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January 31st, 2024
WS 2023/24

Problem 14.1: Quantum Ising model with transverse field: Part 1

[**Oral** | 4 pt(s)]

ID: ex_quantum_Ising_model_with_transverse_field_part_1:sm2324

Learning objective

In the classical Ising model, a spin has two states (up/down), no quantum superposition. On the other hand, in the quantum Ising model, spin can exist with quantum superpositions of up and down states. In this exercise, you will solve the quantum Ising model in the presence of magnetic field using mean-field theory.

We consider a system of Ising-coupled (quantum) spins on a lattice where each spin has z nearest neighbors (z is known as the coordination number of the lattice). A magnetic field of strength Ω is applied perpendicular to the preferential direction of the spins. The Hamiltonian is given by

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z + \Omega \sum_i \hat{\sigma}_i^x \tag{1}$$

where $\hat{\sigma}_i^\alpha$ are the spin-$1/2$ Pauli matrices and $\langle i, j \rangle$ describes the sum over nearest neighbors. We consider a ferromagnetic coupling $J > 0$.

In this exercise we are exploring the physics of this model within a mean-field analysis. We define the mean-field $m \equiv \langle \hat{\sigma}^z \rangle$ as the average magnetization in z-direction.

a) We can always write the spin operators as $\hat{\sigma}_i^z = m + \hat{\delta}_i^z$ where $\hat{\delta}^z$ contains the residual operator character and describes the deviation from the mean-field. **1**pt(s)

Transform the Hamiltonian into a sum of uncoupled spins by assuming that the deviations from the mean-field are small, such that we can neglect terms of second order in $\hat{\delta}^z$. Substitute all occurrences of $\hat{\delta}^z$ with $\hat{\sigma}^z - m$ after making the approximation.

b) Diagonalize the resulting single-spin Hamiltonian. Let $|\nearrow\rangle$ and $|\swarrow\rangle$ be the eigenstates. **1**pt(s)

c) Compute the probability $p_\nearrow(T)$ to be in the state $|\nearrow\rangle$ at temperature T. Then, we can express the magnetization as **1**pt(s)

$$\begin{aligned} m = \langle \hat{\sigma}^z \rangle &= p_\nearrow(T)\ \langle \nearrow | \, \hat{\sigma}^z | \nearrow \rangle + p_\swarrow(T)\ \langle \swarrow | \, \hat{\sigma}^z | \swarrow \rangle \\ &= (2p_\nearrow(T) - 1) \, \langle \nearrow | \, \hat{\sigma}^z | \nearrow \rangle \, . \end{aligned} \tag{2}$$

Compute the right hand side of this equation (as a function of m). The resulting equation is called a *self-consistency equation.*

d) Derive the phase diagram as a function of $\omega = \Omega/J$ and $t = k_B T/J$. To this end, derive an analytic expression for the critical temperature t_c as a function of ω. **1**pt(s)

Hint: While the self-consistency equation can not be solved analytically, you can get the idea of how to derive the phase boundary by inspecting the solutions graphically.

Problem 14.2: Quantum Ising model with transverse field: Part 2 [**Oral** | 3 pt(s)]

ID: ex_quantum_Ising_model_with_transverse_field_part_2:sm2324

Learning objective

In this exercise you will solve the quantum Ising model using an variational ansatz.

We consider the model from the first part at zero temperature. Intuitively, for vanishing field $\Omega/J \longrightarrow 0$, the system favors a configuration where all spins point in either positive or negative z-direction. On the other hand, for $\Omega/J \longrightarrow \infty$, the external field aligns all spins in x-direction.

Ignoring correlations between the spins, we can use these observations to devise a variational wave function for the system

$$|\Psi_\alpha\rangle = \prod_{i=1}^{N} |\swarrow_\alpha\rangle_i = \prod_{i=1}^{N} R_y(\alpha)\,|\downarrow\rangle_i\,. \tag{3}$$

Here, $R_y(\alpha)$ describes a rotation around the y-axis in spin-space. An explicit representation is given by

$$R_y(\alpha) = \mathrm{e}^{-i\frac{\alpha}{2}\sigma_y} = \mathbb{1}\cos\left(\frac{\alpha}{2}\right) - i\sigma_y \sin\left(\frac{\alpha}{2}\right). \tag{4}$$

a) Calculate the energy per spin of the variational state $E(\alpha) = \langle\Psi_\alpha|\,\mathcal{H}\,|\Psi_\alpha\rangle\,/N$. 1pt(s)

b) Show that the variational ansatz yields the true ground state in the limits described above. 1pt(s)

c) Visualize the change in the energy landscape $E(\alpha)$ as Ω/J crosses the critical value for the phase transition. 1pt(s)

Problem 14.3: Black body radiation [**Oral** | 5 pt(s)]

ID: ex_black_body_radiation:sm2324

Learning objective

The goal of this exercise is to derive Planck's law of black body radiation.

We consider a gas of photons at thermal equilibrium. For simplicity, we consider a box of volume $V = L^3$ and periodic boundary conditions. Due to the boundary conditions, there are discrete energy levels (modes). Each mode is labeled by a set of quantum numbers $\boldsymbol{k} = \frac{2\pi}{L}\boldsymbol{z}$ with $\boldsymbol{z} = (z_x, z_y, z_z) \in \mathbb{Z}^3$ and a polarization $\sigma \in \{\pm 1\}$. The energy of the mode $(\boldsymbol{k}, \sigma)$ is given by $E_{\boldsymbol{k},\sigma} = \hbar\omega_{\boldsymbol{k}} = \hbar c|\boldsymbol{k}| = \hbar ck$.

a) Each mode can be occupied by $n_{\boldsymbol{k},\sigma} = 0, 1, 2, \ldots$ photons. Determine the partition function $Z_{\boldsymbol{k},\sigma}(\beta, V)$ of a single mode and the total partition function $Z(\beta, V)$. 1pt(s)

b) In the following, we assume that the modes are 'dense' such that we can go over to a continuum description. In $\boldsymbol{k}$-space there are exactly two modes (polarization) per 'volume' $(2\pi)^3/V$. Show that there are $N(k)\mathrm{d}k \equiv \frac{V}{\pi^2}k^2\mathrm{d}k$ states in a spherical shell of radius k and thickness $\mathrm{d}k$. The quantity $N(k)$ is called the density of states in k-space. Use the relation $D(k)\mathrm{d}k = D(\omega)\mathrm{d}\omega$ to transform to the (frequency) density of states $D(\omega)$, i.e. the number of states in the frequency interval $[\omega, \omega + \mathrm{d}\omega]$. **1**pt(s)

c) Show that we can use the density of states to write a sum $\sum_{\boldsymbol{k},\sigma} f(k)$ in the continuum limit as $\int \mathrm{d}\omega\, D(\omega) f(\omega)$. **1**pt(s)

d) Write the energy density $u \equiv U/V = -(\partial_\beta \ln Z)/V$ in the form $u = \int \mathrm{d}\omega\, u(\omega)$. The expression for the spectral energy density $u(\omega)$ is known as Planck's law. **1**pt(s)

e) Find the Stefan-Boltzmann law $u = \sigma T^4$ by integrating the spectral energy density over the frequency. Determine the value of σ. **1**pt(s)