Problem 9.1: Properties of the linear operator

ID: ex_properties_scattering_operator:sm2324

**Learning objective**

In the lecture you saw that while deriving an approximate solution to the Maxwell-Boltzmann transport equation we linearize both the flow and scattering terms and introduce the operator $L$. We then proceed to solving the corresponding eigenvalue problem, here, you will show that $L$ is indeed Hermitian and negative.

Consider the following definition of a scalar product,

$$\langle g_1, g_2 \rangle = \int d^3p \, f_{lo}(p) \, g_1(p) \, g_2(p). \quad (1)$$

where $f_{lo}$ is the local Maxwell-Boltzmann distribution function.

a) Show that Eq. (1) satisfies the properties of a scalar product.

The linear operator is then defined as

$$L \Psi = - \int d^3p' \, d^3p_1 \, d^3p'_1 \, w_{p,p';p_1,p'_1} \, f_{lo}(p_1) \times [\Psi(p) + \Psi(p_1) - \Psi(p') - \Psi(p'_1)]. \quad (2)$$

b) Show that $L$ is Hermitian and negative.

Problem 9.2: Heat transport in an ideal gas

ID: ex_heat_transport:sm2324

**Learning objective**

In this problem we consider heat transport in a temperature gradient field and derive the thermal conductivity constant $\kappa$. This problem is a typical example for solving the Maxwell-Boltzmann transport equation while being not far from equilibrium.

Given the local Maxwell-Boltzmann distribution function,

$$f_{lo} = \frac{n(r)}{[2\pi mk_B T(r)]^{3/2}} \exp \left[ -\frac{p^2}{2mk_B T(r)} \right], \quad (3)$$
where \( n(r) \) and \( T(r) \) are local equilibrium parameters. \( f_{lo} \) is no longer a solution of the Maxwell-Boltzmann transport equation.

To find a solution, we propose the following ansatz:

\[
f = f_{lo}(1 + \Psi),
\]

where \( \Psi \) is small and \( f_{lo} = f(t = 0) \).

a) First, linearize the scattering term. Show that \( \partial_t f \big|_s = f_{lo}(p) L \Psi \).

b) Now consider the flow term, show that for a stationary solution and in the absence of external drive, one can write

\[
D f_{lo} = f_{lo} \frac{\varepsilon_p - h}{k_B T^2} \mathbf{v} \cdot \nabla T = f_{lo} L \Psi,
\]

where \( h \) is the molar enthalpie. Define \( X = -\frac{\varepsilon_p - h}{k_B T^2} \mathbf{v} \cdot \nabla T \).

Hints:

- We can write \( f_{lo} \) in terms of the local chemical potential \( \mu(r) \),

\[
f_{lo} = \exp \left[ -\frac{\varepsilon_p - \mu(r)}{k_B T(r)} \right].
\]

- Assume constant pressure \( p \) such that \( \mu = \mu(T) \).
- Use enthalpie \( H = G + TS \), with \( \mu = G/N \) and \( \partial G/\partial T = -S \).
- Apply the relaxation time approximation to determine \( L \Psi \).

c) The temperature gradient \( \nabla T \) induces a heat flux density \( w = -\kappa \nabla T \), where \( \kappa \) is the coefficient of thermal conductivity. Assume that \( \nabla T = (\partial_x T, 0, 0) \) and that \( \mathbf{v} \) is isotropic to derive \( \kappa \) for an ideal gas.

\[
\kappa = k_B T \frac{T^2}{(\nabla T)^2} \left( X, \Psi \right) = nk_B T \tau \frac{C_p}{m}.
\]

Hint: Start with the equation for the heat flow,

\[
w = \int d^3p f v \varepsilon_p = \int d^3p f_{lo} \Psi v \varepsilon_p.
\]
Problem 9.3: Barometric height formula

Learning objective

In this exercise, you will derive the formula which describes the variation of pressure with altitude.

To model the atmosphere we assume a homogeneous ideal gas in equilibrium.

a) Use a linear temperature dependence \( T(z) = T(0) - Lz \) to show that the pressure \( p(z) \) satisfies the following relation,

\[
p(z) = p(0) \left( 1 - \frac{Lz}{T(0)} \right)^{\frac{m_g}{k_B T}}. \tag{9}
\]

b) Show that the linear dependence of temperature on altitude can be motivated by the adiabatic behavior of air. What would be the lapse rate \( L \)?