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Problem 13.1: Fermi gas at finite temperatures

[**Written** | 3 pt(s)]

ID: ex_fermi_gas_finite_temperatures:sm2324

Learning objective

In this exercise you will derive and use the Sommerfeld expansion, which is used to approximate Fermi gases near $T = 0$.

We consider a quantum gas in 3 dimensions with spin $1/2$ particles.

a) Derive the *Sommerfeld expansion* **1**pt(s)

$$I\{H\} \equiv \int_0^\infty H(E)\, f_{\mathrm{FD}}(E)\, \mathrm{d}E = \int_0^\mu \mathrm{d}E\, H(E) + \frac{\pi^2}{6}(k_{\mathrm{B}}T)^2 H'(\mu) + \mathcal{O}(T^4). \tag{1}$$

Here, $H(E)$ is an arbitrary function that can be Taylor-expanded around $E = \mu$ with the single-particle energy E. $f_{\mathrm{FD}}(E) = (e^{\beta(E-\mu)} + 1)^{-1}$ is the Fermi-Dirac distribution.

Hints:

- Separate the $T = 0$ behaviour from $f_{\mathrm{FD}}(E)$.
- $\int_0^\infty \mathrm{d}y\, y/(\exp y + 1) = \pi^2/12$. Be careful when shifting integral boundaries.

b) We know that at $T = 0$, the Fermi energy E_{F} is equal to the chemical potential $\mu(T = 0)$, whereas the average occupation f_{FD} jumps from 1 to 0 at $E = E_{\mathrm{F}} = \mu(T = 0)$. Given a small temperature $k_{\mathrm{B}}T \ll E_{\mathrm{F}}$, use the Sommerfeld expansion to calculate the temperature corrections of the chemical potential $\mu(T)$ from the condition (why?) **1**pt(s)

$$\langle N\rangle(T, \mu) \equiv \int_0^\infty \mathrm{d}E\, \rho(E)\, f_{\mathrm{FD}}(E) \stackrel{!}{=} \int_0^{E_{\mathrm{F}}} \mathrm{d}E\, \rho(E). \tag{2}$$

Here, $\rho(E)$ is the density of states in 3 dimensions. Justify, that you can use the relations $\int_0^\mu \mathrm{d}E\, \rho(E) \approx \int_0^{E_{\mathrm{F}}} \mathrm{d}E\, \rho(E) + (\mu - E_{\mathrm{F}})\rho(E_{\mathrm{F}})$ and $\rho'(\mu) \approx \rho'(E_{\mathrm{F}})$.

Solution:

$$\mu = E_{\mathrm{F}}\left[1 - \frac{\pi^2}{12}\left(\frac{k_{\mathrm{B}}T}{E_{\mathrm{F}}}\right)^2 + \mathcal{O}(T^4)\right] \tag{3}$$

c) Now, calculate the temperature corrections of the internal energy and from that the heat capacity. **1**pt(s)

Solution:

$$U(T) = \frac{3}{5}N E_{\mathrm{F}}\left[1 + \frac{5\pi^2}{12}\left(\frac{k_{\mathrm{B}}T}{E_{\mathrm{F}}}\right)^2 + \mathcal{O}(T^4)\right] \tag{4}$$

Problem 13.2: Absence of Bose-Einstein condensation in 2D [**Oral** | 1 pt(s)]

ID: ex_absence_bose_einstein_condensation_2d:sm2324

Learning objective

In this exercise, we will show, that the condensation of Bose-gases occurs only in 3 dimensions.

Determine the grand-canonical partition function $\mathcal{Z}(z, V, T)$ for the ideal Bose gas. By $z \equiv \mathrm{e}^{\beta\mu}$ we denote the *fugacity*. Use the partition function to calculate the mean density $n = n(z, T) = \langle N \rangle / V$ of the gas for $d = 2, 3$ dimensions.

Show that the ideal Bose gas does not condense in two dimensions at any $T > 0$.

Notes:

- A powerful generalization of this result is known as Mermin-Wagner theorem. It states that a continuous symmetry cannot be spontaneously broken at finite temperature in $d \leq 2$ dimensions. A Bose-Einstein condensate has a broken $U(1)$ symmetry due to the overall phase of the wave function — and is therefore forbidden in $d \leq 2$ dimensions.
- The above result is only valid for a uniform system. A two dimensional Bose gas which is harmonically trapped condenses at a finite critical temperature.

Problem 13.3: Bose-Einstein condensate inside a harmonic trap [**Oral** | 3 pt(s)]

ID: ex_bose_einstein_condensate_harmonic_trap:sm2324

Learning objective

In experiments on Bose-Einstein condensation, atoms are brought into optical traps, which can be modeled as 3-d harmonic oscillators. In this exercise, we will calculate the influence of this trap on the critical temperature T_c of the condensate.

Consider a Bose-Einstein condensate in an isotropic 3-d harmonic trap with frequency ω.

a) Determine the density of states $\rho(E)$ inside the trap. **1** pt(s)

b) Assume that $k_\mathrm{B}T \gg \hbar\omega$. Show, that when replacing $\sum_i \to \int \mathrm{d}E\, \rho(E)$ the average number of occupied states is limited by $N_\text{max} = c_1 (k_\mathrm{B}T/\hbar\omega)^3$. Determine c_1. **1** pt(s)

c) For a given N you will get a temperature T_c, below which the ground state occupation **1** pt(s)

$$\langle N_0 \rangle = N\, h(T/T_\mathrm{c}) \tag{5}$$

is macroscopic. Determine $h(T/T_c)$ by separating the ground state.