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Problem 5.1: Opening a freezer

[**Oral** | 5 pt(s)]

ID: ex_how_to_open_a_freezer:sm2324

Learning objective

You may have noticed that after opening a freezer door for a while and then closing it, it is much harder to open them afterwards. We will study what causes this effect, considering thermodynamic processes that take place at every stage.

Consider a freezer of volume $V_F = 0.1m^3$ and depth $D = 40cm$.The room temperature is $T_H = 20^oC$, the temperature of the freezer working liquid is $T_C = -20^oC$. The air pressure at ground level $P_{air} = 1.01 \times 10^5 \left[\frac{kg}{ms^2}\right]$. Assume that the room is much bigger than the freezer and full of an ideal diatomic gas. The ideal gas undergoing adiabatic process satisfies $pV^\gamma =$ *constant*, where for diatomic molecules the adiabatic constant $\gamma = 1.4$.

a) Describe what happens while the freezer door is open. What happens with the air inside the freezer after the door is closed? **1**pt(s)

b) Calculate the force that keeps the door closed. Please give both an analytic expression and a numeric result. **1**pt(s)

Assume that the freezer door is a piston (see Figure 1), which travels the distance $d = 2cm$ before the outside and inside air get into contact and the door opens.

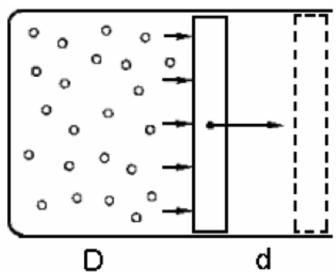

Figure 1: Freezer door as a piston.

The work W_p done by a person opening the door depends on the way of doing it.

c) First, consider opening the door very slowly. Show that in this case, the work W_p is equal to the change of the free energy $\Delta F(T, V)$ of the gas inside the freezer. Calculate W_p. **1**pt(s)

d) Second, consider opening the door very quickly. Change of which thermodynamical potential is equal to the work W_p in this case? Calculate W_p. **1**pt(s)

e) The work W_p depends on the speed of opening the door. Why? Where does the work come from? No equations required. **1**pt(s)

Problem 5.2: Legendre Transformation

[**Oral** | 7 pt(s)]

ID: ex_legendre_transformation_statmech:sm2324

> **Learning objective**
>
> In the following problem we will learn more about Legendre transforms.

The Legendre transform of the function $f(x)$ is defined by

$$f^*(p) := \mathcal{L}f(p) = \sup_x [xp - f(x)] \tag{1}$$

Here, we will show that the Legendre transformation of the strictly convex functions is involutive, i.e. $f^{**} := (f^*)^* = f$.

The function f is convex if it satisfies the following inequality,

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) \qquad \forall \lambda \in [0,1], \tag{2}$$

and strictly convex if inequality is saturated only for $\lambda = 0, 1$ or $x_1 = x_2$.

a) How can we understand the Legendre transformation geometrically? How to reconstruct (geometrically) the original function $f(x)$ from the Legendre transform $f^*(p)$? 1pt(s)

b) Show, that $f^*(p)$ is convex. For that purpose consider $f^*(\lambda p_1 + (1-\lambda)p_2)$. 1pt(s)

c) Assuming that $f(x)$ is continuously differentiable and strictly convex, show that 1pt(s)

$$f^*(p) = \tilde{x}p - f(\tilde{x}), \tag{3}$$

where $\tilde{x}$ is defined by the relation $p = f'(\tilde{x})$. The Legendre transformation changes the independent variable from x to p.

d) Assuming that $f(x)$ is continuously differentiable and strictly convex, show that $f^*(p)$ is strictly convex. 1pt(s)

e) If $f(x)$ is continuously differentiable and strictly convex, show that 1pt(s)

$$f^{**}(x) = f(x). \tag{4}$$

In order to do this apply the Legendre transformation to $f^*(p)$.

f) Calculate the Legendre transform $f^*(p)$ of the function 1pt(s)

$$f(x) = \begin{cases} x^2/2 & x \leq 1 \\ x - 1/2 & 1 \leq x \leq 2 \\ x^2 - 3x + 7/2 & 2 \leq x \end{cases}$$

g) Calculate the Legendre transform $f^*(p)$ of the function 1pt(s)

$$f(x) = \begin{cases} x^2/2 & x \leq 1 \\ -x^2 + 4x - 5/2 & 1 \leq x \leq 2 \\ x^2 - 3x + 7/2 & 2 \leq x \end{cases}$$

which is not convex. How does the backwards transformed function $f^{**}(x)$ look like?

Problem 5.3: Joule-Thomson Effect

[**Written** | 3 pt(s)]

ID: ex_joule_thomson_effect_statmech:sm2324

Learning objective

In this exercise, we will learn about Joule-Thomson effect, where the temperature of a gas or liquid changes as it passes through a valve or porous medium while preventing heat exchange with the surroundings.

A gas (not necessarily ideal) is forced to slowly flow adiabatically (no heat exchange with the environment) and irreversibly through a porous plug from volume 1 to 2 (see Figure 2), in such a way that during the process p_1, p_2 are constant.

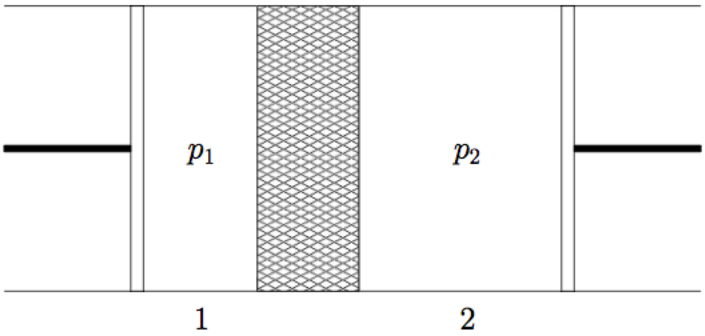

Figure 2: Cylinder with porous plug between two pistons.

a) Show, that during the process the enthalpy $H = U + pV$ is conserved. **1** pt(s)

b) The temperature of the gas changes when it is forced through the porous plug. Depending on the sign of $(\partial T/\partial p)_H$ it warms up or a cools down (Joule-Thomson effect). The curve, which separates two regimes in the $p - T$ diagram, is called the inversion curve. Show that it satisfies **1** pt(s)

$$T \left(\frac{\partial V}{\partial T} \right)_p - V = 0. \tag{5}$$

Under which circumstances does the cooling take place?

Hint: Calculate $(\partial T/\partial p)_H$ with the help of (Problem 1.1: Partial Derivative). Use the Maxwell relation for the Gibbs free energy.

c) Show that the temperature of the ideal gas cannot be changed using the Joule-Thomson effect. **1** pt(s)
Note: The Joule-Thomson expansion is one of the steps in the cycle of conventional refrigerators.