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Problem 9.1: Properties of the linear operator

[**Written** | 2 pt(s)]

ID: ex_properties_scattering_operator:sm2324

> **Learning objective**
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> In the lecture you saw that while deriving an approximate solution to the Maxwell-Boltzmann transport equation we linearize both the flow and scattering terms and introduce the operator L. We then proceed to solving the corresponding eigenvalue problem, here, you will show that L is indeed Hermitian and negative.

Consider the following definition of a scalar product,

$$(g_1, g_2) = \int d^3p\, f_{lo}(\boldsymbol{p})\, g_1(\boldsymbol{p})\, g_2(\boldsymbol{p}). \tag{1}$$

where f_{lo} is the local Maxwell-Boltzmann distribution function.

a) Show that Eq. (1) satisfies the properties of a scalar product. **1**pt(s)

The linear operator is then defined as

$$L\,\Psi = -\int d^3p'\, d^3p_1\, d^3p_1'\, w_{p,p_1;p',p_1'}\, f_{lo}(\boldsymbol{p_1}) \times \left[\Psi(\boldsymbol{p}) + \Psi(\boldsymbol{p_1}) - \Psi(\boldsymbol{p'}) - \Psi(\boldsymbol{p_1'})\right]. \tag{2}$$

b) Show that L is Hermitian and negative. **1**pt(s)

Problem 9.2: Heat transport in an ideal gas

[**Oral** | 3 pt(s)]

ID: ex_heat_transport:sm2324

> **Learning objective**
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> In this problem we consider heat transport in a temperature gradient field and derive the thermal conductivity constant κ. This problem is a typical example for solving the Maxwell-Boltzmann transport equation while being not far from equilibrium.

Given the local Maxwell-Boltzmann distribution function,

$$f_{lo} = \frac{n(\mathbf{r})}{[2\pi m k_B T(\mathbf{r})]^{3/2}} \exp\left[-\frac{\mathbf{p}^2}{2mk_BT(\mathbf{r})}\right], \tag{3}$$

where $n(\mathbf{r})$ and $T(\mathbf{r})$ are local equilibrium parameters. f_{lo} is no longer a solution of the Maxwell-Boltzmann transport equation.

To find a solution, we propose the following ansatz:

$$f = f_{lo}(1 + \Psi), \tag{4}$$

where Ψ is small and $f_{lo} = f(t = 0)$.

a) First, linearize the scattering term. Show that $\partial_t f\big|_s = f_{lo}(\mathbf{p})L\Psi$. **1**pt(s)

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b) Now consider the flow term, show that for a stationary solution and in the absence of external drive, one can write **1**pt(s)

$$D\, f_{lo} = f_{lo}\, \frac{\varepsilon_p - h}{K_B T^2}\, \mathbf{v} \cdot \boldsymbol{\nabla} T = f_{lo} L\Psi, \tag{5}$$

where h is the molar enthalpie. Define $X = -\frac{\varepsilon_p - h}{K_B T^2}\, \mathbf{v} \cdot \boldsymbol{\nabla} T$.

Hints:

- We can write f_{lo} in terms of the local chemical potential $\mu(\mathbf{r})$,

$$f_{lo} = \exp\left[-\frac{\varepsilon_p - \mu(\mathbf{r})}{k_B T(\mathbf{r})}\right]. \tag{6}$$

- Assume constant pressure p such that $\mu = \mu(T)$.
- Use enthalpie $H = G + TS$, with $\mu = G/N$ and $\partial G/\partial T = -S$.
- Apply the relaxation time approximation to determine $L\Psi$.

c) The temperature gradient $\boldsymbol{\nabla} T$ induces a heat flux density $\boldsymbol{w} = -\kappa \boldsymbol{\nabla} T$, where κ is the coefficient of thermal conductivity. Assume that $\boldsymbol{\nabla} T = (\partial_x T, 0, 0)$ and that $\boldsymbol{v}$ is isotropic to derive κ for an ideal gas. **1**pt(s)

$$\kappa = k_B \frac{T^2}{(\nabla T)^2}(X, \Psi) = n k_B T \tau \frac{C_p}{m}. \tag{7}$$

Hint: Start with the equation for the heat flow,

$$\boldsymbol{w} = \int d^3p f \boldsymbol{v} \varepsilon_p = \int d^3p f_{l\,0} \Psi \boldsymbol{v} \varepsilon_p. \tag{8}$$

Problem 9.3: Barometric hight formula

[**Oral** | 2 pt(s)]

ID: ex_barometric_hight_formula:sm2324

Learning objective

In this exercise, you will derive the formula which describes the variation of pressure with altitude.

To model the atmosphere we assume a homogeneous ideal gas in equilibrium.

a) Use a linear temperature dependence $T(z) = T(0) - Lz$ to show that the pressure $p(z)$ satisfies the following relation, **1**pt(s)

$$p(z) = p(0) \left(1 - \frac{Lz}{T(0)}\right)^{\frac{mg}{k_B L}} . \tag{9}$$

b) Show that the linear dependence of temperature on altitude can be motivated by the adiabatic behavior of air. What would be the lapse rate L? **1**pt(s)