.

7 | → Many questions ...

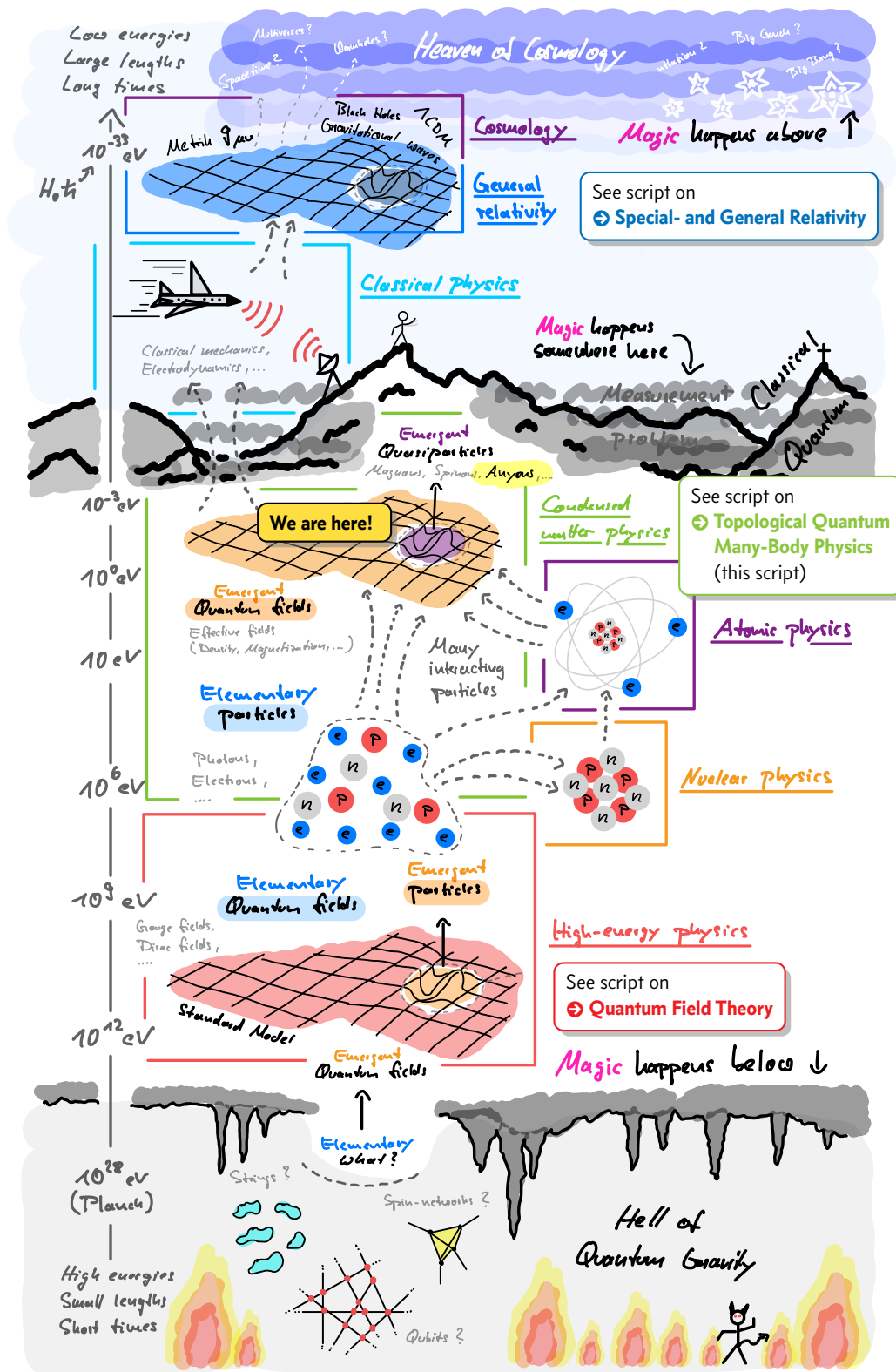
1. Why does **Variante 2** work? Where do these “boundary modes” come from?
2. Why is this procedure robust against one type of disorder, but not the other?
3. What has this to do with *topology*?
4. What has this to do with *quantum mechanics*?

Answers: → *Later*

For the impatient: The first three questions will be answered in Chapter 4. How some features of topological quantum phases translate to classical systems is discussed in Chapter 7.

## 0.2. The Big Picture

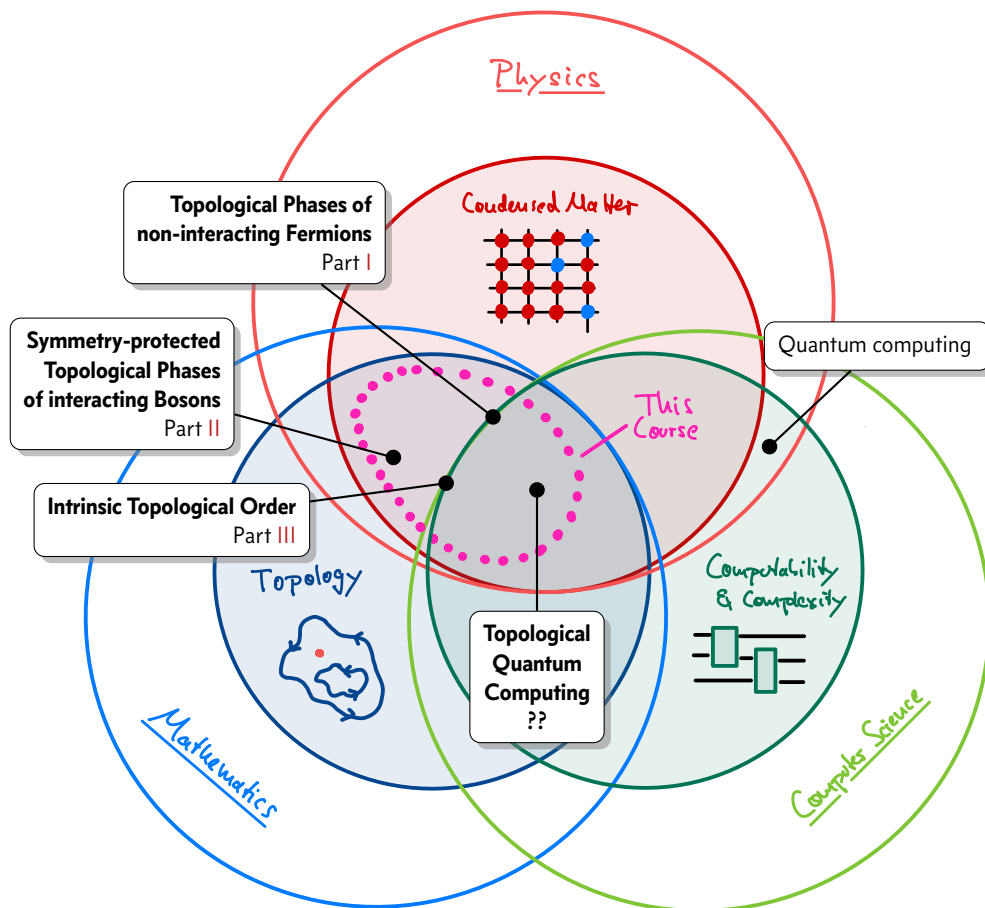
Physics strives for an objective, operational description of nature. To do so systematically, it is convenient to slice reality into layers separated by energy-, time- and length scales:



Comments:

- In this course we neither study the very small ( $\uparrow$  *high-energy physics*) nor the very large ( $\downarrow$  *relativity*). Thus we are not concerned with *fundamental* physics but with *emergent* phenomena.
- While this course is clearly focused on the *mathematics* and *conceptual foundations* that underlie the phenomenology of topological quantum phases, there will be connections to both experiments and applications along the lines. In particular the applications differentiate this course from more fundamental topics close to the extremes of the energy scale.
- At the very end ( $\rightarrow$  ??), we will briefly discuss a scenario where some of these emergent properties (related to topological order) might be of relevance for fundamental questions of high-energy physics. Maybe the realm of particle physics is emergent as well, and the theory of topological quantum many-body physics has something to say about questions that conventional high-energy physics is silent about? (For example, why there are fermions to begin with?)

The topics covered in this course can also be located with respect to adjacent scientific disciplines:



- $\uparrow$  *Topology* is the area of mathematics that deals with properties of spaces (*e.g.* manifolds) that are robust against smooth deformations of these spaces. For example, the topology of a torus (= donut) is characterized by the fact that it has a single “hole”; its exact shape (*e.g.* its size and local bumps on the surface) are part of its *geometry* but not relevant for its *topology*.
- Which, why, and how concepts of topology are instantiated in particular quantum phases is the main focus of this course.

### 0.3. Quantum phases and quantum phase transitions

1 | In this course, we are interested in the following concepts:

#### ✱ Definition: Quantum phases and phase transitions

- **Quantum phase**  $\Leftrightarrow$  Phase of matter at  $T = 0$  (= no thermal fluctuations)
  - $\Leftrightarrow$  Ground state manifold of
 
$$\left\{ \begin{array}{c} \text{scalable} \\ \text{local} \\ \text{many-body} \end{array} \right\} \text{ Hamiltonian in the } \textit{thermodynamic limit}$$
    - “*Scalable*”: The Hamiltonian is actually a *family* of Hamiltonians  $H_L$  parametrized by the system size  $L$  (e.g., number of modes/atoms/spins in each spatial direction).
    - “*Local*”: The Hamiltonian is a sum of operators that act only on a finite number of adjacent degrees of freedom (i.e., no long-range interactions).
    - “*Many-body*”: The Hamiltonian describes the interactions of extensively many degrees of freedom (spins, particles).
    - “*Thermodynamic limit*”: We are interested in the ground state properties for infinitely large systems, i.e., in the limit  $L \rightarrow \infty$ .
- In this course, we are mostly interested in a particular subclass of quantum phases:
  - Gapped quantum phase**  $\Leftrightarrow$  Ground state manifold of  
Hamiltonian with a *stable bulk gap*
    - “*Bulk gap*”: Spectral gap between the ground state manifold and the first excited states of a system with periodic boundaries. Systems *with* boundaries may have eigenstates that cross this gap.
    - “*Stable*”: The gap remains finite in the thermodynamic limit  $L \rightarrow \infty$ .
- Naturally, we are also interested in transitions *between* quantum phases:
  - Quantum phase transition**  $\Leftrightarrow$  Transition between different quantum phases  
(in the thermodynamic limit)
    - $\Leftrightarrow$  *Qualitative* change of *macroscopic* properties triggered by *small* changes of *microscopic* parameters

#### Comments:

- ¡! In this course we consider exclusively quantum phases; hence we drop the term “quantum” in the expressions defined above in many cases.
- Quantum phases are characterized by properties that *emerge* from many particles that interact quantum-mechanically. We are therefore interested how *macroscopic* quantum properties emerge from *microscopic* quantum interactions.
- Thus, the study of quantum phases and phase transitions is particularly challenging, because computing the ground state(s) of large, interacting quantum systems is hard if not impossible. (The Hilbert space dimension grows exponentially with the system size  $L$ !)

Broadly speaking, there are four attack vectors:

- (1) Solve models *analytically* ...
  - (a) ...with exact methods.  
(↑ *Bethe ansatz*, → *Stabilizer formalism*, → *Quadratic theories*, ...)
  - (b) ...with approximate methods.  
(↓ *Perturbation theory*, ↑ *Mean-field theory*, ↑ *Quantum field theory*, ...)
- (2) Solve models *numerically* (on classical computers).  
(↓ *Exact diagonalization*, → *DMRG*, ↑ *Quantum Monte Carlo*, ...)
- (3) Perform *quantum simulations*.  
(→ *Analog quantum simulation*, ↑ *Digital quantum simulation*, ...)
- (4) Last but not least: conduct *experiments*.

Here we focus on approach (1a); in some exercises you will make contact with approach (2).

- Quantum phase transitions are triggered by changes of parameters in the Hamiltonian (*e.g.* interaction strengths, chemical potentials, hopping rates, ...). (Quantum) phase *diagrams* are therefore plotted as functions of *parameters* of the Hamiltonian, and not temperature or pressure [as you learned in your course on ↓ (*classical*) *statistical physics*].
- Quantum phases at  $T = 0$  are properties of *pure quantum states* without entropy. By contrast, *classical phases* (like crystalline phases of solids, or the liquid phase of water), are properties of statistical *ensembles* of states with finite entropy; in the framework of quantum mechanics, these are described by *density matrices* [for example, the Gibbs state  $\rho = e^{-\beta H} / Z$  of the ↓ *canonical ensemble*].
- Classical *thermodynamic* phase transitions (*e.g.*, the boiling of water) are driven by *thermal fluctuations* that modify the statistical ensemble of microstates, such that its macroscopic observables change qualitatively. By contrast, *quantum* phase transitions are driven by *quantum fluctuations* (due to non-commuting terms in the Hamiltonian, → *below*). These modify the *amplitudes* of basis states in the (pure!) ground state of the system, thereby changing its quantum-mechanical properties qualitatively (correlations, entanglement structure, ...).
- ¡! Quantum fluctuations are *not* dynamical fluctuations in time. The ground state is an *eigenstate* and therefore *time-independent*. However, if you would initialize the system in a classical product state which is not an eigenstate (in particular, not the ground state), then it would fluctuate in time, because the ground state is actually a *superposition* of many different such classical product states.

## 2 | Examples of quantum phases that exist in nature and/or can be experimentally realized:

- ↓ *Superconductors*
- ↓ *Superfluids* (*e.g.* superfluid Helium ...)
- ↑ *Supersolids* (have been recently realized in experiments [22–24])
- ↓ *Bose-Einstein condensates* (BEC)
- ↓ *Fermi liquids*
- → *Quantum Hall states*
- ...

While these are important examples, they are typically hard to describe and understand theoretically. It is therefore advisable to focus on a simple “toy model” that is exactly solvable:

## 3 | Paradigmatic example: