Topological Quantum Many-Body Physics

Lecture Notes • Summer Term 2025

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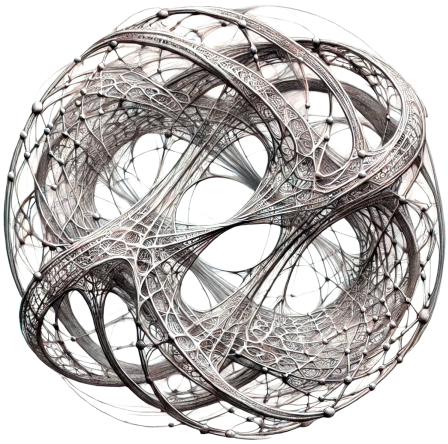
Check for Updates: itp3.info/tqp



How DALL·E imagines "the braiding of anyons" in 2025.

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Preliminaries

Important

This script is in development and continuously updated. To download the latest version:

itp3.info/tqp

If you spot mistakes or have suggestions, send me an email:

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Requirements for this course

I assume that students are familiar with the following concepts:

- Non-relativistic quantum mechanics and second quantization
  Fermions, bosons, spins, …
- Basics of condensed matter theory
  Band theory, quasi particles, Fermi sea, …
- Basics of quantum information theory
  Qubits, quantum gates, …
- Basics of group theory
  (Non-)abelian groups, linear representations, …

Literature recommendations

This course follows no particular textbook but draws its inspiration from various sources.

Topological phases of non-interacting fermions (Part I):

- Bernevig & Hughes: *Topological Insulators and Topological Superconductors* [1]
  ISBN 978-0691151755
  Accessible introduction to topological phases of non-interacting fermions.
- Asbóth *et al.*: *A Short Course on Topological Insulators: Band Structure and Edge States in One and Two Dimensions* [2]
  ISBN 978-3319256054
  Accessible brief introduction to topological phases of non-interacting fermions.
- Franz & Molenkamp *et al.*: *Topological insulators* [3]
  ISBN 978-0444633187
  Accessible and comprehensive introduction to topological insulators in two and more dimensions.

- Shen: *Topological Insulators: Dirac Equation in Condensed Matters* [4]
  ISBN 978-3642328589
  Accessible introduction to topological insulators and superconductors.
- Moessner & Moore: *Topological Phases of Matter* [5]
  ISBN 978-1107105539
  Comprehensive introduction to topological phases of matter (including topological order).

Symmetry-protected topological phases of interacting bosons (Part II):

- Chen *et al.*: *Classification of gapped symmetric phases in one-dimensional spin systems* [6]
  Original research on the classification of interacting spin systems in one dimension.
  (Quite accessible, in particular the first sections of the paper.)
- Verresen *et al.*: *One-dimensional symmetry protected topological phases and their transitions* [7]
  Original research on the classification of interacting systems of spins and fermions in one dimension.
  (Quite accessible, in particular the introduction to the paper.)

Intrinsic topological order and long-range entanglement (Part III):

- Simon: *Topological Quantum* [8]
  ISBN 978-0198886723
  Thorough and modern introduction to many aspects of topological order (anyons, TQFTs, …).
  I highly recommend this book!
- Pachos: *Topological Quantum Computation* [9]
  ISBN 978-1107005044
  Accessible introduction to anyon models and topological quantum computation.
- Moessner & Moore: *Topological Phases of Matter* [5]
  ISBN 978-1107105539
  Contains chapters on topological order and topological quantum computing (among others).
- Wen: *Quantum Field Theory of Many-Body Systems* [10]
  ISBN 978-0199227259
  Focus on field theory methods to describe quantum many-body systems.
- Wang: *Topological Quantum Computation* [11]
  ISBN 978-0821849309
  Very mathematical treatment of anyon models and topological quantum computation.

Mathematical Background

- Nakahara: *Geometry, Topology and Physics* [12]
  ISBN 978-0750306065
  Extensive, mathematically rigorous treatment of topology for physicists.

Milestones & Nobel Prizes

There are three nobel prizes directly related to the subject of this course:

- **Nobel Prize in Physics 1985:**

  > *The Nobel Prize in Physics 1985 was awarded to* Klaus von Klitzing *"for the discovery of the quantized Hall effect."*

  Related milestone paper:

[13] K. v. Klitzing, G. Dorda, M. Pepper
New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance
Physical Review Letters, Vol. 45, p. 494-497 (1980)
Von Klitzing discovers the quantized Hall effect.

- **Nobel Prize in Physics 1998:**

  The Nobel Prize in Physics 1998 was awarded jointly to Robert B. Laughlin, Horst L. Störmer *and* Daniel C. Tsui *"for their discovery of a new form of quantum fluid with fractionally charged excitations."*

  Related milestone papers:

[14] D. C. Tsui, H. L. Störmer, A. C. Gossard
Two-Dimensional Magnetotransport in the Extreme Quantum Limit
Physical Review Letters, Vol. 48, p. 1559-1562 (1982)
Tsui and Störmer discover the fractional quantum Hall effect.

[15] R. B. Laughlin
Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations
Physical Review Letters, Vol. 50, p. 1395-1398 (1983)
Laughlin describes the fractional quantum Hall effect in terms of fractional charges.

- **Nobel Prize in Physics 2016:**

  The Nobel Prize in Physics 2016 was awarded with one half to David J. Thouless, *and the other half to* F. Duncan M. Haldane *and* J. Michael Kosterlitz *"for theoretical discoveries of topological phase transitions and topological phases of matter."*

  Related milestone papers:

[16] J. M. Kosterlitz, D. J. Thouless
Ordering, metastability and phase transitions in two-dimensional systems
Journal of Physics C: Solid State Physics, Vol. 6, No. 7 (1973)
Kosterlitz and Thouless use methods from topology to describe the KT phase transition.

[17] D. J. Thouless *et al.*
Quantized Hall Conductance in a Two-Dimensional Periodic Potential
Physical Review Letters, Vol. 49, p. 405-408 (1982)
Thouless and coworkers explain the quantization of the Hall conductivity.

[18] F. D. M. Haldane
Nonlinear Field Theory of Large-Spin Heisenberg Antiferromagnets: Semiclassically Quantized Solitons of the One-Dimensional Easy-Axis Néel State
Physical Review Letters, Vol. 50, p. 1153-1156 (1983)
Haldane uses methods from topology to describe the 1D Heisenberg antiferromagnet.

[19] F. D. M. Haldane
Model for a Quantum Hall Effect without Landau Levels: Condensed-Matter Realization of the "Parity Anomaly"
Physical Review Letters, Vol. 61, p. 2015-2018 (1988)
Haldane predicts the anomalous quantum Hall effect without external magnetic fields.

Goals of this course

The goal of this course is to gain a thorough understanding of topological concepts in modern quantum many-body physics. You aquire the mathematical tools needed to describe topological quantum phases, understand the physical features that characterize these systems, and learn about potential applications.

In particular (⋆ optional):

(Gray topics are not yet covered by the script.)

Topological phases of non-interacting fermions (Part I):

- Integer quantum Hall effect
- Berry connection, Berry holonomy, Chern number
- Anomalous quantum Hall effect (Haldane model)
- Quantum spin Hall effect, topological insulators (Kane-Mele model)
- Pfaffian topological invariant
- Winding numbers, sublattice symmetry, edge modes (SSH model)
- Topological superconductivity (Majorana chain)
- Tenfold way and periodic table of topological insulators/superconductors
- Effects of interactions
- Topological bands in classical systems (topological metamaterials …) ⋆

Symmetry-protected topological phases of interacting bosons (Part II):

- Tensor network states, matrix product states, PEPS
- Projective representations and (twisted) cohomology groups
- Classification of bosonic topological phases in one dimension
- Haldane chain and AKLT model

Intrinsic topological order and long-range entanglement (Part III):

- Statistics of indistinguishable particles in 2+1 dimensions (Braid group)
- Toric code (anyonic excitations, topological entanglement entropy, …)
- Topological quantum memories
- Fibonacci anyons
- String-net condensates ⋆
- Topological quantum computation (non-abelian anyons, braiding, fusion, …)
- Mathematical framework
  (Modular tensor categories, pentagon & hexagon relations, quantum dimension, topological spin, …)
- Application to foundational questions of high-energy physics (fermions, …) ⋆

Notes on this document

- This document is not an extension of the material covered in the lectures but the script that I use to prepare them.
- Please have a look at the given literature for more comprehensive coverage. References to primary and secondary resources are also given in the text.
- The content of this script is color-coded as follows:
  - Text in black is written to the blackboard.
  - Notes in red should be mentioned in the lecture to prevent misconceptions.
  - Notes in blue can be mentioned/noted in the lecture if there is enough time.
  - Notes in green are hints for the lecturer.
- One page of the script corresponds roughly to one covered panel of the blackboard.
- Enumerated lists are used for more or less rigorous chains of thought:
  1. This leads to …
  2. this. By the way:
     i. This leads to …
     ii. this leads to …
     iii. this.
  3. Let's proceed …
- In the bibliography (p. 71 ff.) you can find links (Download) to download most papers referenced in this script. As most of these papers are not freely available, you need a password to do so; this password is made available to students of my classes. Papers that are open access are highlighted green (Download) and do not require a password.
- This document has been composed in **`Vim`** on **`Arch Linux`** and is typeset by LuaLaTeX and BibTeX. Thanks to all contributors to free software!
- This document is typeset in **Equity**, **Concourse** and MathTimeProfessional.

Acknowledgements

- Your name goes here if you let me know about typos & mistakes!

Symbols & Scientific abbreviations

The following abbreviations and glyphs are used in this document:

cf	confer (“compare”)
dof	degree(s) of freedom
eg	exempli gratia (“for example”)
etc	et cetera (“and so forth”)
et al	et alii (“and others”)
ie	id est (“that is”)
viz	videlicet (“namely”)
vs	versus (“against”)
wlog	without loss of generality
wrt	with respect to
iff	if and only if
$\lhd$	“consider”
$\rightarrow$	“therefore”
¡!	“Beware!”
$\overset{\circ}{=}$	non-obvious equality that may require lengthy, but straightforward calculations
$\overset{*}{=}$	non-trivial equality that cannot be derived without additional input
$\overset{\circ}{\rightarrow}$	“it is easy to show”
$\overset{*}{\rightarrow}$	“it is *not* easy to show”
$\Rightarrow$	logical implication
$\wedge$	logical conjunction
$\vee$	logical disjunction
□	repeated expression
■	anonymous reference
w/o	“without”
w/	“with”
→	internal forward reference (“see below/later”)
←	internal backward reference (“see above/before”)
↑	external reference to advanced concepts (“have a look at an advanced textbook on…”)
↓	external reference to basic concepts (“remember your basic course on…”)
➔	reference to previous or upcoming exercises
⋆	optional choice/item
⁂	implicit or explicit definition of a new technical term (“so called …”)
‡	Aside
≡	Synonymous terms
:=	Definition

The following scientific abbreviations are used in this document:

2DEG	2-Dimensional Electron Gas
AC	Alternating Current
BEC	Bose-Einstein Condensate
CFT	Conformal Field Theory
CS	Chern-Simons
DC	Direct Current
DMRG	Density Matrix Renormalization Group
FQHE	Fractional Quantum Hall Effect
IQH	Integer Quantum Hall
IQHE	Integer Quantum Hall Effect
ITO	Invertible Topological Order
KT	Kosterlitz-Thouless
LL	Luttinger Liquid / Landau Level
LLL	Lowest Landau Level
LU	Local Unitary
MPS	Matrix Product State
PTB	Physikalisch Technische Bundesanstalt
QCD	Quantum Chromo Dynamics
QFT	Quantum Field Theory
QHE	Quantum Hall Effect
SET	Symmetry-Enriched Topological
SI	Système International (d'unités)
SPT	Symmetry-Protected Topological
SSB	Spontaneous Symmetry Breaking
SSH	Su-Schrieffer-Heeger
TIM	Transverse-field Ising Model
TKNN	Thouless-Kohmoto-Nightingale-Nijs
TO	Topological Order
TP	Topological Phase
TQC	Topological Quantum Computation
TQFT	Topological Quantum Field Theory
TQM	Topological Quantum Memory
TQO	Topological Quantum Order
YM	Yang-Mills

0. Setting the Stage

◊ **Topics**

- Motivation: A classical system with topological edge modes
- Localization within physics: Where we are on the energy ladder
- Introduce our objects of interest: Quantum phases and phase transitions
- Sketch the Landau paradigm: Spontaneous symmetry breaking
- Concepts beyond the Landau paradigm: Topological phases
- Sketch different types of topological phases

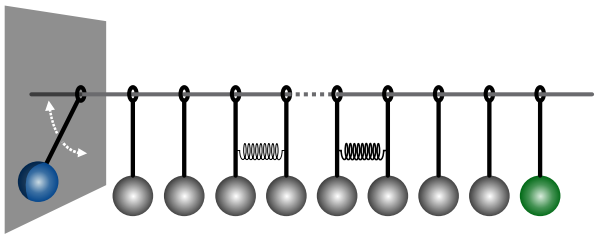
0.1. Motivation: Transferring energy with pendulums

To get you hooked (hopefully!), we start with a series of simple *classical mechanics* "experiments" (= computer simulations). The point of this adventure is to highlight some of the surprising effects *topological* features can have (where exactly topology enters is not obvious and will be discussed in due time):

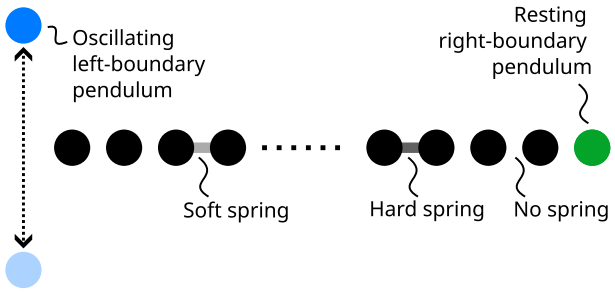
The following is inspired by on one of my papers [20].

Beamer and internet connection required!

1 | ◁ 1D chain of N identical pendulums, coupled by *tunable* springs:



→ Schematic view from bottom:



We encode the strength of springs by their color:
White: no spring / Light: soft spring / Dark: stiff spring

2 | Goal: Transfer oscillation energy from one boundary to the other:

$$\underbrace{\vec{x}(t<0) = (1,0,\ldots,0)\cdot e^{i\omega t}}_{\text{Left pendulum excited}} \xrightarrow[\text{How??}]{\text{Time evolution}} \underbrace{\vec{x}(t>T) = (0,\ldots,0,1)\cdot e^{i\omega t}}_{\text{Right pendulum excited}} \tag{0.1}$$

Here, $x_i(t)$ denotes the displacement of pendulum i at time t; our protocol starts at $t = 0$ and ends at $t = T$. The eigenfrequency of the (identical) pendulums is ω.

3 | Time evolution → Classical equation of motion:

$$\ddot{\vec{x}} + \mathbb{D}(t)\vec{x} = 0 \tag{0.2}$$

This is the Newtonian equation of motion for N coupled harmonic oscillators.

$\mathbb{D}(t) \in \mathbb{R}^{N\times N}$: Time dependent coupling matrix

4 | Rules:

- We can choose the stiffness for each spring independently.
- We can modify the stiffness of an arbitrary subset by *a single* time dependent factor.
- We can choose the time dependence of this factor freely.

→ Allowed form of the coupling matrix:

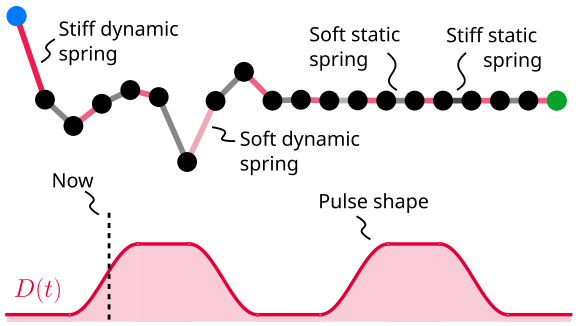
$$\mathbb{D}(t) = \underbrace{\begin{pmatrix} \omega_1^2 & s_1 & & \\ s_1 & \omega_2^2 & s_2 & \\ & s_2 & \omega_3^2 & \\ & & & \ddots \end{pmatrix}}_{\text{Static springs \& pendulums}} + D(t) \underbrace{\begin{pmatrix} 0 & d_1 & & \\ d_1 & 0 & d_2 & \\ & d_2 & 0 & \\ & & & \ddots \end{pmatrix}}_{\text{Time dependent springs}} \tag{0.3}$$

With …

- $\omega_i = \sqrt{g/l_i} \equiv \omega$: Frequency of pendulums (uniform and fixed)
- s_i: Static stiffness of spring coupling pendulums i and $i+1$
- $D(t)\cdot d_i$: Time dependent stiffness of spring coupling pendulums i and $i+1$
- Global time dependence of spring stiffness:

$$D(t) = \begin{cases} 0 & t < T \\ P(t) & 0 \le t \le T \\ 0 & t > T \end{cases} \quad \text{with pulse shape} \quad P(t) : [0,T] \to [0,1] \tag{0.4}$$

→ Schematic view:



We color static (tunable) springs in shades of black (red). The shape of $D(t)$ is plotted below the pendulum chain; the current point in time is marked by a vertical line in this plot.

5 | Questions:

- How to choose the spring couplings s_i and d_i?
- How to choose the pulse shape $P(t)$?

6 | Experiments:

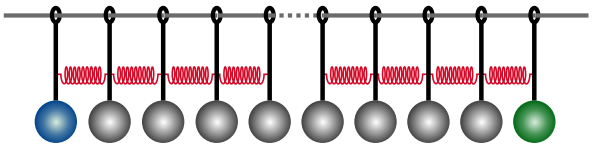
The simulations below are based on numerical integration of Eq. (0.2) with initial configuration (0.1):

Download `Mathematica` notebook

i | ◁ **Variant 1:**

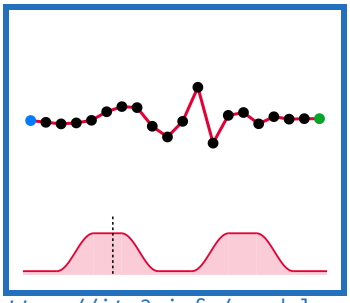
$$s_i = 0 \quad \text{and} \quad d_i = d > 0 \quad \text{for all} \quad i = 1, \dots, N-1 \tag{0.5}$$

In this approach, we couple all pendulums *uniformly* by springs of *time dependent* stiffness:



As pulse $P(t)$ we choose a smoothed-out rectangular double pulse to transfer the excitation from left to right and back. The latter is of course not necessary; it allows us to amplify the effects of a single transfer. We normalize the pulse such that $\max_t P(t) \approx 1$.

→ Simulation: Click on figure (internet required).



`https://itp3.info/pendulumv1`

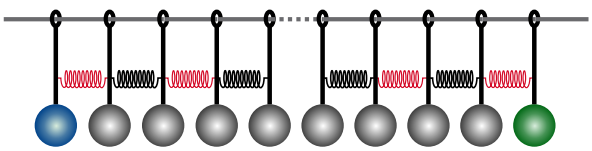
→ Result: No perfect transfer possible! ☹

The reason is quite obvious: The boundary excitation is transferred via an elastic wave that travels through the bulk. Because of ↓ *dispersion*, this excitation cannot be relocalized on the other boundary; we loose inevitably energy to bulk excitations.

ii | ◁ **Variant 2:** (We assume N to be even!)

$$s_i = 0 \quad \text{and} \quad d_i = d > 0 \quad \text{for } \textit{odd} \quad i = 1, 3, \dots, N-1 \tag{0.6a}$$
$$s_i \approx 2 \times d \quad \text{and} \quad d_i = 0 \quad \text{for } \textit{even} \quad i = 2, 4, \dots, N-2 \tag{0.6b}$$

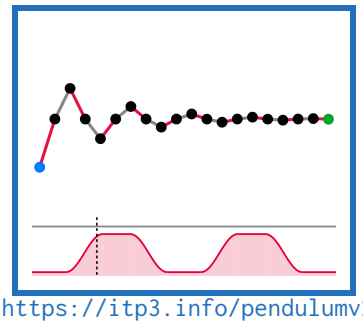
Now we couple pendulums *alternating* with weak & dynamic and strong & static springs:



- We use the same pulse $P(t)$ as for **Variant 1** above. Now it affects only every other spring, of course!

- If you wonder *how* one might come up with this contraption: This is why you should attend this course ☺.

→ Simulation:



https://itp3.info/pendulumv2

→ Result: (Almost) perfect transfer possible! ☺☺

- The video above is "stroboscopic", i.e., the pendulums are oscillating with a much higher frequency; the visible oscillations are therefore determined by the actual frequency and the chosen frame times (↓ *beat frequency*). The transfer also works with lower frequencies (as in the **Variant 1** video above), but would then take much longer.
- The reason why this approach works perfectly is not obvious. In particular, its robustness to certain types of disorder (→ *next*) are not trivial to understand. We need to introduce quite a bit of machinery to tackle this problem (→ *much later*).

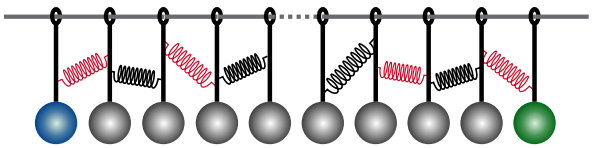
What happens to this method if the constituents of our contraption have *Imperfections*?

a | ◁ Imperfect springs:

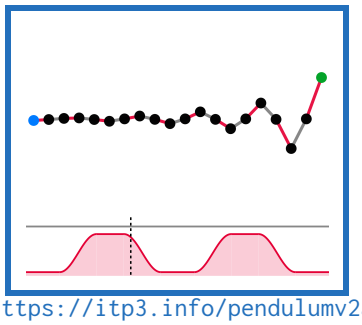
$$s_i = 0 \quad \text{and} \quad d_i \in \mathcal{N}(d, \sigma_d) \quad \text{for } \textit{odd } i \tag{0.7a}$$
$$s_i \in \mathcal{N}(2d, \sigma_s) \quad \text{and} \quad d_i = 0 \quad \text{for } \textit{even } i \tag{0.7b}$$

- $\mathcal{N}(\mu, \sigma)$ denotes the ↓ *normal distribution* with mean μ and standard deviation σ.
- We choose $\sigma_d \approx 0.1 \times d$ and $\sigma_s \approx 0.1 \times 2d$, i.e., tolerances of about 10%.

→ We modify all non-zero spring couplings randomly by a small amount:



→ Simulation:



https://itp3.info/pendulumv2a

→ Result: Still perfect transfer possible! ☺

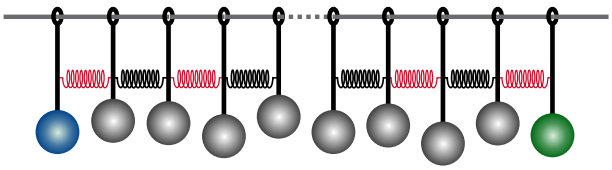
- ¡! This is not what one typically expects for an imperfect system. In particular for rather large imperfections of about 10%.
- To achieve perfect transfer, one has to tune the pulse slightly (either its height or its duration). However, one always finds an appropriately tuned pulse that achieves (almost) perfect transfer.
- Note that even if the pulse is not tuned, there is (almost) no energy loss to bulk modes. A non-optimal pulse therefore leads to an incomplete transfer but not to losses.
- If you look closely, there actually are weak excitations of the pendulum pairs in the bulk after the double transfer. This is a consequence of weak adiabaticity breaking; an ideal transfer would take infinitely long.

b | ◁ Imperfect pendulums:

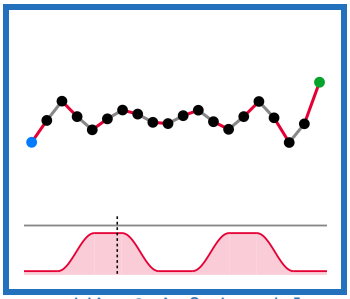
$$\text{Eq. (0.6) together with} \quad \omega_i \equiv \omega \mapsto \omega_i \in \mathcal{N}(\omega, \sigma_\omega) \tag{0.8}$$

We choose $\sigma_\omega \approx 0.1 \times \omega$, i.e., tolerances of about 10%.

→ We modify all frequencies (= lengths of pendulums) ω_i randomly by a small amount:



→ Simulation:



https://itp3.info/pendulumv2b

→ Result: No perfect transfer possible! ☹

- This is the typical effect one might expect for an imperfect system.
- If one optimizes over the pulse length (or height), one typically does *not* find a pulse that achieves perfect transfer.
- Note that there is still no energy loss to bulk modes. This means that for small-enough imperfections, the time-evolution remains almost adiabatic.
- That the two boundary pendulums oscillate with drastically different periods is a consequence of the frequency imperfections (which of course also affect the boundary pendulums) in combination with the "stroboscopic" visualization.

7 | → Many questions …

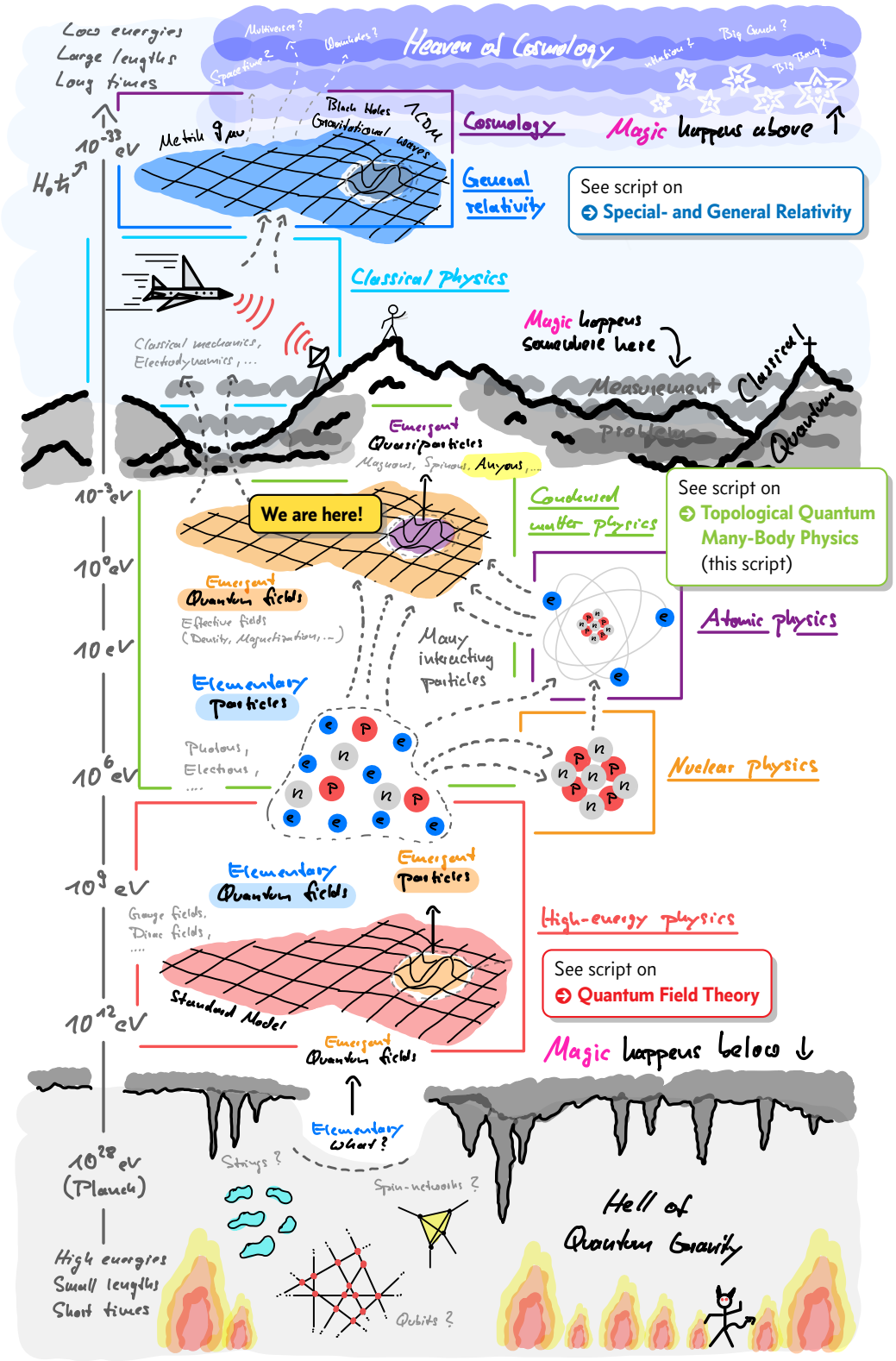
1. Why does **Variant 2** work? Where do these "boundary modes" come from?
2. Why is this procedure robust against one type of disorder, but not the other?
3. What has this to do with *topology*?
4. What has this to do with *quantum mechanics*?

Answers: → *Later*

For the impatient: The first three questions will be answered in **??**. How some features of topological quantum phases translate to classical systems is discussed in **??**.

0.2. The Big Picture

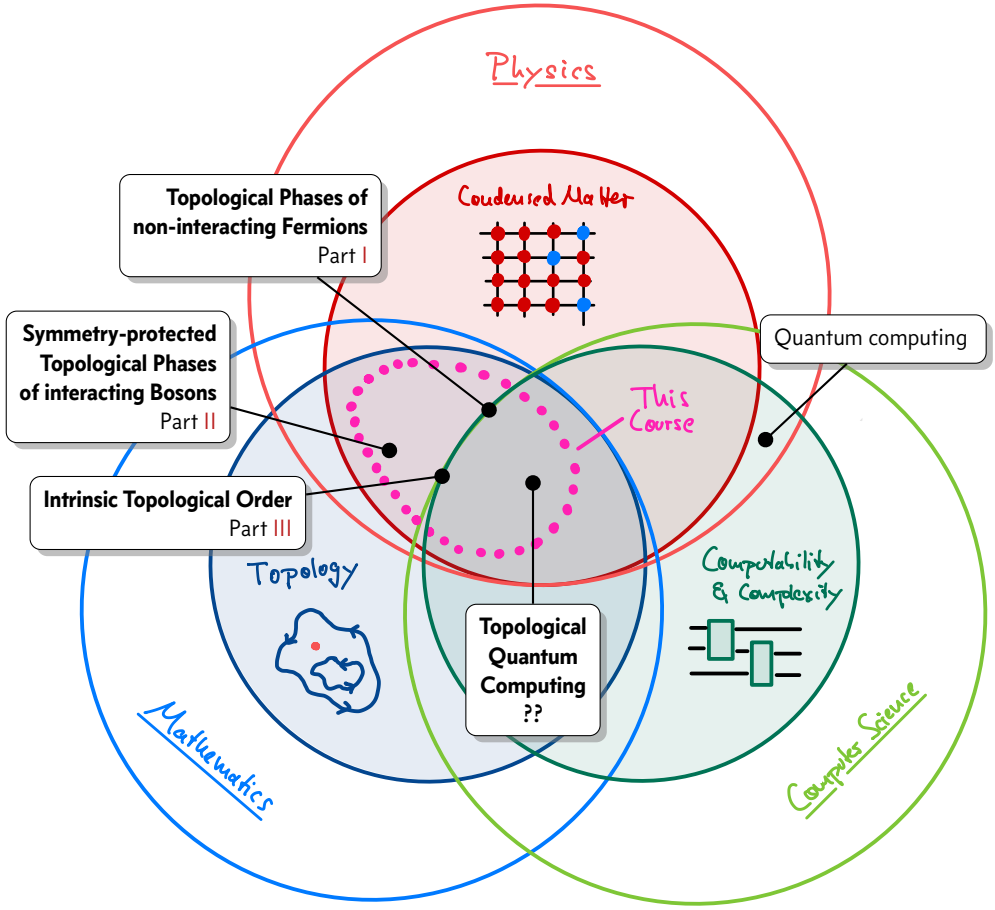
Physics strives for an objective, operational description of nature. To do so systematically, it is convenient to slice reality into layers separated by energy-, time- and length scales:



Comments:

- In this course we neither study the very small (↑ *high-energy physics*) nor the very large (↓ *relativity*). Thus we are not concerned with *fundamental* physics but with *emergent* phenomena.
- While this course is clearly focused on the *mathematics* and *conceptual foundations* that underlie the phenomenology of topological quantum phases, there will be connections to both experiments and applications along the lines. In particular the applications differentiate this course from more fundamental topics close to the extremes of the energy scale.
- At the very end (→ ??), we will briefly discuss a scenario where some of these emergent properties (related to topological order) might be of relevance for fundamental questions of high-energy physics. Maybe the realm of particle physics is emergent as well, and the theory of topological quantum many-body physics has something to say about questions that conventional high-energy physics is silent about? (For example, why there are fermions to begin with?)

The topics covered in this course can also be located with respect to adjacent scientific disciplines:



- ↑ *Topology* is the area of mathematics that deals with properties of spaces (e.g. manifolds) that are robust against smooth deformations of these spaces. For example, the topology of a torus (= donut) is characterized by the fact that it has a single "hole"; its exact shape (e.g. its size and local bumps on the surface) are part of its *geometry* but not relevant for its *topology*.
- Which, why, and how concepts of topology are instantiated in particular quantum phases is the main focus of this course.

0.3. Quantum phases and quantum phase transitions

1 | In this course, we are interested in the following concepts:

⁂ Definition: Quantum phases and phase transitions

- **Quantum phase** $\Leftrightarrow$ Phase of matter at $T = 0$ (= no thermal fluctuations)
  $\Leftrightarrow$ Ground state manifold of
  $$\left\{ \begin{array}{c} \textit{scalable} \\ \textit{local} \\ \textit{many-body} \end{array} \right\} \text{ Hamiltonian in the } \textit{thermodynamic limit}$$
  - *"Scalable"*: The Hamiltonian is actually a *family* of Hamiltonians H_L parametrized by the system size L (e.g., number of modes/atoms/spins in each spatial direction).
  - *"Local"*: The Hamiltonian is a sum of operators that act only on a finite number of adjacent degrees of freedom (i.e., no long-range interactions).
  - *"Many-body"*: The Hamiltonian describes the interactions of extensively many degrees of freedom (spins, particles).
  - *"Thermodynamic limit"*: We are interested in the ground state properties for infinitely large systems, i.e., in the limit $L \to \infty$.

- In this course, we are mostly interested in a particular subclass of quantum phases:

  Gapped quantum phase $\Leftrightarrow$ Ground state manifold of Hamiltonian with a *stable bulk gap*

  - *"Bulk gap"*: Spectral gap between the ground state manifold and the first excited states of a system with periodic boundaries. Systems *with* boundaries may have eigenstates that cross this gap.
  - *"Stable"*: The gap remains finite in the thermodynamic limit $L \to \infty$.

- Naturally, we are also interested in transitions *between* quantum phases:

  Quantum phase **transition** $\Leftrightarrow$ Transition between different quantum phases (in the thermodynamic limit)
  $\Leftrightarrow$ *Qualitative* change of *macroscopic* properties triggered by *small* changes of *microscopic* parameters

Comments:

- ¡! In this course we consider exclusively quantum phases; hence we drop the term "quantum" in the expressions defined above in many cases.
- Quantum phases are characterized by properties that *emerge* from many particles that interact quantum-mechanically. We are therefore interested how *macroscopic* quantum properties emerge from *microscopic* quantum interactions.
- Thus, the study of quantum phases and phase transitions is particularly challenging, because computing the ground state(s) of large, interacting quantum systems is hard if not impossible. (The Hilbert space dimension grows exponentially with the system size L!)

  Broadly speaking, there are four attack vectors:

(1) Solve models *analytically* …

(a) …with exact methods.
(↑ *Bethe ansatz*, → *Stabilizer formalism*,→ *Quadratic theories*, …)

(b) …with approximate methods.
(↓ *Perturbation theory*, ↑ *Mean-field theory*, ↑ *Quantum field theory*, …)

(2) Solve models *numerically* (on classical computers).
(↓ *Exact diagonalization*,→ *DMRG*, ↑ *Quantum Monte Carlo*, …)

(3) Perform *quantum simulations*.
(→ *Analog quantum simulation*, ↑ *Digital quantum simulation*, …)

(4) Last but not least: conduct *experiments*.

Here we focus on approach (1a); in some exercises you will make contact with approach (2).

- Quantum phase transitions are triggered by changes of parameters in the Hamiltonian (e.g. interaction strengths, chemical potentials, hopping rates, …). (Quantum) phase *diagrams* are therefore plotted as functions of *parameters* of the Hamiltonian, and not temperature or pressure [as you learned in your course on ↓ *(classical) statistical physics*].
- Quantum phases at $T = 0$ are properties of *pure quantum states* without entropy. By contrast, *classical phases* (like crystalline phases of solids, or the liquid phase of water), are properties of statistical *ensembles* of states with finite entropy; in the framework of quantum mechanics, these are described by *density matrices* [for example, the Gibbs state $\rho = e^{-\beta H}/Z$ of the ↓ *canonical ensemble*].
- Classical *thermodynamic* phase transitions (e.g., the boiling of water) are driven by *thermal fluctuations* that modify the statistical ensemble of microstates, such that its macroscopic observables change qualitatively. By contrast, *quantum* phase transitions are driven by *quantum fluctuations* (due to non-commuting terms in the Hamiltonian, → *below*). These modify the *amplitudes* of basis states in the (pure!) ground state of the system, thereby changing its quantum-mechanical properties qualitatively (correlations, entanglement structure, …).
- ¡! Quantum fluctuations are *not* dynamical fluctuations in time. The ground state is an *eigenstate* and therefore *time-independent*. However, if you would initialize the system in a classical product state which is not an eigenstate (in particular, not the ground state), then it would fluctuate in time, because the ground state is actually a *superposition* of many different such classical product states.

2 | Examples of quantum phases that exist in nature and/or can be experimentally realized:

- ↓ *Superconductors*
- ↓ *Superfluids* (e.g. superfluid Helium …)
- ↑ *Supersolids* (have been recently realized in experiments [21–23])
- ↓ *Bose-Einstein condensates* (BEC)
- ↓ *Fermi liquids*
- → *Quantum Hall states*
- …

While these are important examples, they are typically hard to describe and understand theoretically. It is therefore advisable to focus on a simple "toy model" that is exactly solvable:

↓ **Lecture 2** [11.04.25]

3 | Paradigmatic example:

i | ◁ Periodic 1D chain of L spin-$\frac{1}{2}$ with Hamiltonian:

⁂ *Transverse-field Ising model* (TIM):

$$H_{\text{TIM}} = -J \sum_{i=1}^{L} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{L} \sigma_i^x \tag{0.9}$$

where …

- $J \geq 0$: ferromagnetic coupling strength
- $h \geq 0$: *transverse* magnetic field

  "*Transverse*" since h points in x-direction, which is transverse to the z-direction of the ferromagnetic Ising interactions.

ii | Observation:

$$\left[\sigma_i^z \sigma_{i+1}^z, \sigma_i^x\right] \neq 0 \tag{0.10}$$

→ The Ising interactions and the magnetic field terms *cannot* be diagonalized simultaneously!

→ *Quantum fluctuations*

→ Ground state(s) = (entangled) *superpositions* of product states $|\uparrow\downarrow \ldots\rangle$ for $h \neq 0$

Product states of the form $|\uparrow\downarrow \ldots\rangle$ are eigenstates of the classical Ising interaction $\sigma_i^z \sigma_{i+1}^z$.

iii | Two qualitatively different parameter regimes:

a | $J \ll h$:

$J \approx 0 \rightarrow$ *Gapped* phase with *unique* ground state:

$$|\Omega_+\rangle \approx |+ + \cdots +\rangle \tag{0.11}$$

◁ Spin-spin correlations:

$$\langle \Omega_+ | \sigma_i^z \sigma_j^z | \Omega_+ \rangle \xrightarrow{|i-j| \to \infty} 0 \tag{0.12}$$

→ ⁂ *Paramagnetic phase* (=*disordered* phase)

- Note that $\langle \Omega_+ | \sigma_i^z | \Omega_+ \rangle = 0$, i.e., measuring any spin yields ± 1 with equal probability. The vanishing of spin-spin correlations (0.12) means that there is no correlation between these random outcomes for distant spins. That is, there is *no order* in the ground state.
- For $J = 0$ and $h > 0$ the system has a stable bulk gap of $\Delta E = 2h$, independent of L (the energy cost of flipping a single spin from $|+\rangle$ to $|-\rangle$).

b | $J \gg h$:

$h \approx 0 \to$ *Gapped* phase with *two-fold degenerate* ground state manifold:

$$|\Omega\rangle \approx \alpha \underbrace{|\uparrow\uparrow \dots \uparrow\rangle}_{|\Omega_\uparrow\rangle} + \beta \underbrace{|\downarrow\downarrow \dots \downarrow\rangle}_{|\Omega_\downarrow\rangle} \tag{0.13}$$

$\lhd$ Spin-spin correlations:

$$\langle\Omega|\sigma_i^z \sigma_j^z|\Omega\rangle \xrightarrow{|i-j|\to\infty} 1 \tag{0.14}$$

This is true for arbitrary amplitudes α and β!

$\to$ ⁂ *Ferromagnetic phase* (*ordered* phase)

- Note that now $\langle\Omega|\sigma_i^z|\Omega\rangle \gtrless 0$ depends on the particular values of α and β; for the "classical" product states it is $\langle\Omega_{\uparrow\downarrow}|\sigma_i^z|\Omega_{\uparrow\downarrow}\rangle = \pm 1$. However, the non-vanishing correlations (0.14) imply in any case that z-measurements of distant spins are correlated. That is, there is *order* in the ground state.
- For $J > 0$ and $h = 0$ and periodic boundaries, the system has a stable bulk gap of $\Delta E = 4J$, independent of L (the energy cost of flipping a contiguous domain of spins, e.g., $|\uparrow\uparrow\uparrow\uparrow\rangle \mapsto |\uparrow\downarrow\downarrow\uparrow\rangle$).

iv | $\to$ The z-magnetization σ_i^z is a ⁂ *local order parameter* for the ferromagnetic phase:

$$\lim_{|i-j|\to\infty} \langle\sigma_i^z \sigma_j^z\rangle = 0 \quad \text{in the paramagnetic (disordered) phase} \tag{0.15a}$$

$$\lim_{|i-j|\to\infty} \langle\sigma_i^z \sigma_j^z\rangle \neq 0 \quad \text{in the ferromagnetic (ordered) phase} \tag{0.15b}$$

- The very fact that there is a local order parameter that characterizes the ferromagnetic phase makes this particular kind of order locally testable, i.e., by looking at a finite patch of the system, you can decide whether you are in the ferromagnetic or the paramagnetic phase. This makes the ferromagnetic phase a *counterexample* of a *topological* phase (→ *later*).
- Note that $[H, \sigma_i^z] \neq 0$, i.e. correlations of this observable at two distant points are a non-trivial phenomenon.

v | Comments:

- So far we only made heuristic arguments regarding the ground states of the TIM Hamiltonian (0.9). Fortunately, this model *can* be solved exactly! Despite the simplicity of the Hamiltonian, this calculation is not straightforward and requires quite a bit of machinery; you solve the model on ➲ Problemset 7 → *later*.
- While the TIM Hamiltonian clearly has a stable bulk gap in the two extreme cases ($J = 0$ and $h > 0$ vs. $J > 0$ and $h = 0$), it is not clear what happens when one adds small perturbations. For example, whether the gap stays open for $J > 0$ and $0 < h \ll J$ is not obvious. The problem is that the gap ΔE is of order unity, but the total operator norm of the magnetic field perturbation goes like $h \times L$, which diverges in the thermodynamic limit $L \to \infty$ for *arbitrarily small* perturbations $h > 0$. In general, bulk gaps can therefore vanish under infinitesimally small perturbations! [For the TIM this does *not* happen, and the gap remains open for up to some critical value h_c of the magnetic field, but this must be proven (➲ Problemset 7).]

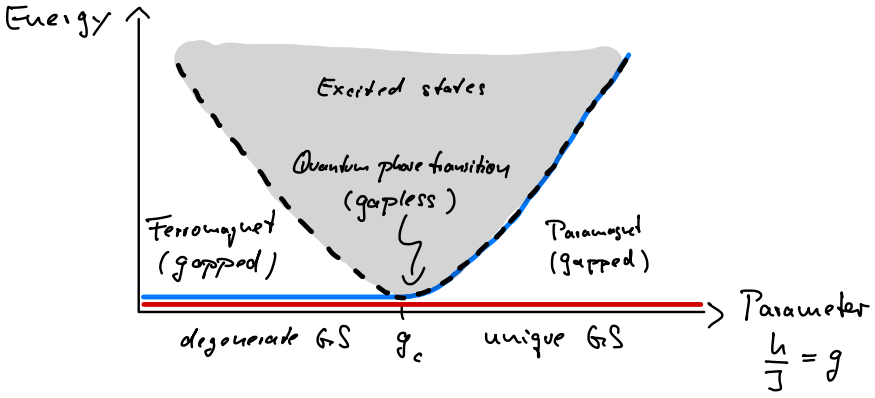
0.4. Spontaneous symmetry breaking

vi | What happends between the two gapped phases for $J \ll h$ and $J \gg h$?

Since the ground state degeneracy of the two gapped phases is different, the gap must close at some critical ratio $g_c = h/J$.

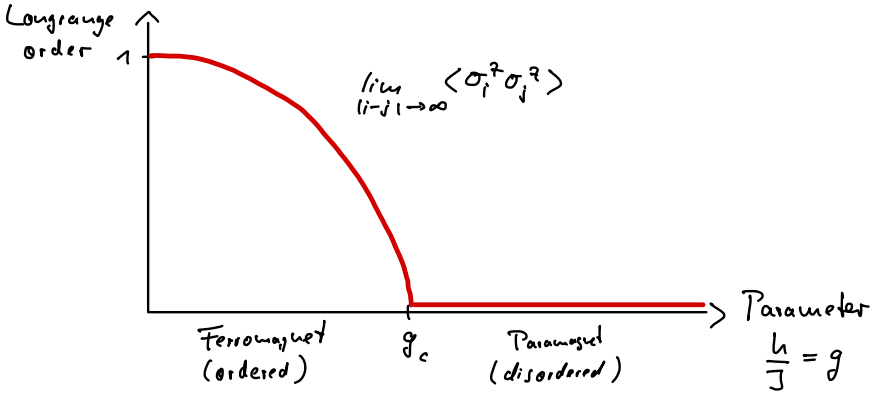
As noted above, we cannot exclude $g_c = 0$ or $g_c = \infty$ with our current knowledge. Here we assume that $0 < g_c < \infty$ (which turns out to be correct).

→ Schematic spectrum:



You compute this spectrum exactly later on ➔ Problemset 7.

vii | Tentative Phase diagram:



→ Order parameter *continuous* at phase transition

Again, this is not obvious; but solving the model exactly shows that it is true.

viii | → ⁂ *Continuous (second-order) phase transition*:

This is the most typical situation (at least for the models studied in this course), with the following features at the phase transition:

- Bulk gap closes
- Long-range fluctuations and self-similarity
  (= quantum fluctuations on all length scales)
- Effective conformal field theory (CFT) description
- Algebraic decay of correlations
  (As compared to exponential decay in gapped phases.)

ix | What characterizes the phase transition?

LEV LANDAU: *Spontaneous symmetry breaking*!

LANDAU was awarded the Nobel Prize in Physics 1962 for his pioneering work on describing quantum phases of matter, especially the superfluid phase of liquid Helium.

(1) ◁ Symmetry group G_S of the TIM *Hamiltonian* (0.9):

$$G_S = \{\mathbb{1}, X\} \simeq \mathbb{Z}_2 \quad \text{with} \quad X := \prod_{i=1}^{L} \sigma_i^x \tag{0.16}$$

X realizes a global flip of all spins: $|\uparrow\rangle \leftrightarrow |\downarrow\rangle$.

Check that $[H_{\text{TIM}}, X] = 0$. Note that $X^2 = \mathbb{1}$ so that $G_S \simeq \mathbb{Z}_2$.

(2) ◁ Symmetry groups G_E of the TIM *ground states* Eqs. (0.11) and (0.13):

- Paramagnetic phase:

$$G_E^{(\text{para})} = \{\mathbb{1}, X\} = G_S \quad \text{since} \quad X|\Omega_+\rangle = |\Omega_+\rangle \tag{0.17}$$

→ ⁂ *Symmetric phase*

- Ferromagnetic phase:

$$G_E^{(\text{ferro})} = \{\mathbb{1}\} \subsetneq G_S \quad \text{since} \quad X|\Omega_\uparrow\rangle = |\Omega_\downarrow\rangle \neq |\Omega_\uparrow\rangle \tag{0.18}$$

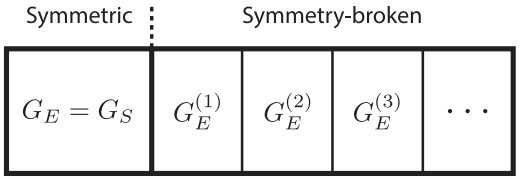
→ ⁂ *Symmetry-broken phase*

> **¡! Important**
>
> In the ferromagnetic phase, the ground states $|\Omega_{\uparrow/\downarrow}\rangle$ *spontaneously break* the symmetry group G_S of the Hamiltonian H_{TIM}.
>
> → ⁂ *Spontaneous symmetry breaking (SSB)*

Landau's paradigm (Spontaneous symmetry breaking)

4 | This concept extends to many quantum phases and their phase transitions (e.g. superconductors/-superfluids where the particle number symmetry $G_S = \mathrm{U}(1)$ is spontaneously broken) and is also applicable to classical phases and phase transitions (e.g. the transition from liquid to solid where rotation and translation symmetry are broken down to crystallographic subgroups).

→ Phases are characterized by the symmetries they break & preserve:



Labels of phases = Subgroups $G_E^{(i)}$ of symmetry group G_S

- This concept covers many (quantum and classical) phases and phase transitions, but in the realm of quantum mechanics there is more than just symmetry breaking—there is *entanglement*! This will become important → *below* …

- For the TIM the symmetry group G_S has only itself and the trivial group as subgroups. In general, G_S can be much larger so that many non-trivial subgroups exists (and therefore many different phases are possible). For example, if $G_S = E(3)$ is the Euclidean group of three-dimensional space (continuous rotations and translations), then G_S contains all possible space groups (symmetry groups of crystals) as subgroups.

5 | Comments:

- Note that the spontaneous symmetry breaking of the TIM in 1D is *not* forbidden by the ↑ *Mermin-Wagner theorem* because the broken symmetry is *discrete* ($\mathbb{Z}_2$).
- In *one* dimension, the spontaneous symmetry breaking (and the ferromagnetic phase) does *not* survive at finite temperatures $T > 0$. (Recall that the *classical* Ising model does not have a thermodynamic phase transition in one dimension, i.e., there is no ferromagnetic phase in a classical 1D Ising chain since domain walls can move without energy penalty.) The quantum phase transition of the 1D TIM is therefore a genuine quantum phenomenon, without classical counterpart.
- By contrast, in *two* dimensions (and above) the spontaneous symmetry breaking (and the ferromagnetic phase) *does* survive at finite temperatures $T > 0$. (Recall that the classical 2D Ising model has a thermodynamic phase transition at a critical temperature T_c below which it enters a ferromagnetic phase that breaks ergodicity.)
- A note on "symmetry breaking" in the quantum case:

  The ground state (for $h = 0$ and $J > 0$)

  $$|\Omega_s\rangle := \tfrac{1}{\sqrt{2}}|\Omega_\uparrow\rangle + \tfrac{1}{\sqrt{2}}|\Omega_\downarrow\rangle = \tfrac{1}{\sqrt{2}}|\uparrow\uparrow\ldots\uparrow\rangle + \tfrac{1}{\sqrt{2}}|\downarrow\downarrow\ldots\downarrow\rangle \tag{0.19}$$

  is clearly *symmetric* under global spin-flips: $X|\Omega_s\rangle = |\Omega_s\rangle$. So what about the symmetry *breaking*? (Note that this is something without analog in a classical setting where you cannot superimpose arbitrary ground states to form new ground states.)

  Mathematically, the two symmetry breaking states $|\Omega_\uparrow\rangle$ and $|\Omega_\downarrow\rangle$ belong to different ↑ *superselection sectors* in the thermodynamic limit (they don't live in the same Hilbert space). As a consequence, the "symmetric state" $|\Omega_s\rangle$ is not a state in the Hilbert space of the *infinite* system (strictly speaking, this is the mathematical manifestation of SSB); ↑ Refs. [24–26].

  Physically, the symmetry-broken states $|\Omega_{\uparrow/\downarrow}\rangle$ behave very differently than the symmetry-invariant states $|\Omega_\uparrow\rangle \pm |\Omega_\downarrow\rangle$: Local measurements (of σ_i^z) immediately collapse the latter into a mixture of the former. I.e. the symmetric states are extremely *fragile* (in contrast to the symmetry-broken states). Thus, in an experiment, one would always observe the *symmetry-broken* states, so that the notion of "spontaneous symmetry breaking" effectively carries over to the quantum realm.

0.5. Extending Landau's paradigm: Topological phases

6 | To understand the deficits of Landau's paradigm, and the conceptual possibility of topological phases, we first need a mathematically more rigorous definition of quantum phases (without spontaneous symmetry breaking!):

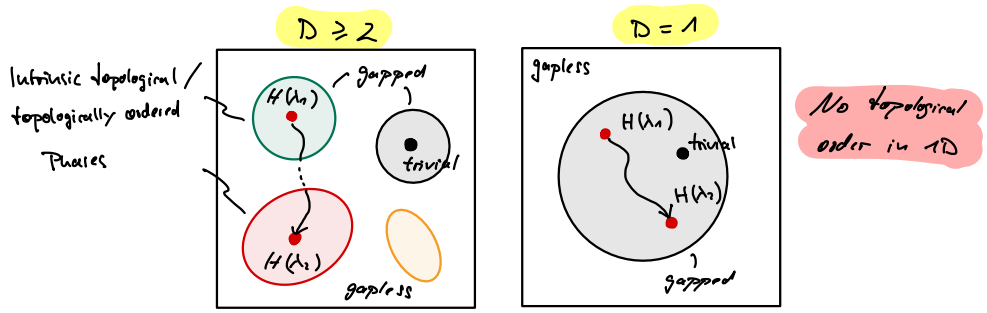
> ⁂ **Definition: Gapped quantum phases (formal version)**
>
> ◁ Gapped, local Hamiltonians H_a and H_b with unique ground states $|\Omega_a\rangle$ and $|\Omega_b\rangle$.
>
> These two many-body ground states belong to the same quantum phase if and only if there is a family of *gapped* and *local* Hamiltonians $\hat{H}(\alpha)$ (which depends continuously on a parameter

$\alpha \in [0,1]$) such that

$$H_a = \hat{H}(0) \quad \text{and} \quad H_b = \hat{H}(1)\,. \tag{0.20}$$

- The two constraints *"gapped"* and *"local"* ensure that the macroscopic properties of the ground states only change gradually along the path. (This precludes the traversal of phase boundaries where macroscopic properties change qualitatively.)
- ¡! Note that, strictly speaking, the two Hamiltonians H_a and H_b [and the family $\hat{H}(\alpha)$] are meant to be *sequences* of Hamiltonians for increasing system sized $L \to \infty$. The condition that the gap remains open along the parameter path thus refers to the *thermodynamic limit* $L \to \infty$, and not to any finite system. (Note that every finite system has a trivial gap that separates its ground state manifold from the first excited states!)
- The above definition can be extended in a straightforward way to systems with finite (but non-extensive) ground state degeneracies. This allows for an extension of the following concepts to symmetry-broken phases as well (→ *below*).

7 | ◁ Parameter-space of local Hamiltonians (without SSB, $G_E = G_S$):



$\overset{*}{\to}$ In $D \geq 2$ dimensions the parameter space decomposes into "islands" of gapped Hamiltonians that cannot be connected without closing the gap:

- *Trivial phase:* Ground state = disentangled product state
  (e.g. $|\Omega_+\rangle = |+\rangle \otimes |+\rangle \otimes \cdots$ or $|\Omega_\uparrow\rangle = |\uparrow\rangle \otimes |\uparrow\rangle \otimes \cdots$)
- *Topological phase:* Ground state = long-range entangled state
  (different patterns of long-range entanglement = different topological phases)

Comments:

- The fact that one-dimensional systems cannot have intrinsic topological order is not obvious.

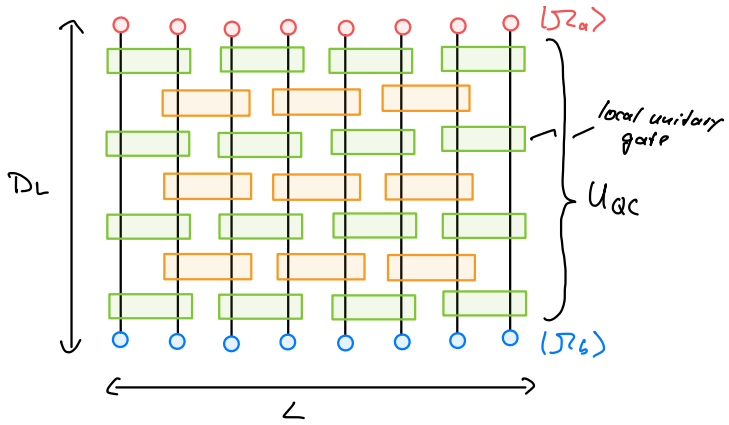
  It follows because the ground states of gapped 1D Hamiltonians (without SSB) are short-range correlated and feature an area law (the entanglement entropy between different parts of the sytem is constant) [27, 28]. One can therefore encode these states as (short-range correlated) → *matrix-product states* (MPS) with finite → *bond dimension*. It then follows that states of this form can always be mapped to a product state by a quantum circuit of finite depth (→ *below*) [6].
- ¡! In this course, we often dinstinguish between *fermionic* systems and *bosonic* systems. Since in our context bosonic systems make only sense with interactions (→ *below*), we also count *spin* systems to this class and often use the terms interchangeably. What makes (interacting) systems *bosonic* is therefore not so much the existence of an infinite-dimensional bosonic Fock space, but rather that the operator algebras of local degrees of freedom commute. Note

that spin-$\frac{1}{2}$ (or ↓ *qubits*) are equivalent to ↓ *hard-core bosons* (➲ Problemset 1), i.e., bosons with an infinite, repulsive on-site interaction.

- This splitting can also occur for Hamiltonians *with* SSB and a fixed subgroup G_E. We will not discuss this case in this course (→ *below*).
- Strictly speaking, the statement that there is no topological order in 1D is only true for *bosonic* systems (or spin systems). For 1D systems of *fermions*, there is a single non-trivial topological phase realized by the → *Majorana chain* (**??**) [29]. The subtle distinction between 1D bosonic and fermionic systems can be traced back to the non-locality of the → *Jordan-Wigner transformation* that translates between them, and the fact that parity is a locality constraint for fermionic systems.

8 | There is an alternative (but mathematically equivalent) definition of quantum phases in terms of local unitary circuits with constant depth:

◁ ⁂ *Local unitary (LU) circuit of depth D_L:*



$\overset{*}{\to}$ $|\Omega_a\rangle$ and $|\Omega_b\rangle$ belong to the same quantum phase, if and only if

$$|\Omega_a\rangle = U_{\mathrm{QC}}|\Omega_b\rangle \tag{0.21}$$

where U_{QC} is a local quantum circuit of constant depth $D_L = \text{const}$ for $L \to \infty$.

- This characterization clarifies that two states belong to the same quantum phase if they share the same "pattern of long-range entanglement" since this pattern can only be modified by long-range unitary gates (and not a LU-circuit of constant depth).
- With this characterization, it follows that a ground state $|\Omega_a\rangle$ is long-range entangled (= topologically ordered) iff it cannot be transformed into a trivial product state $|\uparrow\uparrow\uparrow\ldots\rangle$ by a constant-depth quantum circuit that is local wrt. the geometry of the system.
- This definition can be shown to be equivalent to the one given in the definition above [30]. The unitary can be explicitly expressed as

$$U_{\mathrm{QC}} = \mathcal{P}\exp\left[-i\int_0^1 \mathrm{d}\alpha\,\tilde{H}(\alpha)\right] \tag{0.22}$$

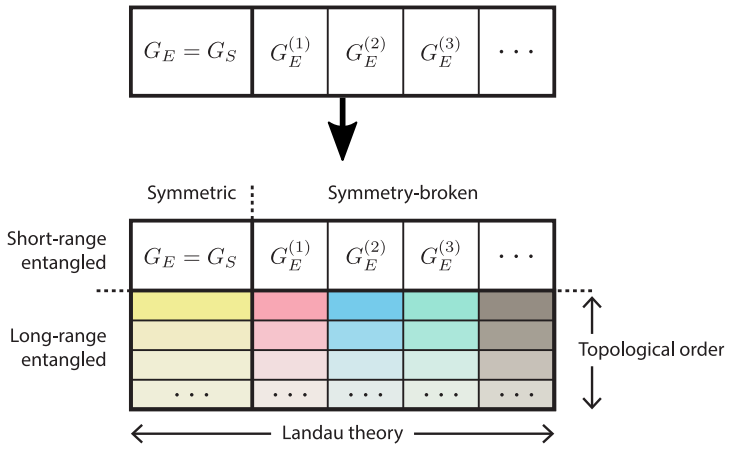
where $\mathcal{P}$ denotes the path-ordered exponential and $\tilde{H}(\alpha)$ is a sum of local Hermitian operators that is related (but generally not identical) to the gapped path $\hat{H}(\alpha)$.

- This makes the preparation of topologically ordered states experimentally challenging for quantum computers and quantum simulators with locality constraints: Quantum computers

must apply quantum circuits with a depth (= run time) that scales with the systems size. Similarly, quantum simulators that rely on adiabatic preparation schemes must cross a topological phase transition – which requires the duration of the preparation protocol to scale with the system size as well.

First extension of Landau's paradigm: (Intrinsic) Topological order

9 | The concept of long-range entanglement and equivalence via LU-circuits suggests the following extension of the classification of (gapped) quantum phase of matter:



This motivates the definition:

> ⁂ **Definition: Topological order (TO)**
>
> ⁂ *(Intrinsic) Topological order* := Patterns of long-range entanglement

- We discuss this concept at the end of this course: → *Part III*
- ¡! Sometimes the term "topological order" is used sloppily in the literature to refer to any phase of matter with some topological characteristics (e.g., → *symmetry-protected topological phases*). Then the modifier "intrinsic" is used to refer to states with non-trivial long-range entanglement. In this course "topological order" *always* refers to long-range entangled states; however, we still might add "intrinsic" to emphasize this point. By contrast, the term "topological phase" is used much broader and refers to any quantum phase with topological features.

↓ **Lecture 3** [17.04.25]

10 | Examples for topologically ordered systems that exist in nature (or in laboratories):

- Fractional quantum Hall states

Yes, this is all we actually know of (except for some special cases, see below)! There is a plethora of *theoretical* models, some of which are actively studied in labs; but none of them have been experimentally realized and characterized to the degree that fractional quantum Hall states have. Examples of promising models that are theoretically known to be topologically ordered and actively experimentally studied include ↑ *topological quantum spin liquids* like the → *toric code* (→ ??) [31, 32], ↑ *Kitaev materials* [33], and ↑ *fractional Chern insulators* [34].

- But actually, that's not quite correct:

  Integer quantum Hall states, first observed in 1980 by Klaus von Klitzing [13], are also long-range entangled, i.e., cannot be transformed into product states via constant-depth LU circuits [35]. However, their long-range entanglement is of a particularly simple type (so called *invertible* topological order, → *below*) that does *not* give rise to anyonic excitations and topological ground state degeneracies (→ *Part III*, see also Ref. [36]) which makes non-invertible topological orders like fractional quantum Hall states so interesting. This is why some use a different nomenclature where "topologically ordered" only refers to non-invertible topological order with anyonic excitations and non-vanishing → *topological entanglement entropy* [37].

- But *aactually* …that's also not quite correct:

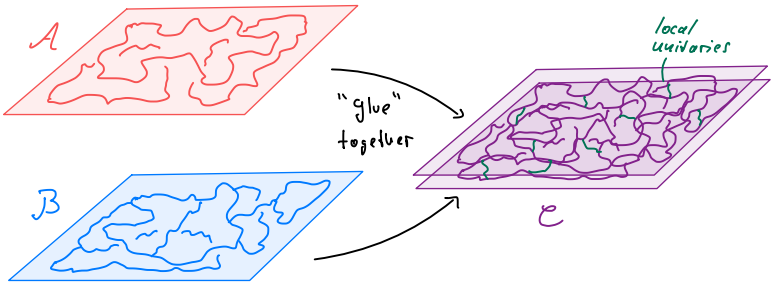
  Surprisingly, conventional s-wave superconductivity in 3D systems (discovered in 1911 by Heike Kamerlingh Onnes) is also an example of intrinsic topological order [35, 38]. This is not true for simplistic models like the BCS-Hamiltonian where the electromagnetic gauge field is treated as a non-dynamical background. In the real world, however, the gauge field *is* dynamical and a superconductor is described by the interactions between charged particles (electrons) with themselves (which gives rise to pairing) and with the dynamical electromagnetic field (which gives rise to string-like excitations, namely quantized ↑ *flux tubes*, and massive photons). The combined system of charges and electromagnetic field turns out to be topologically ordered [39, 40] and is described by a ↑ *topological quantum field theory* called ↑ *BF-theory* [41]. The excitations of such systems are (1) Bogoliubov quasiparticles ("broken Cooper pairs"), (2) flux lines/loops, and (3) massive photons. The Bogoliubov quasiparticles have non-trivial braiding statistics with the flux lines/loops – which demonstrates the (non-invertible) topological order of such systems.

11 | ‡ Invertible topological orders (ITO):

With our definition of gapped quantum phases, one can define a "multiplication" of such phases:

◁ Two topological phases $\mathcal{A}$ and $\mathcal{B}$ (in the same dimension):

$$\underbrace{\mathcal{A} \boxtimes \mathcal{B}}_{\text{Stacking two TOs}} = \underbrace{\mathcal{C}}_{\text{New TO}} \tag{0.23}$$



Observation: Trivial phase $\mathcal{E}$ (= product states) acts as identity:

$$\mathcal{A} \boxtimes \mathcal{E} = \mathcal{A} \tag{0.24}$$

→ Commutative monoid of quantum phases [42]

In mathematics, a ↓ *monoid* is a set with an associative binary operation and an identity. It is commutative if the binary operation is abelian. Elements are not required to have inverse elements (if all elements have inverse elements, the monoid becomes a group).

◁ Class of topological phases that have an *inverse element*:

⁂ Definition: Invertible topological order (ITO)

$$\text{⁂ } \textit{Invertible TOs (ITO)} := \left\{ \mathcal{A} \,|\, \exists \mathcal{A}^{-1} \,:\, \mathcal{A} \boxtimes \mathcal{A}^{-1} = \mathcal{E} \right\}$$

- The class of ITOs forms a *group* within the monoid of TOs.
- If they exist, the inverse phases are given by a time-reversal operation [43].
- In words: A topologically ordered ground state is invertible if and only if you can find another ground state (of a gapped, local Hamiltonian) such that the combination of both can be transformed into a product state by a constant-depth LU circuit.

$\overset{*}{\rightarrow}$ The entanglement patters of ITOs are of a particularly simple kind [42, 43]:

$$\text{ITO} \quad \Leftrightarrow \quad \begin{cases} \text{No} \rightarrow \textit{anyons} \\ \text{Vanishing} \rightarrow \textit{topological entanglement entropy} \end{cases}$$

In that sense, ITOs are a rather "boring" type of long-range entanglement, which is why some do not refer to ITOs as topologically ordered in the first place (in this course, we do).

Examples [35, 44]:

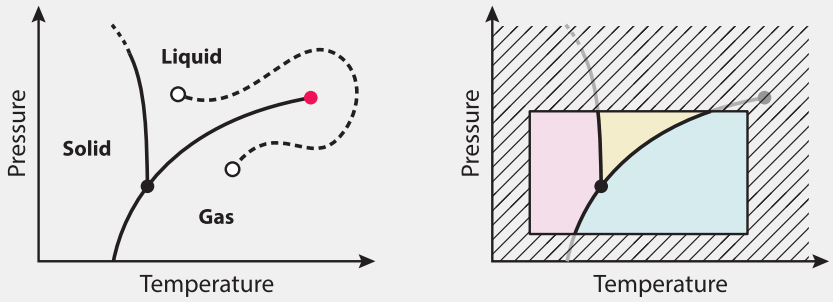
- → *Integer quantum Hall states* in 2D (Chapter 1)
- → *Haldane model* in 2D (**??**)
- → *Kane-Mele model (Topological insulator)* in 2D (**??**)
  (This is the only example in this list that is truly short-range entangled.)
- → *Majorana chain* in 1D (**??**),

So despite their lack of fancy anyonic statistics, ITOs are not so boring after all and we will study them in detail (and discover much interesting physics).

12 | But wait! There is more …

Adding patterns of long-range entanglement to our labeling scheme produces a more fine-grained classification of quantum phases. However, it can be useful to make this classification *even more fine-grained* by adding additional symmetry constraints.

We can motivate this rationale by a classical analog:

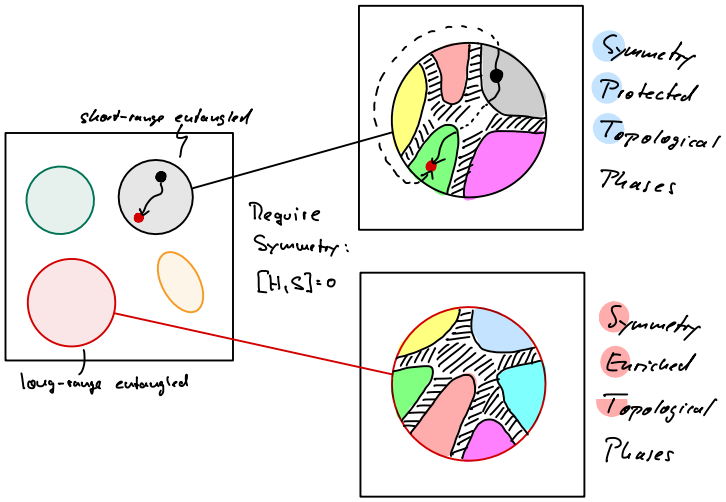
> ⋆ **Interlude: Restrictions on perturbations**
>
> ◁ Phase diagram of water:
>
> 
>
>
> (This argument draws inspiration from classical thermodynamic phases, not quantum phases.)
>
> → Restrictions on allowed paths can be useful! (depending on the system …)

13 | → Restrict Hamiltonians by *(protecting) symmetries* $G_P \subseteq G_S$:

> ⁂ **Definition: Symmetry-protected quantum phases**
>
> ◁ Gapped, local Hamiltonians H_a and H_b with unique ground states $|\Omega_a\rangle$ and $|\Omega_b\rangle$, and a symmetry group G_P (represented by unitaries U_g on the Hilbert space) with $[H_x, U_g] = 0$ for all $g \in G_P$ and $x = a, b$.
>
> The ground states belong to the same *symmetry-protected quantum phase* if and only if there exists a family of *gapped* and *local* Hamiltonians $\hat{H}(\alpha)$ that depends continuously on $\alpha \in [0, 1]$ such that
>
> $$H_a = \hat{H}(0) \quad \text{and} \quad H_b = \hat{H}(1) \tag{0.25}$$
>
> and
>
> $$\left[\hat{H}(\alpha), U_g\right] = 0 \quad \text{for all} \quad g \in G_P \text{ and } \alpha \in [0, 1]\,. \tag{0.26}$$

14 | → Phases with the *same entanglement pattern* can *split* further:

In particular short-range entangled phases that belong to the trivial phase!



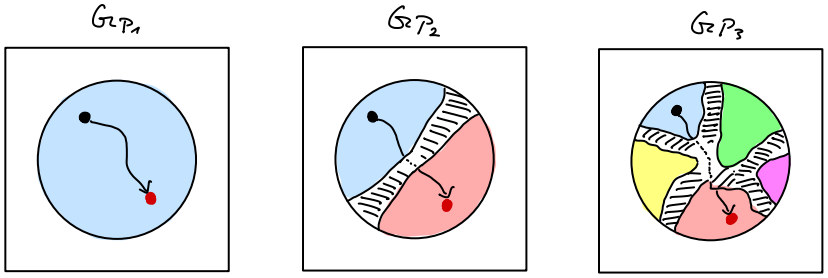
→ Conventional nomenclature:

> ⁂ **Definition: SPT and SET phases**
>
> - ⁂ *Symmetry-protected topological (SPT) phases* :=
> Short-range entangled phases protected by symmetries
> - ⁂ *Symmetry-enriched topological (SET) phases* :=
> Long-range entangled phases with additional symmetries
>
> We use the terms "SPT(s)" and "SET(s)" to mean "symmetry-protected/enriched topological quantum phase(s)" whenever the context requires it.

Comments:

- Typical examples for SPT phases are the → *SSH chain* in 1D (??) and the → *topological insulator* in 2D (??).
- Typical examples for SET phases are ↑ *factional quantum Hall states* with protected U(1) symmetry (= particle number conservation) [35].
- We will not study SET phases in this course!
- Since SPT phases are LU-equivalent to product states (when ignoring symmetries), they belong to the equivalence class $\mathcal{E}$ of "trivial" phases; in particular, they are ← *invertible topological orders* (of a special kind, namely without any long-range entanglement).

15 | Important: Possible SPTs (and SETs) depend on the protecting symmetry G_P:

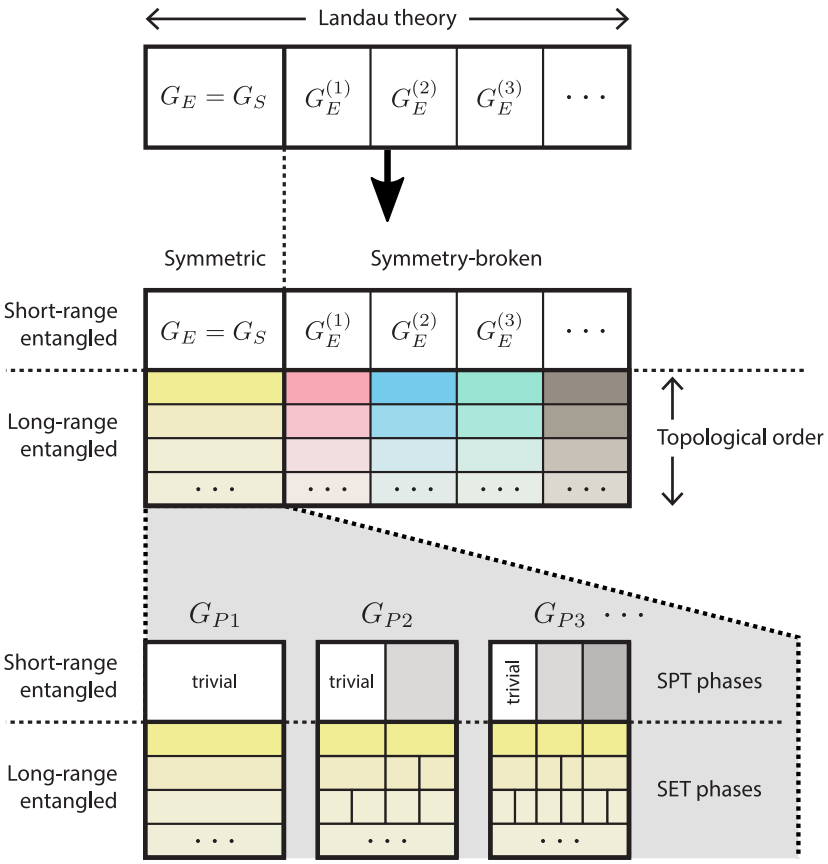


¡! Note that SPTs (and SETs) are not properties of ground states (like intrinsic topological order) but rather classifications of ground states with respect to a prescribed class of allowed Hamiltonian

deformations (or a restricted class of constant-depth LU circuits). The relevance of this class typically derives from physical considerations (recall the motivation above).

Second extension of Landau's paradigm: Symmetry-protected topological phases

16 | These insights lead us to a second extension of our classification scheme:



¡! In this course we only study phases without SSB.

17 | *Question:* How to characterize SPT phases?

We cannot label them by patterns of long-range entanglement nor by the symmetries they break!

Answer: Complicated! (also: subject of ongoing research!)

→ Make simplifying assumptions: Consider restricted classes of models/Hamiltonians:

- ◁ Non-interacting fermions → Part I

  The benefit of non-interacting fermions is that such models can be solved exactly. This provides us with powerful tools to classify them systematically. Note that some of these models will turn out to be invertible topological orders (we will not find *non*-invertible TOs with anyons etc. for this class of Hamiltonians).

- ◁ Interacting bosons/spins (in 1D) → Part II

  Interacting systems of bosons/spins in 1D are usually not exactly solvable. However, since there is no topological order in these systems, such models realize true SPT phases with a powerful description in terms of matrix-product states (which again makes a systematic classification possible).

Let me comment on a few questions that might come up at this point:

- Why not *interacting* fermions?

  In 1D, this classification derives (via Jordan-Wigner transformation) from the classification of interacting spin systems (also in 1D) [29]. In higher dimensions, there are approaches to classify interacting fermions (this is ongoing research [45]), but this goes beyond the scope of this course.

- Why not *non-interacting* bosons?

  Because non-interacting bosons form a Bose-Einstein condensate (which is a well-understood non-topological quantum phase). Thus topological phases for bosons *require* interactions (in contrast to fermions).

- Why not interacting bosons in *higher dimensions*?

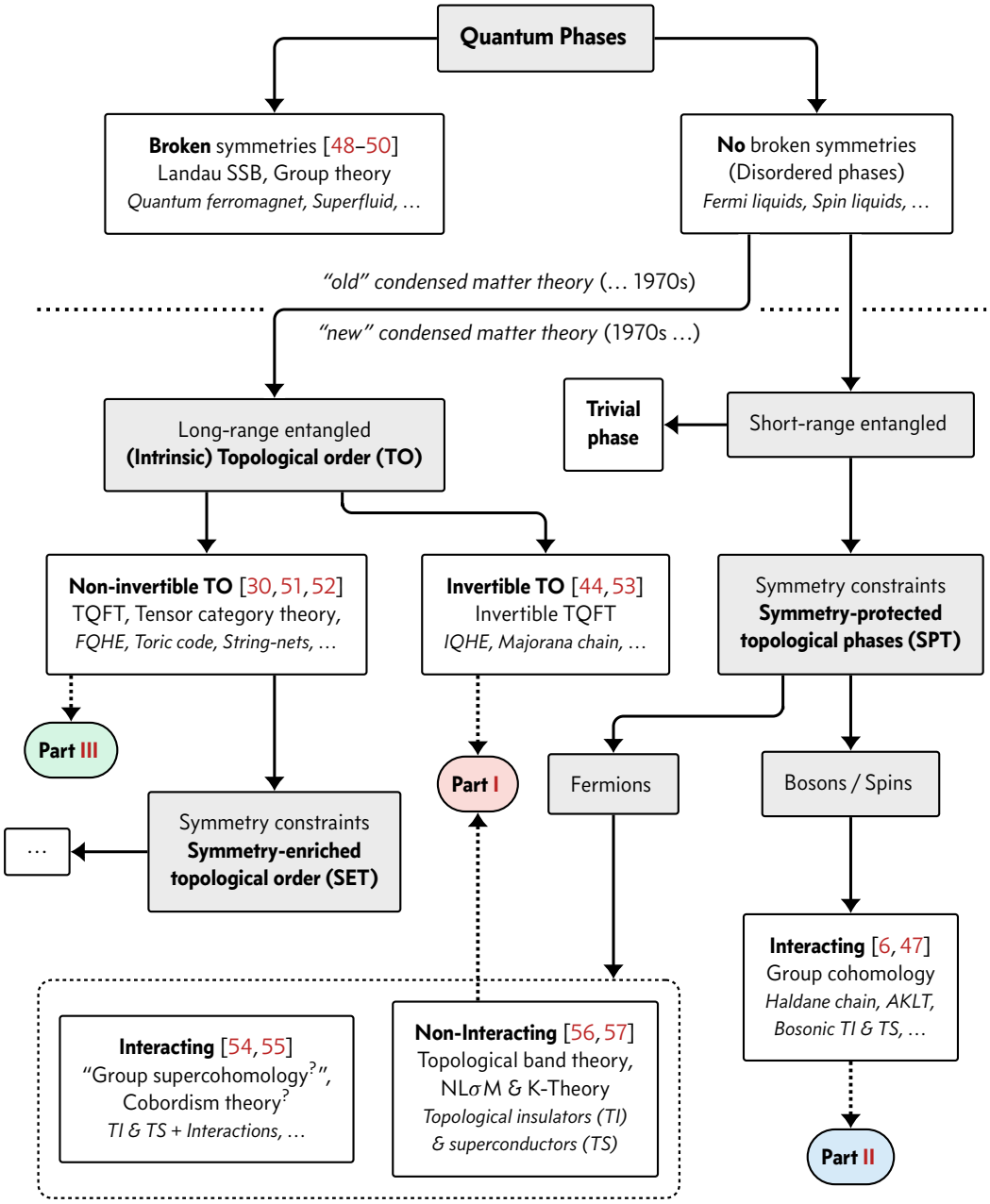
  This can be done with a generalization of the mathematical methods that we will discuss for the 1D case (Part II) [46, 47]. We will not discuss these generalizations in this course.

0.6. Overview and Outlook

We can combine all these concepts into a flowchart:

Highlight the three classes that we will discuss in this course.

(A few important original references are given that establish the various concepts.)



⁂ Definition: Nomenclature

In this course, gapped quantum phases that …

(1) do not break any symmetries

(2) and are either …

(2a) *topologically ordered (TO)* or
(invertible and non-invertible, with and without additional protecting symmetries)

(2b) *symmetry-protected (SPT)*

…will jointly be referred to as *topological phases (TP)*.

Note the difference between "topologically ordered phases" (which are long-range entangled) and "topological phases" (which can also refer to short-range entangled SPT phases)!

So far, we discussed topological phases on an abstract level. To further motivate these concepts, let us briefly list a few features of these intriguing phases of matter. Note that not every topological phase exhibits all these features! Some of them necessarily require long-range entanglement, some don't. Conversely, long-range entanglement does not necessarily imply all these features. It is quite a mess and we will study various models in this course to shed light on these features and their origins.

Some features of topological phases

- TPs cannot be characterized by local order parameters
(all correlations of local operators decay exponentially, cf. our discussion of the TIM).

  This is the defining property of topological phases and applies to intrinsic topological order and SPT phases alike.

- For some TPs, the ground state degeneracy on closed manifolds depends on their topology (whether it is a sphere, a torus, etc.) and is robust in the presence of perturbations [58].

  This is true for non-invertible topological orders (like fractional quantum Hall states), but not for SPT phases and invertible topological orders (topological insulators, integer quantum Hall states).

- Some TPs feature exponentially localized excitations (*quasiparticles*) that obey neither fermionic nor bosonic statistics – they are *anyons* and obey *fractional* or *anyonic statistics* [59, 60].

  The presence of anyons is closely related to the topological degeneracy mentioned above. For example, integer quantum Hall states and topological insulators do not have anyonic quasiparticles, but fractional quantum Hall states do.

- These quasiparticles can carry fractionalized charges (e.g. a fraction of the electron charge) [15].

  Fractional charges are a consequence of additional symmetries (for example, particle number conservation). Fractional charges are therefore a consequence of anyonic excitations in the presence of a conserved symmetry, i.e., SET order. Fractional quantum Hall states with U(1) particle number conservation are an example.

- Some TPs have an effective low-energy description in terms of a *topological quantum field theory (TQFT)* [61] (a quantum field theory defined by an action that is a topological invariant).

This is closely related to features of intrinsic topological orders (topological degeneracy, anyonic excitations). Invertible topological orders are described by ↑ *invertible TQFTs*.

- In some TPs, (lattice) defects can exhibit anyonic statistics as well (under continuous deformations of the Hamiltonian).

  This can happen even for invertible topological phases like the Majorana chain in 1D and the $p_x + i\,p_y$ superconductor in 2D. Note that such defects are *not* intrinsic quasiparticle excitations but deformations of the Hamiltonian.

- Some TPs feature robust, gapless edge states on manifolds with boundaries that allow for scattering-free transport [62].

  This can happen for invertible topological phases and even SPT phases. Examples are the famous chiral edge states of integer quantum Hall systems.

- The linear response of TPs can be *quantized* to a remarkable degree (even in the presence of disorder!).

  This can happen for invertible and non-invertible topological orders (like integer and fractional quantum Hall states) and even SPT phases (like topological insulators). The quantization requires some additional symmetry (like particle number conservation), even if the phase itself does not require any symmetry protection.

† Note: Why topology?

After this introductory info-dump you might wonder:

Why are topological phases called "topological"?

Where does topology enter the picture?

The answer to these questions is, as usual, complicated and cannot be fully appreciated at this point (answering these questions is the purpose of this course). However, we can make a few high-level comments to set your expectations:

First, remember that ↓ *topology* is the field of mathematics that is concerned with the formalization of deformation-invariant properties of spaces. Topology is therefore considered with rather "robust" qualities of (potentially abstract) objects – in contrast to *geometry* that is concerned with concepts like distances and angles (for which one requires a metric). For instance, a donut (bagel) is topologically equivalent to a coffee mug because you can continuously deform these two shapes into each other (such a deformations is called a ↓ *homotopy*):

➲ Example of a homotopy (Source: Wikipedia)

We say that the donut and the coffee mug are *topologically equivalent* but *geometrically distinct*. Formalizing the concept of "topological equivalence" and studying its implications is the core subject of topology.

Generally speaking, topological phase are called "topological" because various (!) concepts from topology play a role in their description. It is important to appreciate that the term "topology" can refer to *different* topological concepts and their application in physics. Broadly speaking, there are two very distinct such applications in the realm of topological quantum many-body physics:

- *"Classical topology":*

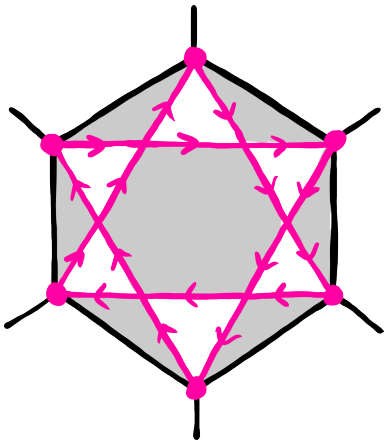
  Our first encounter of topological concepts will concern so called → *topological invariants* that can be used to characterize certain manifolds (parametric paths, Brillouin zones, band structures, …) that describe *non-interacting* quantum systems (Part I). This is an application of topology to *single-particle* quantum mechanics (with many-body ramifications like the quantized Hall conductivity). As such, some (not all!) of these phenomena translate to classical systems (→ *topological edge modes*) with several applications in engineering (**??**). The SPT phases of non-interacting fermions (topological insulators and superconductors) are an example of "classical topology" at play.

- *"Quantum topology":*

  A completely different application of methods from topology concerns the description of long-range entangled quantum many-body phases, i.e., *intrinsic topological order*. The low-energy physics of quantum phases in general can be described by ↑ *quantum field theories* (QFT). It turns out that the quantum field theories of systems with topological order are of a particularly elegant type: they are so called ↑ *topological quantum field theories* (TQFT). These are QFTs that do not depend on the metric of the space on which the fields live; hence their degrees of freedom only depend on the *topology* of this space. The algebraic properties of TQFTs capture all the fascinating properties of topologically ordered systems (anyonic excitations, topological ground state degeneracies, …). This application of topology is a genuine feature of *quantum* systems and has no classical counterpart, hence "quantum topology".

Part I.

Topological Phases of Non-Interacting Fermions



↓ **Lecture 4** [24.04.25]

1. The Integer Quantum Hall Effect

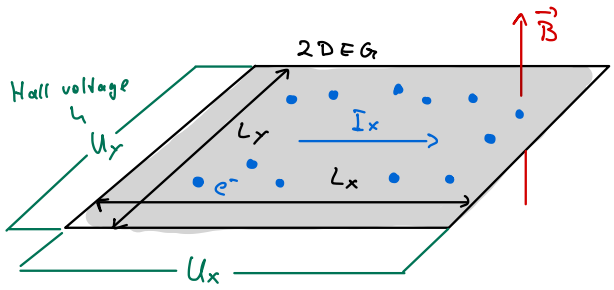
◊ **Topics**

- Review the classical & (integer) quantum Hall effect
- Derive Landau levels
- Motivate & define the Berry connection & holonomy
- Motivate & define the Berry curvature & phase
- Motivate & define the Chern number as a topological invariant
- Derive the Kubo formula and the TKNN formula for the Hall conductivity
- Comment on the role of disorder and edge states
- Locate the integer quantum Hall states in our classification of topological phases

- We start our discussion with topological phases that can be realized by *non-interacting fermions*. Such systems can be solved exactly in terms of single-particle Hamiltonians, the spectrum of which defines a ↓ *band structure*. The many-body ground states are then given by a ↓ *Fermi sea* of "filled bands" (in first-quantized language, the ground state is given by a ↓ *Slater determinant* of single-particle eigenstates). The resulting quantum phases will be symmetry-protected (SPT) phases and invertible topological orders. We will not encounter non-invertible topological orders (with anyons etc.) within this family of quantum many-body systems.
- Historically, the study of topological phases was kick-started by the experimental observation of the integer quantum Hall effect by Klaus von Klitzing in 1980 [13] who was awarded the 1985 Nobel Prize in Physics for his seminal discovery. The theoretical explanation of the effect by Thouless *et al.* in 1982 [17] highlighted the pivotal role that topological concepts can play in quantum many-body physics. For these theoretical contributions (among others) David J. Thouless (jointly with F. Duncan M. Haldane and J. Michael Kosterlitz) was awarded the 2016 Nobel Prize in Physics. We will use the integer quantum Hall effect and its theoretical description as motivation and starting point for the exploration of topological phases of non-interacting fermions in general.
- In the following, we have a quite detailed look at some aspects of the integer quantum Hall effect, especially the mathematics that underlies the quantization of the Hall conductivity. However, the integer quantum Hall effect is not the main focus of this course, and we will not cover the subject to its full extend (to do so would merit its own dedicated course!). If you are interested in more details, have a look at the textbook *Field Theories of Condensed Matter Systems* by Fradkin [63] (Chapter 12 and 13) or the *Lectures on the Quantum Hall Effect* by David Tong [64] on which parts of this chapter are based. You may also have a look at the collection [65] by Prange *et al.*.

1.1. From the classical to the quantum Hall effect

1 | ≺ 2D electron gas (2DEG) in perpendicular magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$:



Our sample is wired such that a current I_x can flow from a connection on the left boundary to a connection on the right boundary (there is no source/drain on the boundaries in y-direction, $I_y = 0$). There are voltage probes on all four boundaries to measure the voltages U_x and U_y.

2 | Drude model: (= Electrons as billard balls)

$$m\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = \underbrace{-e\boldsymbol{E} - e\,\boldsymbol{v}\times\boldsymbol{B}}_{\text{Lorentz force}} \underbrace{-\frac{m}{\tau}\boldsymbol{v}}_{\text{Scattering}} \tag{1.1}$$

τ: scattering time (due to electrons bouncing of much heavier crystal ions)

3 | ≺ Equilibrium $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v} = \boldsymbol{0}$

Define the current density $\boldsymbol{J} = -ne\boldsymbol{v}$ (n: electron density) $\overset{\circ}{\rightarrow}$

Note that $I_x = L_y J_x$ and $U_y = L_y E_y$.

$$\underbrace{\boldsymbol{J} = \sigma\boldsymbol{E}}_{\text{Ohm's law}} \quad \text{with} \quad \underbrace{\sigma = \begin{bmatrix}\sigma_{xx} & \sigma_{xy}\\ \sigma_{yx} & \sigma_{yy}\end{bmatrix} \overset{\circ}{=} \frac{\sigma_0}{1+\omega_B^2\tau^2}\begin{bmatrix}1 & -\omega_B\tau\\ \omega_B\tau & 1\end{bmatrix}}_{\text{Conductivity tensor}} \tag{1.2}$$

Note that $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$ is a consequence of the *rotation symmetry* of the system about the perpendicular z-axis.

with

$$\omega_B = \frac{eB}{m} \quad \text{⁂ cyclotron frequency} \tag{1.3}$$

and $\sigma_0 = ne^2\tau/m$ the ⁂ *DC conductivity* (conductivity w/o magnetic field).

4 | $\overset{\circ}{\rightarrow}$ ⁂ *Resistivity tensor:*

$$\rho \equiv \begin{bmatrix}\rho_{xx} & \rho_{xy}\\ \rho_{yx} & \rho_{yy}\end{bmatrix} := \sigma^{-1} = \frac{1}{\sigma_0}\begin{bmatrix}1 & \omega_B\tau\\ -\omega_B\tau & 1\end{bmatrix} \quad \text{with} \quad \boldsymbol{E} = \rho\boldsymbol{J} \tag{1.4}$$

Note that Hall *resistance* and Hall *resistivity* are (up to a sign depending on convention) the same:

$$R_{xy} := \frac{U_y}{I_x} = \frac{\cancel{L_y}E_y}{\cancel{L_y}J_x} = \frac{E_y}{J_x} = -\rho_{xy} \tag{1.5}$$

(Here we used $J_y = 0$ due to our experimental setup.)

This is *not* true for longitudinal resistance and resistivity:

$$R_{xx} := \frac{U_x}{I_x} = \frac{L_x E_x}{L_y J_x} = \frac{L_x}{L_y}\rho_{xx} \tag{1.6}$$

This already suggests that the Hall resistance is in some sense more robust than the longitudinal (ohmic) resistance as the former is independent of the sample geometry whereas the latter is not.

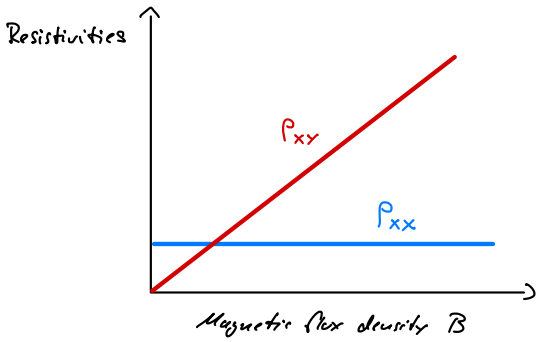
→ In particular:

$$\rho_{xy} = \frac{\omega_B \tau}{\sigma_0} = \frac{B}{ne} \quad \text{Independent of } \tau \text{ (= no dissipation) !} \tag{1.7a}$$

$$\rho_{xx} = \frac{m}{ne^2\tau} \tag{1.7b}$$

This implies that the Hall resistivity [and via Eq. (1.5) the Hall resistance] does not depend on the microscopic interactions of electrons with crystal ions and lattice defects (which determines the scattering time τ).

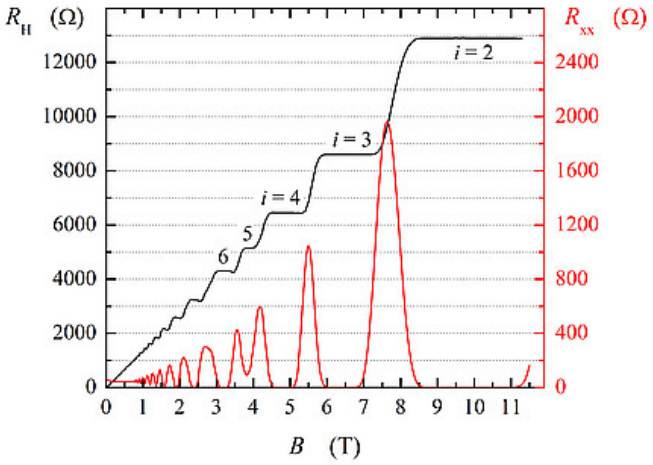
5 | → Classical prediction:



6 | Observation:

✓ Valid for high temperatures & weak magnetic fields ($\hbar\omega_B \ll k_B T$).

✗ *Not* valid for low temperatures & strong magnetic fields ($\hbar\omega_B \gg k_B T$):



– Note how the plot resembles the classical predictions in the lower-left corner (i.e., for weak magnetic fields).

- These results are from the electrical quantum metrology division of the PTB (the national metrology institute of Germany) and taken from this website; here $R_H = R_{xy} = L_y/L_y \cdot \rho_{xy} = \rho_{xy}$ and $R_{xx} = L_x/L_y \cdot \rho_{xx}$ and $i = \nu$ (see below). This phenomenon was first observed by KLAUS VON KLITZING in [13] for which he was awarded the 1985 Nobel Prize in Physics.
- The oscillations of the longitudinal resistance R_{xx} are called ↑ *Shubnikov-de Haas oscillations*. Here we are interested in the → *Hall plateaus* of the transversal resistance R_{xy} (→ *below*).

7 | → *Quantized* plateaus for Hall resistivity:

$$\rho_{xy} = \underbrace{\frac{\overbrace{2\pi\hbar}^{h/e^2}}{e^2}}_{R_K} \frac{1}{\nu} \quad \text{with} \quad \nu \in \{1, 2, 3, \dots\} \tag{1.8}$$

R_K: ⁂ *von Klitzing constant* or *quantum of resistivity* ($R_K \approx 25.8\,\mathrm{k\Omega}$)

At this point, Eq. (1.8) is an observational fact and a theoretical miracle!

Note: By the revision of the SI system of units in 2019, the numerical values of h and e are now fixed. Consequently, the value of the von Klitzing constant R_K is also fixed by definition and does not have to be measured. The integer quantum Hall effect can then be used as a universal (and defining) resistance measurement device (that's why the BTP is measuring the Hall resistance, see above). In particular, the quantization of the first → *Landau level* is perfect by definition: $\nu = 1.000\dots$. (Using your ohmmeter to measure this quantization would be as if using your balance to measure the weight of the primary kilogram in Paris before the revision of the SI. With one big difference: the primary kilogram was a unique artifact. By contrast, the integer quantum Hall effect is a universal phenomenon that can be reproduced everywhere with the right equipment. Thus "bootstrapping" universal units for measurements is much easier when artifacts are not involved. This was the motivation behind the 2019 revision of the SI system in the first place.)

8 | Historically, the miracle of the quantized Hall response and its "topological explanation" [17] (→ *below*) kick-started the study of topological phases in the first place:

> **¡! Important**
>
> The *exact quantization* of the (macroscopic) Hall resistivity in *disordered samples* of a 2DEG is a remarkable and unexpected feature that demands for an explanation!

With "exact quantization" one refers to the extraordinary precision to which the experimentally measured Hall resistivity of *different* samples coincides: the relative variations are of order 10^{-10}! A miracle indeed.

1.2. Landau levels

Up to now we used *classical* physics to describe the Hall effect – and we failed to explain the quantization of the Hall resistance. It is time for quantum mechanics to flex it's muscles …

> **¡! Important**
>
> The *integer* quantum Hall effect can be understood in the context of *non-interacting fermions*. Therefore we focus on *single-particle wavefunctions* in the following.
>
> This is *not* true for the ↑ *fractional quantum Hall effect*!

1 | ◁ Same setup as before, but now we qantize the system!

→ Single-particle Hamiltonian of an electron in a magnetic field:

$$H = \frac{1}{2m}\underbrace{(\boldsymbol{p} + e\boldsymbol{A})}_{\boldsymbol{\pi}}{}^2 \tag{1.9}$$

$\boldsymbol{\pi}$: *kinetic* momentum (gauge independent)
$\boldsymbol{p}$: *canonical* momentum (gauge dependent)
$\boldsymbol{A}(\boldsymbol{x})$: gauge potential with $\nabla \times \boldsymbol{A} = B\boldsymbol{e}_z$ (we do not yet fix a gauge!)

2 | Canonical quantization:

$$\left[x_i, p_j\right] = i\hbar\delta_{ij} \tag{1.10}$$

→ $[\pi_x, \pi_y] \mathrel{\hat{=}} -ie\hbar B$

Remember that the (static) gauge potential $\boldsymbol{A}(\boldsymbol{x})$ depends on the position (operator) $\boldsymbol{x}$, and that the canonical momentum (operator) that satisfies Eq. (1.10) is $p_i = -i\hbar\frac{\partial}{\partial x_i}$ (↑ *Stone-von Neumann theorem*).

→ The magnetic field couples the movement in x- and y-direction, so that the kinetic momenta form a pair of conjugate observables.

3 | This immediately suggests the introduction of ↓ *ladder operators*:

$$a := \frac{1}{\sqrt{2e\hbar B}}(\pi_x - i\pi_y) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\pi_x + i\pi_y) \tag{1.11}$$

$\overset{\circ}{\rightarrow}$ These satisfy as usual $[a, a^\dagger] = 1$ and we find with Eq. (1.9) $\overset{\circ}{\rightarrow}$

$$\boxed{H \mathrel{\hat{=}} \hbar\omega_B\left(a^\dagger a + \frac{1}{2}\right)} \tag{1.12}$$

→ Discrete spectrum $E_n = \hbar\omega_B\left(n + \frac{1}{2}\right)$ with $n = 0, 1, 2, \ldots$

→ ⁂ *Landau levels* (LL)

The term "Landau levels" refers to both the quantized *eigenenergies* E_n and the corresponding (degenerate) *eigenspaces* within the single-particle Hilbert space.

4 | Eigenstates? Degeneracy?

Note that we only used *one* degree of freedom (= one harmonic oscillator) although we started with *two* independent degrees of freedom (an electron moving in a 2D plane). The Landau levels must therefore be extensively degenerate to harbor all the needed states! So see this, we must first fix a gauge …

We stress that here the gauge field $\boldsymbol{A}$ is *not* a dynamical degree of freedom (like when you quantize the electromagnetic field). Thus gauge fixing is really just a classical inconvenience and does not lead to fundamental problems like negative norm states etc.

1.2.1. Landau gauge

Here we proceed with the particularly simple *Landau gauge* (which comes with a price: it breaks the rotational symmetry of the problem); the alternative *symmetric gauge* is discussed in Section 1.2.2 → *below*. Since these are gauges, their choice does not affect physical conclusions; however, they lead to different basis states in the Landau levels that paint different (but equivalent) pictures of the physics within them.

5 | ◁ Gauge choice $\boldsymbol{A} := xB\boldsymbol{e}_y$

This gauge breaks translation symmetry in x-direction and rotation symmetry in the plane. This is of course a mathematical artifact: the physics remains completely invariant under these transformations.

Eq. (1.9) → Hamiltonian:

$$H = \frac{1}{2m}\left[p_x^2 + (p_y + eBx)^2\right] \tag{1.13}$$

6 | ◁ Translation symmetry in y-direction

Here we assume either periodic boundaries in y-direction or take the limit $L_y \to \infty$.

→ Ansatz: $\Psi_k(x,y) = e^{iky} f_k(x)$

In Eq. (1.13) → *Shifted* harmonic oscillator:

$$H_k \mathrel{\overset{\circ}{=}} \frac{1}{2m}p_x^2 + \frac{m\omega_B^2}{2}(x + kl_B^2)^2 \tag{1.14}$$

with

$$l_B = \sqrt{\frac{\hbar}{eB}} \quad ⁂ \textit{ magnetic length} \tag{1.15}$$

The magnetic length is the relevant length scale for electrons in a magnetic field (it is the length scale of their cyclotron orbits).

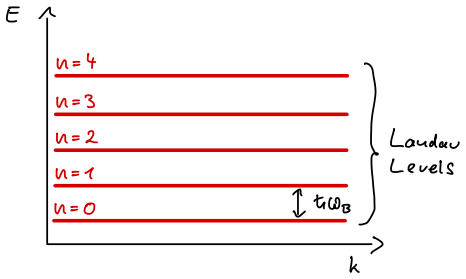
7 | $\overset{\circ}{\to}$ Eigenfunctions: (of H_k for each y-momentum k)

$$\Psi_{n,k}(x,y) = \mathcal{N} \underbrace{e^{iky}}_{\substack{\text{Plane wave}\\ \text{in } y\text{-direction}}} \underbrace{\underbrace{H_n\left(xl_B^{-1} + kl_B\right)}_{\text{Hermite polynomials}} e^{-\frac{1}{2}\left(xl_B^{-1}+kl_B\right)^2}}_{\text{Harmonic oscillator wavefunctions in } x\text{-direction}} \tag{1.16}$$

with $n = 0, 1, 2, \ldots$ the Landau level index and $k = \frac{2\pi}{L_y}\mathbb{Z}$ the y-momentum.

Note that the eigenspaces of H (and the eigenfunctions) are *physical* and therefore gauge independent. What is *unphysical* is the choice of a basis (and the labeling of the basis wavefunctions by "good" quantum numbers). Since the Landau gauge preserves translation symmetry in y-direction, the basis above can be labeled by momenta in y-direction. In other gauges (see below), this is not the case. However, the eigenspaces that are spanned by these wavefunctions are the same for all gauges (of course) and you can linearly combine basis functions of one gauge with basis functions of another.

8 | Spectrum: $E_n = \hbar\omega_B\left(n + \frac{1}{2}\right)$ (degenerate in the k quantum number!)



The Landau levels are prototypes for *perfectly flat bands*. If a LL is only partially filled, the many-body properties of the electrons that occupy this level are determined by their (Coulomb) interactions. This is crucial to understand the long-range entanglement (topological order) of ↑ *fractional quantum Hall states*.

9 | Degeneracy: $0 \le x \le L_x \to$ Restricted y-momenta k: $-L_x/l_B^2 \le k \le 0$

(Since the wavefunctions (1.16) are exponentially localized around $x_k = -kl_B^2$.)

→ Number of states in each Landau level:

$$N = \frac{L_x/l_B^2 - 0}{2\pi/L_y} = \frac{L_xL_y}{2\pi l_B^2} = \frac{AB}{\Phi_0} = \frac{\Phi}{\Phi_0} \tag{1.17}$$

$\Phi_0 = 2\pi\hbar/e$: ⁂ *quantum of flux* (cf. $R_K = 2\pi\hbar/e^2$ the *quantum of resistivity*)

$A = L_xL_y$: area of the sample

→ Extensive degeneracy of each Landau level (as expected)

In particular, the number of electrons N than can be crammed into each Landau level increases with the magnetic flux through the sample (one electron per quantum of flux). This implies that if we fix the electron density and increase the magnetic flux density, fewer and fewer Landau levels will be needed to distribute all electrons, until for very large B all electrons fit into the lowest Landau level (LLL). Conversely, at "every day" weak-field conditions, Landau levels up to very large indices n are occupied by fermions.

1.2.2. ‡ Symmetric gauge

You will do these calculations on ➲ Problemset 2.

In contrast to the ← *Landau gauge*, the *symmetric gauge* breaks translation invariance in *both* directions but retains the two-dimensional rotation invariance of the system. Thus, instead of k, we should expect a basis labeled by *angular momentum quantum numbers* m:

10 | ◁ Gauge choice

$$\boldsymbol{A} := -\frac{1}{2}\boldsymbol{r} \times \boldsymbol{B} = -\frac{yB}{2}\boldsymbol{e}_x + \frac{xB}{2}\boldsymbol{e}_y \tag{1.18}$$

11 | Hamiltonian: [recall Eq. (1.12)]

$$H = \frac{\boldsymbol{\pi}^2}{2m} = \hbar\omega_B \left(a^\dagger a + \frac{1}{2}\right) \tag{1.19}$$

with $a, a^\dagger$ defined via π_x and π_y [recall Eq. (1.11)]

So far, this procedure does not depend on the gauge choice since the kinetic momentum is a gauge independent quantity.

12 | Define additional "momentum": (which does not show up in the Hamiltonian!)

$$\tilde{\boldsymbol{\pi}} := \boldsymbol{p} - e\boldsymbol{A} \quad \Rightarrow \quad \left[\tilde{\pi}_x, \tilde{\pi}_y\right] \doteq ie\hbar B \tag{1.20}$$

(Recall that $\boldsymbol{\pi} = \boldsymbol{p} + e\boldsymbol{A}$.)

Important: In *symmetric gauge* (1.18) the two momenta are independent: $[\pi_i, \tilde{\pi}_j] \doteq 0$

This is not so in other gauges!

13 | → Define additional ladder operators:

$$b := \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x + i\tilde{\pi}_y) \quad \text{and} \quad b^\dagger = \frac{1}{\sqrt{2e\hbar B}}(\tilde{\pi}_x - i\tilde{\pi}_y) \tag{1.21}$$

→ $[b, b^\dagger] = 1$ and $[a, b] = 0$ (The latter is only true in symmetric gauge!)

14 | → Eigenstates:

$$|n, m\rangle := \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n!m!}}|0, 0\rangle \quad \text{with} \quad a|0, 0\rangle = b|0, 0\rangle = 0 \tag{1.22}$$

$n = 0, 1, 2, \ldots$: Landau level index
$m = 0, 1, 2, \ldots$: Angular momentum index (→ *below*)
In symmetric gauge, m replaces the y-momentum k and generates the degeneracy of the LLs.

15 | ◁ Complex coordinates:

The unconventional sign makes the functions below holomorphic instead of antiholomorphic.

$$z := x - iy \quad \text{and} \quad \bar{z} := x + iy \tag{1.23}$$

and the corresponding ↓ *Wirtinger derivatives*

$$\partial := \frac{1}{2}(\partial_x + i\,\partial_y) \quad \text{and} \quad \bar{\partial} := \frac{1}{2}(\partial_x - i\,\partial_y) \tag{1.24}$$

Then $\partial z = \bar{\partial}\bar{z} = 1$ and $\partial\bar{z} = \bar{\partial}z = 0$. A function of complex variables is then holomorphic (= satisfies the Cauchy-Riemann equations) if and only if $\bar{\partial} f = 0$, i.e., $f = f(z)$.

16 | Use $p_i = -i\hbar\partial_i$ & Eqs. (1.18), (1.20), (1.21), (1.23) and (1.24) →

$$a = -i\sqrt{2}\left(l_B\bar{\partial} + \frac{z}{4l_B}\right) \quad \text{and} \quad a^\dagger = -i\sqrt{2}\left(l_B\partial - \frac{\bar{z}}{4l_B}\right) \tag{1.25a}$$

$$b = -i\sqrt{2}\left(l_B\partial + \frac{\bar{z}}{4l_B}\right) \quad \text{and} \quad b^\dagger = -i\sqrt{2}\left(l_B\bar{\partial} - \frac{z}{4l_B}\right) \tag{1.25b}$$

17 | ◁ *Lowest* Landau level wave functions $\Psi_0(z,\bar{z})$:

$$a\Psi_0 = 0 \quad \Leftrightarrow \quad \bar{\partial}\Psi_0 = -\frac{z}{4l_B^2}\Psi_0 \quad \Leftrightarrow \quad \Psi_0(z,\bar{z}) = f(z)e^{-z\bar{z}/4l_B^2} \tag{1.26}$$

$f(z)$: *arbitrary* holomorphic function

18 | ◁ Unique state with $m = 0$: (within the lowest Landau level)

$$b\Psi_0 = 0 \;\Leftrightarrow\; \partial\Psi_0 = -\frac{\bar{z}}{4l_B^2}\Psi_0 \quad \overset{(1.26)}{\Longleftrightarrow} \quad \partial f(z) = 0 \;\Leftrightarrow\; f(z) = \text{const} \tag{1.27}$$

→ $\Psi_{0,0}(z,\bar{z}) \propto e^{-|z|^2/4l_B^2}$ (Gaussian state)

19 | ◁ Other states in the LLL → Apply $b^\dagger$ to $\Psi_{0,0}$: (Remember that $\bar{\partial} z = 0$.)

$$\Psi_{0,m} \propto b^{\dagger m}\Psi_{0,0} \propto \left(l_B\bar{\partial} - \frac{z}{4l_B}\right)^m e^{-z\bar{z}/4l_B^2} \propto \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4l_B^2} \tag{1.28}$$

→ Holomorphic monomials × Gaussian

Since all wave functions $\Psi_{0,m}$ are degenerate, one can form arbitrary linear combinations of these holomorphic monomials (times a Gaussian) to form more general holomorphic polynomials.

20 | → In the LLL, m is an *angular momentum* quantum number:

$$J\Psi_{0,m} = \hbar m\Psi_{0,m} \quad \text{with} \quad \underbrace{J = i\hbar\left(x\partial_y - y\partial_x\right)}_{\text{Angular momentum operator}} = \hbar\left(z\partial - \bar{z}\bar{\partial}\right) \tag{1.29}$$

with $m = 0, 1, 2, \ldots$

Note: In 2D there is only one generator of angular momentum $J = J_z$ and the Lie algebra that generates the rotation group $\mathrm{SO}(2) \simeq U(1)$ (namely $\mathfrak{u}(1) \simeq \mathbb{R}$) is abelian. Consequently, there is no algebraic reason for spin to be quantized (as in 3D where spin can take only half-integer values) and all irreducible representations are one-dimensional. Thus there is only *one* spin quantum number needed (to label the irrep) but none to label distinct basis states of an irrep, i.e., $J = m$. So Eq. (1.29) is all there is to say about spin in this context. Note that the abelian "angular momentum algebra" in 2D has also consequences for particles with anyonic statistics which do not only feature "fractional charges" and "fractional statistic" but also "fractional spin" (→ Part III).

1.3. Berry connection and Berry holonomy

We now take a step back and discuss some rather abstract (and high-level) concepts of quantum mechanics. We return to the integer quantum Hall effect → *later*.

The following concepts are very generic and play a role in many areas of modern physics; they are also important throughout this course. Their application to the quantized Hall conductivity → *below* is only one of many examples.

The following derivation is quite common and leads to physically important (and valid) conclusions. However, it is mathematically not rigorous and uses hidden assumptions on the ↑ *connection* of the full ↑ *Hilbert bundle* on which the parametric family of Hamiltonians is defined, see Ref. [66] for a mathematical treatment of the problem geared towards physicists.

1 | ◁ Setting:

- Continuous family of gapped Hamiltonians $H(\mathbf{\Gamma})$ with k parameters $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \ldots, \Gamma_k)$ and n-fold degenerate ground state space $\mathcal{V}(\mathbf{\Gamma}) \equiv \mathcal{V}(H(\mathbf{\Gamma}))$

  Since $H(\mathbf{\Gamma})$ is continuous and gapped, the dimension of $\mathcal{V}(\mathbf{\Gamma})$ is constant.
  We set $H(\mathbf{\Gamma})|\Psi\rangle = 0$ for $|\Psi\rangle \in \mathcal{V}(\mathbf{\Gamma})$ and all $\mathbf{\Gamma}$, i.e., the ground state energy is zero.

- Slow "parameter path" $\mathbf{\Gamma}(t)$ for $0 \leq t \leq T$

  "Slow" compared to the (inverse) of the smallest energy gap along the path $\mathbf{\Gamma}(t)$.

- Initial ground state $|\Psi_0\rangle \in \mathcal{V}(\mathbf{\Gamma}_0)$

2 | Question: What happens with $|\Psi_0\rangle$ as $H(\mathbf{\Gamma}(0))$ evolves to $H(\mathbf{\Gamma}(T))$?

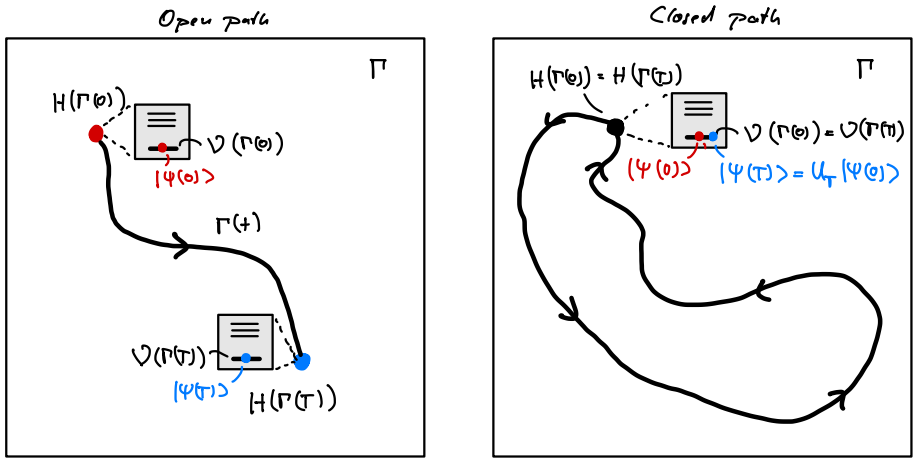
3 | To answer this question, we use the following well-known fact:

> **¡! Important: Adiabatic theorem**
>
> A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum.

This fundamental insight is due to MAX BORN and VLADIMIR FOCK [67].

4 | Solution:

Here is a sketch of the scenario/task that we want to solve:



We proceed step by step:

i | Pick a basis $\{|v_i(\mathbf{\Gamma})\rangle\}_{i=1,\ldots,n}$ of $\mathcal{V}(\mathbf{\Gamma})$ for every $\mathbf{\Gamma}$

This choice is not unique and leads to a $\mathrm{U}(n)$ gauge degree of freedom (→ *below*). Here we assume that the choice is differentiable (and therefore continuous) along the path $\mathbf{\Gamma}$. This makes it less arbitrary but leaves a lot arbitrariness to choose from. Note that a choice that is globally continuous is often impossible. Then one follows the arguments below on local patches in parameter space on which such a choice is possible.

ii | ◁ Time-dependent Schrödinger equation:

$$\underbrace{i\hbar\partial_t|\Psi(t)\rangle}_{\text{(L)}} = \underbrace{H(\mathbf{\Gamma}(t))|\Psi(t)\rangle}_{\text{(R)}} \tag{1.30}$$

↓ Lecture 5 [25.04.25]

iii | ◁ Adiabatic theorem

→ Initial state remains in the ground state manifold: $|\Psi(t)\rangle \in \mathcal{V}(\mathbf{\Gamma}(t)) \rightarrow$

(L) $|\Psi(t)\rangle = \sum_{i=1}^{n} \Psi_i(t)|v_i(\mathbf{\Gamma}(t))\rangle \rightarrow$

$$\partial_t|\Psi(t)\rangle = (\partial_t\Psi_i(t))\,|v_i(\mathbf{\Gamma}(t))\rangle + \Psi_i(t)\,[\partial_{\Gamma_l}|v_i(\mathbf{\Gamma}(t))\rangle]\,(\partial_t\Gamma_l(t)) \tag{1.31}$$

We omit sum symbols; sums over repeated indices are implied (Einstein notation).

(R) $H(\mathbf{\Gamma}(t))|\Psi(t)\rangle = 0$ (Remember that we set the ground state energy to zero.)

This assumption is not crucial for the derivation that follows; it simply removes any dynamical phase from the evolution, so that only a geometric phase remains (which is what we are interested in). If you do not set the ground state energy to zero, use $H(\mathbf{\Gamma}(t))|\Psi(t)\rangle = E_0(\mathbf{\Gamma}(t))|\Psi(t)\rangle$ instead and track the additional term. Its effect is to add an additional, energy-dependent dynamical phase to the evolution of the wave function (which is not a new & interesting insight …).

iv | Apply $\langle v_j(\mathbf{\Gamma}(t))|$ and use Eq. (1.30):

$$\partial_t\Psi_j(t) = -\Psi_i(t)\,\langle v_j(\mathbf{\Gamma}(t))|\partial_{\Gamma_l}|v_i(\mathbf{\Gamma}(t))\rangle\,(\partial_t\Gamma_l(t)) \tag{1.32}$$

v | This suggests the definition of the

$$\text{⁂ } \textit{Berry connection} \quad [\mathcal{A}_l(\mathbf{\Gamma})]_{ji} := -i\,\langle v_j(\mathbf{\Gamma})|\partial_{\Gamma_l}|v_i(\mathbf{\Gamma})\rangle \quad \in \mathfrak{u}(n) \tag{1.33}$$

Think of the $\mathcal{A}_l$ as $\mathbf{\Gamma}$-dependent Hermitian $n \times n$-matrices, one for each of the $l = 1, \ldots, k$ parameters.

vi | With this definition, we can write [$\mathbf{\Psi} \equiv (\Psi_j)_{j=1,\ldots,n}$]

$$\partial_t\mathbf{\Psi}(t) = -i\,\underbrace{(\partial_t\Gamma_l(t))\,\mathcal{A}_l(\mathbf{\Gamma}(t))}_{\text{Time-dependent matrix}}\,\mathbf{\Psi}(t) \tag{1.34}$$

vii | This equation can be solved with a ↓ *Time- ($\mathcal{T}$) or path-ordered ($\mathcal{P}$) exponential*:

$$\mathbf{\Psi}(T) = \mathcal{T}\exp\left[-i\int_0^T \mathcal{A}_l(\mathbf{\Gamma}(t))\,\partial_t\Gamma_l(t)\,\mathrm{d}t\right]\mathbf{\Psi}_0 \tag{1.35a}$$

$$= \underbrace{\mathcal{P}\exp\left[-i\int_\Gamma \boldsymbol{\mathcal{A}}\mathrm{d}\mathbf{\Gamma}\right]}_{\equiv U_\Gamma\ \text{(Unitary matrix)}}\mathbf{\Psi}_0 \tag{1.35b}$$

Here, $\boldsymbol{\mathcal{A}} = (\mathcal{A}_l)$ should be seen as a $\mathfrak{u}(n)$-valued vector field on the parameter space (a 1-form). I.e., $\boldsymbol{\mathcal{A}}$ can be integrated along parameter paths which, after (path ordered) exponentiation, produces a unitary $\mathrm{U}(n)$ that describes the geometric part of the adiabatic evolution on the ground state space.

5 | ◁ Change of local basis by unitary $\Omega(\mathbf{\Gamma}) \in \mathrm{U}(n)$: $|v_i'(\mathbf{\Gamma})\rangle := \Omega_{ij}(\mathbf{\Gamma})|v_j(\mathbf{\Gamma})\rangle$

Note that the choice of basis is a gauge choice: it cannot have physical significance!

$$\overset{\circ}{\to} \quad \mathcal{A}_l' = \Omega \mathcal{A}_l \Omega^\dagger - i \frac{\partial \Omega}{\partial \Gamma_l} \Omega^\dagger \tag{1.36}$$

If you attended a course on quantum field theory, you might recognize this as the gauge transformation of a non-abelian $\mathrm{U}(n)/\mathrm{SU}(n)$ Yang-Mills gauge theory (like QCD). The difference is that here the gauge (Berry) connection $\mathcal{A}_l$ does not live on Minkowski spacetime but on an abstract "parameter space." Gauge transformations arise from "parameter-local" basis transformations in the degenerate ground state space of a Hamiltonian (family).

$$\overset{\circ}{\to} \quad U_\Gamma' = \Omega(\mathbf{\Gamma}(T))\, U_\Gamma\, \Omega^\dagger(\mathbf{\Gamma}(0)) \tag{1.37}$$

To show this, consider an infinitesimal piece $\mathrm{d}\mathbf{\Gamma}$ of the path Γ and linearize U_Γ along this piece to derive the above transformation. Then use that the path-ordered exponential is defined as the product of such infinitesimal pieces. The identity $\Omega \frac{\partial \Omega^\dagger}{\partial \Gamma_l} = -\frac{\partial \Omega}{\partial \Gamma_l} \Omega^\dagger$ might help (prove this!).

6 | ◁ Open path $\Gamma \to U_\Gamma$ is gauge dependent → Cannot contain physical information!

To see this let $\Omega(\mathbf{\Gamma}(0)) = \mathbb{1}$. Then $U_\Gamma' = \Omega(\mathbf{\Gamma}(T))\, U_\Gamma$ can be chosen (almost) *arbitrary* since $\mathrm{U}(n)$ is a group and $\Omega(\mathbf{\Gamma}(T))$ can be chosen (almost) arbitrary (just connect it smoothly to the identity, i.e., its determinant must be one). This means that U_Γ cannot contain physical information as it can be transformed into any other unitary U_Γ' (with the same determinant) by parameter-local basis transformations.

→ ◁ Closed loops Γ in parameter space

I.e., $H(\mathbf{\Gamma}(0)) = H(\mathbf{\Gamma}(T))$ and $\mathcal{V}(\mathbf{\Gamma}(0)) = \mathcal{V}(\mathbf{\Gamma}(T))$ such that U_Γ is an *automorphism* on $\mathcal{V}(\mathbf{\Gamma}(0))$ and described the geometric transformation of ground states due to cyclic (and adiabatic) deformations of the Hamiltonian.

7 | Then the

$$\text{⁂ } \textit{Berry holonomy} \quad U_\Gamma = \mathcal{P} \exp\left[-i \oint_\Gamma \boldsymbol{\mathcal{A}} \mathrm{d}\mathbf{\Gamma}\right] \quad \in \mathrm{U}(n) \tag{1.38}$$

is gauge covariant: [This follows from the continuity of $\Omega(\mathbf{\Gamma})$ and $\mathbf{\Gamma}(T) = \mathbf{\Gamma}(0)$.]

$$U_\Gamma' = \Omega(\mathbf{\Gamma}(0))\, U_\Gamma\, \Omega^\dagger(\mathbf{\Gamma}(0)) \tag{1.39}$$

Note that the argument from above breaks down since both unitaries $\Omega(\mathbf{\Gamma}(T)) = \Omega(\mathbf{\Gamma}(0))$ are necessarily the same (since the parameter path is closed). U_Γ can still be changed, but not arbitrarily: It is unique up to unitary basis transformations (for instance, its spectrum is independent of basis changes!). This quantity *can* encode physical properties of the system. Note the difference between gauge *invariant* ($U_\Gamma' = U_\Gamma$) and gauge *covariant* [Eq. (1.39)].

8 | There is another important *gauge covariant* quantity (that we will use → *below*):

$$\text{⁂ } \textit{Berry curvature} \quad \mathcal{F}_{lm} := \frac{\partial \mathcal{A}_l}{\partial \Gamma_m} - \frac{\partial \mathcal{A}_m}{\partial \Gamma_l} - i\,[\mathcal{A}_l, \mathcal{A}_m] \quad \in \mathfrak{u}(n) \tag{1.40}$$

This is the "field-strength" of the gauge field $\boldsymbol{\mathcal{A}}$, the non-abelian generalization of the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ in electrodynamics (where $A_\mu \in \mathfrak{u}(1) \simeq \mathbb{R}$ so that the commutator vanishes identically).

→ $\mathcal{F}_{lm}$ is gauge covariant:

$$\mathcal{F}'_{ij}(\boldsymbol{\Gamma}) \triangleq \Omega(\boldsymbol{\Gamma})\, \mathcal{F}_{ij}(\boldsymbol{\Gamma})\, \Omega^\dagger(\boldsymbol{\Gamma}) \tag{1.41}$$

Notes:

- This is the field strength tensor known from ↑ *non-abelian Yang-Mills gauge theories*. The Yang-Mills Lagrangian takes the trace of the field strength tensor, thereby converting a gauge *covariant* quantity into a gauge *invariant* quantity: $\mathrm{Tr}[F_{\mu\nu} F^{\mu\nu}]$. (Note that the summation over μ and ν is not related to gauge but Lorentz invariance for YM theories; as we do not have generic symmetries on the *parameter space*, we do not have an analog of this symmetry in the current situation.)
- If Eq. (1.40) seems abstract but you know about ↓ *general relativity*, there is some insightful connection (☺) to be drawn. Remember that the ↓ *Riemann curvature tensor* can be expressed as [68, Section 10.2.3]

  $$R^i{}_{jlm} = \partial_l \Gamma^i{}_{jm} - \partial_m \Gamma^i{}_{jl} + \Gamma^i{}_{nl}\, \Gamma^n{}_{jm} - \Gamma^i{}_{nm}\, \Gamma^n{}_{jl} \tag{1.42}$$

  in terms of ↓ *Christoffel symbols* $\Gamma^i{}_{jm}$, which are the (coordinate-dependent) connection coefficients of the (metric-induced) ↓ *Levi-Civita connection* on the spacetime manifold. Let us interpret the first two indices of the Christoffel symbols as indices of a $D \times D$ matrix (where D is the spacetime dimension), $[\boldsymbol{\Gamma}_m]_{ij} \equiv \Gamma^i{}_{jm}$, and do the same for the Riemann curvature tensor: $[\boldsymbol{R}_{lm}]_{ij} \equiv R^i{}_{jlm}$. In this notation, Eq. (1.42) reads

  $$\boldsymbol{R}_{lm} = \partial_l \boldsymbol{\Gamma}_m - \partial_m \boldsymbol{\Gamma}_l + \boldsymbol{\Gamma}_l \boldsymbol{\Gamma}_m - \boldsymbol{\Gamma}_m \boldsymbol{\Gamma}_l = \partial_l \boldsymbol{\Gamma}_m - \partial_m \boldsymbol{\Gamma}_l - [\boldsymbol{\Gamma}_m, \boldsymbol{\Gamma}_l] \tag{1.43}$$

  which is (up to prefactors) completely analogous to Eq. (1.40). This explains why the Berry curvature is called "curvature": it describes a generalized (and rather abstract) curvature of the vector bundle defined by the ground state spaces $\mathcal{V}(\boldsymbol{\Gamma})$ on the parameter manifold $\mathcal{M}$.

  Note that in general relativity, the vector space at each point of the spacetime manifold is given by the ↓ *tangent space* – which has the same dimension as the manifold itself. This is why it is covenient to treat all four indices of the Riemann tensor on the same footing. In our context, the parameter manifold is k-dimensional and has nothing to do with the attached ground state spaces $\mathcal{V}(\boldsymbol{\Gamma})$ that are n-dimensional. Hence we prefer the matrix notation in Eq. (1.40) where the indices that correspond to the Hilbert space are suppressed.

1.3.1. Berry phase and Chern number

We now want to focus on the important special case w/o degeneracy ($n = 1$). In this case, we can make use of the Berry curvature to calculate the Berry holonomy (which is for $n = 1$ just a phase known as → *Berry phase*):

9 | ◁ Special case $n = 1$: $\mathcal{V}(\boldsymbol{\Gamma}) = \mathrm{span}\,\{|v(\boldsymbol{\Gamma})\rangle\}$ (= systems w/o ground state degeneracy)

In this special case, the quantities introduced above simplify as follows:

$$\text{Berry connection:} \quad \mathcal{A}_l(\boldsymbol{\Gamma}) = -i \langle v(\boldsymbol{\Gamma}) | \partial_{\Gamma_l} | v(\boldsymbol{\Gamma}) \rangle \qquad \in \mathfrak{u}(1) \simeq \mathbb{R} \tag{1.44a}$$

$$\text{Berry holonomy:} \quad U_\Gamma = \exp\left[-i \oint_\Gamma \boldsymbol{\mathcal{A}} \mathrm{d}\boldsymbol{\Gamma} \right] \equiv e^{i\gamma(\Gamma)} \qquad \in \mathrm{U}(1) \tag{1.44b}$$

$$\text{Berry curvature:} \quad \mathcal{F}_{lm} = \frac{\partial \mathcal{A}_l}{\partial \Gamma_m} - \frac{\partial \mathcal{A}_m}{\partial \Gamma_l} \qquad \in \mathfrak{u}(1) \simeq \mathbb{R} \tag{1.44c}$$

→ Ground state can only change by a phase!

10 | Gauge transformation: $\Omega(\mathbf{\Gamma}) = e^{i\xi(\mathbf{\Gamma})} \rightarrow$

The gauge transformation of the Berry connection is similar to electrodynamics:

$$\boldsymbol{\mathcal{A}}' = \boldsymbol{\mathcal{A}} + \nabla_{\mathbf{\Gamma}}\xi \tag{1.45a}$$

$$U'_\Gamma = U_\Gamma \qquad \text{(gauge } \textit{invariant}\text{)} \tag{1.45b}$$

$$\mathcal{F}'_{lm} = \mathcal{F}_{lm} \qquad \text{(gauge } \textit{invariant}\text{)} \tag{1.45c}$$

11 | This motivates the following definition:

> ⁂ **Definition: Berry phase**
>
> For $n = 1$, the exponent of the Berry holonomy is called ⁂ *Berry phase*:
>
> $$\gamma(\Gamma) = -\oint_\Gamma \boldsymbol{\mathcal{A}}\mathrm{d}\mathbf{\Gamma} = i\oint_\Gamma \langle v(\mathbf{\Gamma})|\partial_{\Gamma_l}|v(\mathbf{\Gamma})\rangle\,\mathrm{d}\Gamma_l \quad \in \mathbb{R} \tag{1.46}$$
>
> The nomenclature is sometimes a bit vague: $\gamma(\Gamma)$ and $e^{i\gamma(\Gamma)}$ are both called "Berry phase."

- The Berry phase is a ⁂ *geometric phase* – as compared to the usual ↓ *dynamical phases* accumulated by wave functions in quantum mechanics. Remember that an eigenstate with energy E collects the phase $e^{-\frac{i}{\hbar}E\,\Delta t}$ in the time interval Δt due to the unitary evolution governed by the Schrödinger equation. Such phases are called *dynamical phases*. By contrast, the Berry phase is *not* a consequence of the energy of the system (recall that we set the ground state energy to zero for all parameters!); it is rather a *geometric property* of the parametric path Γ over the ↑ *vector bundle* $\mathcal{V}$ of ground state spaces.
- The Berry phase was first discussed by Michael Berry in 1984 [69].
- The Berry phase follows from the Berry connection. But where does the Berry connection "come from"? It seems that it is somehow hidden in the Hamiltonian family $H(\mathbf{\Gamma})$, but this can only be partially true as the latter only defines a projector onto its ground state manifold. This provides us with the Hilbert sub-bundle $\mathcal{V}(\mathbf{\Gamma})$ on which the Berry connection is defined. But a projection does not magically produce a connection. Actually, we start from the full Hilbert bundle (its fibers are the Hilbert spaces on which the Hamiltonians act) und (silently) assume that it is trivialized $\mathcal{M} \times \mathcal{H}_0$ with some reference Hilbert space $\mathcal{H}_0$. A trivialized bundle has a natural connection, namely the trivial (or constant) connection. Starting from this connection, the ground state projection provided by a Hamiltonian then induces a connection on the sub-bundle $\mathcal{V}(\mathbf{\Gamma})$ – and this is the Berry connection. If there is no canonical (or physically motivated) trivialization of the full Hilbert bundle, the choice of the connection on this bundle leads to potentially distinct Berry connections and thereby distinct Berry phases; for details on this subtlety see Ref. [66].

12 | Examples of systems with non-trivial Berry phase:

- Spin-$\frac{1}{2}$ in a variable magnetic field (➲ Problemset 2 and ↑ Ref. [69])
- Aharonov-Bohm effect (↑ [69])
- Focault pendulum (↑ [70, 71])

  The concept of parallel transport with non-trivial holonomies is not restricted to quantum mechanical systems!

13 | ◁ Effect of gauge transformations on the Berry phase:

$$\gamma'(\Gamma) = -\oint_\Gamma \mathcal{A}' \mathrm{d}\mathbf{\Gamma} = -\oint_\Gamma (\mathcal{A} + \nabla_{\mathbf{\Gamma}} \xi) \mathrm{d}\mathbf{\Gamma} = \gamma(\Gamma) - [\xi(\mathbf{\Gamma}(T)) - \xi(\mathbf{\Gamma}(0))] \tag{1.47}$$

Note that here $\xi(\mathbf{\Gamma}(T))$ should be read as $\lim_{\varepsilon\to 0} \xi(\mathbf{\Gamma}(T-\varepsilon))$ and $\xi(\mathbf{\Gamma}(0))$ is shorthand for $\lim_{\varepsilon\to 0} \xi(\mathbf{\Gamma}(0+\varepsilon))$, which explains why Eq. (1.48) below makes sense even though $\mathbf{\Gamma}(T) = \mathbf{\Gamma}(0)$.

Continuity of the gauge transformation: $\Omega(\mathbf{\Gamma}(0)) = \Omega(\mathbf{\Gamma}(T)) \to$

Recall that Γ is a closed path: $\mathbf{\Gamma}(T) = \mathbf{\Gamma}(0)$. Note that continuity of the gauge transformation $e^{i\xi(\mathbf{\Gamma}(0))} = \Omega(\mathbf{\Gamma}(0)) = \Omega(\mathbf{\Gamma}(T)) = e^{i\xi(\mathbf{\Gamma}(T))}$ does not imply continuity of $\xi(\mathbf{\Gamma})$!

$$\text{Eq. (1.45b)} \quad \Rightarrow \quad \xi(\mathbf{\Gamma}(T)) - \xi(\mathbf{\Gamma}(0)) = 2\pi m \quad \text{for} \quad m \in \mathbb{Z} \tag{1.48}$$

$\to \gamma(\Gamma)$ is gauge invariant *up multiples of* 2π

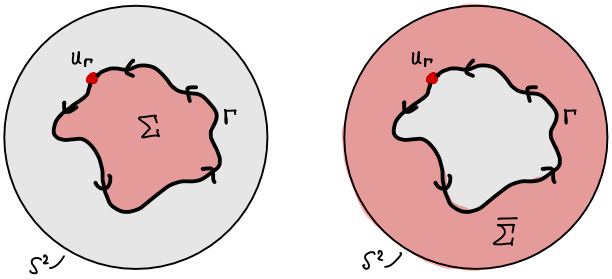
$\to$ For $\gamma(\Gamma) \notin 2\pi\mathbb{Z}$, the Berry phase *cannot* be gauged away and can have physical consequences!

14 | ◁ Special case k=2: $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2)$

This is the most important case for us because the parameter space we are interested in will be the 2D ↓ *Brillouin zone* (which is a torus).

→ Computation of the Berry phase for $k = 2$ on a compact manifold $\mathcal{M}$ (sphere, torus):

i | ◁ Closed path Γ on sphere $\mathcal{M} = S^2$
◁ Submanifolds with $\Sigma \cup \bar{\Sigma} = \mathcal{M}$ and $\partial\Sigma = \Gamma = \partial\bar{\Sigma}$:



> **¡! Important**
>
> In general it is *not* possible to choose a gauge that is continuous (= non-singular) everywhere on $\mathcal{M}$!

Hence we must be careful when integrating the Berry connection $\mathcal{A}$ along paths on $\mathcal{M}$! In the following, we assume that we *can* find continuous gauges for every simply connected, open submanifold of $\mathcal{M}$ though:

ii | ◁ Continuous gauge $\mathcal{A}_1$ on Σ → Stokes' theorem valid on Σ →

$$\oint_\Gamma \mathcal{A}_1 \mathrm{d}\mathbf{\Gamma} \overset{\text{Stokes}}{=} \int_\Sigma \mathcal{F}_{lm} \mathrm{d}\sigma^{lm} \tag{1.49}$$

σ^{lm} is the differential area element (a 2-form that is antisymmetric in l and m, just as $\mathcal{F}_{lm}$).

For a reformulation in terms of differential forms see the comments → *below*.

iii | ◁ Continuous gauge $\boldsymbol{\mathcal{A}}_2$ on $\bar{\Sigma}$ → Stokes' theorem valid on $\bar{\Sigma}$ →

$$\oint_\Gamma \boldsymbol{\mathcal{A}}_2 \mathrm{d}\boldsymbol{\Gamma} \overset{\text{Stokes}}{=} -\int_{\bar{\Sigma}} \mathcal{F}_{lm} \mathrm{d}\sigma^{lm} \tag{1.50}$$

The sign is due to the opposite orientation of the boundary for $\bar{\Sigma}$.

iv | Using Eq. (1.48) ∧ Eq. (1.49) ∧ Eq. (1.50) →

$$\int_{\mathcal{M}} \mathcal{F}_{lm} \mathrm{d}\sigma^{lm} = \underbrace{\oint_\Gamma \boldsymbol{\mathcal{A}}_1 \mathrm{d}\boldsymbol{\Gamma}}_{\gamma(\Gamma)+2\pi m_1} - \underbrace{\oint_\Gamma \boldsymbol{\mathcal{A}}_2 \mathrm{d}\boldsymbol{\Gamma}}_{\gamma(\Gamma)+2\pi m_2} = 2\pi m \quad \text{with} \quad m \in \mathbb{Z} \tag{1.51}$$

Here we used that the closed loop integrals of the Berry connection are unique up to integer multiples of 2π.

15 | This motivates the following definition:

> ⁂ **Definition: Chern number**
>
> For a compact, closed two-dimensional parameter space $\mathcal{M}$ with Berry curvature $\mathcal{F}$, the ⁂ *(first) Chern number* is an integer and defined as
>
> $$C := \frac{1}{2\pi} \int_{\mathcal{M}} \mathcal{F}_{lm} \mathrm{d}\sigma^{lm} \quad \in \mathbb{Z} \tag{1.52}$$
>
> This is our first example of a *topological invariant*.

- We will meet the Chern number again in Section 1.4 where we compute the Hall conductivity.
- ¡! Following the argument above, it is clear that whenever there exists a gauge that is non-singular on the *complete* parameter space, the Chern number is necessarily zero. [Because you can then choose $\boldsymbol{\mathcal{A}}_1 = \boldsymbol{\mathcal{A}}_2$ such that the difference in Eq. (1.51) vanishes.] Conversely, whenever the Chern number does *not* vanish, there must be singularities in all gauges! You will encounter an example of this in ➔ Problemset 2.

16 | ‡ Comments:

- Differential forms:

  The proper way to formulate the application of Stokes' theorem is in terms of *differential forms*. In this framework

  $$\mathcal{A} := \sum_{l=1}^{k} \mathcal{A}_l \mathrm{d}\Gamma_l \tag{1.53}$$

  is a *1-form* that can be integrated along paths:

  $$\gamma(\Gamma) = -\oint_\Gamma \mathcal{A} \,. \tag{1.54}$$

  The Berry curvature is then the *2-form* given by the *exterior derivative* of $\mathcal{A}$ (this is only true for $n = 1$, i.e., abelian gauge fields):

  $$\mathcal{F} := \mathrm{d}\mathcal{A} = \sum_{1 \le l,m \le k} \underbrace{(\partial_m \mathcal{A}_l - \partial_l \mathcal{A}_m)}_{\mathcal{F}_{lm}} \underbrace{\frac{1}{2} \mathrm{d}\Gamma_m \wedge \mathrm{d}\Gamma_l}_{\mathrm{d}\sigma^{lm}} = \mathcal{F}_{lm} \mathrm{d}\sigma^{lm} \tag{1.55}$$

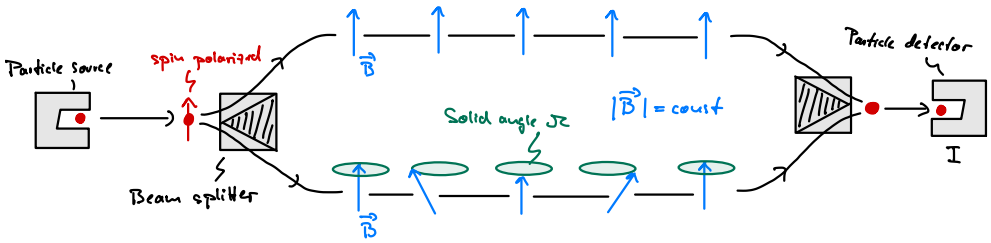
where the last expression is just a shorthand notation. (For *non-abelian* gauge fields it is $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$; note that $\mathcal{A}$ is a 1-form with values in a non-abelian Lie algebra so that the wedge product does not vanish in general.)

Finally, Stoke's theorem for differential forms states that

$$\oint_{\Gamma=\partial\Sigma} \mathcal{A} = \int_\Sigma \mathrm{d}\mathcal{A} = \int_\Sigma \mathcal{F} \,. \tag{1.56}$$

- Observation of the Berry phase:

  ◁ Spin-polarized particles on beam splitter in magnetic field with constant amplitude:

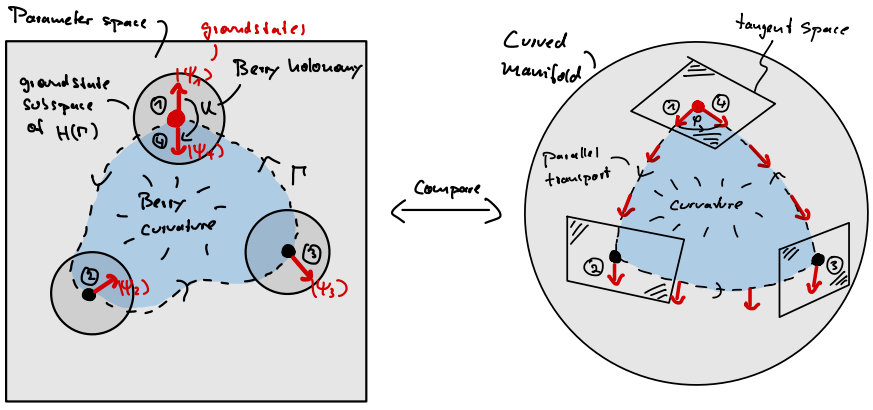


  → Interference pattern: $I = |1 + e^{i\gamma(\Gamma)}|^2$ where $e^{i\gamma(\Gamma)} = e^{i\Omega/2}$ with solid angle $0 \leq \Omega \leq 4\pi$. You will calculate the dependency of the Berry phase on the solid angle traced out by the magnetic field in ➲ Problemset 2. This experiment was already proposed and studied by Berry in his original work [69].

  To the best of my knowledge, there has been no experiment that implemented exactly Berry's proposal (due to experimental issues controlling additional dynamical phases). However, there have been multiple other experimental verifications of the Berry phase in quantum systems since its prediction in 1984 [72, 73]. (Note that the historically first reporting [74] was later disputed [75] because it can be explained classically, without invoking quantum mechanics.)

- Geometric interpretation of the Berry curvature:

  In general, the parameter space can be multi-dimensional. For obvious reasons we only draw two of them:



  The Berry holonomy can be compared to the rotation of a vector when carried ("parallel transported") around a closed curve on a curved space (like the shown sphere). The analog to the ↓ *Riemann curvature* is the Berry curvature, the role of the ↓ *Levi-Civita connection* is played by the Berry connection. The Chern number equals the ↑ *Euler characteristic* of a compact 2D manifold, and the relation that gives the Chern number in terms of the

Berry curvature is then known as ↑ *Gauss-Bonnet theorem* (more precisely: ↑ *Chern-Gauss-Bonnet theorem*, a generalization of the classic Gauss-Bonnet theorem to even-dimensional Riemannian manifolds). This "real space analog" may be known from your lectures on ↓ *general relativity*. Note that in general relativity one is interested in the ↑ *tangent bundle* where a tangential space is attached to every point of the (spacetime) manifold. Here we are *not* interested in the tangent bundle of the parameter manifold but more general ↑ *fiber bundles* where the local fibers are given by ground state spaces $\mathcal{V}(\mathbf{\Gamma})$ or Lie groups $U(n)$ that act on them.

1.4. Quantization of the Hall conductivity

With these new mathematical insights, we now return to the integer quantum Hall effect and its Hall plateaus. Our goal is to find a relation between the Hall conductivity and the Chern number. This remarkable relation between a *physical quantity* and a *topological invariant* is one of the most important insights in contemporary condensed matter physics and explains the quantization of the Hall conductivity.

The following discussion is based on David Tong's lecture notes on the quantum Hall effect [64]. For a more detailed (and much more technical) discussion, see Chapter 3 of Bernevig's textbook [1]; another account can be found in Chapter 12 of Fradkin's textbook [63]. You might also want to have a look at the original manuscript by Thouless *et al.* [17] and the follow-up [76].

1.4.1. The Kubo formula

As a preparation, we compute the linear response of a quantum mechanical system at $T = 0$ for a time-dependent, external perturbation. Here we focus on the special case where the perturbation is a time-dependent electric field and the response is a current of charged particles. The approach is generic and valid for general (in particular: interacting) Hamiltonians. The resulting → *Kubo formula* has many applications beyond computing the quantized Hall conductivity.

1 | ◁ Unperturbed Hamiltonian H_0 with Eigenstates $|m\rangle$ and Eigenenergies E_m

◁ Time-dependent perturbation $\Delta H(t)$

→ $H(t) = H_0 + \Delta H(t)$ (Schrödinger picture!)

2 | It is convenient to absorb the unperturbed time evolution into operators:

→ ↓ *Interaction picture:*

$$\Delta H_I(t) := U_0^\dagger(t)\Delta H(t)U_0(t) \quad \text{and} \quad |\Psi(t)\rangle_I := U(t,t_0)|\Psi(t_0)\rangle_I \tag{1.57}$$

with unperturbed time evolution operator $U_0(t) := e^{-\frac{i}{\hbar}H_0 t}$ and

$$U(t,t_0) := \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\Delta H_I(t')\mathrm{d}t'\right] \tag{1.58}$$

Here $\mathcal{T}$ denotes the time-ordered exponential. It is easy to check that the states $|\Psi(t)\rangle_I$ satisfy the Schrödinger equation in the interaction picture:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle_I = \Delta H_I(t)|\Psi(t)\rangle_I\,. \tag{1.59}$$

To show the unitary equivalence between the interaction picture and the conventional Schrödinger picture, you must show that $U(t,t_0) \triangleq U_0^\dagger(t-t_0)U_S(t,t_0)$ with the full Schrödinger evolution

$$U_S(t,t_0) := \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} H(t')\mathrm{d}t'\right]. \tag{1.60}$$

3 | Prepare system for $t_0 \to -\infty$ in ground state $|0\rangle$ of H_0 (or some other eigenstate)

4 | ◁ Expectation value of arbitrary (interaction picture) operator $\mathcal{O}_I(t) = U_0^\dagger \mathcal{O} U_0$:

$$\langle \mathcal{O}(t)\rangle = \underbrace{\langle 0|U_S^\dagger(t,-\infty)\mathcal{O}U_S(t,-\infty)|0\rangle}_{\text{Schrödinger picture}} \tag{1.61a}$$

$$= \underbrace{\langle 0|U^\dagger(t,-\infty)\mathcal{O}_I(t)U(t,-\infty)|0\rangle}_{\text{Interaction picture}} \tag{1.61b}$$

$$\overset{1.58}{\approx} \langle 0|\left\{\mathcal{O}_I(t) + \frac{i}{\hbar}\int_{-\infty}^{t}\left[\Delta H_I(t'),\mathcal{O}_I(t)\right]\mathrm{d}t'\right\}|0\rangle \tag{1.61c}$$

This linearization is the core of *linear response* theory.

Note that time ordering is not important in linear order (only one time integral!).

→

⁂ *Kubo formula:*

$$\delta\langle\mathcal{O}(t)\rangle \equiv \langle\mathcal{O}(t)\rangle - \langle\mathcal{O}\rangle = \frac{i}{\hbar}\int_{-\infty}^{t}\langle 0|\left[\Delta H_I(t'),\mathcal{O}_I(t)\right]|0\rangle\,\mathrm{d}t' \tag{1.62}$$

- This is the linear response of the system to the perturbation $\Delta H(t)$. Note that $\langle\mathcal{O}\rangle = \langle 0|\mathcal{O}|0\rangle = \langle 0|\mathcal{O}_I(t)|0\rangle$ is not a dynamic response but the static expectation value of $\mathcal{O}$ in the initial state (remember that $|0\rangle$ is a eigenstate of H_0). In the following, we will set it to zero.
- The Kubo formula was first presented by Ryogo Kubo in 1957 [77].

↓ Lecture 6 [02.05.25]

5 | ◁ Special case: Coupling to uniform electric field $\boldsymbol{E}(t) = \boldsymbol{E} e^{-i\omega t}$

i | Choose gauge such that $\boldsymbol{E}(t) = -\partial_t \boldsymbol{A}(t)$ (i.e. $A_t = \phi = \text{const}$)

Remember that in general $\boldsymbol{E} = -\nabla\phi - \partial_t \boldsymbol{A}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

$\rightarrow \boldsymbol{A}(t) = \boldsymbol{E} e^{-i\omega t}/(i\omega)$

ii | ◁ Perturbation Hamiltonian:

$$\Delta H_I(t) = -\boldsymbol{J}(t) \cdot \boldsymbol{A}(t) \tag{1.63}$$

with (total) current operator $\boldsymbol{J}(t)$

- At this point we do not want to fix the unperturbed Hamiltonian H_0 that describes the charge carriers without the field. Hence we do not know the form of $\boldsymbol{J}(t)$ in the interaction picture. We therefore play it safe and carry a potential time-dependence along.
- This is a linearized version of the true coupling Hamiltonian that describes the effect of the electromagnetic field on electrical charges. For instance, a free particle with charge q (and with $\phi = \text{const} = 0$) is described by the Hamiltonian

$$H = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 = \underbrace{\frac{\boldsymbol{p}^2}{2m}}_{\sim H_0} \underbrace{- \overbrace{\frac{q\boldsymbol{p}}{m}}^{\sim \boldsymbol{J}} \cdot \boldsymbol{A}}_{\sim \Delta H(t)} + \cancel{\mathcal{O}(\boldsymbol{A}^2)}. \tag{1.64}$$

There is also a quadratic term $\boldsymbol{A}^2$ which does not contribute to the Hall conductance (so we can safely drop it).

- In therms of the ↓ *current density* $\boldsymbol{j}(\boldsymbol{r}, t)$ the Hamiltonian reads

$$\Delta H_I(t) = -\int \mathrm{d}^2 r\, \boldsymbol{j}(\boldsymbol{r}, t) \cdot \boldsymbol{A}(\boldsymbol{r}, t) \tag{1.65}$$

with the usual current density $\boldsymbol{j} = \frac{q}{2m}\sum_i [\boldsymbol{p}_i \delta(\boldsymbol{r} - \boldsymbol{r}_i) + \delta(\boldsymbol{r} - \boldsymbol{r}_i)\boldsymbol{p}_i]$ for many particles indexed by i. With a homogeneous electric field, this becomes

$$\Delta H_I(t) = -\boldsymbol{J}(t) \cdot \boldsymbol{A}(t) \quad \text{with total current} \quad \boldsymbol{J}(t) = \int \mathrm{d}^2 r\, \boldsymbol{j}(\boldsymbol{r}, t). \tag{1.66}$$

For a homogeneous current, the total current is $\boldsymbol{J} = L_x L_y\, \boldsymbol{j} = A\boldsymbol{j}$ where $A = L_x L_y$ denotes the area of the sample.

iii | ◁ Current as observable: $\mathcal{O} = J_i \rightarrow$

(Remember that we set the static expectation value to zero: $\langle 0|J_i|0\rangle = 0$.)

$$\langle J_i(t)\rangle \overset{1.62}{=} -\frac{1}{\hbar\omega}\int_{-\infty}^{t} \langle 0|\left[J_j(t'), J_i(t)\right]|0\rangle E_j e^{-i\omega t'}\,\mathrm{d}t' \tag{1.67a}$$

Time-translation invariance of H_0; Substitution $t'' = t - t'$

$$\doteq \underbrace{\left\{-\frac{1}{\hbar\omega}\int_0^{\infty} \langle 0|\left[J_j(0), J_i(t'')\right]|0\rangle\, e^{i\omega t''}\,\mathrm{d}t''\right\}}_{=:\sigma_{ij}(\omega)\,A} E_j e^{-i\omega t} \tag{1.67b}$$

with ⁂ *conductivity tensor* $\sigma_{ij}(\omega)$

The sample area $A = L_x L_y$ shows up because the conductivity tensor relates, by definition, the current *density* j_i to the electric field, and not the total current $J_i = A j_i$.

To show the second equality, use that $J_j(t') = e^{\frac{i}{\hbar}H_0 t'} J_j e^{-\frac{i}{\hbar}H_0 t'}$ [and similar for $J_i(t)$] and that $|0\rangle$ is an eigenstate of H_0.

iv | → Hall conductivity:

$$\sigma_{xy}(\omega) = -\frac{1}{\hbar\omega A}\int_0^\infty \langle 0|\left[J_y(0), J_x(t)\right]|0\rangle\, e^{i\omega t}\,\mathrm{d}t \tag{1.68}$$

This is the *AC Hall conductivity* as it is still frequency dependent.

v | Set $t_0 = 0$ and use $U_0(t) = \sum_n e^{-iE_n t/\hbar}|n\rangle\langle n|$ and $J_i(t) = U_0^\dagger(t) J_i U_0(t)$:

→

$$\sigma_{xy}(\omega) = -\frac{1}{\hbar\omega A}\int_0^\infty \sum_n \begin{Bmatrix} \langle 0|J_y|n\rangle\langle n|J_x|0\rangle e^{i(E_n-E_0)t/\hbar} \\ -\langle 0|J_x|n\rangle\langle n|J_y|0\rangle e^{i(E_0-E_n)t/\hbar} \end{Bmatrix} e^{i\omega t}\,\mathrm{d}t \tag{1.69a}$$

Integrate (using a regularization $\omega + i\varepsilon$ to make the integral convergent)

$$= -\frac{i}{\omega A}\sum_{n\neq 0}\left\{\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle}{\hbar\omega + E_n - E_0} - \frac{\langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{\hbar\omega + E_0 - E_n}\right\} \tag{1.69b}$$

vi | Take DC limit $\omega \to 0$ and use $\frac{1}{\hbar\omega + E_n - E_0} = \frac{1}{E_n - E_0} - \frac{\hbar\omega}{(E_n-E_0)^2} + \mathcal{O}(\omega^2)$:

(Note the i/ω that must be canceled to render the expression finite!)

$$\sigma_{xy} \mathrel{\overset{\circ}{=}} \frac{i\hbar}{A}\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle - \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{(E_n - E_0)^2} \tag{1.70}$$

This is the Hall conductivity expressed in terms of current matrix elements. Our → *next* project will be a (quite tedious) reformulation of this expansion with the goal to re-express it in terms of a topological invariant, namely the ← *Chern number*.

vii | Comment on the constant term:

For the derivation of Eq. (1.70) it is crucial that

$$\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle + \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{E_n - E_0} = 0 \tag{1.71}$$

which makes the constant terms of the Taylor expansion cancel (this avoids the divergence for $\omega \to 0$!).

One way to see this is from *rotation invariance* of the system in the x-y-plane (a quantum Hall system should be rotation invariant about the axis of the magnetic field). In particular, σ_{xy} should be invariant under the $\pi/2$-rotation $J_x \mapsto J_y$ and $J_y \mapsto -J_x$ (note that $\boldsymbol{J}$ is a vector operator). This means that

$$\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle + \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{E_n - E_0} \overset{!}{=} -\sum_{n\neq 0}\frac{\langle 0|J_x|n\rangle\langle n|J_y|0\rangle + \langle 0|J_y|n\rangle\langle n|J_x|0\rangle}{E_n - E_0} \tag{1.72}$$

which implies Eq. (1.71) so that only the *antisymmetric* part of σ_{xy} survives.

Note that this is a quite general argument: If we decompose the 2D conductivity tensor into symmetric and antisymmetric parts, $\sigma = \sigma_s + \sigma_a$, and demand rotational invariance of the tensor, i.e., $\sigma = R\sigma R^T$ for a 2D rotation matrix R, we have $\sigma_s = R\sigma_s R^T$ and $\sigma_a = R\sigma_a R^T$ separately. The only *symmetric* matrix invariant under rotations is proportional to the identity, $\sigma_s = \sigma_{xx} \cdot \mathbb{1}$, so that there cannot be a symmetric contribution to the off-diagonals (that is, the Hall conductivity σ_{xy}). Thus the most general form of a *rotation invariant* conductivity tensor is

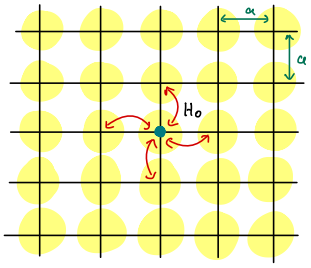
$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix} . \tag{1.73}$$

1.4.2. The TKNN invariant

Here we want to connect the Hall conductivity [given by the Kubo formula Eq. (1.70)] to the Chern number and thereby explain the quantization of the former. To do so, we consider non-interacting electrons in a two-dimensional periodic potential, so that the momentum space is a torus.

The rationale of the following discussion is similar to the original approach by Thouless *et al.* [17].

1 | ◁ Single electron in a periodic potential with Hamiltonian H_0:



System size: $L_x \times L_y$ & periodic boundaries

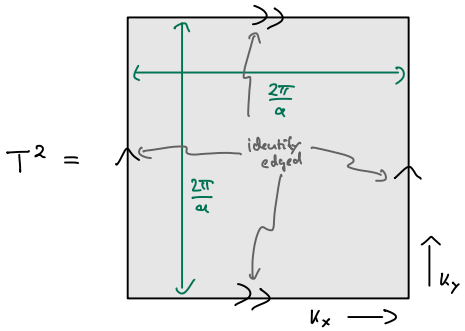
We take the thermodynamic limit $L_x, L_y \to \infty$ later.

2 | ↓ *Bloch theorem*:

- Eigenfunctions: $\Psi_{n\boldsymbol{k}} = e^{i\boldsymbol{k}\boldsymbol{x}} u_{n\boldsymbol{k}}(\boldsymbol{x})$
  with $u_{n\boldsymbol{k}}(\boldsymbol{x} + \boldsymbol{R}) = u_{n\boldsymbol{k}}(\boldsymbol{x})$ for lattice vectors $\boldsymbol{R}$ and band index $n = 1, 2, \ldots$
- Eigenenergies $\varepsilon_n(\boldsymbol{k})$ continuous in $\boldsymbol{k}$ → "Bands"
- $\Psi_{n\boldsymbol{k}+\boldsymbol{K}} = \Psi_{n\boldsymbol{k}}$ for reciprocal lattice vectors $\boldsymbol{K}$

  If $\boldsymbol{R} = an_x\boldsymbol{e}_x + an_y\boldsymbol{e}_y$ describes a square lattice with lattice constant a, the reciprocal lattice is $\boldsymbol{K} = m_1\boldsymbol{k}_1 + m_2\boldsymbol{k}_2$ with $\boldsymbol{k}_i = \frac{2\pi}{a}\boldsymbol{e}_i$.

  → Brillouin zone = Torus T^2



Since our system is finite, momenta are discrete. The size of the Brillouin zone is determined by the inverse lattice constant and remains fixed in the following.

3 | ◁ Many-body Fock states with Fermi energy E_F:

¡! While we can understand the integer quantum Hall effect within the framework of non-interacting fermions, the quantization of the Hall conductivity is a genuine quantum many-body phenomenon. It is crucial that you understand the difference (and relation) between these concepts.

$$\text{Ground state} = |0\rangle \;\mapsto\; \mathbf{|0\rangle} = \text{Filled Fermi sea} \tag{1.74a}$$

$$\text{Excited states} = |n\rangle \;\mapsto\; \mathbf{|n\rangle} = \text{Fermi sea with particle-hole excitations} \tag{1.74b}$$

$$\text{Current operator} = J_i \;\mapsto\; \mathfrak{J}_i = \text{Second-quantized current operator} \tag{1.74c}$$

In the following, **bold states** live in the fermionic Fock space (= many-body states), whereas states in normal font live in the single-particle Hilbert space.

4 | Eq. (1.70) → Hall conductivity of fermionic many-body system:

$$\sigma_{xy} \triangleq \frac{i\hbar}{A} \sum_{\boldsymbol{n}\neq\mathbf{0}} \frac{\langle\mathbf{0}|\mathfrak{J}_y|\boldsymbol{n}\rangle\langle\boldsymbol{n}|\mathfrak{J}_x|\mathbf{0}\rangle - \langle\mathbf{0}|\mathfrak{J}_x|\boldsymbol{n}\rangle\langle\boldsymbol{n}|\mathfrak{J}_y|\mathbf{0}\rangle}{(E_{\boldsymbol{n}} - E_{\mathbf{0}})^2} \tag{1.75}$$

Note that the sum goes over all possible excited many-body states (which are all states except the Fermi sea ground state). However, below we will see that only states with a single particle-hole excitation contribute.

5 | Current operator = Single-particle operator:

$$\mathfrak{J}_i = \sum_{n\boldsymbol{k},m\boldsymbol{q}} \langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{q}}\rangle\, c^\dagger_{n\boldsymbol{k}} c_{m\boldsymbol{q}} \tag{1.76}$$

$c^\dagger_{n\boldsymbol{k}}$: Creation operator for fermion in Bloch state $|\Psi_{n\boldsymbol{k}}\rangle$

Remember that this recipe produces an operator on Fock space that acts like the single-particle operator J_i within the one-fermion subspace.

6 | Eq. (1.75) → [Here nk' is short for $(nk)' = n'k'$.]

$$\sum_{n\neq 0} \frac{\langle 0|\mathfrak{J}_y|n\rangle\langle n|\mathfrak{J}_x|0\rangle}{(E_n - E_0)^2} = \sum_{nk',mq'} \sum_{nk,mq} \langle \Psi_{nk}|J_y|\Psi_{mq}\rangle\langle\Psi_{nk'}|J_x|\Psi_{mq'}\rangle \underbrace{\sum_{n\neq 0} \frac{\langle 0|c^\dagger_{nk} c_{mq}|n\rangle\langle n|c^\dagger_{nk'} c_{mq'}|0\rangle}{(E_n - E_0)^2}}_{\frac{\delta_{nk=mq'}\delta_{mq=nk'}\delta_{\varepsilon_m(q)>E_F}\delta_{\varepsilon_n(k)<E_F}}{[\varepsilon_m(q)-\varepsilon_n(k)]^2}} \tag{1.77}$$

$$\triangleq \sum_{\substack{nk,mq \\ \varepsilon_n(k)<E_F<\varepsilon_m(q)}} \frac{\langle\Psi_{nk}|J_y|\Psi_{mq}\rangle\langle\Psi_{mq}|J_x|\Psi_{nk}\rangle}{[\varepsilon_m(q) - \varepsilon_n(k)]^2} \tag{1.78}$$

To evaluate the sum $\sum_{n\neq 0}$ over all excited many-body states, convince yourself that you can *w.l.o.g.* replace the denominator by $[\varepsilon_m(q) - \varepsilon_n(k)]^2$ (which is independent of n!). Then $\sum_{n\neq 0} |n\rangle\langle n|$ can be written as $\mathbb{1} - |0\rangle\langle 0|$ and the rest follows.

7 | Assume $\varepsilon_n(k) \lneq E_F$ for all $k \in T^2$

¡! This means that the Fermi energy falls into a *band gap*. This is absolutely crucial for what follows.

(Note that statements like "$\varepsilon_n < E_F$" are now well-defined since $\varepsilon_n(k) < E_F$ is true for all momenta and only depends on the band index n.)

→

$$\sigma_{xy} \triangleq \frac{i\hbar}{A} \sum_{\substack{n,m \\ \varepsilon_n<E_F<\varepsilon_m}} \sum_{k,q\in T^2} \frac{\left\{\begin{array}{l} \langle\Psi_{nk}|J_y|\Psi_{mq}\rangle\langle\Psi_{mq}|J_x|\Psi_{nk}\rangle \\ -\langle\Psi_{nk}|J_x|\Psi_{mq}\rangle\langle\Psi_{mq}|J_y|\Psi_{nk}\rangle \end{array}\right\}}{[\varepsilon_m(q) - \varepsilon_n(k)]^2} \tag{1.79}$$

8 | As a first simplification, we want to get rid of one of the two momentum summations. To do so, we must show that the current operator cannot change the momentum of a state:

i | Define the *single-particle current operator*

$$J := e\dot{x} = i\frac{e}{\hbar}[H_0, x] \tag{1.80}$$

Here we use the ↓ *Heisenberg equation of motion* to express the velocity operator in terms of a commutator. Remember that we are in the *interaction picture*, i.e., operators evolve in time under the unperturbed Hamiltonian H_0.

ii | ◁ Translation operator T_R with lattice vector R:

$$T_R x T_R^{-1} = x + R \tag{1.81a}$$

$$T_R H_0 T_R^{-1} = H_0 \tag{1.81b}$$

$$T_R|\Psi_{nk}\rangle = e^{ikR}|\Psi_{nk}\rangle \tag{1.81c}$$

- The first equation follows from the definition of the translation operator.
- The commutativity with the Hamiltonian follows from our assumption that the system features a discrete translation invariance ("periodic potential").
- The energy eigenstates of such a Hamiltonian are Bloch states $|\Psi_{nk}\rangle$ which are also eigenstates of these lattice translations (this is just the statement of ← *Bloch's theorem*).

iii | Consequently

$$T_{\boldsymbol{R}}\boldsymbol{J}T_{\boldsymbol{R}}^{-1} = i\frac{e}{\hbar}[H_0, \boldsymbol{x}+\boldsymbol{R}] = i\frac{e}{\hbar}[H_0,\boldsymbol{x}] = \boldsymbol{J} \tag{1.82}$$

→ $\boldsymbol{J}$ cannot change lattice momenta

Formally: $\langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{q}}\rangle = \langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{k}}\rangle\delta_{\boldsymbol{k},\boldsymbol{q}}$

iv | Thus Eq. (1.79) →

$$\sigma_{xy} \mathrel{\overset{\circ}{=}} \frac{i\hbar}{A}\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\sum_{\boldsymbol{k}\in T^2}\frac{\left\{\begin{matrix}\langle\Psi_{n\boldsymbol{k}}|J_y|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_x|\Psi_{n\boldsymbol{k}}\rangle\\ -\langle\Psi_{n\boldsymbol{k}}|J_x|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_y|\Psi_{n\boldsymbol{k}}\rangle\end{matrix}\right\}}{[\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.83}$$

9 | ◁ Continuum limit: $L_x, L_y \to \infty$

In the thermodynamic limit, the sum over momenta turns into an integral over the Brillouin zone T^2:

$$\sigma_{xy} \mathrel{\overset{\circ}{=}} i\hbar\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\left\{\begin{matrix}\langle\Psi_{n\boldsymbol{k}}|J_y|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_x|\Psi_{n\boldsymbol{k}}\rangle\\ -\langle\Psi_{n\boldsymbol{k}}|J_x|\Psi_{m\boldsymbol{k}}\rangle\langle\Psi_{m\boldsymbol{k}}|J_y|\Psi_{n\boldsymbol{k}}\rangle\end{matrix}\right\}}{[\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.84}$$

- The continuum limit is convenient because we can now use tools from calculus to simplify this expression further.
- Here we used the usual approximation of a Riemann sum:

$$\frac{1}{L_i}\sum_{k_i} = \frac{1}{2\pi}\sum_{k_i}\frac{2\pi}{L_i} \xrightarrow{L_i\to\infty} \int\frac{\mathrm{d}k_i}{2\pi} \tag{1.85}$$

Remember that $A = L_xL_y$.

10 | Our next goal is to get rid of the current operators:

i | Use $|\Psi_{n\boldsymbol{k}}\rangle = e^{i\boldsymbol{k}\boldsymbol{x}}|u_{n\boldsymbol{k}}\rangle$ (← *Bloch theorem*) and define $\tilde{\boldsymbol{J}}(\boldsymbol{k}) := e^{-i\boldsymbol{k}\boldsymbol{x}}\boldsymbol{J}e^{i\boldsymbol{k}\boldsymbol{x}}$ so that

$$\langle\Psi_{n\boldsymbol{k}}|J_i|\Psi_{m\boldsymbol{k}}\rangle = \langle u_{n\boldsymbol{k}}|\tilde{J}_i(\boldsymbol{k})|u_{m\boldsymbol{k}}\rangle \tag{1.86}$$

¡! Note that in $e^{i\boldsymbol{k}\boldsymbol{x}}$, $\boldsymbol{x}$ is the position *operator*.

ii | Define $\tilde{H}_0(\boldsymbol{k}) := e^{-i\boldsymbol{k}\boldsymbol{x}}H_0e^{i\boldsymbol{k}\boldsymbol{x}}$ so that

$$H_0|\Psi_{n\boldsymbol{k}}\rangle = \varepsilon_n(\boldsymbol{k})|\Psi_{n\boldsymbol{k}}\rangle \;\Leftrightarrow\; \tilde{H}_0(\boldsymbol{k})|u_{n\boldsymbol{k}}\rangle = \varepsilon_n(\boldsymbol{k})|u_{n\boldsymbol{k}}\rangle \tag{1.87}$$

iii | With these preliminaries, we can write:

$$\tilde{J}_i \mathrel{\overset{\circ}{=}} \frac{e}{\hbar}\tilde{\partial}_i\tilde{H}_0 \quad\text{with}\quad \tilde{\partial}_i := \frac{\partial}{\partial k_i} \tag{1.88}$$

To show this use the definition of $\tilde{H}_0(\boldsymbol{k})$ and show that $\tilde{\partial}_i\tilde{H}_0 = i[\tilde{H}_0, x]$.

iv | Eqs. (1.84), (1.86) and (1.88) →

$$\sigma_{xy} \mathrel{\overset{\circ}{=}} i\frac{e^2}{\hbar}\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\frac{\left\{\begin{matrix}\langle u_{n\boldsymbol{k}}|\tilde{\partial}_y\tilde{H}_0|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde{\partial}_x\tilde{H}_0|u_{n\boldsymbol{k}}\rangle\\ -\langle u_{n\boldsymbol{k}}|\tilde{\partial}_x\tilde{H}_0|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde{\partial}_y\tilde{H}_0|u_{n\boldsymbol{k}}\rangle\end{matrix}\right\}}{[\varepsilon_m(\boldsymbol{k})-\varepsilon_n(\boldsymbol{k})]^2} \tag{1.89}$$

11 | Use

$$\langle u_{n\boldsymbol{k}}|\tilde{\partial}_y \tilde{H}_0|u_{m\boldsymbol{k}}\rangle = \langle u_{n\boldsymbol{k}}|\tilde{\partial}_y\left(\tilde{H}_0|u_{m\boldsymbol{k}}\rangle\right) - \langle u_{n\boldsymbol{k}}|\tilde{H}_0|\tilde{\partial}_y u_{m\boldsymbol{k}}\rangle \tag{1.90a}$$

$$= [\varepsilon_m(\boldsymbol{k}) - \varepsilon_n(\boldsymbol{k})]\langle u_{n\boldsymbol{k}}|\tilde{\partial}_y u_{m\boldsymbol{k}}\rangle \tag{1.90b}$$

$$= [\varepsilon_n(\boldsymbol{k}) - \varepsilon_m(\boldsymbol{k})]\langle \tilde{\partial}_y u_{n\boldsymbol{k}}|u_{m\boldsymbol{k}}\rangle \tag{1.90c}$$

The first line is just the product rule, in the second line we used that $\tilde{H}_0 = \tilde{H}_0^\dagger$ and that $\langle u_{n\boldsymbol{k}}|u_{m\boldsymbol{k}}\rangle = 0$ for $n \neq m$ (which is the case in our expression for the Hall conductivity). The last line follows if in the first line the derivative acts on the bra to the left instead on the ket to the right.

→

$$\sigma_{xy} \doteq i\frac{e^2}{\hbar}\sum_{\substack{n,m\\ \varepsilon_n<E_F<\varepsilon_m}}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\begin{Bmatrix}\langle\tilde{\partial}_y u_{n\boldsymbol{k}}|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde{\partial}_x u_{n\boldsymbol{k}}\rangle\\ -\langle\tilde{\partial}_x u_{n\boldsymbol{k}}|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}|\tilde{\partial}_y u_{n\boldsymbol{k}}\rangle\end{Bmatrix} \tag{1.91}$$

Yay! The denominator is gone …☺

12 | Use

$$\sum_m |u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}| = \mathbb{1} \tag{1.92a}$$

$$\Rightarrow \sum_{m:\varepsilon_m>E_F}|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}| = \mathbb{1} - \sum_{m:\varepsilon_m<E_F}|u_{m\boldsymbol{k}}\rangle\langle u_{m\boldsymbol{k}}| \tag{1.92b}$$

These statements are true on the subspace spanned by the Bloch functions $|u_{n\boldsymbol{k}}\rangle$ for fixed $\boldsymbol{k}$.

More rigorously, one should replace $\mathbb{1}$ by the projector $P_{\boldsymbol{k}}$ onto states with lattice momentum $\boldsymbol{k}$ and do the derivatives in the expression for σ_{xy} properly; the result will be the same, though.

→

$$\boxed{\sigma_{xy} \doteq i\frac{e^2}{\hbar}\sum_{n:\varepsilon_n<E_F}\int_{T^2}\frac{\mathrm{d}^2k}{(2\pi)^2}\left\{\langle\tilde{\partial}_y u_{n\boldsymbol{k}}|\tilde{\partial}_x u_{n\boldsymbol{k}}\rangle - \langle\tilde{\partial}_x u_{n\boldsymbol{k}}|\tilde{\partial}_y u_{n\boldsymbol{k}}\rangle\right\}} \tag{1.93}$$

Only the term with $\mathbb{1}$ survives. The second term vanishes as it replaces the sum over empty bands by a sum over filled bands. But then the sum in the expression for the Hall conductance vanishes identically if one shifts the derivatives to the states with $m\boldsymbol{k}$ in the first term [using Eq. (1.90)] and substitutes $n \leftrightarrow m$ in the sums (the last step only works because m and n now run over the same range of filled bands).

13 | Finally, we can relate our findings to the geometrical quantities introduced in Section 1.3:

i | Define the Berry connection of band n:

$$\boxed{\mathcal{A}_i^{[n]}(\boldsymbol{k}) := -i\langle u_{n\boldsymbol{k}}|\tilde{\partial}_i u_{n\boldsymbol{k}}\rangle} \tag{1.94}$$

This is a U(1) connection on the Brillouin zone which is the compact 2D manifold T^2. The parameters are the momenta ($\boldsymbol{\Gamma} = \boldsymbol{k}$) and the local Hilbert spaces are one dimensional: $\mathcal{V}^{[n]}(\boldsymbol{k}) = \mathrm{span}\{|u_{n\boldsymbol{k}}\rangle\}$; these are the non-degenerate eigenspaces (no band crossings!) of the Hamiltonian family $\tilde{H}_0(\boldsymbol{k})$ with discrete spectrum $\varepsilon_n(\boldsymbol{k})$ (fix $\boldsymbol{k}$ as a parameter!). Thus $n = 1$ and $k = 2$ in the context of our general discussion in Section 1.3; in the present context, n denotes the band index.

ii | → Berry curvature of band n:

$$\begin{aligned}\mathcal{F}_{ij}^{[n]}(\boldsymbol{k}) &= \tilde{\partial}_j \mathcal{A}_i^{[n]} - \tilde{\partial}_i \mathcal{A}_j^{[n]} \\ &= -i\langle \tilde{\partial}_j u_{n\boldsymbol{k}} | \tilde{\partial}_i u_{n\boldsymbol{k}} \rangle + i\langle \tilde{\partial}_i u_{n\boldsymbol{k}} | \tilde{\partial}_j u_{n\boldsymbol{k}} \rangle\end{aligned} \tag{1.95}$$

The cross terms cancel.

iii | → Chern number of band n:

$$\begin{aligned}C^{[n]} &= \frac{1}{2\pi}\int_{T^2} \mathcal{F}_{ij}\,\mathrm{d}\sigma^{ij} = -\frac{1}{2\pi}\int_{T^2} \mathcal{F}_{xy}\,\mathrm{d}^2k \\ &= \frac{i}{2\pi}\int_{T^2} \left\{\langle \tilde{\partial}_y u_{n\boldsymbol{k}} | \tilde{\partial}_x u_{n\boldsymbol{k}} \rangle - \langle \tilde{\partial}_x u_{n\boldsymbol{k}} | \tilde{\partial}_y u_{n\boldsymbol{k}} \rangle\right\} \mathrm{d}^2k\end{aligned} \tag{1.96}$$

The integral is best evaluated with differential forms where $\mathcal{F} = \mathrm{d}\mathcal{A}$ is a 2-form and $\mathcal{A} = A_x \mathrm{d}k_x + A_y \mathrm{d}k_y$ is a 1-form. Then

$$C = \frac{1}{2\pi}\int_{T^2} \mathcal{F} = \frac{1}{2\pi}\int_{T^2} \left(\tilde{\partial}_y A_x\, \mathrm{d}k_y \wedge \mathrm{d}k_x + \tilde{\partial}_x A_y\, \mathrm{d}k_x \wedge \mathrm{d}k_y\right) \tag{1.97a}$$

$$= -\frac{1}{2\pi}\int_{T^2} \underbrace{\left(\tilde{\partial}_y A_x - \tilde{\partial}_x A_y\right)}_{\mathcal{F}_{xy}} \underbrace{\mathrm{d}k_x \wedge \mathrm{d}k_y}_{\mathrm{d}^2k} \tag{1.97b}$$

where we used $\mathrm{d}k_i \wedge \mathrm{d}k_j = -\mathrm{d}k_j \wedge \mathrm{d}k_i$.

14 | Compare Eq. (1.93) with Eq. (1.96) →

¡! Important: TKNN formula

$$\sigma_{xy} = \frac{e^2}{2\pi\hbar}\sum_{n:\varepsilon_n<E_F} C^{[n]} = \frac{e^2}{h}\nu \quad \text{with} \quad \nu := \sum_{n:\varepsilon_n<E_F} C^{[n]} \in \mathbb{Z} \tag{1.98}$$

- In summary: The Hall conductivity of a system with non-degenerate bands that are either completely filled or completely empty is an integer multiple ν of $e^2/2\pi\hbar = e^2/h$, where ν is the sum of the Chern numbers of the filled bands. This quantization is robust and independent of microscopic details because the Chern numbers are topological invariants that are necessarily integer, as long as they are well-defined (= no gaps close).
- ¡! If the Fermi energy lies *within* a (then partially filled) band, our proof of the quantization of the Hall conductivity breaks down (where?). In this situation, we cannot make any statements about the value of σ_{xy}.
- ¡! You might wonder: Where is the magnetic field? In our derivation of the TKNN formula we didn't use it. But in experiments, the quantized Hall plateaus arise when tuning the magnetic flux through the sample. The answer is that the quantization of the Hall conductivity itself has nothing to do with a magnetic field. The statement is very clear: Whenever the Fermi energy lies within a gap, the Hall conductivity is quantized and given by the sum of Chern numbers of the filled bands. Note that our result is perfectly consistent with these Chern

numbers (and thereby the Hall conductivity) being *zero*! In that sense we didn't prove the exact "staircase" shape of the Hall resistance observed in 2DEGs penetrated by a magnetic field. We only showed that *if* the Hall conductivity happens to be non-zero, then it must come in steps. The role of the magnetic field is twofold: First, it opens gaps $\hbar\omega_B$ between the Landau levels, so that the conditions for a quantization of σ_{xy} are met (namely when all Landau levels are either full or empty). Second, and this is both crucial and not obvious, it makes the Landau levels "topological" in that their Chern number is $C^{[n]} = \pm 1$ (the same for all n, the sign depends on conventions and the direction of the perpendicular magnetic field). This then explains the exact structure of the famous Hall resistance plots. One can study the emergence of Landau levels and their Chern numbers in the ↑ *Hofstadter model* [17, 78] (➲ Problemset 4). Two different approaches to explicitly compute the Chern numbers of Landau levels are discussed by Fradkin [63, Chapter 12].)

- In our proof, we explicitly used that the many-body ground state is given by a Fermi sea. This description is invalidated by interactions between the fermions (e.g. Coulomb interactions). Similarly, our use of Bloch wave functions is invalidated by disorder in the system. Remarkably, it can be shown that the quantization Eq. (1.98) remains robust under general perturbations (that break translation invariance and/or add interactions) if these perturbations are not too strong [76, 79].
- Another subtlety is that all our calculations refer to *bulk properties* (namely the linear response of the bulk to a homogeneous electric field). This is *not* what one measures in experiments where one attaches point contacts to the *boundary* of a "Hall bar" (which hosts the 2DEG). The conductivity (both longitudinal and transversal) is then determined by the properties of the system boundary and not the bulk. However, due to the → *bulk-boundary correspondence*, the topological nature of the bulk directly influences the property of the edge (→ *below*); in particular, the total Chern number of the bulk (= filled Landau levels) correlates one-to-one with gapless chiral edge modes on the boundary. It is the scattering-free transport in these edge modes that one measures in actual experiments, and the quantized Hall resistance is due to the number of edge modes that contribute (= are partially filled). Formally, this is described by the ↑ *Landauer-Büttiger formalism* [80].
- This formula was first derived by **T**houless, **K**ohmoto, **N**ightingale, and **N**ijs in Ref. [17]; hence the name. It is one of the achievements that earned D. J. Thouless the 2016 Nobel Prize in Physics. Since Thouless got a half-share of the prize, and the Nobel Committee cited both his description of the KT phase transition and the TKNN result as motivation, one can put a Prize tag on Eq. (1.98): $^1/_4$ of a Nobel Prize. I hope you are duly impressed (you can also be a bit proud of having followed the derivation to this point ☺).
- One can show that, without adding additional symmetry constraints, the TKNN invariant (Chern number) is the *only* quantized topological invariant that can be used to distinguish gapped bands [81].
- Historically, the first convincing (but more heuristic) argument for the quantization of the Hall plateaus was already given by Robert Laughlin in 1981 [82]. However, from this derivation one cannot establish a connection to the Chern number as a topological invariant.

15 | Closing remarks:

The salient feature of the integer quantum Hall effect is that a quantity that describes a macroscopic response of system (the Hall conductivity) is exactly quantized and hence impervious to microscopic disorder. This magic turns into comprehension when we go back [to Eq. (1.70)] and realize that we only showed that the *antisymmetric* part of the conductivity tensor has a topological character

(remember that we argued the symmetric part away to evade a divergence in the DC limit). Note that in a conventional conductor (w/o magnetic field) the conductivity tensor is *not* antisymmetric but symmetric. So in general we should start with the decomposition

$$\sigma = \sigma_s + \sigma_a \tag{1.99}$$

with $\sigma_s^T = \sigma_s$ and $\sigma_a^T = -\sigma_a$. W/o magnetic field σ_a vanishes (this is an example of an *Onsager relation* [83]). Strictly speaking, we have only shown that the contribution of this antisymmetric part is topologically quantized. But this contribution is also special in another way. The current $\boldsymbol{J}$ is the response due to an external electric field: $\boldsymbol{J} = \sigma \boldsymbol{E}$. The power that is dissipated in an equilibrium setting (through bumps of the charge carriers with the crystal structure) is then $P = \boldsymbol{J} \cdot \boldsymbol{E}$ (if $\boldsymbol{J}$ is the current *density* this is of course the power *density*); this is known as *Joule's law*. Putting everything together, we find

$$P = \boldsymbol{E}^T \sigma \boldsymbol{E} = \boldsymbol{E}^T \sigma_s \boldsymbol{E} \tag{1.100}$$

since $\boldsymbol{E}^T \sigma_a \boldsymbol{E} = (\boldsymbol{E}^T \sigma_a \boldsymbol{E})^T = \boldsymbol{E}^T \sigma_a^T \boldsymbol{E} = -\boldsymbol{E}^T \sigma_a \boldsymbol{E} = 0$. Thus only the *symmetric* part of the conductivity tensor plays a role for dissipation! But we didn't show that this part is quantized, only the "non-dissipative" contribution σ_a is. So our intuition that a *dissipative* quantity should depend on microscopic details and hence *not* be quantized was right, after all. What we missed is that not everything about the conductivity *tensor* is dissipative; there is also a topological (or geometric) contribution that has nothing to do with microscopic physics. It is this contribution that gives rise to the integer quantum Hall effect.

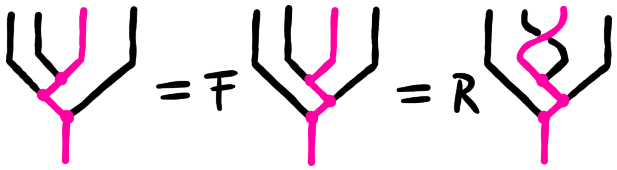
There is much more to be said about the physics of the integer quantum Hall effect. Since this a course on the broader topic of topological phases, we should not linger too long, though. However, there are three last topics that must be mentioned to prevent misconceptions and embed the IQHE into the Big Picture. For students who want to dig deeper into quantum Hall physics, I can highly recommend the lecture notes by David Tong [64].

Part II.

Symmetry-Protected Topological Phases of Interacting Bosons

Part III.

Intrinsic Topological Order and Long-Range Entanglement



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