Special and General Relativity

Lecture Notes • Winter Term 2023/24

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How Stable Diffusion imagines a light cone.

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Preliminaries

Important

This script is in development and continuously updated. To download the latest version:

🔗 itp3.info/rt

If you spot mistakes or have suggestions, send me an email:

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Requirements for this course

We assume that students are familiar with the following concepts:

- Classical mechanics (Lagrangian and Hamiltonian formalism …)
- Non-relativistic quantum mechanics (Schrödinger equation …)
- Classical electrodynamics (Maxwell equations …)
- Basics of algebra & linear algebra (groups, linear maps, …)
- Second quantization and path integrals ★
  This is only required for the excursions on quantum gravity!

Literature recommendations

Special relativity

- Schröder: *Spezielle Relativitätstheorie* [1]
  ISBN 978-3-808-55653-5
  Compact, pedagogic, mathematically precise introduction (in German).

  Extensive standard textbook on special and general relativity (in English).

General relativity

- Schröder: *Gravitation: Einführung in die Allgemeine Relativitätstheorie* [3]
  ISBN 978-3-817-11874-8
  Compact, pedagogic, mathematically precise introduction (in German).

  Extensive standard textbook on special and general relativity (in English).
Preliminaries

- Schutz: *A First Course in General Relativity* [4]
  Extensive, pedagogic, mathematically precise introduction.

  Very high level and compact overview with links to quantum gravity.

Quantum gravity

  Extensive, pedagogic introduction with many detailed calculations.

- Rovelli: *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory* [7]
  Compact, pedagogic introduction, omitting some technical details.

This course follows roughly the textbook *Spezielle Relativitätstheorie* by Ulrich Schröder [1] in the first part on special relativity (with admixtures from Schutz [4] and Straumann [8]). The second part on general relativity follows roughly the textbook *Gravitation* by Ulrich Schröder [3] (with admixtures from Schutz [4] and Rovelli [5]). The excursions on quantum gravity at the end draw from Barton Zwiebach’s *A First Course in String Theory* [6] for the primer on bosonic string theory, and Carlo Rovelli’s *Covariant Loop Quantum Gravity* [7] for the sneak peek at loop quantum gravity.

Original literature

  Annalen der Physik, 17, p. 891–921, (1905)
  Einstein bootstraps SPECIAL RELATIVITY (in German).

  Annalen der Physik, 18, p. 639–641, (1905)
  Einstein derives the famous mass-energy equivalence (in German).

  Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 778–786, 799–801, (1915)
  A. Einstein: *Die Feldgleichungen der Gravitation* [12]
  Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 844–847, (1915)
  Einstein bootstraps the field equations of general relativity (in German).

  Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 831–839, (1915)
  Einstein explains Mercury’s apsidal precession (in German).

- A. Einstein: *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie* [14]
  Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 142–152, (1917)
  Einstein kickstarts relativistic cosmology and introduces the cosmological constant (in German).

  Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 154–167, (1918)
  Einstein predicts and studies gravitational waves (in German).
Goals of this course

The goal of this course is to gain a thorough understanding of relativity, our modern theory of space and time (“spacetime”). This includes both the symmetries and the dynamics of spacetime; the former being described by special relativity, the latter by general relativity. We close with an (optional) excursion into the quantization of gravity, and briefly discuss the two most prominent contenders: string theory and loop quantum gravity.

In particular (★ optional):

Special relativity
• Conceptual foundations special relativity
• Galileian and Einsteinian relativity principles
• Lorentz transformations and the principle of invariance
• Kinematical consequences of Lorentz transformations
• Tensor calculus and the metric tensor
• Special relativity in Minkowski space
• Lorentz- and Poincaré group
• Relativistic mechanics
• Lagrange function and principle of least action
• Electrodynamics as a relativistic field theory
• Noether theorem and the energy momentum tensor
• Relativistic quantum mechanics (Klein-Gordon- and Dirac equation)
• Limitations of special relativity

General relativity
• Incompatibility of gravitation and special relativity
• Mathematical toolbox:
  Riemannian manifolds, metric tensor, Levi-Civita connection, curvature, …
• Conceptual framework of general relativity:
  Metric field, general covariance vs. background independence, …
• Classical mechanics in curved spacetime
• Electrodynamics in curved spacetime ★
• Dynamics of general relativity (Einstein field equations)
• Implications of the Einstein field equations:
  Newtonian limit, Gravitational time dilation, Apsidal precession, Light deflection …
• Application: Gravitational waves (linearized Einstein equations)
• Application: Black holes (Schwarzschild solution)
• Application: The standard model of cosmology (FLRW metric, ΛCDM, …)
• Limitations of general relativity:
  Einstein-Hilbert action, quantum field theory, (non-)renormalizability, …
Quantum gravity (excursion)

- The bosonic string ★:
  Quantization, Virasoro algebra, anomalies, Hilbert space, gravitons, tachyons, …

- Concepts of quantum loop gravity ★:
  Discretized gravity, spin networks, vertex amplitude, transition amplitudes, …

Notes on this document

- This document is not an extension of the material covered in the lectures but the script that I use to prepare them.
- Please have a look at the given literature for more comprehensive coverage. References to primary and secondary resources are also given in the text.
- The content of this script is color-coded as follows:
  - Text in black is written to the blackboard.
  - Notes in red should be mentioned in the lecture to prevent misconceptions.
  - Notes in blue can be mentioned/noted in the lecture if there is enough time.
  - Notes in green are hints for the lecturer.
- One page of the script corresponds roughly to one covered panel of the blackboard.
- Enumerated lists are used for more or less rigorous chains of thought:
  1 | This leads to …
  2 | this. By the way:
     i | This leads to …
     ii | this leads to …
     iii | this.
  3 | Let’s proceed …
- In the bibliography (p. 285 ff.) you can find links to download most papers referenced in this script (they look like this: [Download]). Because most of these papers are not freely available, you need a username & password to access them. These credentials are made available to students of my classes.
- This document has been composed in Vim on Arch Linux and is typeset by LuaLATEX and BeLATEX. Thanks to all contributors to free software!
- This document is typeset in Equity, Concourse and MathTime Professional.

Acknowledgements

- Several students and colleagues spotted typos in the script. Thanks!
Symbols & Scientific Abbreviations

The following abbreviations and glyphs are used in this document:

- **cf**: confer (“compare”)
- **dof**: degree(s) of freedom
- **eg**: exempli gratia (“for example”)
- **etc**: et cetera (“and so forth”)
- **et al**: et alii (“and others”)
- **ie**: id est (“that is”)
- **viz**: videlicet (“namely”)
- **vs**: versus (“against”)
- **wlog**: without loss of generality
- **wrt**: with respect to
- **≪**: “consider”
- **→**: “therefore”
- **¡!**: “Beware!”
- **≤**: non-obvious equality that may require lengthy, but straightforward calculations
- **≡**: non-trivial equality that cannot be derived without additional input
- **→**: “it is easy to show”
- **→**: “it is not easy to show”
- **⇒**: logical implication
- **∧**: logical conjunction
- **∨**: logical disjunction
- **□**: repeated expression
- **■**: anonymous reference
- **w/o**: “without”
- **w/**: “with”
- **→**: internal forward reference (“see below/later”)
- **←**: internal backward reference (“see above/before”)
- **↑**: external reference to advanced concepts (“have a look at an advanced textbook on…”)
- **↓**: external reference to basic concepts (“remember your basic course on…”)
- **⊕**: reference to previous or upcoming exercises
- **∗**: optional choice/item
- **⁂**: implicit or explicit definition of a new technical term (“so called…”)
- **‡**: Aside
- **≡**: Synonymous terms
- **:=**: Definition
The following scientific abbreviations are used in this document:

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>BRT</td>
<td>Belinfante-Rosenfeld tensor</td>
</tr>
<tr>
<td>CERN</td>
<td>European Organization for Nuclear Research</td>
</tr>
<tr>
<td>COE</td>
<td>Center of energy</td>
</tr>
<tr>
<td>COM</td>
<td>Center of mass</td>
</tr>
<tr>
<td>CO</td>
<td>Continuity</td>
</tr>
<tr>
<td>DFST</td>
<td>Dual field-strength tensor</td>
</tr>
<tr>
<td>EMT</td>
<td>Electromagnetic</td>
</tr>
<tr>
<td>EOM</td>
<td>Equation of motion</td>
</tr>
<tr>
<td>ES</td>
<td>Einstein synchronization</td>
</tr>
<tr>
<td>FLRW</td>
<td>Friedmann–Lemaître–Robertson–Walker (metric)</td>
</tr>
<tr>
<td>FST</td>
<td>Field-strength tensor</td>
</tr>
<tr>
<td>GR</td>
<td>GENERAL RELATIVITY</td>
</tr>
<tr>
<td>HME</td>
<td>Homogeneous Maxwell equations</td>
</tr>
<tr>
<td>HO</td>
<td>Homogeneity</td>
</tr>
<tr>
<td>IC</td>
<td>Invariance of coincidence</td>
</tr>
<tr>
<td>IME</td>
<td>Inhomogeneous Maxwell equations</td>
</tr>
<tr>
<td>IN</td>
<td>Inertial (test)</td>
</tr>
<tr>
<td>IRF</td>
<td>Instantaneous rest frame</td>
</tr>
<tr>
<td>IRS</td>
<td>Instantaneous rest system</td>
</tr>
<tr>
<td>IS</td>
<td>Inertial system</td>
</tr>
<tr>
<td>ISS</td>
<td>International space station</td>
</tr>
<tr>
<td>IT</td>
<td>Infinitesimal transformation</td>
</tr>
<tr>
<td>KG</td>
<td>Klein-Gordon</td>
</tr>
<tr>
<td>KGE</td>
<td>Klein-Gordon equation</td>
</tr>
<tr>
<td>LT</td>
<td>Lorentz transformation</td>
</tr>
<tr>
<td>ME</td>
<td>Maxwell equation(s)</td>
</tr>
<tr>
<td>OC</td>
<td>Orthonormal Cartesian (coordinates)</td>
</tr>
<tr>
<td>PDE</td>
<td>Partial differential equation</td>
</tr>
<tr>
<td>QED</td>
<td>Quantum electrodynamics</td>
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<tr>
<td>QFT</td>
<td>Quantum field theory</td>
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<tr>
<td>RI</td>
<td>Reparametrization invariance</td>
</tr>
<tr>
<td>SE</td>
<td>Schrödinger equation</td>
</tr>
<tr>
<td>SI</td>
<td>Système international d’unités</td>
</tr>
<tr>
<td>SL</td>
<td>Speed of light</td>
</tr>
<tr>
<td>SR</td>
<td>SPECIAL RELATIVITY</td>
</tr>
<tr>
<td>UV</td>
<td>Ultraviolett</td>
</tr>
</tbody>
</table>
Setting the Stage

Terminology

The most important terms in this course and their German correspondence:

- RELATIVITY = Relativitätstheorie
- SPECIAL RELATIVITY = Spezielle Relativitätstheorie (SRT)
- GENERAL RELATIVITY = Allgemeine Relativitätstheorie (ART)

Relation of the theories:

\[
\text{RELATIVITY} \Bigg\{ \text{SPECIAL RELATIVITY} \quad \text{GENERAL RELATIVITY} \Bigg\}
\]

Motivation

RELATIVITY is arguably the most popular of scientific theories, for it speaks about an entity of every day experience: space and time. This popularity comes with a caveat:

The “Mona Lisa perspective”

The popular status of RELATIVITY in physics parallels that of the Mona Lisa in arts: Einstein’s magnum opus inherits an aura of perfection and finality.

The “Puzzle Perspective”

RELATIVITY is interesting because it describes some, but not all facets of reality. Its incompatibility with quantum mechanics hints at a reality even stranger than its pieces.

¡ You should not view RELATIVITY as the “Mona Lisa of physics” but as the harbinger of quantum gravity\(^1\) that, most likely, will come with a reformulation of reality so profound that the “strangeness” of quantum mechanics and RELATIVITY alike will pale in comparison (→ Excursions).

\(^1\)I use the term “quantum gravity” here very loosely and essentially synonymous with “theory of everything”.
Ontology

1. The **ontology of physics** is the collection of “things that exist” (**entities**):

\[
\text{Ontology} = \{ \text{Leptons, Hadrons, Higgs, Gauge bosons} \}
\]

Matter: Atoms… \hspace{1cm} Interactions: Photons…

Standard Model of Particle Physics

2. Physical theories are **models** that describe how these entities behave.

   **Examples:**

   - **Classical mechanics** describes the dynamics of matter on macroscopic scales.
   - **Quantum mechanics** describes the dynamics of matter on microscopic scales.
   - **Electrodynamics** describes the dynamics of electromagnetic fields on macroscopic scales.

   Note that these can be **effective** (approximate) descriptions that are restricted to finite scales of validity (length, energy, time).

3. **What is relativity** a theory of?

   i. Two notions of space and time:

      **Relational space & time**

      ![Relational space & time diagram]

      **Newtonian space & time**

      ![Newtonian space & time diagram]

   ii. Delete all entities from the world:

      Nothing! \hspace{1cm} Newtonian space & time left!

   Question: Which notion describes reality?
Newton’s bucket:

Question: Rotation with respect to what determines the shape of the water surface?

Tentative answer: Rotation with respect to Newtonian space!

[!] Today, Newtonian space & time (sometimes called neo-Newtonian or Galilean spacetime) is not seen as a preferred (“absolute”) coordinate system, with respect to which absolute positions, times and velocities can be measured; it is the entity that is responsible for the absolute notion of acceleration in Newtonian physics (which is also present in RELATIVITY). It is “the thing” that determines the reference frames that are inertial [5].

→ Space & time (Spacetime) is an independent “thing that exists.”

The correct answer to the bucket experiment in RELATIVITY will be: The rotation with respect to the local inertial frame—which is determined by the local gravitational field—determines the shape of the water surface. This field is determined by the large-scale distribution of mass and energy in the universe, i.e., the fixed stars; the (rotating) mass of the earth has a non-zero but tiny effect as well (→ Frame dragging).

Thus we should extend our ontology:

Extended Ontology = { Leptons, Hadrons, Gauge bosons, Higgs, Spacetime } RELATIVITY

The Core Theory [16] (→ below) is an effective (quantum) field theory that encompasses the standard model and RELATIVITY. It describes all entities know to us on our scales—but is expected to fail on the Planck scale (in the “UV limit”). The theory that the Core Theory renormalized to in this UV limit is the famous “Theory of Everything”. This is uncharted territory and we do not know what this theory looks like.

The extended ontology above is known as substantivalism in the philosophy of science, see [17] for a review and [18] for a supportive account of this ontology. Opposing substantivalism is relationalism, which defends the view that spacetime is not an independent entity but an emergent description of relations between entities (→ The Hole Argument). Relationalism is exemplified by Mach’s principle, which has been historically influential in the development of general relativity (though Einstein later changed his views). In the light of non-trivial solutions (of the Einstein field equations) for “empty” universes in general relativity, and the (now experimentally confirmed) existence of gravitational waves, I take a substantivalist stance in this course.
This extended ontology allows us to answers the question:

**RELATIVITY** is the theory of spacetime (on macroscopic scales), just as electrodynamics is the theory of the electromagnetic field.

Despite these conceptual similarities, there is a fundamental difference between **RELATIVITY** and electrodynamics (→ below): Whereas electrodynamics describes the dynamics of the electromagnetic field on spacetime, the gravitational field of RELATIVITY does not evolve on spacetime; it is spacetime!

‡ **The Core Theory**

The ⊡ **Core Theory** $S_*$ is the ⊡ **effective field theory** that describes all entities on the energy scales relevant for our everyday life [16]. As typical for a field theory, it is best expressed as a ⊡ **path integral**:

$$A_* \equiv \int \mathcal{D}g \mathcal{D}G \mathcal{D}\psi \mathcal{D}\phi \exp \left( \frac{i}{\hbar} S_*(g, G, \psi, \phi) \right).$$

What makes this an **effective** theory is the momentum cutoff $\Lambda$: The theory describes the dynamics of the fields only up to some finite momentum/energy cutoff $\Lambda$. In [16] it is argued that $\Lambda \sim 10^{11}$ eV is a reasonable cutoff; since this is well below the Planck scale of $10^{28}$ eV, $A_*$ does not describe the physics on these energy scales (e.g., what happens in black holes or near the Big Bang is not encoded in $A_*$). This reflects the lack of a consistent theory of quantum gravity.

The action $S_*$ splits into two parts (plus one additional, technical term that we can safely ignore here):

$$S_*[g, G, \psi, \phi] = S_{EH}[g] + S_{SM}[g, G, \psi, \phi].$$

The first part is the famous ⊡ **Einstein-Hilbert action** ($G$ is the gravitational constant) and describes the gravitational field $g$:

$$S_{EH}[g] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R(g).$$

We will encounter this action in the second part of this course as it encodes the (source-free) ⊡ **Einstein field equations**; there you will learn what $R(g)$ is.

The second part is the action of the ⊡ **standard model of particle physics** (coupled to gravity via $g$) and describes all the stuff in our world (matter and interactions) except gravity:

$$S_{SM}[g, G, \psi, \phi] = \int d^4x \sqrt{-g} \left[ i \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{1}{4} G^2 + |D\phi|^2 - V(\phi) + \left( \bar{\psi}^\dagger Y_{ij} \psi^i \psi^j + \text{h.c.} \right) \right].$$

**Dirac** (Fermion kē&i)
**Yang-Mills** (Gauge boson kē&i)
**Klein-Gordon** (Higgs boson kē&i)
**Higgs potential** (Symmetry breaking)
**Yukawa coupling** (Fermion masses)
Here “ke&i” stands for kinetic energy and interactions (with gauge bosons). The standard model action $S_{\text{SM}}[G, \psi, \phi]$ on a static, flat spacetime $g = \eta$ is typically discussed in a course on quantum field theory with focus on high energy physics (↑ Section 10.2 of my script on QFT [19]). In this course on RELATIVITY, the existence of $S_{\text{SM}}$ will leave its (classical) mark on the Einstein field equations in form of the $\mathbb{E}$ energy-momentum tensor.

Relation to other theories

1 | RELATIVITY is similar to other theories in that it is a theory of an entity that makes up reality. However, it is also different in that this very entity makes an appearance in most other theories:

Classical mechanics describes the macr. dynamics of matter on spacetime: $\vec{x}(t)$.
Quantum mechanics describes the micr. dynamics of matter on spacetime: $\Psi(\vec{x}, t)$.
Electrodynamics describes the macr. dynamics of EM fields on spacetime: $E(\vec{x}, t)$. $B(\vec{x}, t)$.

In the light of the extended ontology (where spacetime is an independent entity described by RELATIVITY), it can be useful to reframe the objective of various theories as follows:

Classical mechanics describes the macr. dynamics of matter interacting with a (static) spacetime.
Quantum mechanics describes the micr. dynamics of matter interacting with a (static) spacetime.
Electrodynamics describes the macr. dynamics of EM fields interacting with a (static) spacetime.

Note that this reading is manifest in the background-independent formulation of the Core Theory $S_{\text{G}}[g, G, \psi, \phi]$ where the metric $g$ and the other fields are treated on the same footing.

$\rightarrow$ The properties of spacetime (as posited by RELATIVITY) must be reflected by these theories!

This means that we might have to modify known theories to be consistent with RELATIVITY. These modifications must adhere to the correspondence principle: The “old” (non-relativistic) versions of the theories must be included in the “new” (relativistic) versions as limiting cases.

2 | Incorporating the tenets of SPECIAL RELATIVITY leads to …

- Relativistic mechanics
- Relativistic quantum mechanics (Dirac equation, Klein-Gordon equation)
- Relativistic electrodynamics (= classical electrodynamics)
Luckily, classical electrodynamics is already consistent with special relativity and needs no modification. By contrast, both classical mechanics and the quantum mechanics you learned in your previous courses must be modified to reflect the symmetries of spacetime posited by special relativity.

3 | Incorporating the tenets of general relativity leads to …

- (Relativistic) Mechanics on curved spacetimes
- (Relativistic) Quantum mechanics on curved spacetimes
- (Relativistic) Electrodynamics on curved spacetimes

In this course, we will discuss the modifications needed for mechanics and electrodynamics to fit the framework of general relativity. We won’t discuss quantum mechanics on curved spacetimes.

Quantum mechanics (describing matter and gauge bosons) on a curved spacetime is not “quantum gravity!” Quantum gravity is a theory where the metric field \( g \) itself is quantized (which we do not know how to do).

**Spoiler**

The gist of relativity can be summarized as follows:

\[
\text{Spacetime} \leftrightarrow \text{Four dimensional Lorentzian manifold } (M, g) \\
\text{Gravitational field} \leftrightarrow \text{Metric tensor field } g
\]

This is what is meant by the popular statement that gravity “is not a force” but a geometrical deformation (“curvature”) of spacetime.

and

**SPECIAL RELATIVITY**: \( g \) has signature \((1, 3)\) (Lorentz symmetry)

**GENERAL RELATIVITY**: \( g \) is a dynamical field (Background independence)

You most likely do not understand these statements at this point. That’s fine! To provide you with the background knowledge to do so is the purpose of this course.

So let’s start …
Part I.

Special Relativity
1. Conceptual Foundations

◊ Concepts

- Events, Observations, Coincidences, Observers, Reference frames, Einstein synchronization, Cartesian coordinates, Inertial frames, Inertial coordinate systems, Coordinate transformations, Laws of nature, Physical models and theories
- Newtonian mechanics, Form-invariance and covariance, Invariance group, Active and passive transformations, Galilei transformations, Galilei group, Galilean principle of relativity
- Maxwell equations, Aether, Michelson Morley experiment, Principle of Special Relativity
- Isotropy, Homogeneity, Affine transformations
- Special Lorentz transformations, Lorentz Boosts, Lorentz group, Lorentz factor, Limiting velocity, Lorentz covariance, Addition of collinear velocities, Finite speed of causality
- Relativity principles, Symmetries of spacetime, Simplicity of nature, Compressibility, Anthropic principle

1.1. Events, frames, laws, and models

Events:

A. Einstein writes in his 1905 paper “Zur Elektrodynamik bewegter Körper” [9]:


And in his 1916 review “Die Grundlage der allgemeinen Relativitätstheorie” [20]:


We condense this into the following postulate:
§ Postulate: Invariance of coincidence (IC)

- Observations are coincidences of events local in space and time.
- Coincidences of events are absolute and observer independent.

Example:

- Event $e_1 = \text{(Clock A shows time 11:30)}$
- Event $e_2 = \text{(Detector B detects electron)}$
- Event $e_3 = \text{(Clock C shows time 9:45)}$

If detector B and clock A are at the same location (spatial coincidence), and clock A shows 11:30 when detector B detects and electron (temporal coincidence), we say that the events $e_1$ and $e_2$ coincide: $e_1 \sim e_2$.

→ Collect all events $e_i$ that coincide into an equivalence class $E$:

$$e_1 \sim e_2 \sim e_3 \sim \ldots \rightarrow E = \{e_1, e_2, e_3, \ldots\}$$

In a slight abuse of nomenclature we call the coincidence class $E$ also event.

Note that this abuse of nomenclature is also used in everyday life: What makes up an “event” (like a party) is the set of all “little events” (like you meeting your friend) that happen (roughly) at the same location and the same time.

Assumption:

The set $E = \{E_1, E_2, \ldots\}$ of all coincidence classes is a complete, observer independent record of reality.

We call the information stored in $E$ absolute because all observers agree on it.

Observer $O = \text{(Reference) Frame } O$:

Goal: Systematic description of physical phenomena in terms of models.

Question: How to systematically observe reality and encode these observations?

:= Experimental setup to collect data about events in space & time:
Assumptions:

- The rods and clocks are conceptual: they do not affect physical experiments.
- All rods and clocks are identical (when brought together, the rods have the same, time-independent length and the clocks tick with the same rate).
- The lattice is “infinitely dense”: there is a clock at every point in space.
- Each clock is assigned a unique position label $\vec{x}$ and the reference frame label $\mathcal{O}$.

For example, a unique position label $\vec{x}$ for a clock can be obtained by counting the rods in $x$-, $y$- and $z$-direction that one has to traverse to reach the clock from the origin. The origin $O$ is, by definition, a “special” clock that is assigned the position label $\vec{x}_O = \vec{0}$.

Observers are not sitting at the origin, looking at their wristwatch, and observing the events with binoculars! They are simply collecting and processing the data that is accumulated by the contraption we call a reference frame.

Since we assume that (ideally) there is one clock at every point in space:

$\rightarrow$ For every observer $\mathcal{O}$ and every coincidence class $E$ there is a unique event $e_\mathcal{O}$

$$E \ni e_\mathcal{O} = \text{(Clock with frame label } \mathcal{O} \text{ and position label } \vec{x} \text{ shows time } t) \quad (1.1a)$$

$$\iff [E]_\mathcal{O} = (t, \vec{x}) \quad (1.1b)$$

for some position label $\vec{x}$ and clock reading $t$.

We refer to the event $(t, \vec{x})_\mathcal{O}$ as the spacetime coordinates of $E$ with respect to frame $\mathcal{O}$. A different observer $\mathcal{O}'$ will use its own clocks and therefore other events (“coordinates”) $(t', \vec{x}')_{\mathcal{O}'} \in E$ to refer to $E$.

In the real world, the tracking detectors of particle colliders are reminiscent of this ideal setup: They are comprised of 3D arrangements of semiconductor-based particle detectors that all report to a central computer that then reconstructs the trajectories of scattering products from the combination of all detection events.

3 | Inertial (coordinate) systems:

The setup of a reference frame $\mathcal{O}$ above is incomplete and actually very hard to work with: Without additional constraints on the geometry of the lattice and the correlations of clocks (their “calibration”), the record of events is essentially arbitrary. Let us therefore impose some deterministic “calibration procedure” (the same for all frames) that determines how to lay out the rod lattice and how to synchronize the clocks. This procedure endows our reference frame with a specific coordinate system, a labeling scheme to describe events.

Clock calibration: (Poincaré-)Einstein synchronization

The conventional synchronization procedure (which is actually in practical use) is (Poincaré-)Einstein synchronization:

$$t_\mathcal{O} \equiv \frac{1}{2} (t_\mathcal{A} + \tilde{t}_\mathcal{A}) \quad (1.2)$$
You will study this particular procedure and its properties in Problemset 1.

In brief, the procedure goes as follows: Consider a reference clock $O$ and some other clock $A$ you wish to synchronize with $O$.

1. To do so, you send a light signal from $A$ to $O$ and note the time $t_A$ your clock $A$ reads when the signal is emitted.

2. When the signal arrives at $O$, it is immediately reflected back to $A$ together with the reading $t_O$ of clock $O$ at this very moment.

3. When the signal arrives back at your clock $A$ (together with the timestamp $t_O$), you note again the reading of your clock as $Q t_A$.

4. You are now in the possession of three timestamps: $t_A$, $t_O$, $Q t_A$. The idea of Einstein’s synchronization is to postulate the reciprocity of the speed of light: We declare that the speed of the signal from $A$ to $O$ is the same as on its way back from $O$ to $A$ (note that we cannot measure this reciprocity because we would need already synchronized clocks to do so!). Under this assumption, the readings of synchronized clocks must satisfy:

$$\Delta t_{A\rightarrow O} \equiv t_O - t_A \equiv t_A - t_O \equiv \Delta t_{O\rightarrow A} \iff t_O = \frac{1}{2}(t_A + \tilde{t}_A), \quad (1.3)$$

which you can locally check with your data $(t_A, t_O, \tilde{t}_A)$. Note that you do not need to know the distance from $O$ to $A$, nor the numerical value of the speed of light $c$ for this procedure to work!

5. Now if you just powered on your shiny new clock $A$ for the first time, it is very unlikely that the condition Eq. (1.3) will be satisfied:

$$t_O = \frac{1}{2}(t_A + \tilde{t}_A) + \delta t = \frac{1}{2}[(t_A + \delta t) + (\tilde{t}_A + \delta t)]. \quad (1.4)$$

Here $\delta t$ is an offset that you might encounter. But then you can just recalibrate your clock $A$ by $\delta t$ such that the new readings are $t_A + \delta t$ and $t_A + \delta t$.

Repeating this procedure for all clocks of the frame $\mathcal{O}$ allows you to establish a synchronization relation between arbitrary pairs of clocks. The fact that (under some reasonable and experimentally verified assumptions) the order in which you synchronize your clocks does not matter (the established relation is an equivalence relation, Problemset 1 and Ref. [21]) makes Einstein synchronization a very useful and peculiar convention [22–24]. However, one can show that it is the only convention that yields a non-trivial equivalence relation of simultaneity that is consistent with the causal structure on $\mathcal{E}$ (later) [25].

Lattice calibration: Orthonormal Cartesian coordinates $\mathcal{EC}$:

The layout of the lattice of rods assigns coordinates $\vec{x} = (x, y, z)$ to each clock. Depending on the actual shape of the lattice, we will denote events by different position labels. (Note that even with rigid rods connected in the topology of a cubic lattice the geometry is not fixed; for example, you can shear the lattice.) If we assume that space (not spacetime!) is a flat Euclidean space where all the facts of Euclidean geometry hold good (angles of triangles add up $\pi$, the Pythagorean theorem holds, the area of circles is $\pi r^2$, etc.), we can parametrize it without loss of generality by orthonormal Cartesian coordinates. In these coordinates, distances can be calculated by the Pythagorean formula:
Spatial distance between clocks at $\vec{x}$ and $\vec{y}$:

$$d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

The fact that the coordinates of a point $(x, y, z)$ are distances along paths parallel to the coordinate axes makes the coordinates Cartesian. The fact that Eq. (1.5) holds makes them orthonormal (i.e., the axes are orthogonal and have the same scale, as suggested by the sketch above). Coordinates are an intrinsically mathematical concept, they are “labels” to identify points on a manifold of physical points (or events, if you consider spacetime coordinates). By contrast, distances carry physical significance: You can measure them with light signals or rods. The prevalence of Cartesian coordinates makes it easy to conflate these two concepts (this will become particularly important in general relativity).

Here is a way to check whether your lattice satisfies the DC condition using the clocks of $\mathcal{O}$ (and the assumption of the isotropy of the two-way speed of light):

iii | “Inertial Test” ($\star$ law of inertia):

Once you have arranged your rods and synchronized your clocks and thereby established a Cartesian coordinate system and a (allegedly) well-defined notion of simultaneity, you can perform the following test and check whether your particular reference frame $\mathcal{O}$ passes it or not:

**IN** Free particles move at constant velocity and in straight lines.  
($\star$ Homogeneity of Inertia)

- It is implied that this statement is true everywhere, anytime, and in all directions.
- Velocities are computed as the time derivative of trajectories in the frame: $\frac{d\vec{x}(t)}{dt}$.
- The property **IN** implies a certain form of homogeneity in space and time (since free particles must move in straight lines anywhere and anytime) and isotropy in space (they must move in straight lines in any direction). Without additional empirical input, this does not automatically imply that every experiment yields the same result anywhere, anytime and in any direction. This more general form of homogeneity and isotropy will be introduced later as **HO** and **IS**. Empirical evidence shows that spacetime indeed is homogeneous **HO** and space isotropic **IS** (in the absence of gravity). With this additional input, the “Inertial Test” to establish **IN** can be simplified to only one particle moving in a straight line at one place for some finite time (which is actually doable). If you presuppose homogeneity **HO** but not isotropy **IS**, you could observe multiple free particles starting at the same point but moving in different (linearly independent) directions.

Frames equipped with a coordinate system defined by $\mathcal{ES}+\mathcal{OC}$ which satisfy **IN** are called $\star$ inertial coordinate systems.
To distinguish arbitrary frames \( \mathcal{O} \) (with arbitrary coordinates) from the special frames (equipped with Cartesian coordinates and synchronized clocks) that passed the inertial test, we label these coordinate systems by \( K, K', K'' \) etc. (if we refer to arbitrary inertial systems) and by \( A, B, C \) etc. (if we refer to specific inertial systems); the set of all inertial systems is denoted \( J \).

**Alternative definitions:**

There seem to be as many definitions of inertial systems as there are texts on *special relativity*. Some are equivalent, some are not. Some more useful, others less so (none are "wrong", though, because definitions cannot be wrong). Some are operational in nature (like the one above), some purely mathematical. Here I only want to point out two ways one can modify the above definition without changing the concept of an inertial system:

- The "inertial test" is crucial to the concept of an inertial frame. It rules out accelerated frames (both linear or rotating). An alternative to throwing test masses in different directions and recording their trajectories is to *repeat* the ES procedure periodically to *test* whether the clocks stay in sync. That is, to setup the coordinate system one synchronizes the clocks *once* (by recalibrating the clocks) and then repeats the procedure periodically to check whether the Einstein-synchronization condition remains valid (\( \Delta t = 0 \) in our description above). As it will turn out in *general relativity*, your clocks will not stay in sync in frames that do not pass IN (and vice versa). This is essentially the definition given by Schutz [4].

- Instead of "hiding" the law of inertia in the synchronization of clocks, one can do a somewhat reverse modification and "hide" the synchronization of clocks in (an extension of) the law of inertia. To this end one extends the "inertial test" by a second class of tests/experiments, namely:

\[ \text{IN*} \]

Two identical particles that are initially adjacent and at rest, and then interact to repel each other, fly apart with the same velocity in opposite directions. (**Isotropy of Inertia**)

This statement about the isotropy of inertia implies an operational definition of simultaneity that is (empirically) equivalent to ES: You synchronize your clocks such that \( \text{IN*} \) is satisfied, for example by performing the experiment described by \( \text{IN*} \) equidistant between two clocks. When the particles reach the clocks, you reset both to \( t = 0 \). In this synchronization \( \text{IN*} \) is satisfied *by construction*; experiments show that clocks synchronized in this way are also synchronized according to ES (and vice versa).

---

### Spacetime diagram

\[ \text{WD} \]

Data structure that encodes the collected data of an inertial coordinate system \( K \):

- Often we draw only one dimension of space for the sake of simplicity.
• Because it will prove useful later, we measure time in units of length by multiplying $t$ with the speed of light $c$. The choice of $c$ is arbitrary at this point.

**Notation:** Two inertial systems $K$ and $K'$:

We use the following shorthand notations to refer to the coordinates of events in the spacetime diagrams of $K$ and $K'$, respectively:

$$(t', x')_K \equiv (x')_K \equiv x' \equiv (t', x')$$  \hspace{1cm} (1.6)

When it is clear to which inertial system the coordinates belong we drop the subscripts $K$ and $K'$.

**Interlude: Reconstructing spacetime diagrams from $E$**

If you are given the set $E$ of events you can reconstruct the spacetime diagram of an inertial system $K$ by looking in each coincidence class $E \in E$ for the clock event $(t, \bar{x})_K \in E$. You then place $E$ (or some sub-event you are interested in) graphically at the coordinate $(t, \bar{x})$ on a sheet of paper. The resulting picture is the spacetime diagram of $K$. In another inertial system $K'$ the events are arranged differently because different clock events $(t', \bar{x}')_K' \in E$ and hence coordinates $(t', \bar{x}')$ are used to draw the spacetime diagram. How $(t, \bar{x})$ and $(t', \bar{x}')$ are related is unclear at this point.

5 | Empirical facts:

The following facts cannot be bootstrapped from logical thinking alone. They are facts about our physical reality that we have strong experimental evidence for.

• Inertial systems exist (at least in some approximation).

Examples would be an unaccelerated spaceship floating far away from the solar system or the interior of the international space station (if you do not measure too precisely). In special relativity we assume that these systems can be extended to encompass all of spacetime.

• Constructing inertial systems (of arbitrary size) is not possible everywhere.

→ **General Relativity**

We will find in our discussion of general relativity that in a gravitational field the construction of inertial systems is only possible locally. For example: If you extend the ISS inertial system rigidly beyond the ISS itself, at some point you will find the trajectories of free particles to deviate from straight lines due to the inhomogeneity of the gravitational field. We will also see that the synchronization procedure used to calibrate the clocks fails in gravitational fields (you cannot keep your clocks in sync). For our discussion of special relativity we ignore this and assume that our inertial systems cover all of spacetime.

6 | Relations between inertial systems:

There are three straightforward ways to construct a new inertial system $K'$ from a given one $K$. They have in common that the two observers do not move with respect to one another so that pairs of clocks from $K$ and $K'$ spatially coincide for all times (this implies in particular that you can check that these pairs of clock run at the same rate):

(1) Translation in time by $s \in \mathbb{R}$ ($\rightarrow$ 1 parameter)

**Procedure:**

Duplicate all clocks & rods in place. Label the new clocks with $K'$ and the old position labels. Shift the reading of all clocks by a constant value $-s$: $$(t', \bar{x}')_K \sim (t, \bar{x})_K \quad \text{with} \quad t' = t - s \quad \text{and} \quad \bar{x}' = \bar{x}.$$  \hspace{1cm} (1.7)
It is easy to see that this modification does not invalidate \( ES, OC \) or \( IN \). In particular, the Einstein synchronization condition Eq. (1.2) remains valid:

\[
t_O = \frac{1}{2} (t_A + \bar{t}_A) \Leftrightarrow (t_O - s) = \frac{1}{2} [t_A - s + (\bar{t}_A - s)].
\]  

(1.8)

**How to check from \( K \):**

At \( (t)_K = 0 \) the reading of the origin clock of \( K' \) is shifted by \(-s \in \mathbb{R}\).

(2) Translation in space by \( \vec{b} \in \mathbb{R}^3 \) (\( \rightarrow 3 \) parameters)

**Procedure:**

Duplicate all clocks & rods and translate the whole lattice by \( \vec{b} \) (since all clocks are type-identical, you can again simply modify the position labels without moving anything). Label the new clocks with \( K' \) and keep their synchronization:

\[
(t', \vec{x}')_{K'} \sim (t, \vec{x})_K \quad \text{with} \quad t' = t \quad \text{and} \quad \vec{x}' = \vec{x} - \vec{b}.
\]  

(1.9)

\( \triangleright \) If you move the lattice \( K' \) in direction \( \vec{b} \), the origin clock of \( K \) with position label \( \vec{x} = 0 \) will spatially coincide with a clock of \( K' \) with position label translated in the opposite direction, namely \(-\vec{b} \). The same happens for rotations (\( \rightarrow \) below) and translations in time (\( \leftarrow \) above).

It is easy to see that this modification does not invalidate \( ES, OC \) or \( IN \). In particular, distances can still be computed with Eq. (1.5) since

\[
d(\vec{x}, \vec{y}) = d(\vec{x} - \vec{b}, \vec{y} - \vec{b}) \quad \text{for} \quad \vec{b} \in \mathbb{R}^3.
\]  

(1.10)

**How to check from \( K \):**

At \( (t)_K = 0 \) the origin of \( K' \) is translated by \( \vec{b} \in \mathbb{R}^3 \) wrt. the origin of \( K \).

(3) Rotation in space by \( R \in SO(3) \) (\( \rightarrow 3 \) parameters)

**Procedure:**

Duplicate all clocks & rods and rotate the whole lattice by the axis and angle defined by the rotation matrix \( R \) (since all clocks are type-identical, you can again simply modify the position labels without moving anything). Label the new clocks with \( K' \) and keep their synchronization:

\[
(t', \vec{x}')_{K'} \sim (t, \vec{x})_K \quad \text{with} \quad t' = t \quad \text{and} \quad \vec{x}' = R^{-1} \vec{x}.
\]  

(1.11)

It is easy to see that this modification does not invalidate \( ES, OC \) or \( IN \). In particular, distances can still be computed with Eq. (1.5) since

\[
d(\vec{x}, \vec{y}) = d(R^{-1} \vec{x}, R^{-1} \vec{y}) \quad \text{for} \quad R^{-1} \in SO(3).
\]  

(1.12)

**How to check from \( K \):**

The spatial axes of \( K' \) are rotated by \( R \in SO(3) \) wrt. the spatial axes of \( K \).

\( \triangleright \) You can add spatial reflections to these transformations (\( \rightarrow \) improper rotations), i.e., \( R \in O(3) \) instead of \( R \in SO(3) \). In our discussions we will omit these and only comment on them where necessary.

The combination of spatial rotations (proper and improper, i.e., including reflections) and spatial translations form the \( \uparrow \) *Euclidean group* \( E(3) = ISO(3) \).

However, experiments (and everyday experience) tell us that there is a fourth possibility how two inertial systems can be related:

**Empirical fact:**
(4) Uniform linear motion (Boost) by $\vec{v} \in \mathbb{R}^3$ ($\rightarrow$ 3 parameters)

You experience this fact whenever you have a very smooth flight: If you don’t look out the window (and cover your ears) everything behaves just as if the airplane were standing still on the ground; there is no evidence that you move with several hundred kilometers per hour relative to the ground.

**How to check from $K$:**

The origin of $K'$ moves with constant velocity $(\vec{v})_K = \frac{dx(t)}{dt} \in \mathbb{R}^3$.

Note that just from this observation one cannot distinguish between a pure boost and a boost combined with a spatial rotation of the axes (because one probes only for the trajectory of a single point). We will → later be more precise about this distinction.

We cannot write down the coordinate transformations for this relation (yet). The fundamental difference to (1)-(3) is that now the clocks of $K'$ move wrt. the clocks of $K$. We cannot interpret this as a simple relabeling of fixed clocks. We cannot even be sure that the $K$- and $K'$-clocks “run at the same rate” (even if they are type-identical) because to check this we would have to compare the reading of a pair of clocks (one in $K$ and one in $K'$) at two consecutive points in time. To do this, however, the two clocks must be at the same place (remember that we can only observe coincidences!). But this is not possible: Since the two frames move uniformly, two clocks can never meet twice! As it will turn out, it is this relation (4) [and its concatenations with (1)-(3)] that harbors the essence of special relativity.

**Empirical fact:** The relations (1)-(4) are exhaustive.

With this we mean that whenever you encounter two inertial systems $K$ and $K'$ (i.e., both observers certify that they satisfy our definition of an inertial system, in particular, the “Inertial Test” IN), then you will find that the relation between the two is one of the four relations (1)-(4) or a combination of them.

→ The relation of two inertial systems $K$ and $K'$ is given by 10 parameters:

Note that all these relations can be operationally defined and measured within the frame $K$.

The first three sketches can be taken at face value: For example, a translation in time really corresponds to the situation where all clocks are shifted by $s$ an all spatial labels (in particular the axes) remain unaffected. However, for the boost (the last sketch on the right) we do not know (yet) how the coordinates transform (neither time nor space) except that the origin clock of $K'$ follows a trajectory in $K$ with uniform velocity $\vec{v}$. This implies that you should not take the sketch for a boost at face value: For example, we do not know whether the axes remain parallel as suggested by the sketch (spoiler: in general they will not).

iii | Since the transformations (1)-(3) do not change the state of motion of the observer (and can therefore be interpreted as a simple relabeling of the position labels and clock readings), it makes sense to collect all inertial frames $K$ that can be connected in this way into an equivalence class $[K]$ which we call …
**Inertial frame**: Equivalence class \([K]\) of all inertial coordinate systems \(K\) related by spacetime translations and spatial rotations.

Inertial frames \([K]\) therefore correspond to the physical notion of a “state of motion.” Physically, an inertial frame corresponds to the class of all freely moving particles in the universe that are mutually at rest. Given such a “state of motion” (e.g., by declaring one of the particles as reference point), you can then construct various Cartesian coordinate systems (e.g., using said reference particle as your origin) to describe events; these are the inertial systems that make up the equivalence class \([K]\).

**Notation:**
We denote these relations between two inertial systems with the following shorthand notations:

\[
\begin{align*}
& K \xrightarrow{R, \vec{v}, s, \vec{b}} K', & K \xrightarrow{R, \vec{u}} K', & K \xrightarrow{\vec{v}} K', & K \xrightarrow{v_x} K' \\
& (1.13)
\end{align*}
\]

From left to right the relations become increasingly specialized.

These relations are not symmetric (as indicated by the arrow). For example, \(K \xrightarrow{v_x} K'\) specifies the situation where the (origin of) system \(K'\) moves with velocity \(v_x\) in \(x\)-direction as measured in system \(K\).

**Coordinate transformations:**

\(< \)** Two descriptions of the same events:

\[
\varphi(s, \vec{s}, \vec{b}) = \mathcal{P}(s, \vec{s}, \vec{b})
\]

\(\rightarrow\) Transformation between these descriptions?

\[
\varphi(K \rightarrow K') : (t, \vec{x})_K \mapsto (t', \vec{x}')_K' \quad \text{Coordinate transformation}
\]
Finding the functional form of $\varphi$ (for the non-trivial case $\bar{v} \neq 0$) will be our main goal and central result of this chapter. However, before we can tackle this problem, we first have to introduce a few more concepts.

### Interlude: Relative information

We called the data in $E$ *absolute* because all observers agree on the coincidence of events. However, this data cannot include arbitrary statements, e.g., the event “the particle has velocity $\bar{v}$” cannot be part of $E$ because we know from experience that different observers in general do not agree on the velocity of an object. However, following Einstein, we postulated that coincidences are all we can ever observe; thus all there is to know must be encoded in $E$! How is this consistent with the fact that velocities (for example) cannot show up in $E$?

To understand this, it is instructive to think about quantities that can be *derived* from the absolute data in $E$ by means of prescribed algorithms. An algorithm $A$ is simply a program using data from $E$ to compute other data (it can use potentially multiple events $E_1, E_2, \ldots, E_N \in E$ to do so). Furthermore, we allow the algorithm to take the label of an inertial system $K$ as input:

\[
A : E^N \times J \rightarrow \text{Output data}
\]  

(1.14)

As a constraint, we require that the algorithm *must not* use any (static) labels $A, B, \ldots \in J$ of inertial systems. The only reference to a frame it can use is the *variable* $K$. This somewhat arbitrary sounding restriction formalizes the notion that there are no inertial systems that are “special”. Since all inertial systems must be treated equal, the algorithm cannot refer to any specific frame. (This → principle of relativity will take the center stage later and turns out to be crucial for the derivation of the transformation $\varphi$.)

Let us now contrive two algorithms to compute two quantities that are clearly physically relevant but are *not* contained in $E$:

- **Example 1: Velocity**
  
  First think about how you would measure the velocity of a particle in the lab: You would detect the particle at two different (but nearby) locations, measure the time it requires to get from one to the other, and then compute the difference quotient of distance traveled by the time needed. Note that there is no way to measure the velocity at one point in space and time; you always need two points!

  To formalize this, consider two events $E_1$ and $E_2$ that both contain the sub-event “particle detected”. The algorithm $V(E_1, E_2; K)$ computes the (average) velocity between the two events as follows:

  1. Select the event $(t_1, \bar{x}_1)_K \in E_1$.
  2. Select the event $(t_2, \bar{x}_2)_K \in E_2$.
  3. Compute and return the value $\bar{v} = \frac{\bar{x}_2 - \bar{x}_1}{t_2 - t_1}$.

  It is important that this algorithm can be used *without modifications* by all observers $K \in J$. To do so, each observer $K$ plugs into $V$ the two events (which are objective) an its *own* label $K$ (since this is the only non-random choice possible).

  But then two *different* observers $K$ and $K'$ will pick *different* coordinates $(t_i, \bar{x}_i)$ (measured by different clocks) to compute their value of $\bar{v}$, which obviously can yield different outcomes (as expected for velocities). Note that for the velocities to be really different it must be $[K'] \neq [K]$, i.e., the two inertial systems *must* belong to different frames.

  - **Example 2: Duration & Simultaneity**

  A very natural question is how much time passed between two events $E_1$ and $E_2$. The formal prescription how to answer this question is given by the algorithm $T(E_1, E_2; K)$:

    1. Select the event $(t_1, \bar{x}_1)_K \in E_1$.
    2. Select the event $(t_2, \bar{x}_2)_K \in E_2$. 
3. Compute and return the value $\Delta t = t_2 - t_1$.

For the very same reason as for the velocity algorithm above, the return value of course will depend on the chosen “clock events” $(t_i, \vec{x}_i)$. And so for the very same reason that velocities can be observer-dependent, time intervals can be as well. Since we define “simultaneity” as the property $\Delta t = 0$, this possibility for observer-dependent results directly transfers to our notion of simultaneity!

Note that we did not make quantitative statements about the outcomes for different observers. We neither showed how velocities depend on the frame nor whether simultaneity really is relative. (It could just be the case that in our world $t_2 - t_1$ always equals $t'_2 - t'_1$ for a fixed event.) This depends on the actual numbers of the coordinates. Such statements therefore require quantitative statements about the relation of $(t, \vec{x})_K \in E$ and $(t', \vec{x}')_{K'} \in E$, which we do not know at this point (this is exactly the question for the functional form of the coordinate transformation $\varphi$).

However, what we did show is the possibility that simultaneity is relative, just as we already expect velocities to be! So when we later find the correct transformation $\varphi$ and (surprise!) that indeed simultaneity is not an observer independent fact, you should not be surprised.

**Question:** Can the values of the electric and magnetic fields $\vec{E}$ and $\vec{B}$ be included in $E$? If not, can you think of an algorithm that determines the electric and magnetic fields $\vec{E}$ and $\vec{B}$ using only coincidence data available in $E$? Do you expect the electromagnetic field to be observer-dependent?

---

**Henceforth:**

Unless noted otherwise, all frames will be inertial (with Cartesian coordinates).

$\rightarrow$ We will (almost exclusively) work with inertial coordinate systems.

We use the concept of inertial systems because to describe physics by equations, coordinates are a useful tool. As it turns out, Cartesian coordinates allow for particularly simple equations (at least if space is Euclidean). So our concept of inertial systems as defined above is the most useful one.

**Physical Models:**

Let us fix a bit of terminology:

- **(Physical) laws** are ontic features of reality (↑ scientific realism).

  Physical laws can only be discovered; they can neither be invented nor modified.

- **(Physical) models** are algorithms used to describe reality.

  These algorithms are typically encoded in the language of mathematics.

  Physical models are invented and can be modified; I will use the terms model and theory interchangeably.

↑ These definitions are by no means conventional and you will find many variations in the literature. For the following discussion, it is only important that the terms we use have precise meaning.
The validity of models cannot be proven; we can only gradually increase our trust in a model by repeated observations (experiments) – or reject it as invalid by demonstrating that its predictions contradict reality (↑ Karl Popper). Note that models might describe reality only approximately and in specific parameter regimes and still be useful.

You may dismiss this focus on terminology as “philosophical banter.” Conceptual clarity, however, is absolutely crucial for science – in particular for RELATIVITY. Whenever there is confusion in physics, it is often rooted in the conceptual fuzziness of our thinking.

1.2. Galilei’s principle of relativity

Example: Newtonian mechanics

Definition of the model:
- Closed system of $N$ massive particles with masses $m_i$ and positions $\vec{x}_i$.
- Force exerted by $k$ on $i$:

$$F_{k \rightarrow i}(\vec{x}_k - \vec{x}_i) = (\vec{x}_k - \vec{x}_i) f_{k \rightarrow i}(|\vec{x}_k - \vec{x}_i|) \quad (1.15)$$

It is $f_{k \rightarrow i} = f_{i \rightarrow k}$ and therefore $F_{k \rightarrow i}(\vec{x}_k - \vec{x}_i) = -F_{i \rightarrow k}(\vec{x}_i - \vec{x}_k)$.

Newtonian equations of motion (in some inertial system $K$):

$$m_i \frac{d^2 \vec{X}_i}{dt^2} = \sum_{k \neq i} \vec{F}_{k \rightarrow i}(\vec{X}_k - \vec{X}_i) \quad (1.16)$$

We denote with $\vec{X}_i(t)$ coordinate-valued functions; i.e., $\vec{x}_i = \vec{X}_i(t)$ determines a spatial point $\vec{x}_i$ for given $t$.

Remember: This model fully implements “Newton’s laws of motion”:

1. Lex prima:

   A body remains at rest, or in motion at a constant speed in a straight line, unless acted upon by a force.

This is the principle of inertia. It is part of the definition of the concept of a Newtonian force used in Eq. (1.16). Note that it is not a consequence of Eq. (1.16) for $F_{k \rightarrow i} = 0$. It rather defines (together with the lex tertia below) the frames and coordinate systems (inertial systems) in which Eq. (1.16) is valid (recall IN).
2. Lex secunda:

*When a body is acted upon by a net force, the body’s acceleration multiplied by its mass is equal to the net force.*

This is just the functional form of Eq. (1.16) in words.

3. Lex tertia:

*If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.*

This is guaranteed by the property $F_{k\rightarrow i} = -F_{i\rightarrow k}$ of the forces. Together with the lex secunda this is an expression of momentum conservation. For two particles:

$$m_1 \frac{dv_1}{dt} + m_2 \frac{dv_2}{dt} = \frac{dp_1}{dt} + \frac{dp_2}{dt} = F_{2\rightarrow 1} + F_{1\rightarrow 2} = 0$$

(1.17)

This implies in particular that two identical particles ($m_1 = m_2$) that are both at rest at $t = 0$ must obey $v_1(t) = -v_2(t)$ for all times (recall IN*).

Application of the model:

As a working hypothesis, let us assume that the model Eq. (1.16) describes the dynamics of massive particles perfectly (from experience we know that there are at least regimes where it is good enough for all practical purposes).

Symmetries of Newtonian mechanics:

To understand the solution space of Eq. (1.16) better, it is instructive to study transformations that map solutions to other solutions.

Galilei transformations:

We define the following coordinate transformation:

$$G : \mathbb{R}^4 \rightarrow \mathbb{R}^4 : \begin{cases} t' = t + s \\ \vec{x}' = R\vec{x} + \vec{v}t + \vec{b} \end{cases}$$

(1.18)

A Galilei transformation $G$ is characterized by 10 real parameters:

- $s \in \mathbb{R}$: Time translation (1 parameter)
- $\vec{b} \in \mathbb{R}^3$: Space translation (3 parameters)
- $\vec{v} \in \mathbb{R}^3$: Boost (3 parameters)
- $R \in \text{SO}(3)$: Spatial rotation (3 parameters; rotation axis: 2, rotation angle: 1)
The set of all transformations forms (the matrix representation of) a group:

\[ \mathcal{G} = \{ G(R, \tilde{v}, s, \tilde{b}) \} \quad \text{Proper orthochronous Galilei group} \quad (1.19) \]

with group multiplication

\[ G_3 = G_1 \cdot G_2 = G(R_1 R_2, R_1 \tilde{v}_2 + \tilde{v}_1, s_1 + s_2, R_1 \tilde{b}_2 + \tilde{b}_1) \quad (1.20) \]

You derive this multiplication in \( \text{Problemset 1} \) and show that the group axioms are indeed satisfied.

As a special case, the multiplication yields the rule for addition of velocities in Newtonian mechanics:

\[ G(\tilde{v}_1, 0, \tilde{b}) \cdot G(\tilde{v}_2, 0, \tilde{b}) = G(\tilde{v}_1 + \tilde{v}_2, 0, \tilde{b}) \quad (1.21) \]

The full Galilei group is generated by the proper orthochronous transformations together with space and time inversion:

\[ \mathcal{G} = \{ \mathcal{G}_+ \cup \{ P, T \} \} \quad \text{Galilei group} \quad (1.22a) \]

\[ P : (t, \vec{x}) \mapsto (t, -\vec{x}) \quad \text{Space inversion (parity)} \quad (1.22b) \]

\[ T : (t, \vec{x}) \mapsto (-t, \vec{x}) \quad \text{Time inversion} \quad (1.22c) \]

\[ \text{b) Galilei covariance & Form-invariance:} \]

Details: \( \text{Problemset 1} \)

\( \triangleright \) Coordinate transformation Eq. (1.18)

We express the total differential and the trajectory in the new coordinates:

\[ \frac{d}{dt} = \frac{dt'}{dt} \frac{d}{dt'} = \frac{d}{dt'} \quad (1.23) \]

and

\[ \ddot{X}'_i(t') = R \dddot{X}_i(t) + \dddot{\tilde{v}} + \dddot{\tilde{b}} = R \dddot{X}_i(t - s) + \dddot{\tilde{v}}(t' - s) + \dddot{\tilde{b}} \quad (1.24a) \]

\[ \Leftrightarrow \quad \dddot{X}_i(t) = R^{-1} \left[ \dddot{X}'_i(t') - \dddot{\tilde{v}}(t' - s) - \dddot{\tilde{b}} \right] \quad (1.24b) \]

Thus the left-hand side of the Newtonian equation of motion Eq. (1.16) reads in new coordinates:

\[ m_i \frac{d^2 \dddot{X}_i(t)}{dt^2} = m_i \frac{d^2 \dddot{X}'_i(t')}{dt'^2} = R^{-1} \left[ \dddot{X}'_i(t') - \dddot{\tilde{v}}(t' - s) - \dddot{\tilde{b}} \right] = R^{-1} m_i \frac{d^2 \dddot{X}'_i(t')}{dt'^2} \quad (1.25) \]

Note that the quantity \( m_i \frac{d^2}{dt^2} \dddot{X}_i(t) \) is not \textit{invariant}; it transforms with an \( R^{-1} \in \text{SO}(3) \).

And the right-hand side:

\[ \sum_{k \neq i} \dddot{F}_{k \rightarrow i} (\dddot{X}_k(t) - \dddot{X}_i(t)) = R^{-1} \sum_{k \neq i} \dddot{F}_{k \rightarrow i} (\dddot{X}'_k(t') - \dddot{X}'_i(t')) \quad (1.26a) \]
Here we used the form of the force Eq. (1.15), that \( \vec{X}_k(t) - \vec{X}_i(t) = R^{-1}[\vec{X}_k(t') - \vec{X}_i(t')] \) and \( |\vec{X}_k(t) - \vec{X}_i(t)| = |\vec{X}_k(t') - \vec{X}_i(t')| \) because of \( R \in SO(3) \).

Note that the force on the right-hand side is not invariant either; luckily, it transforms with the same \( R^{-1} \in SO(3) \); it “co-varies” with the left-hand side!

In conclusion, Newton’s equation of motion Eq. (1.16) reads in the new coordinates:

\[
R^{-1} m_i \frac{d^2 X'_i(t')}{dt'^2} = R^{-1} \sum_{k \neq i} \vec{F}_{k \rightarrow i}(\vec{X}'_k(t') - \vec{X}'_i(t')) \quad (1.27a)
\]

\[
\rightarrow \text{Covariance}
\]

\[
\times R \leftrightarrow m_i \frac{d^2 X'_i(t')}{dt'^2} = \sum_{k \neq i} \vec{F}_{k \rightarrow i}(\vec{X}'_k(t') - \vec{X}'_i(t')) \quad (1.27b)
\]

\[
\rightarrow \text{Form-invariance}
\]

(You can easily check that this holds for \( P \) and \( T \) as well.)

Newton’s EOMs Eq. (1.16) are form-invariant under Galilei transformations.

Or: Newton’s EOMs Eq. (1.16) are Galilei-covariant.

\[\downarrow\] Nomenclature

Let \( X \) be some group of coordinate transformations (here: \( X = \mathcal{G} \) the Galilei group).

- A quantity is called \( X \)-invariant if it does not change under the coordinate transformation. Such quantities are called \( X \)-scalars.
  - An example is the mass \( m \) in Eq. (1.16) (which is also constant).
- A quantity is called \( X \)-covariant if it transforms under some given representation of the \( X \)-group. If this representation is the trivial one (i.e., the quantity does not change at all) this particular \( X \)-covariant quantity is then also an \( X \)-scalar.
  - An example of a Galilei-covariant (but not invariant) quantity is the force \( \vec{F}_{k \rightarrow i} \) which transforms under a representation of \( \mathcal{G} \).
- An equation is called \( X \)-covariant if the quantity on the left-hand side and on the right-hand side are \( X \)-covariant (under the same \( X \)-representation).
  - An example is Newton’s lex secunda Eq. (1.16) where \( m_i \frac{d^2 x_i(t)}{dt^2} \) transforms in the same (non-trivial) representation as \( \vec{F}_{k \rightarrow i} \).
- \( X \)-covariant equations have the feature that a \( X \)-transformation leaves them form-invariant, i.e., they “look the same” after \( X \)-transformations because their left- and right-hand side vary in the same way (they “co-vary”). Note that the quantities in a form-invariant equation do not have to be invariant.
  - An example is again Eq. (1.16) as we just showed. Note that \( \vec{x}'_i(t') \) and \( \vec{x}_i(t) \) are different vectors such that the two sides of the equation as not invariant (but covariant).
Active symmetries:

There is something additional and particularly useful to be learned from the coordinate transformation above. We showed:

\[
\text{If } \ddot{X}_i(t) \text{ satisfies } m_i \frac{d^2 \ddot{X}_i(t)}{dt^2} = \sum_{k \neq i} \bar{F}_{k \rightarrow i}(\ddot{X}_k(t) - \ddot{X}_i(t)) \quad (1.28a)
\]

then \( \ddot{X}'_i(t') \) satisfies

\[
\frac{d^2 \ddot{X}'_i(t')}{dt'^2} = \sum_{k \neq i} \bar{F}_{k \rightarrow i}(\ddot{X}'_k(t') - \ddot{X}'_i(t')) \quad (1.28b)
\]

But \( t' \) in the lower statement is just a dummy variable that can be renamed to whatever we want:

\[
\text{If } \ddot{X}_i(t) \text{ satisfies } m_i \frac{d^2 \ddot{X}_i(t)}{dt^2} = \sum_{k \neq i} \bar{F}_{k \rightarrow i}(\ddot{X}_k(t) - \ddot{X}_i(t)) \quad (1.29a)
\]

then \( \ddot{X}'_i(t) \) satisfies

\[
\frac{d^2 \ddot{X}'_i(t)}{dt'^2} = \sum_{k \neq i} \bar{F}_{k \rightarrow i}(\ddot{X}'_k(t) - \ddot{X}'_i(t)) \quad (1.29b)
\]

Use colors to highlight the changes.

\[
\rightarrow \ddot{X}'_i(t) = R\ddot{X}_i(t - s) + \ddot{v}(t - s) + \ddot{b} \text{ is a new solution of Eq. (1.16)!}
\]

Note that for \( s = 0 \) it is \( \ddot{X}'_i(0) = R\ddot{X}_i(0) + \ddot{b} \) and \( \ddot{X}_i(0) = R\ddot{X}_i(0) + \ddot{v} \), i.e., the solution \( \ddot{X}'_i(t) \) satisfies different initial conditions.

\[
\rightarrow \text{ We say:}
\]

The Galilei group \( G \) is an \( \#$* \) invariance group or an (active) symmetry of Eq. (1.16).

Interlude: Active and passive transformations

It is important to understand the conceptual difference between the two last points:

- In the previous step we took a specific trajectory (solution of Newton’s equation) and expressed it in different coordinates. We then found that the differential equation obeyed by the same physical trajectory in these new coordinates “looks the same” as in the old coordinates. We called this peculiar feature of the differential equation “Galilei-covariance” or “form-invariance”. This type of a transformation is called passive because we keep the physics the same and only change our description of it.

- In the last step, we have shown that there is a dual interpretation to this: If a differential equation is form-invariant under a coordinate transformation, then we can exploit this fact to construct new solutions from given solutions (in the same coordinate system!). This type of transformation is called active because we keep the coordinate frame fixed and actually change the physics. You can therefore think of active transformations/symmetries as “algorithms” to construct new solutions of a differential equation (a quite useful feature since solving differential equations is often tedious).
Remember:

The law of inertia holds (by definition) in all inertial systems.

→ The “inertial test” cannot be used to distinguish inertial systems.

This is a tautological statement because we define inertial systems in this way!

Empirical fact:

Every mechanical experiment (not just the “inertial test”) yields the same result in all inertial systems.

This is not a tautology but an empirically tested feature of reality.

This motivates the following postulate (first given by Galileo Galilei):

§ Postulate: Galilei’s principle of Relativity

No mechanical experiment can distinguish between inertial systems.

¡! In this formulation, GR encodes a (so far uncontested) empirical fact. In particular, it does neither refer nor rely on (the validity of) any physical model, e.g., Newtonian mechanics. As such we should expect that it survives our transition to special relativity.

Here is a more operational formulation of GR: You describe a detailed experimental procedure using equipment governed by mechanics (springs, pendula, masses, . . .) that can be performed in a closed (but otherwise perfectly equipped) laboratory. Then you copy these instructions without modifications and hand them to scientists with labs in different inertial systems. They all perform your instructions and get some results (e.g. the final velocities of a complicated contraption of pendula). When they report back to you, their results will all be identical. This is the essence of GR.

In the language of models that describe the mechanical laws faithfully, GR can be reformulated:

§ Postulate: Galilei’s principle of Relativity

The equations that describe mechanical phenomena faithfully have the same form in all inertial systems.

If this would not be the case you could distinguish between different inertial systems by checking which formula you have to use to describe your observations. Imagine a rotating (non-inertial) frame where you have to use a modified version of Newton’s EOMs (that include additional terms for the Coriolis force) to describe your observations.

Note that “the same form” actually means that the models are functionally equivalent (have the same solution space). Functional equivalence is equivalent to the possibility to formulate the model (= equation of motion) in the same form.

Under the assumption (!) that Newtonian physics (in particular Eq. (1.16)) describes mechanical phenomena faithfully, this implies:

Newton’s equations of motion have the same form in all inertial systems.
This statement is not equivalent to GR or GR’ as it relies on an independent empirical claim (namely the validity of Newton’s equation as a model of mechanical phenomena).

We can now combine this claim with our (purely mathematical!) finding concerning the invariance group of Newton’s equations:

→ Preliminary/Historical conclusion:

Recall that rotating the coordinate axes by $R$ makes the coordinates of fixed events rotate in the opposite direction $R^{-1}$; the same is true for the other transformations.

Since this is a course on relativity, we should be skeptical (like Einstein) and ask:

Is this true?

1.3. Einstein’s principle of special relativity

Mathematical fact:

The Maxwell equations of electrodynamics are not Galilei-covariant.

Proof: Problemset 1

Here for your (and my) convenience the Maxwell equations in vacuum (in cgs units):

Gauss’s law (electric): \[ \nabla \cdot E = 0 \] \hspace{1cm} (1.30a)
Gauss’s law (magnetic): \[ \nabla \cdot B = 0 \] \hspace{1cm} (1.30b)

Law of induction: \[ \nabla \times E = -\frac{1}{c} \partial_t B \] \hspace{1cm} (1.30c)
Ampère’s circuital law: \[ \nabla \times B = \frac{1}{c} \partial_t E \] \hspace{1cm} (1.30d)

“Handwavy explanation” for the absence of Galilei symmetry:

The Maxwell equations imply the wave equation for both fields:

\[ \left( \nabla^2 - \frac{1}{c^2} \partial_t^2 \right) X = 0 \quad \text{for } X \in \{ E, B \} \] \hspace{1cm} (1.31)
Here the speed of light \( c \) plays the role of the phase and group velocity of the waves; i.e., all light signals propagate with \( c \). Form-invariance under some coordinate transformation \( \varphi \) implies that the same light signal propagates with the same velocity \( c \) in all coordinate systems related by \( \varphi \). This is clearly incompatible with the Galilean law for adding velocities (according to which a signal with velocity \( u'_x \) in frame \( K' \) propagates with velocity \( u_x = u'_x + v_x \) in frame \( K \) if \( K \rightarrow K' \)).

The simplest escape from our predicament:

**Maybe there is no relativity principle for electrodynamics?**

*Reasoning:* If we cling to the validity of Newtonian mechanics and Galilean relativity \( GR \), we are forced to assume \( \varphi = G \) as the transformation between inertial systems. Since the Maxwell equations are not form-invariant under these transformations, they look differently in different inertial systems. So there must be a (class of) designated inertial coordinate systems \( \{K_0\} \) in which the Maxwell equations in the specific form Eq. (1.30) you’ve learned in your electrodynamics course are valid.

\[ \{K_0\} = \text{Frame in which the “luminiferous aether” is at rest (?)} \]

**Michelson Morley experiment (plots from [26, 27]):**

\[ \rightarrow \text{The speed of light is the same in all directions.} \]

\[ \rightarrow \text{There is no “luminiferous aether” \( \{K_0\} \).} \]

(Or it is pulled along by earth – which contradicts the observed \( \uparrow \) aberration of light.)

\[ \rightarrow \text{The speed of light} \, c, \text{cannot be fixed wrt. some designated reference frame \( \{K_0\} \).} \]

\[ \rightarrow \text{No experimental evidence that the Maxwell equations do not hold in all inertial systems.} \]

\[ \rightarrow \text{Relativity principle for electrodynamics?!} \]

*Historical note:*

A. Einstein writes in a letter to F. G. Davenport (see Ref. [28]):

[... ] In my own development Michelson’s result has not had a considerable influence. I even do not remember if I knew of it at all when I wrote my first paper on the subject (1905). The explanation is that I was, for general reasons, firmly convinced how this could be reconciled with our knowledge of electro-dynamics. One can therefore understand why in my personal struggle Michelson’s experiment played no role or at least no decisive role.

\[ \rightarrow \text{The Michelson Morley experiment did not kickstart special relativity.} \]

*Modern Michelson-Morley like tests of the isotropy of the speed of light achieve much higher precision than the original experiment. The authors of Refs. [29, 30], for example, report an upper bound of \( \Delta c/c \sim 10^{-17} \) on potential anisotropies of the speed of light by rotating optical resonators.*

**Two observations:**

(1) No evidence that there is no relativity principle for electrodynamics.
(2) Why does Galilean relativity treat mechanics differently anyway?

Put differently: Why should mechanics, a branch of physics artificially created by human society, be different from any other branch of physics? This is not impossible, of course, but it certainly lacks simplicity! (To Galilei’s defence: At his time “mechanics” was more or less identical to “physics”.)

→ A. Einstein writes in §2 of Ref. [9] as his first postulate:

1. Die Gesetze, nach denen sich die Zustände der physikalischen Systeme ändern, sind unabhängig davon, auf welches von zwei relativ zueinander in gleichförmiger Translationsbewegung befindlichen Koordinatensystemen diese Zustandsänderungen bezogen werden.

We reformulate this into the following postulate:

§ Postulate: (Einstein’s principle of) Special Relativity SR

No mechanical experiment can distinguish between inertial systems.

Note the difference to Galilean relativity according to which no experiment governed by classical mechanics can distinguish between inertial systems. Einstein simply extended this idea to all of physics – no special treatment for mechanics!

¡! There are various names used in the literature to refer to SR. Here we call it the principle of special relativity, where the “special” refers to its restriction on inertial systems – as compared to the principle of general relativity in general relativity that refers to all frames (→ later). To emphasize its difference to Galilean relativity, some authors call SR the universal principle of relativity, where “universal” refers to its applicability on all laws of nature (not just the realm of classical mechanics).

But now that there are more contenders (mechanics, electrodynamics, quantum mechanics) all of which must be invariant under the same transformation \( \varphi \), we have to open the quest for \( \varphi \) again:

The differently colored/shaped trajectories symbolize phenomena of mechanics (red), electrodynamics (blue), and quantum mechanics (green). According to SR, all of them must be form-invariant under a common coordinate transformation \( \varphi \).

¡! To reiterate: This is not a question about symmetry properties of equations or models! It is an experimentally testable fact about reality. There is only one correct \( \varphi \) and it is just as real as the three-dimensionality of space.
1.4. Transformations consistent with the relativity principle

Since this is a theory lecture, so we cannot do experiments. Let us therefore weaken the question slightly:

What is most general form of \( \varphi \) consistent with reasonable assumptions about reality?

§ Assumptions

- \( \text{SR} \) Special Relativity: There is no distinguished inertial system.
- \( \text{IS} \) Isotropy: There is no distinguished direction in space.
- \( \text{HO} \) Homogeneity: There is no distinguished place in space or point in time.
- \( \text{CO} \) Continuity: \( \varphi \) is a continuous function (in the origin).

Something is “distinguished” if there exists an experiment that can be used to identify it unambiguously.

This derivation follows Straumann [8] with input from Schröder [1] and Pal [31].

Detailed calculations: Problemset 2

1 | Setup:

\< Two inertial systems \( K \xrightarrow{R, \vec{v}, s, \vec{b}} K' \).

\< Event \( E \in \mathcal{E} \) with coordinates \( x \equiv (t, \vec{x})_K \in E \) and \( x' \equiv (t', \vec{x}')_{K'} \in E \):

We are interested in the transformation \( \varphi \equiv \varphi_{R, \vec{v}, s, \vec{b}} \) with

\[ x' = \varphi(x) \quad (1.32) \]

Note that \( \text{SR} \) forbids us to use the inertial system labels \( K \) or \( K' \) in the definition of \( \varphi \)! We can only use the relative parameters \( (R, \vec{v}, s, \vec{b}) \) measured in \( K \) wrt \( K' \).

2 | Affine structure:

Our first goal is to show that \( \varphi \) must be an affine map.

\< Event \( \tilde{E} \in \mathcal{E} \) with coordinates \( \tilde{x} = x + a \) in \( K \) for some shift \( a \in \mathbb{R}^4 \).

\< Homogeneity \( \text{HO} \) →

\[ \varphi(x + a) - \varphi(x) = a'(\varphi, a) \quad (1.33) \]
For \( x = 0 \): \( a'(\varphi, a) = \varphi(a) - \varphi(0) \rightarrow \)
\[
\varphi(x + a) = \varphi(x) + \varphi(a) - \varphi(0) .
\] (1.34)

This would be satisfied if \( \Psi \) were linear! But we do not know this yet …

Claim: \( \Psi(x) \) continuous at \( x = 0 \) (follows from \( \mathcal{C}^0 \) ⇒ \( \Psi \) is linear.

\( \Psi(x) \) continuous everywhere.

Show this using the definition of continuity, i.e., \( \lim_{x \to 0} \Psi(x) = \Psi(0) \! \).
We already know from our discussion of inertial systems [recall Eq. (1.11)]:

Rotation group $\text{SO}(3)$ must be part of the transformations $\varphi$ with representation

$$x' = \Lambda R^{-1} x \quad \text{with} \quad \Lambda R := \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \quad \text{where} \quad R \in \text{SO}(3). \quad (1.40)$$

This is just a fancy way to rewrite Eq. (1.11).

**Pure boost $K \xrightarrow{1\,\bar{v},\,0,\,0} K'$:**

1. $\triangleleft \ (t)_{K} = 0 \rightarrow \bar{x}' = \mathcal{M} \bar{x}$ for an invertible matrix $\mathcal{M} \in \mathbb{R}^{3\times3}$:
   
   This is the most general transformation for the position labels of the $K$- and $K'$-clocks at $t = 0$. Note that we make no statements on the times $t'$ displayed by the $K'$-clocks at $t = 0$.

   $$\mathcal{M} = R_{1} D R_{2} = R_{1} D R_{1}^{T} R = MR \quad (1.41)$$

   with $R \in \text{O}(3)$ and $M^{T} = M$.

   This follows from the $\dagger$ singular value decomposition of real matrices with $R_{1}, R_{2} \in \text{O}(3)$ and $D$ a diagonal matrix.

2. With spatial rotations Eq. (1.40) we can always transform the $K$-coordinates by $\bar{x} \mapsto R^{-1} \bar{x}$ such that $\bar{x}' = \mathcal{M} \bar{x} = M \bar{x}$ at $t = 0$.

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   **Pure boost $K \xrightarrow{1\,\bar{v},\,0,\,0} K'$:**

   $$x' = \Lambda \bar{v} x \quad \Leftrightarrow \quad \begin{cases} t' = a(\bar{v}) t + \bar{b}(\bar{v}) \cdot \bar{x} \\ \bar{x}' = M(\bar{v}) \bar{x} + \bar{e}(\bar{v}) t \end{cases} \quad (1.42)$$

   - $a$: $\bar{v}$-dependent scalar
   - $\bar{b}, \bar{e}$: $\bar{v}$-dependent vectors
   - $M^{T} = M$: $\bar{v}$-dependent $3 \times 3$-matrix

   Pure boosts are therefore characterized by a symmetric transformation of the spatial coordinates at $t = 0$ in $K$. Geometrically, this implies that there are three (orthogonal) lines through the origin of $K$ which are mapped onto themselves under the boost (spanned by the eigenvectors of $M(\bar{v})$). The only other possibility is that there is a single invariant line, which then coincides with the rotation axis of a spatial rotation mixed into the boost. The pure boosts are therefore those boosts without any rotation mixed in.

   → We focus on pure boosts in the remainder of this derivation:
Our characterization of a pure boost does not imply that at \( t = 0 \) the axes of the two systems \( K \) and \( K' \) align (as suggested by the sketch and naively expected). If this were the case, the eigenbasis of \( M (\vec{u}) \) would be given by the basis vectors \( \hat{e}_i \) in \( K \). Since we do not know the form of \( M (\vec{u}) \) (yet), we cannot make this assumption! So do not take this sketch literally, it only illustrates symbolically the situation of a pure boost in an arbitrary direction.

**6 | Isotropy:**

Here are two lines of arguments that use isotropy \( IS \) to restrict the form of Eq. (1.42) further:

- **Argument A:**
  
  - We claim that isotropy \( IS \) requires the following multiplicative structure for pure boosts and rotations:

    \[
    \Lambda_R \Lambda_{\vec{u}} \Lambda_{R^{-1}} \overset{1}{=} \Lambda_{\vec{u}R} \Leftrightarrow \forall x : \Lambda_R \Lambda_{\vec{u}} x = \Lambda_{R^{-1}} \Lambda_{\vec{u}} \Lambda_R x. \quad \text{(1.43a)}
    \]
    \[
    \Leftrightarrow \forall x : \Lambda_{\vec{u}} x = \Lambda_{R^{-1}} \Lambda_{\vec{R}} (\Lambda_R x). \quad \text{(1.43b)}
    \]

  The reasoning goes as follows:

  1. \( \overset{1}{=} \) Left-hand side of Eq. (1.43b):
     
     \( x = (t, \vec{x}) \) are the coordinates of some event in \( K \) and \( \Lambda_{\vec{u}} x \) of the same event in \( K' \):

        \[
        \begin{array}{c}
        K' \\
        \beta
        \end{array}
        \]

    2. \( \overset{1}{=} \) Right-hand side of Eq. (1.43b):

     We consider \( y = (t', \vec{y}) := \Lambda_R x = (t, R \vec{x}) \) as an active transformation, i.e., \( y \) denotes a different event that is spatially rotated from \( x \) by \( R \). To state our isotropy claim \( IS \), we now rotate the coordinate system \( K'' \) in which we want to express this event *in the same way*. This implies a rotated boost \( \Lambda_{R''} \) and a subsequent rotation of the coordinate axes by \( R \) via \( \Lambda_{R^{-1}} \). (Remember that when rotating the coordinate axes by \( R \), the coordinates of an event transform by \( \Lambda_{R^{-1}} \).)
3. Spatial isotropy is the property that the event $x$ as seen from $K'$ cannot be distinguished from the rotated event $y$ as seen from the rotated system $K''$; this is Eq. (1.43b).

A shorter (but less rigorous) line of arguments goes as follows:

- $a(\vec{v}) \defeq a(R\vec{u}) \Rightarrow a(\vec{v}) = a_v$ with $v = |\vec{v}|$
  - Functions invariant under arbitrary rotations can only depend on the norm $|\vec{v}|$.
- $b(\vec{v}) \defeq R^T b(R\vec{u}) \Rightarrow b(\vec{v}) = b_v \vec{v}$
  - Note that $b(R\vec{u}) \cdot R\vec{x} = [R^T b(R\vec{u})] \cdot \vec{x}$. Let $R_\varepsilon$ be some rotation with axis $\vec{v} = \vec{v}/v$ such that $R_\varepsilon \vec{v} = \vec{v}$; then $b(\vec{v}) \defeq R_\varepsilon^T b(\vec{v})$ and therefore $b(\vec{v}) \propto \vec{v}$ since rotation matrices have only a single eigenvector.
- $RM(\vec{v}) \defeq M(R\vec{u}) \cdot R \Rightarrow M(\vec{v}) = c_v \mathbf{1} + d_v \vec{v}\vec{v}^T$
  - First recall that $M^T(\vec{v}) = M(\vec{v})$ such that $M(\vec{v})$ can be written as sum of orthogonal projectors (projecting onto its eigenspaces). It is in particular $R_\varepsilon M(\vec{v}) R_\varepsilon^T \defeq M(\vec{v})$ such that one of the eigenvectors must be $\vec{v} \propto \vec{v}$. The remaining two eigenvectors are orthogonal to $\vec{v} \propto \vec{v}$ and can therefore be mapped onto each other by $R_\varepsilon$. Since $R_\varepsilon$ commutes with $M(\vec{v})$, their eigenvalues must be degenerate such that the two-dimensional subspace orthogonal to $\vec{v}$ is a degenerate eigenspace. The most general spectral decomposition of $M(\vec{v})$ is then the one given above.
- $R\vec{e}(\vec{v}) \defeq \vec{e}(R\vec{u}) \Rightarrow \vec{e}(\vec{v}) = e_v \vec{v}$
  - This is the same argument as for $b(\vec{v})$.

**Argument B:**

A shorter (but less rigorous) line of arguments goes as follows:

- To define the unknown functions algebraically, we are only allowed to use the vector $\vec{v}$ and constant scalars. We cannot use $\vec{x}$ or $t$ due to linearity, and any other constant vector (like $\vec{e}_\varepsilon = (1,0,0)^T$) would pick out some direction and therefore violate isotropy.
- Since the only scalar one can construct from a single vector is its norm, $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$, it must be $a(\vec{v}) = a_v$.
- Similarly, since the only vector one can construct from a single vector is a scalar multiplied by the vector itself, it must be $b(\vec{v}) = b_v \vec{v}$ and $\vec{e}(\vec{v}) = e_v \vec{v}$.
- Lastly, since $M^T(\vec{v}) = M(\vec{v})$, we can decompose the matrix into orthogonal projectors: $M(\vec{v}) = \sum_i \lambda_i(\vec{v}) P_i(\vec{v})$. The only projectors that can be defined by a single vector are $P_0 = \vec{v}\vec{v}^T$ and $P_1 = \mathbf{1} - P_0 = \mathbf{1} - \vec{v}\vec{v}^T$ which leads to the most general form $M(\vec{v}) = c_v \mathbf{1} + d_v \vec{v}\vec{v}^T$.

Both arguments lead to the same form for pure boosts $\Lambda_\varepsilon$ consistent with isotropy:

\[
\begin{align*}
t' &= a_v t + b_v (\vec{v} \cdot \vec{x}) \\
\vec{x}' &= c_v \vec{x} + \frac{d_v}{v^2} \vec{v}(\vec{v} \cdot \vec{x}) + e_v \vec{v} t
\end{align*}
\]  
with $v = |\vec{v}| = |R\vec{v}|$ and $(R\vec{v} \cdot R\vec{x}) = (\vec{v} \cdot \vec{x})$. 
7 | Trajectory of origin $O'$ of $K'$:

- In $K'$: $\vec{x}_{O'} = 0$ (This is the operational definition of the origin $O'$.)
- In $K$: $\vec{x}_{O'} = \vec{v}t$ (This is the operational definition of $\vec{v}$ in $K$)

In Eq. (1.45b):

$$
\vec{0} = c_v \vec{v}t + \frac{dv}{c_v^2} \vec{v}(\vec{v} \cdot \vec{v})t + e_v \vec{v} t
$$
\[ (1.46a) \]

$$
\vec{v} \neq \vec{0} \& \forall t \Rightarrow 0 = c_v + d_v + e_v
$$
\[ (1.46b) \]

8 | Reciprocity:

i | Inverse transformation $K' \xrightarrow{\vec{v}, \vec{0}, \vec{0}} K$ from $K'$ to $K$:

$$
\Lambda_{\vec{v}}, \Lambda_{\vec{0}} = 1 \Leftrightarrow \Lambda_{\vec{v}} = \Lambda_{\vec{0}}^{-1}.
$$
\[ (1.47) \]

Note that $\vec{v}$ is the velocity of the origin $O$ of $K$ as measured in $K'$.

In general: $\vec{v}' = \vec{V}(\vec{v})$ with unknown function $\vec{V}$.

We assume reciprocity: $\vec{v}' = -\vec{v}$ such that

$$
\Lambda_{\vec{v}}^{-1} = \Lambda_{-\vec{v}}.
$$
\[ (1.48) \]

While this is clearly the most reasonable/intuitive assumption, it is not trivial! Recall that $\vec{v}$ is the speed of the origin $O'$ of $K'$ measured with the clocks in $K$, whereas $\vec{v}'$ is the speed of the origin $O$ of $K$ measured with different clocks in $K'$. So without additional assumptions we cannot conclude that the results of these measurements yield reciprocal results.

However, the assumption of reciprocity can be rigorously derived from relativity SR, isotropy IS and homogeneity HO, see Ref. [32]. Reciprocity is therefore not an independent assumption.

ii | Inverse transformation in Eq. (1.45):

$$
t = a_v t' - b_v (\vec{v} \cdot \vec{x}')
$$
\[ (1.49a) \]

$$
\vec{x} = c_v \vec{x}' + \frac{dv}{c_v^2} \vec{v}(\vec{v} \cdot \vec{x}') - e_v \vec{v} t'
$$
\[ (1.49b) \]

iii | Eq. (1.49) in Eq. (1.45) & Eq. (1.46b) → (we suppress the $v$ dependence)

$$
c^2 = 1,
$$
\[ (1.50a) \]

$$
a^2 - ev^2 = 1,
$$
\[ (1.50b) \]

$$
e^2 - ev^2 = 1,
$$
\[ (1.50c) \]

$$
e(a + e) = 0,
$$
\[ (1.50d) \]

$$
b(a + e) = 0.
$$
\[ (1.50e) \]

To show this, use $\vec{v} = (v_x, 0, 0)^T$ with $v_x \neq 0$ and remember that the equations you obtain from plugging Eq. (1.49) into Eq. (1.45) must be valid for all $t'$ and $\vec{x}'$. Use Eq. (1.46b) to replace $c_v + d_v$ by $-e_v$.

We can conclude:
\[
\begin{align*}
7 & \text{ Collecting results from Eq. (1.50) & Eq. (1.46b):} \\
& \quad c = 1, \quad e = -a, \quad d = a - 1, \quad b = \frac{1-a^2}{av^2}.
\end{align*}
\]

\[d = a - 1 \text{ follows from Eq. (1.46b) and the first two equations.}
\]

Eq. (1.45) \[\rightarrow\] Eq. (1.51)

\[
\begin{align*}
t' & = a_v t + \frac{1-a^2}{av^2} (\hat{v} \cdot \vec{x}) \\
\vec{x}' & = \vec{x} + [a_v - 1] (\hat{v} \cdot \vec{x}) - va_v \hat{v} t
\end{align*}
\]

with \(\hat{v} := \vec{v}/|\vec{v}|\).

\[\langle \text{ Special boost } \vec{v} = (v_x, 0, 0)^T \text{ in } x\text{-direction:}\]

\[
\begin{align*}
t' & = a_v t + \frac{1-a^2}{v_x a_v} x \\
x' & = a_v x - v_x a_v t \\
y' & = y \\
z' & = z
\end{align*}
\]

Note that \(v = |v_x| \text{ with } v_x \in \mathbb{R}\).

Matrix form:

\[
\begin{pmatrix}
t' \\
x' \\
y' \\
z'
\end{pmatrix} =
\begin{pmatrix}
a_v & \frac{1-a^2}{v_x a_v} \\
-v_x a_v & a_v \\
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
t \\
x \\
y \\
z
\end{pmatrix}
\]

In the following, we refer to the upper 2 \times 2-block as \(A(v_x)\).

\[\langle \text{ Group structure:}\]

\[i \quad \text{ Relativity principle } \mathbf{SR} \rightarrow \]

\[
\psi(K' \xrightarrow{R_2, \vec{v}_2, s_2, \vec{b}_2} K'') \circ \psi(K \xrightarrow{R_1, \vec{v}_1, s_1, \vec{b}_1} K') \equiv \psi(K \xrightarrow{R_3, \vec{v}_3, s_3, \vec{b}_3} K'')
\]
for some parameters \((R_3, \vec{v}_3, s_3, \vec{b}_3)\) that are a function of \((R_i, \vec{v}_i, s_i, \vec{b}_i)_{i=1,2}\).

In words:
The concatenation of a coordinate transformations from \(K\) to \(K'\) and from \(K'\) to \(K''\) must be another coordinate transformation that is parametrized by data that relates the reference systems \(K\) with \(K''\) directly (without referring to \(K'\) in any way).

You may ask why Eq. (1.55) is a constraint on \(\psi\) in the first place. After all, we could just define that

\[
\psi(K \xrightarrow{R_3, \vec{v}_3, s_3, \vec{b}_3} K') := \psi(K' \xrightarrow{R_2, \vec{v}_2, s_2, \vec{b}_2} K'') \circ \psi(K \xrightarrow{R_1, \vec{v}_1, s_1, \vec{b}_1} K').
\]

The problem is that the function defined such generically depends on 8 (!) parameters \(R_1, \vec{v}_1, s_1, \vec{b}_1, R_2, \vec{v}_2, s_2, \vec{b}_2\) — it is a non-trivial functional constraint on \(\psi\) that these can be compressed to four parameters \(R_3, \vec{v}_3, s_3, \vec{b}_3\). This "compression" is mandated by the relativity principle \(\text{SR}\) according to which all inertial systems must be treated equally. In particular, the transformation between two systems \(K\) and \(K''\) can only depend on parameters that can be experimentally determined from within these two systems. (The existence of) a third frame \(K'\) cannot be of relevance for this transformation as this would make \(K'\) special.

Combined with the existence of an inverse transformation \((\leftarrow\) above):

→ The set of all transformations forms a \(\odot\) (multiplicative) group.

Note that **associativity** is implicit since we talk about the concatenation of linear/affine maps.

ii In particular:

\[
\Lambda_{v_x} \Lambda_{u_x} \overset{!}{=} \Lambda_{w_x} \iff A(v_x)A(u_x) \overset{!}{=} A(w_x)
\]

(1.57)

where \(w_x = W(v_x, u_x)\) has to be determined.

- \(\leftarrow 1\) ! Using the restricted form of the boost Eq. (1.54) that followed from previous arguments, it follows indeed that the concatenation of two pure boosts \(\text{in the same direction}\) has again the form of a pure boost (in the same direction). For the arguments that follow, this is sufficient.

However, in general, the multiplicative group structure Eq. (1.55) allows for two boosts to concatenate to a **combination** of boosts and rotations. As we will see \(\rightarrow\) later, this is indeed what happens: The concatenation of two pure boosts (in different directions) produces a boost with a rotation mixed in (\(\uparrow\) Thomas-Wigner rotation).

- \(\rightarrow\) Note that due to Eq. (1.43a) all that follows holds for any pair of collinear velocities \(\vec{u}\) and \(\vec{v}\) (there is nothing special about the \(x\)-direction). Indeed, let \(R\) be a rotation that maps \(\vec{u}\) and \(\vec{v}\) to vectors on the \(x\)-axis, \(\vec{v}_x := R\vec{v}\) and \(\vec{u}_x := R\vec{u}\). Then

\[
\Lambda_{\vec{v}} \Lambda_{\vec{u}} \overset{1.43a}{=} \Lambda_R^{-1} \Lambda_{\vec{v}_x} \Lambda_{\vec{u}_x} \Lambda_R \overset{1}{=} \Lambda_{R^{-1} \vec{v}_x} \Lambda_{R} \overset{1.43a}{=} \Lambda_{\vec{u}}
\]

(1.58)

where \(\vec{w}\) is again collinear with \(\vec{v}\) and \(\vec{u}\).

\(\rightarrow\) (use that the diagonal elements of \(A(w_x)\) must be equal)

\[
\forall_{v_x, u_x} : \frac{1 - a_u^2}{v_x^2 a_v^2} = \frac{1 - a_u^2}{u_x^2 a_v^2}
\]

(1.59)

\(\rightarrow\) Universal constant:

\[
\kappa := \frac{a_v^2 - 1}{v_x^2 a_v^2} = \text{const}
\]

(1.60)
Note: $[\kappa] = \text{Velocity}^{-2}$

\[
a_v = \frac{1}{\sqrt{1 - \kappa v^2}}. \tag{1.61}
\]

We use the positive solution for $a_v$ since $\lim_{v \to 0} A \left( \frac{t}{v} \right) \equiv 1$, i.e., $\lim_{v \to 0} a_v \equiv 1$.

iii | With this we check: $A(v_x)A(u_x) \equiv A(w_x)$ with

\[
w_x = W(v_x, u_x) \equiv \frac{v_x + u_x}{1 + u_x v_x \kappa} \tag{1.62}
\]

Eq. (1.62) becomes important later: it tells us how to add velocities in special relativity.

12 | Preliminary result:

Eq. (1.52) & Eq. (1.60) $\rightarrow$ Boost $\Lambda \tilde{v}$ in direction $\tilde{v}$ with velocity $\tilde{v} = v \hat{v}$:

\[
\begin{align*}
t' &= a_v \left[ t - \kappa (\tilde{v} \cdot \tilde{x}) \right] \tag{1.63a} \\
\tilde{x}' &= \tilde{x} + [a_v - 1] \hat{v} (\tilde{v} \cdot \tilde{x}) - a_v \tilde{v} t \tag{1.63b}
\end{align*}
\]

with

\[
a_v = \frac{1}{\sqrt{1 - \kappa v^2}}. \tag{1.64}
\]

This is the most general transformation between two inertial coordinate systems that move with relative velocity $\tilde{v}$ (with coinciding axes at $t = 0$) that is consistent with our basic assumptions stated at the beginning of this section: SR, HO, and IS.

The only undetermined parameter left is $\kappa$.

1.5. The Lorentz transformation

The purpose of this section is to select the value for $\kappa$ that describes our reality.

13 | Since $[\kappa] = \text{Velocity}^{-2}$ define formally: $\kappa \equiv 1/v^2_{\text{max}}$.

Why we subscribe the velocity $v_{\text{max}}$ with "max" will become clear below.

14 | Three cases:

• $\kappa = 0 \iff v_{\text{max}} = \infty$:

\[
\begin{align*}
\text{Eq. (1.63)} \Rightarrow \quad & t' = t \\
& \tilde{x}' = \tilde{x} - \tilde{v} t \quad \{ \text{Galilei boost} \} \tag{1.65a}
\end{align*}
\]

$\rightarrow$ Maxwell equations are not form-invariant under $\varphi$. 

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Maxwell equations cannot be correct and must be modified.

Experiment that shows the invalidity of Maxwell equations?

Note that we cannot conclude the validity of classical mechanics from this; Newton’s equations may still require modifications (without spoiling the Galilean symmetry, of course).

\[ \kappa > 0 \iff v_{\text{max}} < \infty \]

\[
\begin{align*}
\text{Eq. (1.63)} \Rightarrow \\
t' &= \gamma \left( t - \frac{\mathbf{v} \cdot \mathbf{x}}{v_{\text{max}}} \right) \\
\mathbf{x}' &= \mathbf{x} + (\gamma - 1) \frac{\mathbf{v}}{v_{\text{max}}} (\mathbf{v} \cdot \mathbf{x}) - \gamma \mathbf{v} t
\end{align*}
\]

\(*\ast \text{ Lorentz boost} (1.66a)

with the \(*\ast \text{ Lorentz factor}

\[
\gamma_v \equiv \gamma := \frac{1}{\sqrt{1 - \beta^2}} \quad \text{and} \quad \beta := \frac{v}{v_{\text{max}}}
\]

\[ (1.67) \]

→ Newton’s equations are not form-invariant under \( \varphi \).

→ Classical mechanics cannot be correct and must be modified.

→ Experiment that shows the invalidity of Newton’s equations?

Similarly, we cannot conclude the validity of electrodynamics from this; Maxwell equations may still require modifications (without spoiling the Lorentz symmetry).

\[ \kappa < 0 \] Physically not relevant. (Problem 2; we ignore this solution in the following.)

This solution is not self-consistent (see e.g. Ref. [31]) and immediately leads to implications that are not observed in nature.

For example, the rule Eq. (1.62) to compute the velocity \( w_x \) between \( K / K'' \) from the velocities \( v_x \) and \( u_x \) between \( K / K' \) and \( K' / K'' \) reads for \( \kappa < 0 \)

\[
w_x = \frac{v_x + u_x}{1 - u_x v_x |\kappa|}.
\]

(1.68)

Let \( u_x, v_x > 0 \) be positive, i.e., \( K' \) moves in positive \( x \)-direction wrt \( K \) and \( K'' \) moves also in positive \( x \)-direction wrt \( K' \). But for large enough velocities \( u_x v_x > 1/|\kappa| \) we find \( w_x < 0 \) such that \( K'' \) moves in negative \( x \)-direction wrt \( K \).

No such effect has ever been observed; if you do, let us know!

Note that at no point we used or claimed that \( v_{\text{max}} \) is the speed of light!

Which transformation describes reality: \( v_{\text{max}} < \infty \) or \( v_{\text{max}} = \infty \)?

Evidence:

• Maximum velocity \( v_{\text{max}} \approx c < \infty \) for electrons (plot from Ref. [33]):
Newton’s equations are clearly invalid for high velocities!
See Refs. [33,34] for more technical details. Note that these results were obtained decades after Einstein published his seminal paper in 1905.

By contrast:
No evidence for the invalidity of Maxwell equations (on the macroscopic level).
Electrodynamics, as encoded by the Maxwell equations, is of course not a truly fundamental theory as it is the classical limit of a quantum theory: Quantum electrodynamics (QED). For example, the linearity of the Maxwell equations (= EM waves cannot scatter off each other) is an approximation; in QED photons can (weakly) scatter off each other! This is why I emphasize that Maxwell theory is experimentally valid only on the macroscopic level. Note, however, that QED has the same spacetime symmetry group as electrodynamics, namely Lorentz transformations.

Hence it is reasonable stipulate $v_{\text{max}} < \infty$ and postulate:

The transformations $\varphi$ between inertial systems are given by Lorentz transformations.
These transformations must be (part of) the spacetime symmetries of all physical theories.

The last statement is often rephrased as follows:

All (fundamental) theories must be form-invariant (covariant) under Lorentz transformations.

This is just SR all over again: The equations of models that describe reality must “look the same” (more precisely: be functionally equivalent) in all inertial systems. Since the transformations between inertial systems are given by Lorentz transformations (and not Galilean transformations, as historically anticipated), this requires their form-invariance under Lorentz transformations.

→ SPECIAL RELATIVITY restricts the structure of all fundamental theories of physics!
This is what is meant by the statement that SPECIAL RELATIVITY is a theoretical framework (German: Rahmentheorie) or “meta theory”: It provides a “recipe” (ordering principle) of how to construct consistent theories of physics. The Standard Model of particle physics, for example, is form-invariant under Lorentz transformations, and if you propose an extension thereof (for example to give neutrinos a mass) you better make sure that the terms you write down are also form-invariant under Lorentz transformations (otherwise you will not be taken seriously!). Note,
however, that this perspective prevents an important insight: What we really study is an entity called spacetime, and this entity has a property: Lorentz symmetry. Since all our (fundamental) physical theories are formulated on spacetime, it should not come as a surprise that the Lorentz symmetry of spacetime shows up all over the place.

17 | Interpretation of $v_{\text{max}}$:

i | Systems $K \xrightarrow{v_x} K'$ and signal with velocity $\frac{dx'}{dt} = u'_x$:

\[
\varphi(K' \xrightarrow{v_2} K'') \circ \varphi(K \xrightarrow{v_1} K') = \varphi(K \xrightarrow{v_3} K'') \quad \text{with} \quad v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{v_{\text{max}}}}. \quad (1.69)
\]

Let $v_1 = v_x$ and $v_2 = u'_x$ so that $v_3 = u_x$ (i.e., the signal is at rest in the origin of $K''$).

You can also derive this by computing the time derivative of the position of the signal in $K$ using a Lorentz transformation; you will do this properly when you derive a more general addition of velocities ($\otimes$ Problemset 2).

ii | Addition formula for collinear velocities:

\[
u_x = \frac{v_x + u'_x}{1 + \frac{v_x u'_x}{v_{\text{max}}}} \quad (1.70)
\]

Because of isotropy [15] this formula must be true in all directions (not just in $x$-direction) as long as the two velocities to be added are parallel. We still keep the index $x$ to signify that these are not absolute values of velocities.

- Note that for $v_{\text{max}} \to \infty$ we get back the “conventional” (Galilean) additivity of velocities:

\[
u_x = (v_x + u'_x) \left[1 - \frac{v_x u'_x}{v_{\text{max}}} + \ldots \right] \xrightarrow{v_{\text{max}} \to \infty} v_x + u'_x \quad (1.71)
\]

From this expansion and the validity of classical mechanics for small velocities (in particular its law for adding velocities), we can also conclude that $v_{\text{max}}$ must be large compared to everyday experience.

- A historically influential experiment that (in hindsight) can be explained by the relativistic addition of velocities Eq. (1.70) is the Fizeau experiment [35,36] (see also Fresnel drag coefficient). The Fizeau experiment was one of the crucial hints that led Einstein to special relativity.
iv. \( 0 \leq v_x, u_x' \leq v_{\text{max}} \) \( (\tilde{v}_x := v_x / v_{\text{max}} \text{ so that } 0 \leq \tilde{v}_x, \tilde{u}_x \leq 1) \)

\[
\begin{aligned}
 u_x &= v_{\text{max}} \frac{\tilde{v}_x + \tilde{u}_x'}{1 + \tilde{v}_x \tilde{u}_x'} \leq v_{\text{max}} \\
\end{aligned}
\tag{1.72}

Here we used that \( a + b \leq 1 + ab \) for numbers \( 0 \leq a, b \leq 1 \).

→ “Addition” of velocities Eq. (1.70) never exceeds \( v_{\text{max}} \).

→ \( v_{\text{max}} \) plays the role of a maximum velocity.

v. \( \ll 0 \leq u_x, u_x' \leq v_{\text{max}} \) \( (u_x' = v_{\text{max}}) \)

\[
\begin{aligned}
 u_x &= \frac{v_{\text{max}} + v_x}{1 + \frac{v_{\text{max}} v_x}{v_{\text{max}}}} = v_{\text{max}} \frac{v_{\text{max}} + v_x}{v_{\text{max}} + v_x} = v_{\text{max}} \\
\end{aligned}
\tag{1.73}

Note that the result is completely independent of the velocity \( v_x \) of \( K' \)!

→ Whatever moves with the maximum velocity \( v_{\text{max}} \) does so in all inertial systems!

Please appreciate how counterintuitive this effect is from the perspective of everyday experience!
But also notice that we didn’t have to postulate it: The relativity principle \( \text{SR} \) together with the existence of a (finite) maximum velocity is sufficient.

If you think about it: Assuming a maximum velocity (in the absence of a preferred reference frame) automatically invalidates the simple Galilean law of additive velocities. So it is actually not surprising at all that the maximum velocity must be independent of the reference system.
Experiments (in particular: the validity of Maxwell equations) show:

\[ v_{\text{max}} = c = 299792458 \text{ m s}^{-1} \]  

(1.74)

Note that since 1983 the value of \( c \) in the international system of units (SI) is exact by definition.

A. Einstein incorporated this insight in §2 of Ref. [9] as his second postulate:

2. Jeder Lichtstrahl bewegt sich im "ruhenden" Koordinatensystem mit der bestimmten Geschwindigkeit \( V \), unabhängig davon, ob dieser Lichtstrahl von einem ruhenden oder bewegten Körper emittiert ist.

Note that at the time it was conventional to denote the speed of light with a capital \( V \). The convention switched to our now standard lower-case \( c \) just a few years later. For more historical background:

⇔ https://math.ucr.edu/home/baez/physics/Relativity/SpeedOfLight/c.html

We can condense this into:

§ Postulate: Constancy of the speed of light \( \text{SL} \)

The speed of light is independent of the inertial system in which it is measured.

Comments:

• If you take the validity of the Maxwell equations for granted, then \( v_{\text{max}} = c < \infty \) (and thereby \( \text{SL} \)) follows immediately from the relativity principle \( \text{SR} \) because then the Maxwell equations must be valid in all inertial systems. But you’ve learned in your course on electrodynamics that the wavelike solutions of these equations always propagate with group velocity \( c \) in vacuum. This is only possible if the speed of light plays the role of the limiting velocity:

\[ v_{\text{max}} = c. \]

Einstein acknowledges as much at the beginning of Ref. [10]. However, \( \text{SL} \) is empirically weaker than claiming the validity of Maxwell’s equations (after all, there could be alternative equations that also predict the velocity \( c \) of wavelike solutions). At the time when Einstein formulated \( \text{SL} \) in [9], he also worked on the photoelectric effect (another of his \textit{annus mirabilis} papers [37]). The postulation of “quanta of light” is the foundation of quantum mechanics, but cannot be explained by Maxwell’s equations. It is therefore reasonable to assume that Einstein didn’t want to rely on the validity of this specific theory when formulating his \textbf{SPECIAL RELATIVITY}. He therefore opted for the empirically weaker (but still sufficient) assumption \( \text{SL} \).

• If you derive the transformation \( \varphi \) using both postulates \( \text{SR} \) and \( \text{SL} \), the derivation is shorter (see e.g. [1] or [4]); one then of course doesn’t find the Galilei transformations as an option. Note, however, that the relativity principle \( \text{SR} \) is a reasonable and intuitive starting point that doesn’t need much convincing (after all, we witness the relativity of Newtonian mechanics in our everyday life). By contrast, the speed of light postulate \( \text{SL} \) clashes directly with our everyday experience (how velocities add up, that is). Through our elaborate derivation we learned how much is already implied by the simple, reasonable assumption of relativity. We
only had to check whether there is any evidence of a finite maximum velocity \( v_{\text{max}} \). The counterintuitive feature that this velocity is the same viewed from all inertial systems was then a necessary conclusion from our derivation.

† **Note: Finite speed of causality (Locality)**

Another insight from our SR-based derivation of the Lorentz transformation is that the formulation of the speed-of-light postulate SL is conceptually misleading:

- The constant \( v_{\text{max}} \) and its role as a maximum velocity followed *without* referring to light (or electrodynamics) in any way!  
  
  *Put bluntly: Special Relativity is *not* about the “strange behavior” of light!*

- The relevant speed for special relativity is the speed of causality: How fast can information travel, i.e., one event affect another. \( v_{\text{max}} \) is the maximum speed of causal interactions, irrespective of the mediator of these interactions.

  In our world, the fastest and most salient information carrier just happens to be the electromagnetic field (“light”). For example, to synchronize our clocks with light signals, it wasn’t the light *per se* we were interested in; we just used it as carrier of information to correlate the clocks.

- Given the relativity principle SR and our derivation in Section 1.4, we showed that there are only two possibilities: (1) There is no upper bound on velocities (Galilean symmetry) or (2) there is such an upper bound \( v_{\text{max}} \) (Lorentz symmetry). In the latter case, every signal that propagates with \( v_{\text{max}} \) in some frame automatically does so in all inertial systems. (Which immediately leads to the counterintuitive conclusion, akin to SL, that there are signals the velocity of which does not depend on the velocity of the observer.)

- We could replace SL therefore by the (empirically weaker) postulate that there are no instantaneous actions at a distance (this is essentially a statement about locality). This modified postulate implies the existence of a maximal velocity \( v_{\text{max}} < \infty \) which, in turn, selects the Lorentz transformation as the correct symmetry. That \( v_{\text{max}} = c \) is then a fact to be discovered by experiments.

- It turns out that *everything with vanishing rest mass* travels at this maximum speed \( v_{\text{max}} = c \). Since photons are the only elementary particles that are massless and can be easily detected, we just happen to refer to this maximum velocity as “speed of light.”

  For example: Without Higgs symmetry breaking, the \( W^\pm \) and \( Z \) bosons of the weak interaction are massless and would propagated with light velocity, just as the photon (the weak interactions would then be no longer “weak”). For a long time it was believed that neutrinos are massless as well, and therefore would also propagate with the speed of light (today we know that they have a very tiny mass).

Special Lorentz transformations = Lorentz boosts:

Now that everything is settled, let us write down our final result in their conventional form.

† These are not the most general (homogeneous) Lorentz transformations since we omit rotations, parity and time inversion. We will discuss the structure of the full homogeneous Lorentz group and its inhomogeneous generalization (→ Poincaré group) later. To discuss the “fancy” phenomena of special relativity, the transformations below are sufficient.
i | Boost in arbitrary directions ($\vec{v} = v\dot{v}$ with $\dot{v} \equiv \vec{v}/|\vec{v}|$):

\[ \Lambda(K \overset{\vec{v}}{\to} K') : \begin{cases} ct' = \gamma (ct - \beta \vec{x} \cdot \vec{v}) \\ \vec{x}' = \vec{x} + (y - 1)(\vec{x} \cdot \vec{v}) \cdot \vec{v} - \gamma \vec{v}t \end{cases} \]  

(1.75)

(Since we now settled on Lorentz transformation for $\varphi$, we write $\varphi = \Lambda$ henceforth.)

with $\beta \equiv v/c$ and the Lorentz factor

\[ \gamma_v \equiv \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}} . \]  

(1.76)

ii | Special case: Boost in $x$-direction ($\vec{v} = v_x\hat{x}$):

\[ \Lambda(K \overset{v_x}{\to} K') : \begin{cases} ct' = \gamma (ct - \frac{v_x}{c}x) \\ x' = \gamma (x - v_xt) \\ y' = y \\ z' = z \end{cases} \]  

(1.77)

State of affairs:

Now that we know the spacetime symmetry $\varphi$ of reality, we have quite a to-do list:

- We will have to modify Newton’s equations to replace their Galilean by a Lorentz symmetry, without changing their predictions for small velocities $v \ll c$ ($\varphi$ correspondence principle).

  → Relativistic mechanics

- We can keep the Maxwell equations in their current form ☑.

  Note that we still have to check that they are really Lorentz covariant (☑ Problemset ?)!

  In the end we will come up with a neat notation that allows us to rewrite (not modify!) the Maxwell equations in a compact form to make their Lorentz symmetry apparent.

- Similar to classical mechanics, we will have to replace the Schrödinger equation in quantum mechanics by a modified version with Lorentz symmetry.

  → Relativistic quantum mechanics (Klein-Gordon and Dirac equation)

But before we do all the heavy work:

Simple implications of this transformation? (→ below and next lectures)

With “simple” we refer to implications that follow without imposing a model-specific dynamics (= equation of motion). We will refer to these implications as kinematic because they follow from fundamental constraints on the degrees of freedom of all relativistic theories.
1.6. Invariant intervals and the causal partial order of events

Trajectory of a light signal in $x$-direction in $K$:

$$x(t) = ct, \quad y = 0, \quad z = 0$$

(1.78)

Trajectory of the same signal in $K'$ with $K \stackrel{v_x}{\longrightarrow} K'$:

$$x'(t') = ct', \quad y' = 0, \quad z' = 0$$

(1.79)

This follows from our previous discussion: signals propagating with $c = v_{\text{max}}$ do so in all inertial systems!

You can also simply calculate this using the Lorentz boost Eq. (1.77):

$$ct' = \gamma (ct - \frac{v_x}{c}ct)$$

and

$$x' = \gamma (ct - v_x t) = ct'.$$

(1.80a)

(1.80b)

$$\rightarrow$$

$$(ct)^2 - x^2 = 0 = (ct')^2 - (x')^2$$ is a frame-independent quantity.

(1.81)

Note that the separate summands $[(ct)^2$ etc.] are not frame-independent!

This finding motivates the definition of the …

2 | Spacetime interval:

Details: Problemset 2

Two events $E_1 \ni (t_1, \bar{x}_1)_K$ and $E_2 \ni (t_2, \bar{x}_2)_K$ with temporal and spatial separation

$$(\Delta t)_K := t_1 - t_2 \quad \text{and} \quad (\Delta \bar{x})_K := \bar{x}_1 - \bar{x}_2 .$$

(1.82)

Then the spacetime interval between $E_1$ and $E_2$ is denoted $(\Delta s)^2 \equiv \Delta s^2$ and defined as

$$(\Delta s)^2 := (c\Delta t)_K^2 - (\Delta \bar{x})_K^2 .$$

(1.83)

We omit the subscript $K$ from $\Delta s$ because it is frame-independent (→ next).

In our example above it was $\Delta t = t \sim 0$ and $\Delta \bar{x} = (x - 0, 0 - 0, 0 - 0)$, i.e., we considered the interval between the event in the origin $x_O = (0, 0)$ and the events along the trajectory $(ct, x(t), 0, 0)$ of the light signal.
The importance of $\Delta s^2$ stems from the following fact:

The spacetime interval $\Delta s^2$ is independent of the frame in which it is calculated.

This means that given two events, all observers agree on the numerical value of the interval $\Delta s^2$ between these two events.

**Proof:** Use Eq. (1.75) to calculate (Details: → Problemset 2)

\[
(t')^2 = \left( y \left( ct - \beta \cdot \hat{v} \cdot \hat{x} \right) \right)^2
\]

\[
(x')^2 = \left[ \hat{x} + (y - 1)(\hat{v} \cdot \hat{u}) \cdot \hat{v} + \gamma \hat{v} \right]^2
\]

\[
(t')^2 - (x')^2 \equiv (ct)^2 - (\hat{x})^2 + \ldots = 0
\]

Note that we do not have to do the computation for two events and an interval $\Delta t$ and $\Delta \hat{x}$ since the special Lorentz transformations are linear.

This proves the invariance under *special* Lorentz transformations (= Lorentz boosts). It is easy to see that the invariance is also valid for inhomogeneous shifts in time and space (these drop out in the intervals $\Delta t$ etc.) and spatial rotations $\Lambda_R$ [since $(\Delta \hat{x})^2$ is clearly invariant under rotations]. We will come back to this when we discuss the structure of the Lorentz group in more detail (→ later).

Two events $E_1$ and $E_2$ are in one of three possible (frame-independent) relations:

\[
\Delta s^2 \begin{cases} 
> 0 & E_1 \text{ and } E_2 \text{ are } time-like \text{ separated} \\
= 0 & E_1 \text{ and } E_2 \text{ are } light-like \text{ separated} \\
< 0 & E_1 \text{ and } E_2 \text{ are } space-like \text{ separated}
\end{cases}
\]

Note that $\Delta s^2$ can be *negative* so that $\Delta s^2$ should be read as a symbol rather than defining an imaginary number $\Delta s$. For the special case of *time-like* intervals, however, $\Delta s^2$ indeed defines a real number $\Delta s = \sqrt{\Delta s^2}$ which we will later relate to the time measured by moving clocks (the so-called *proper time*).

All events that are light-like separated from an event $E$ (*wlog* in the origin) satisfy

\[
\Delta s^2 = 0 \iff (ct)^2 = (\hat{x})^2 \iff |ct| = |\hat{x}|
\]

which determines the *light cone* of $E$:
Here we show the light cone of an event $E$ in a space time with two spatial dimensions $x$ and $y$. The light cone in our 3 + 1 dimensional space time is a higher-dimensional generalization which obeys the same equations.

- Time-like events satisfy $\Delta s^2 > 0 \iff |ct| > |\vec{x}|$ which characterizes the (disconnected) interior of the light cone. The manifold with $ct > |\vec{x}| \geq 0$ is called the future light cone (of $E$) whereas the events with $-ct > |\vec{x}| \geq 0$ make up the past light cone (of $E$).

- Space-like events satisfy $\Delta s^2 < 0 \iff |ct| < |\vec{x}|$ which characterizes the (connected) spacetime volume outside the light cone.

5 Causality:

The importance of the threefold classification of spacetime intervals stems from the following observations.

i Actions of (homogeneous) Lorentz transformations:

Since $\Delta s^2$ is invariant under Lorentz transformations, the manifold of events characterized by a specific value $\Delta s^2 = \pm C$ ($C \geq 0$) must be mapped onto itself under these transformations: Events on these hyperbolic manifolds cannot leave their manifolds under Lorentz transformations.

**Invariant hyperbolae:**

\[
\begin{align*}
\text{time-like:} & \quad \Delta s^2 = C > 0 \quad \Rightarrow \quad ct = \pm \sqrt{C + |\vec{x}|^2} & (1.87a) \\
\text{light-like:} & \quad \Delta s^2 = C = 0 \quad \Rightarrow \quad ct = \pm |\vec{x}| & (1.87b) \\
\text{space-like:} & \quad \Delta s^2 = -C < 0 \quad \Rightarrow \quad ct = \pm \sqrt{|\vec{x}|^2 - C} & (1.87c)
\end{align*}
\]
This picture leads immediately to two conclusions:

ii | Two distinct events $E_1 \equiv (t_1, \vec{x}_1)_K$ and $E_2 \equiv (t_2, \vec{x}_2)_K$ with coordinates in $K$:

- If $\Delta s^2 \geq 0$ (time-like or light-like), then
  
  
  either $\forall K : (t_1)_K > (t_2)_K$ or $\forall K : (t_1)_K < (t_2)_K$.  \hspace{1cm} (1.88)

  This means that for time-like or light-like separated events all observers agree on their temporal ordering! Note that they do not necessarily agree on the time $(t_1)_K - (t_2)_K$ elapsed between the two events.

  **Proof:** Assume $(t_1)_A < (t_2)_A$ and $(t_1)_B > (t_2)_B$ for two inertial systems $A$ and $B$. Because of the continuity of Lorentz transformations there must exist a frame $C$ with $(t_1)_C = (t_2)_C$. But in this frame $(\Delta s)_C^2 = -(\Delta \vec{x})_C^2 \geq 0$ such that $(\vec{x}_1)_C = (\vec{x}_2)_C$ and therefore $E_1 = E_2$ (which contradicts our assumption that the two events are distinct).

  **Proof by picture!**

- If $\Delta s^2 < 0$ (space-like), then

  $\exists_{A, B} : (t_1)_A > (t_2)_A$ and $(t_1)_B < (t_2)_B$.  \hspace{1cm} (1.89)

  This means that for space-like separated events there are always observers who see $E_1$ happening before $E_2$ while other observers see $E_1$ happening after $E_2$. The temporal order of space-like separated events is therefore observer-dependent!

  **Proof:** \( \blacktriangleright \) Problemset ?

  **Proof by picture!**

iii | Conventional relation of time order and causality:

  $E_1$ can causally affect $E_2 \iff E_1$ happens before $E_2$ \hspace{1cm} (1.90)

  Since causality should be an objective, observer-independent fact, and we just showed that only time- and light-like separated events have an observer-independent temporal order, it is reasonable to define the following …

…\( \blacktriangleright \) (strict) partial order \( \prec \) on the set $E$ of events:

- \( E_1 < E_2 \iff \Delta s^2 \geq 0 \) and \( t_1 < t_2 \) : “$E_1$ can affect $E_2$” \hspace{1cm} (1.91)

- \( E_1 > E_2 \iff \Delta s^2 \geq 0 \) and \( t_1 > t_2 \) : “$E_2$ can affect $E_1$” \hspace{1cm} (1.92)
This is a partial order because for $\Delta s^2 < 0$ there is no relation between $E_1$ and $E_2$ (we denote this by $E_1 \nless E_2$).

To be a partial order, one has to show irreflexivity (which is trivial since $t < t$ is not true) and transitivity. To show transitivity, show that $\Delta s^2_{1,2} \geq 0$ and $\Delta s^2_{2,3} \geq 0$ together with $t_2 > t_1$ and $t_3 > t_2$ implies $\Delta s^2_{1,3} \geq 0$ and $t_3 > t_1$ (use the triangle inequality).

This definition of causality is consistent with our previous findings that no signal can travel faster than the speed of light $c$:

- $E \prec E_1$: There exists a signal trajectory $\vec{x}(t)$ with $\left| \frac{d\vec{x}(t)}{dt} \right| \leq c$ connecting the two events (blue in the sketch).
- $E \nless E_3$: Any trajectory $\vec{x}(t)$ connecting the two events (red in the sketch) has some segment with $\left| \frac{d\vec{x}(t)}{dt} \right| > c$ (yellow in the sketch). Since this is physically impossible, there is no signal of any kind that can mediate causal influence from $E$ to $E_3$ (and vice versa).

This follows from an application of (a generalization of) the ↓ mean value theorem.

Since the causal structure $(\mathcal{E}, \prec)$ is observer independent:

There is no relativity of causality in special relativity!

If one observer states that $E_1$ can causally affect $E_2$, then all observers will agree on this statement.

Fun fact:

If one starts from the causal structure $(\mathcal{E}, \prec)$ and derives the group of ↑ causality-preserving automorphisms $\Phi$,

$$E_1 \prec E_2 \iff \Phi(E_1) \prec \Phi(E_2) \quad (1.93)$$

one again finds the homogeneous Lorentz transformations (boosts & rotations) that we constructed above (plus space-inversion, spacetime dilations and translations), see Ref. [38] for more details. Most interestingly, for the proof neither a continuity assumption on $\Phi$ nor a topology on $\mathcal{E}$ is required; all this follows (at least in $2 + 1$ spacetime dimensions and more) from the partial order $\prec$. 
1.7. Relativity, compressibility, and the anthropic principle

The statements in this section are not specific to Einstein’s relativity principle SR.

1 | Relativity principles …
   • … are statements about (the existence of) symmetries of spacetime.
   • … imply the versatility of models to predict events from many viewpoints.
   • … are statements about an a priori unnecessary simplicity of nature.

2 | Imagine a world without any relativity principle:
   The equations (models) that capture physical laws faithfully are different from frame to frame.
   → Your brain must learn arbitrary many different models adapted to all possible reference frames to anticipate the future in all situations.
   → Biologically impossible (your brain capacity is finite, building models is expensive)

3 | Example: Catching balls:

   ![Diagram of catching balls]

   Notice that most reference frames that we naturally encounter are (approximately) inertial only in x and y direction (the axes that are locally parallel to earth’s surface) and constantly accelerated in z direction (the axis perpendicular to earth’s surface; the acceleration is \( g \approx 9.81 \text{ m/s}^2 \)). The non-relativistic symmetries that relate these frames are a subgroup of the full Galilei group (excluding rotations around the x and y axes as well as “large” translations). Our brain contains only models for these frames (equipped with Cartesian coordinates). Have you ever tried throwing or catching a ball in frames with acceleration in x or y directions (like a centrifuge)?
   ⊗ YouTube Video: The artificial gravity lab (Tom Scott)

   Note that it is not impossible to train specific models for other frames to which the relativity principle of our everyday experience does not apply (after some practice you can throw and catch balls in a centrifuge of constant angular velocity). But this is just one additional model and even this is not implemented in our brains by default!

4 | Relativity principle
   → Descriptions of natural phenomena are highly compressible.
   → Only few models (equations) are necessary to anticipate the future.

5 | Anthropic principle:
   Question: Why are there spacetime symmetries / relativity principles in the first place?
   Answer: Because if there were none, evolution would most likely be impossible, hence we would be unable to ask the question.
Note that evolution relies on the somewhat reliable proliferation of information over time. This seems only possible if the individuals carrying this information survive. Surviving in environments with life-threatening phenomena (thunderstorms, predators, ...) relies on its (approximate) predictability by (approximate) models that are learned evolutionary and/or by experience.

For this argument to work some form of “ensemble interpretation” of reality is required (e.g. ↑ multiverses) [39].
2. Kinematic Consequences

In this chapter we study implications of the special Lorentz transformations Eq. (1.75) and Eq. (1.77) that follow without imposing a model-specific dynamics (= equations of motion). We refer to these implications as **kinematic** because they follow from fundamental constraints on the degrees of freedom of all relativistic theories. The phenomena we will encounter are therefore features of spacetime itself – and not of some entities that live on/in (or couple to) spacetime.

¹ The phenomena we will encounter are not “illusions” (in the sense that we “see” things differently than they “really are”). Remember that we precisely defined what we mean by observers/reference frames; in particular, we emphasized that we do not “look” at anything, we *measure* events in a systematic way, using a well-defined structure called a **inertial system**. All phenomena we will encounter are derived from and to be understood in this operational, physically meaningful context.

2.1. Length contraction and the Relativity of Simultaneity

1. Inertial systems $A \overset{v_x}{\longrightarrow} A'$ with rod on $x'$-axis and at rest in $A'$:

   $\text{A} \overset{v_x}{\longrightarrow} \text{A}'$

   Remember that $A \overset{v_x}{\longrightarrow} A'$ denotes a boost in $x$-direction with $v_x$ (as measured in $A$) where the spatial axes of both $A$ and $A'$ coincide at $t = 0$:

   ![Inertial systems diagram]

   In such situations, we refer to $A'$ as the **rest frame** of the rod and $A$ as the **lab frame** (some call $A$ the **stationary frame**). In the following, coordinates of events in the inertial system $A'$ are marked by primes.

2. First, we have to define what we mean by the “length” of an object:

   “Length” is an intrinsically non-local concept. It is not something you can measure or define at a single point in space. Consequently, there are no “length-events” in $\mathcal{E}$. Thus we need an algorithm (= operational definition) of what we mean by “length”.

   **Two event types**:

   \[
   \{e_L\} = \{\text{Left end of rod detected}\} \quad (2.1a)
   \[
   \{e_R\} = \{\text{Right end of rod detected}\} \quad (2.1b)
   \]

   Think of an event type as a set (equivalence class) of all elementary events that you deem **type-identical** (but not **token-identical**). In the example given here, there will be many events $e_L$ in
spacetime that signify “Left end of rod detected” (if there is one rod, there will be one such event for each time \( t \)); these are different events of the same type \( \{e_L\} \).

One could even declare that the event type \( \{e_L\} \) is what we refer to as “the left end of the rod.”

→ Algorithm LENGTH to compute “Length of Rod” in system \( K \) at time \( t \):

```
LENGTH:
+ Input: Coincidences \( \mathcal{E} \), Inertial system label \( K \), Time \( t \)
+ Output: Length \( l_K \) of rod at time \( t \) as measured in \( K \)
  1. Find (unique) event \( L \in \mathcal{E} \) with \( \{e_L\} \in L \) and \((t,\overline{l})_K \in L\).
  2. Find (unique) event \( R \in \mathcal{E} \) with \( \{e_R\} \in R \) and \((t,\overline{r})_K \in R\).
  3. Return \( l_K := |\overline{l} - \overline{r}| \).
```

Here, \( \{e_L\} \in L \) is shorthand for \( \{e_L\} \cap L \neq \emptyset \). In words: the coincidence class \( L \) contains an event of the type “Left end of rod detected”.

Note that we define “length” as the spatial distance between the two ends of the rod at the same time \( t \) (as measured by the clocks in \( K \)). I hope you agree that this is what one typically means by “length.”
We now apply this algorithm twice, in the lab frame $A$ and the rest frame $A'$:

**Rest frame $A'$:**

**Proper length** $\equiv$ **Rest length** $:=$ Length of rod in $A'$:

$$l_0 := \text{LENGTH}(E, t'_0; A') = |\vec{l}_0 - \vec{r}_0'| = |l'_0 - r'_0|$$  \hspace{1cm} (2.2)

with simultaneous clock events $(t'_0, \vec{l}_0)_{A'} \in L_0$ and $(t'_0, \vec{r}_0')_{A'} \in R_0$.

The time $t'_0$ that we choose is irrelevant since the rod is (by definition) at rest in $A'$. Since the rod lies on the $x'$-axis, it is $\vec{l}_0' = (l'_0, 0, 0)$ and $\vec{r}_0' = (r'_0, 0, 0)$.

The subscript “0” in $L_0$ indicates that this is a specific event (coincidence class) we selected in $A'$ to compute the length of the rod. It does not mean “as seen from the rest frame $A''$” or anything like that. Remember that coincidence classes in $E$ are objective information!

**Lab frame $A$:**

Length of moving rod in $A$:

$$l := \text{LENGTH}(E, t; A) = |\vec{l} - \vec{r}|$$  \hspace{1cm} (2.3)

with simultaneous clock events $(t_l, \vec{l})_A \in L$ and $(t_R, \vec{r})_A \in R$ with $t_l = t_R = t$.

The time $t$ that we choose might be irrelevant as well, but we do not know this yet.

There is no reason to assume that the events $L_0/R_0$ chosen in $A'$ to measure the length of the rod are identical to the events $L/R$ used in $A$: $L_0 \neq L$ and $R_0 \neq R$ in general.

How does $l_0$ relate to $l$?

**In Section 1.5** we did a lot of hard work to compute the transformation $\varphi$ which transforms the coordinates of an event in one inertial system into the coordinates of the same event in another inertial system. We identified the transformation as the Lorentz transformation:

$$\Lambda(A \xrightarrow{v} A') : [E]_A = (t, \vec{x}) \mapsto \Lambda_{v,x} x = x' = (t', \vec{x}') = [E]_{A'}$$  \hspace{1cm} (2.4)

So let us use this tool [namely Eq. (1.77)] to obtain the coordinates of the events $L$ and $R$ (used for the length measurement in $A$) in the rest frame $A'$ of the rod:

$$[L]_{A'} = \begin{cases} 
ct'_l = \gamma \left( ct_l - \frac{v_x}{c^2} l_x \right) \\
'x = \gamma(l_x - v_x t_l) \\
y_b' = y_b \\
z_b' = z_b 
\end{cases} \quad \text{and} \quad [R]_{A'} = \begin{cases} 
ct'_R = \gamma \left( ct_R - \frac{v_x}{c^2} r_x \right) \\
r'_x = \gamma(r_x - v_x t_R) \\
y_b' = y_b \\
z_b' = z_b 
\end{cases}$$  \hspace{1cm} (2.5)

Here we use $\vec{l} = (l_x, l_y, l_z)$ and $\vec{r} = (r_x, r_y, r_z)$. Since we declared that the rod is fixed on the $x'$-axis of $A'$ and $\{e_L\} \in L$ and $\{e_R\} \in R$, it must be $l'_x = l'_z = r'_x = r'_z = 0$, and therefore $\vec{l} = (l_x, 0, 0)$ and $\vec{r} = (r_x, 0, 0)$. That is, the rod is not rotated by the boost and always lies on the $x$-axis of $A$ as well. In particular: $l = |\vec{l} - \vec{r}| = |l_x - r_x|$.

Two immediate conclusions:
a | In $A'$ the two events $L$ and $R$ are no longer simultaneous:

$$t_L = t_R \text{ in } A \quad \text{but} \quad t'_L \neq t'_R \text{ in } A' \quad \text{(since } l_x \neq r_x).$$

$\rightarrow$ The simultaneity of events is observer-dependent.

This ambiguity of simultaneity can be graphically illustrated in a spacetime diagram (for details on how to draw the $(t', x')$-axes in $A$: ☀ Problemset 2):

- As a side note, this calculation implies that not only is it generally not true that $L_0 = L$ and $R_0 = R$, it is actually impossible (at least for both pairs).
- In the sketch above, the “interior of rod”-events are painted gray. One is tempted to ask: Which “line” of these events is the rod? The counterintuitive answer is that this depends on the observer: For $A$-observers, horizontal lines of gray events make up “the rod”, whereas for the $A'$-observer tilted lines are “the rod”. It is actually more reasonable to think of the complete area of gray events as “the rod”, just as the event type $e_L$ is “the left edge” of the rod. This suggests that our intuitive concept of the instantaneous existence of extended objects – which feels so natural to us – is, to some extend, misleading.

b | In $A'$ the coordinate distance is different:

$$|l_x' - r_x'| = \gamma |l_x - r_x| \quad \text{v} \neq 0 \neq |l_x - r_x| = l$$

$\dagger$ The time-dependence cancels so that the expressions are time-independent.

At this point, it is a bit premature to identify the left-hand side as the rest length $l_0$ of the rod because these are spatial coordinates of events that are not simultaneous! (Remember that the length of any object in any frame is defined as the coordinate distance of simultaneous events.)

However, since $A'$ is (by definition) the rest frame of the rod, the position labels of the $A'$-clocks adjacent to the ends of the rod are the same for all events:

$$\left\{ \begin{array}{l}
 l_x' \equiv \{e_L\} \in L \\
 r_x' \equiv \{e_R\} \in R
 \end{array} \right\} \Rightarrow |l_x' - r_x'| = |l_0' - r_0'| = l_0$$

(2.7)
**Length contraction ≡ Lorentz contraction:**

A rod of rest length $l_0$ is shorter if measured from an inertial system in relative motion:

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}} \quad v \neq 0 < l_0 \quad (2.9)$$

- Due to isotropy, this result is true for any length of extended objects in the direction of the boost. A rod along the $y'$-axis, for example, is contracted according to Eq. (2.9) for a boost in $y$-direction, but not for a boost in $x$-direction.
- The rod is just a proxy for any physical object; the Lorentz contraction therefore affects all physical objects in the same way. The contraction is not a dynamical feature of the object itself (like a force that compresses the atomic lattice) but an intrinsic property of space(time).
- Note that we say above “if measured from …” and not “as viewed from ….” This distinction is important: If you ask how you would visually perceive extended objects flying by (or how they look on a picture taken by a camera) you have to factor in that the photons bouncing off the object at different points take different times to reach your eye (out the camera sensor). If you do the math (Problemset 3), this additional optical effect leads to the surprising result that 3D objects actually do not look “squeezed” but rotated. This implies in particular that a moving sphere still looks like a sphere and not like an ellipse (Penrose-Terrell effect [40, 41], see also Ref. [42]).

You can experience this effect (among others) in the educational game “A Slower Speed of Light,” which has been developed by the MIT Game Lab for educational purposes, and can be downloaded here for Windows, Mac, and Linux (Problemset 3):

- Download “A Slower Speed of Light”

You should always keep in mind, however, that this “looking” is not what we refer to as observing in relativity; the latter has been defined operationally as a measurement procedure at the beginning of this course.

### 2.2. Time dilation

1. Inertial systems $A \xrightarrow{v_x} A'$ and a clock $\bar{x}'$ at rest in $A'$:

   ![Diagram](attachment:diagram.png)

2. Two events:

   - $A'$-Clock $\bar{x}'$ meets $A$-clock $\bar{x}_0$: $(t'_0, \bar{x}'_0)_{A'} \sim (t_0, \bar{x}_0)_{A} \in E_0 \quad (2.10a)$
   - $A'$-Clock $\bar{x}'$ meets $A$-clock $\bar{x}_1$: $(t'_1, \bar{x}'_1)_{A'} \sim (t_1, \bar{x}_1)_{A} \in E_1 \quad (2.10b)$
The two events $E_0$ and $E_1$ relate three different clocks: The single $A'$-clock $\bar{x}'$ and two different $A$-clocks $\bar{x}_0$ and $\bar{x}_1$.

As for length, the concept of “duration” cannot be defined locally in spacetime. We therefore need an operational definition (algorithm) of “duration”:

**DURATION:**

- **Input:** Two events $E_0$ and $E_1$, Inertial system label $K$
- **Output:** Time interval $\Delta t_K$ between events as measured in $K$

1. Find (unique) clock event $(t_0, \bar{x}_0)_K \in E_0$.
2. Find (unique) clock event $(t_1, \bar{x}_1)_K \in E_1$.
3. Return $\Delta t_K := t_1 - t_0$.

Hopefully you agree that this is a reasonable definition of the duration (or time interval) between two events.

We can now apply this algorithm to determine the time elapsed between $E_0$ and $E_1$:

- In $A'$: $\Delta t' = \text{DURATION}(E_0, E_1; A') = t'_1 - t'_0$ Measured by a single clock! (2.11a)
- In $A$: $\Delta t = \text{DURATION}(E_0, E_1; A) = t_1 - t_0$ Measured by two clocks! (2.11b)

How does $\Delta t$ relate to $\Delta t'$?

1. Since $(t'_0, \bar{x}')_{A'} \sim (t_0, \bar{x}_0)_A$ and $(t'_1, \bar{x}')_{A'} \sim (t_1, \bar{x}_1)_A$, we can use the Lorentz transformation to translate between the coordinates: Inverse of Eq. (1.77)

Remember that $\Lambda_{\gamma}^{-1} = \Lambda_{-\gamma}$ because of reciprocity; the inverse Lorentz transformation can then be obtained by substituting $v_x \mapsto -v_x$:

$$[E_0]_A = \begin{cases} ct_0 = \gamma (ct_0' + \frac{v_x}{c}x') \\ x_0 = \gamma(x' + v_xx_0') \end{cases} \quad \text{and} \quad [E_1]_A = \begin{cases} ct_1 = \gamma (ct_1' + \frac{v_x}{c}x') \\ x_1 = \gamma(x' + v_xx_1') \end{cases}$$ (2.12)

We omit the other two coordinates since they are invariant anyway; the transformation of the spatial coordinate is also not necessary for the following derivation.

2. Subtracting the equations for the time coordinate of both events yields:

$$c(t_1 - t_0) = \gamma c(t'_1 - t'_0)$$ (2.13)

Note that in the inverse Lorentz transformation Eq. (2.12) the position coordinate in $A'$ is $x'$ for both events because the same $A'$-clock takes part in both coincidences.

3. **Time dilation:**

   - The moving clocks in $A'$ run slower than the stationary clocks in $A$:

   $$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{for} \quad v \neq 0$$ (2.14)

   We renamed $\Delta t' = \Delta t_0$ to emphasize the analogy to the proper length $l_0$:

   - $\Delta t_0$: Proper time elapsed in $A'$ between $E_0$ and $E_1$
   - $\Delta t$: Time elapsed in $A$ between $E_0$ and $E_1$
• The characteristic feature of the proper time $\Delta t_0$ between two (time-like separated) events $E_0$ and $E_1$ is that it can be measured by a single inertial clock that takes part in both events. All other time intervals must be measured by subtracting the reading of two different clocks. Eq. (2.14) tells you that these time intervals are always longer than the proper time $\Delta t_0$.

• Due to isotropy, our result above is true for boosts in any direction. Note that in the derivation above, we did not impose any special constraints on the positions of the clocks (except that they coincide pairwise at $E_0$ and $E_1$). In particular, we did not assume (despite the sketch suggesting this) that the clocks are located on the $x'/x$-axis. All clocks in $A'$ are slowed down in the same way, irrespective of their location!

• This result does not contradict our assumption that all clocks are type-identical (= run with the same rate if put next to each other at rest) because the two events needed to compare the tick rate of moving clocks necessarily describe coincidences between different pairs of clocks.

**6 | Relativity principle:**

Because of the relativity principle SR time dilation must be completely symmetrical: The $A'$-clocks run slower compared to the $A$-clocks, and the $A$-clocks run slower compared to the $A'$ clocks. That this is indeed that case (without being a clock “paradox”) is best illustrated in a symmetric spacetime diagram:

The existence of the “median frame” $A''$ between $A \longrightarrow A'$ can be easily shown with the addition for collinear velocities Eq. (1.70). This symmetric form of a spacetime diagram is sometimes called ↑ Loedel diagram [43] and makes the symmetry between inertial frames manifest; in particular, the units on the axes of $A$ and $A'$ are identical (they are not identical to the units of $A''$, tough). In this symmetric form, the $t'$-axis is orthogonal to the $x$-axis and the $t$-axis to the $x'$-axis. Note that because of the relativistic addition of velocities, it is $A'' \xrightarrow{\vec{v}_x} A'$ and $A'' \xrightarrow{-\vec{v}_x} A$ with $\vec{v}_x = v_x \frac{\vec{v}}{\sqrt{1+\frac{v^2}{c^2}}}$ and $\tan(\varphi) = \frac{\vec{v}_x}{c}$ (Problemset 3). Only in the non-relativistic limit $v_x/c \rightarrow 0$ one finds $\vec{v}_x = \frac{v_x}{c}$ as naively expected.

Note that due to the relativity of simultaneity, the two observers use different pairs of clock-events to decide which of the two origin clocks runs slower:

• For $A$ the two clock events $\tilde{D}$ and $C$ are simultaneous such that one has to conclude that the (blue) $A'$-clock runs slower than the (red) $A$-clock.
• By contrast, for the observer $A'$ the two events $D$ and $C$ are simultaneous such that one has to conclude that the (red) $A$-clock runs slower than the (blue) $A'$-clock.

It is evident from the diagram that there is no disagreement about coincidences of events (or readings of clocks). It is just the observer-dependent concept of simultaneity that leads to the seemingly "paradoxical" reciprocity of time dilation.

### Experiments:

**Muon decay** [44]:

Muons quickly decay into electrons (and neutrinos):

$$\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e.$$  \hspace{1cm} (2.15)

This decay can be readily observed in storage rings of particle colliders like CERN. The lifetime of muons at rest (measured by clocks in an inertial laboratory frame) is $\tau^0_\mu \approx 2.1948(10) \mu$s. However, the lifetime of muons in flight (close to the speed of light) is measured to be $\tau_\mu \approx 64.368(29) \mu$s, i.e., much longer! If one carefully takes into account the speed of the muons and additional experimental imperfections, this result fits Eq. (2.14) with deviations of only $\sim 0.1 \%$ [44].

**Notes:**

- In the rest frame of the flying muons one would measure the usual lifetime $\tau^0_\mu \approx 2.1948(10) \mu$s. However, in this frame, the laboratory is Lorentz contracted such that the muon reaches exactly the same point in space where it decays in this "shorter" lifetime. Note how time-dilation and Lorentz contraction provide different explanations for the same experimental obervation.

- One can also use different particle species to study time dilation, for example pions (a sort of meson, i.e., a hadron with one quark and one antiquark) [45].

**Hafele-Keating experiment** [46, 47]:

In 1971, J.C. Hafele and R. E. Keating took four Cesium atomic clocks along commercial jet flights around the globe twice: once eastward and once westward. Compared to a reference clock on the ground, the clocks on the eastward flight lost on average $\sim 59 \text{ ns}$ (= they ran slower) and the clocks on the westward flight gained $\sim 273 \text{ ns}$ (= they ran faster). To understand this qualitatively, note that the reference clock on the ground is rotating (together with earth) and therefore is not an inertial clock. Therefore imagine an (approximately) inertial reference system flying along earth around the sun, and from this system look down on the north pole; earth is now slowly rotating beneath you. From this inertial system, the eastward flight has higher velocity than the reference clock, which, in turn, has higher velocity than the westward flight. Thus you find that the eastward clock runs slower than the reference clock which runs slower than the westward clock (this is also true if the clocks are accelerated, below). These theoretical considerations are explained in [46].

### 2.3. Addition of velocities

Details: \(\text{Problemset 2}\)

1. \(\triangleleft\) Particle moving with $\vec{u}' = \frac{d\vec{X}'}{dt'}$ in system $K'$ and inertial system $K$ with $K \rightarrow K'$:
2 | Velocity \( \vec{\mu} \) in \( K \):

\[
\vec{\mu} = \frac{d\vec{x}}{dt} = \vec{v} \oplus \vec{\mu}' = \frac{1}{\gamma} \left[ \vec{v} + \gamma' \frac{\vec{u}' \cdot \vec{v}}{c^2} (\vec{u}' \cdot \vec{v}) \right]
\]  

(2.16)

**Proof:** Use Eq. (1.75) (→ Problemset 2).

\[ \vec{v} \oplus (\vec{u} \oplus \vec{\mu}) \neq (\vec{v} \oplus \vec{u}) \oplus \vec{\mu} \]  

As you can easily see from Eq. (2.16), it is also not linear:

\[ \vec{v} \oplus (\lambda \vec{u}) \neq \lambda (\vec{v} \oplus \vec{u}) \]

Be careful: There are different notations (in particular: orderings) used in the literature.

3 | Non-relativistic limit \( (c \to \infty \Rightarrow \gamma \to 1) \):

\[
\lim_{c \to \infty} \vec{v} \oplus \vec{\mu}' = \lim_{c \to \infty} \vec{\mu}' \oplus \vec{v} = \vec{v} + \vec{\mu}'
\]  

(2.17)

\[ \rightarrow \text{Galilean addition of velocities} \]

4 | Special case: \( \vec{v} = (\nu_x, 0, 0) \):

\[
u_x = \frac{\nu_x + u'_x}{1 + \frac{v_x u'_x}{c^2}}, \quad \nu_y = \frac{u'_y / \gamma_v}{1 + \frac{v_x u'_x}{c^2}}, \quad \nu_z = \frac{u'_z / \gamma_v}{1 + \frac{v_x u'_x}{c^2}}.
\]  

(2.18)

\[ \rightarrow \text{Galilean addition of velocities} \]

5 | **Thomas-Wigner rotation** \([48,49]\):

Remember that for **collinear** addition of velocities the concatenation of two boosts yields another boost: \( \Lambda_{\nu_x} \Lambda_{u_x} = \Lambda_{u_x} \) [recall Eq. (1.57)].

As a straightforward (but tedious) calculation using two general boosts Eq. (1.75) shows, this is **not** true in general: \( \Lambda_{\nu} \Lambda_{\vec{u}} \neq \Lambda_{\vec{u}} \) with \( \vec{u} = \vec{u} \oplus \vec{v} \). Rather one finds

\[
\Lambda_{\nu} \Lambda_{\vec{u}} = \Lambda_{\vec{u} \oplus \vec{v}} \Lambda_{R(\vec{u}, \vec{v})}
\]  

(2.19)

with the **Thomas-Wigner rotation** \( R(\vec{u}, \vec{v}) \in \text{SO}(3) \) (we omit the explicit form of \( R(\vec{u}, \vec{v}) \) here).

This is not in contradiction with our general addition for velocities above because there we were only interested in the velocity of a moving particle (which you can identify with the origin of its rest frame \( K'' \)); we completely ignored the axes of \( K'' \). The Thomas-Wigner rotation tells you that the concatenation of two **pure** boosts is **not** a pure boost in general.
2.4. Proper time and the twin “paradox”

1 | < Time-like trajectory $\mathcal{P} \subseteq \mathcal{E}$ of a spaceship with departure $D \in \mathcal{P}$ and arrival $A \in \mathcal{P}$.
   < Coordinate parametrization $\vec{x}(t)$ of $\mathcal{P}$ in system $K$ with
     
     \[
     \text{departure } [D]_K = (t_D, \vec{x}_D) \quad \text{and} \quad \text{arrival } [A]_K = (t_A, \vec{x}_A) : \quad (2.20)
     \]

   Formally, $\mathcal{P}$ is a set of coincidence classes parametrized in $K$ by the clock events $(t, \vec{x}(t))_K$:

   \[
   \mathcal{P} = \{ (t, \vec{x}(t))_K \mid t \in [t_D, t_A] \} \subseteq \mathcal{E} .
   \]

   This suggests the formal notation $[\mathcal{P}]_K = (t, \vec{x}(t))$.

2 | Thought experiment:

The spaceship takes a clock along and resets it to $\tau_D = \tau(t_D)$ at departure $D$.

What is the reading $\tau_A = \tau(t_A)$ of the clock at arrival $A$?

We assume that the clock in the spaceship is type-identical to the clocks used for inertial observers.

3 | Idea:

Approximate the trajectory by a polygon of $N$ segments $i = 1, \ldots, N$ separated by time steps $t_i$ (with $t_0 := t_D$ and $t_N := t_A$):

\[
\text{Let } \Delta t_i := t_{i-1} - t_i \text{ and } \Delta \vec{x}_i := \vec{x}(t_{i-1}) - \vec{x}(t_i)
\]

For each segment, there is an inertial frame $K'$ with a $t'$-axis that follows the spacetime segment (because all segments are time-like!). This is the instantaneous rest frame of the spaceship where the clock in the spaceship and the origin clock of $K'$ are at the same place and at rest relative to each other. Since the clocks are type-identical, the time $\Delta \tau$ accumulated by the spaceship clock on this segment is identical to the time $\Delta t'$ elapsed for the origin clock.
clock of $K'$ on this segment: $\Delta t_i = \Delta t'_i$. This time is equal to the spacetime interval $(\Delta s'_i)^2 = (c \Delta t'_i)^2 - 0$ because the origin clock is at rest in $K'$ (so that $\Delta \bar{x}_i' = 0$). But remember that the spacetime interval $(\Delta s'_i)^2$ is Lorentz invariant so that we can calculate the same number in any inertial system: $(\Delta s_i')^2 = (\Delta s_i)^2 = (c \Delta t_i)^2 - (\Delta \bar{x}_i)^2$.

In summary, on the $i$th interval, the spaceship clock accumulates the time

$$
\Delta t_i = \frac{\Delta s_i}{c} := \sqrt{\frac{\Delta s_i^2}{c^2}} = \sqrt{\frac{(c \Delta t_i)^2 - (\Delta \bar{x}_i)^2}{c^2}} = \Delta t_i \sqrt{1 - \left(\frac{\Delta \bar{x}_i}{\Delta t_i c^2}\right)^2}
$$

The above chain of arguments provided us with a physical interpretation for the Lorentz invariant spacetime interval $(\Delta s_i)^2 > 0$ of time-like separated events: It measures (up to a factor of $c$) the time accumulated by an inertial (= unaccelerated) clock that takes part in both events.

Continuum limit $N \to \infty$ ($v(t) := |\dot{v}(t)| = |\dot{x}(t)|)$:

$$
\frac{dr}{c} = \frac{ds}{c} = \frac{dr}{c} \sqrt{1 - \frac{\dot{x}(t)^2}{c^2}} \Leftrightarrow \frac{dr}{dr} = \gamma v(t)
$$

(2.23)

Note that this is just an infinitesimal version of the time-dilation formula Eq. (2.14) with $\Delta t \to dr$ and $\Delta t_0 \to dr$.

Since $(\Delta s)^2 = (\Delta s')^2$ is Lorentz invariant:

$$
K \xrightarrow{\Lambda} K': \quad dr \sqrt{1 - \frac{\dot{x}(t)^2}{c^2}} = \frac{ds}{c} = \frac{ds'}{c} = dr' \sqrt{1 - \frac{\dot{x}'(t)^2}{c^2}}
$$

(2.24)

You can check this also explicitly using the Lorentz transformation Eq. (1.75).
iii | → Proper time accumulated by the spaceship clock along the trajectory $P$:

$$\Delta \tau[P] = \lim_{N \to \infty} \sum_{i=1}^{N} \Delta \tau_i = \int_{P} \frac{d\tau}{c} = \int_{t_D}^{t_A} dt \sqrt{1 - \left(\frac{\dot{x}(t)}{c}\right)^2}$$  \hspace{1cm} (2.25)

- As constructed, the proper time $\Delta \tau[P]$ of a time-like trajectory $P$, parametrized by $\tilde{x}(t)$ for $t \in [t_0, t_1]$, is the time elapsed by a clock that follows this trajectory in spacetime.

- ! This result is valid for accelerated clocks.

In general, special relativity can described the physics of accelerated objects as long as the description of the process is given in an inertial coordinate system (as is the case here).

- ! The right-most expression in Eq. (2.25) yields the same result in all inertial systems $K$ [recall Eq. (2.24)]. This is why $\tau[P]$ is a function of the event trajectory $P$ and not its coordinate parametrization $\tilde{x}(t)$. This is important: It tells us that all inertial observers will agree on the reading of the spaceship clock at arrival $A$ (although their parametrization $\tilde{x}(t)$ may look different).

- Note that since $\tilde{x}(t)$ is assumed to be time-like, it is $\forall t : |\ddot{x}(t)| < c$ such that the radicand is always non-negative.

- $\tau[*]$ is a functional of the trajectory $P$; this is why we use square-brackets.

4 | Which trajectory $P^*$ between the two events $D$ and $A$ maximizes the proper time $\Delta \tau$?

i | $D$ and $A$ are time-like separated $\rightarrow \exists$ Inertial system $K' = K(D, A)$ with

$$[D]_{K'} = (t_D' = 0, \tilde{x}_D' = \tilde{0}) \hspace{0.5cm} \text{and} \hspace{0.5cm} [A]_{K'} = (t_A', \tilde{x}_A' = \tilde{0})$$  \hspace{1cm} (2.26)

That is, without loss of generality, we can Lorentz transform into an inertial system where the two events happen at the same location (and by translations we can assume that this location is the origin $\tilde{0}$ and that the coordinate time is $t_D' = 0$ at $D$). We label the time and space coordinate in $K'$ by $t'$ and $\tilde{x}'$. Because of the relativity principle SR, $K'$ is as good as any system to describe events.

ii | Time of an arbitrary path $P \ni D, A$ with $[P]_{K'} = (t', \tilde{x}'(t'))$:

$$\Delta \tau[P] = \int_{t_D'}^{t_A'} dt' \sqrt{1 - \left(\frac{\dot{x}'(t')}{c}\right)^2} \leq \int_{t_D'}^{t_A'} dt' = t_A' - t_D' = \Delta \tau[P^*]$$  \hspace{1cm} (2.27)
Here $\mathcal{P}^*$ is the trajectory between $D$ and $A$ that is parametrized by the constant function $\tilde{x}'(t') \equiv 0$ in $K'$. In other inertial systems, this trajectory will not be constant; however, it is inertial, i.e., $\mathcal{P}^*$ is described by a trajectory between $D$ and $A$ with uniform velocity.

Check this by applying a Lorentz transformation to the coordinates $(t', \tilde{0})_{K'}$!

→ Clocks that travel along the inertial trajectory $\mathcal{P}^*$ between $D$ and $A$ collect the largest proper time $\tau^* = \Delta \tau[\mathcal{P}^*]$.

Collecting the “largest time” means that the these clocks run the fastest.

5 | It is important to let this result sink in:

Let $K'$ be the rest frame of earth (which is located in the origin $\tilde{0}$) and consider two twins of age $\tau_D$:

- **Twin S** departs with a spaceship at $D$, flies away from earth, turns around and returns to earth at $A$. **Twin S** therefore follows a trajectory similar to $\mathcal{P}_2$ in the sketches above.
- **Twin E** stays on Earth. He follows the inertial trajectory $\mathcal{P}^*$ in the sketches above.

We just proved above:

\[
\langle \text{Age of Twin S at A} \rangle = \Delta \tau[\mathcal{P}_2] + \tau_D < \Delta \tau[\mathcal{P}^*] + \tau_D = \langle \text{Age of Twin E at A} \rangle
\]

This is the famous **Twin “paradox”**: Twin S aged less than Twin E.

6 | Why there is no paradox:

- If you don’t see why the above result should be paradoxical:
  
  Good! Move along. Nothing to see here! 😊

- Why one **could** conclude that the above result is paradoxical (= logically inconsistent):
  
  - From the view of **Twin E**, **Twin S** speeds around quickly, thus time-dilation tells him that **Twin S** should age slower. And indeed, when **Twin S** returns, he actually didn’t age as much.
  
  - Now, you conclude, due to the relativity principle **SR**, we could also take the perspective of **Twin S** (i.e., our system of reference is now attached to the spaceship). Then **Twin S** would conclude that time-dilation makes **Twin E** (who now, together with earth, speeds around quickly) age more slowly. But this does not match up with the above result that, when both twins meet again at $A$, **Twin S** is the younger one! *Paradox!*

The result is quite straightforward:

The invocation of the relativity principle **SR** in the last point is not admissible! Remember that **SR** only makes claims about the equivalence of *inertial systems*. Now have a look at the trajectory $\mathcal{P}_2$ of the spaceship again: it is clearly accelerated and cannot be inertial. And that there is at least a period where the spaceship (and **Twin S**) is accelerating is a *necessity* for **Twin S** to *return* to **Twin E* (at least in flat spacetimes, but not so in curved ones [50])! This implies that the reunion of both twins at $A$ requires at least one of them to *not* stay in an inertial system. This breaks the symmetry between the two twins and explains why the result can be (and is) asymmetric.

- ¡! For historical (and anthropocentric) reasons, the “twin paradox” is called a “paradox.” We stick to this term because we have to – and not because it is appropriate name. The term “paradox” suggests an intrinsic inconsistency of *relativity*. As we explained above: *This is not the case*. All “paradoxes” in *relativity* are a consequence of unjustified, seemingly “intuitive” reasoning. The root cause is almost always an inappropriate, vague notion of “absolute simultaneity” that cannot be operationalized.
• An overview on different geometric approaches to rationalize the phenomenon can be found in Ref. [51].

Below are two widely used spacetime diagrams of an idealized version where Twin S changes inertial systems only once from $S_D$ to $S_A$ halfway through the journey at $R$. You can think of this as an instantaneous acceleration at the kink. Note, however, that the acceleration itself is dynamically irrelevant for the arguments; it is only important that the inertial frames in which Twin S departs and returns are not the same:

- In the left diagram the slices of simultaneity in the two systems $S_D$ and $S_A$ are drawn. As predicted by time-dilation (and mandated by SR), Twin S observes the clocks of Twin E to run slower during his “inertial periods”, i.e., while he stays in a single inertial system. However, the moment Twin S “jumps” from $S_D$ to $S_A$ at $R$, his notion of simultaneity changes instantaneously: In $S_D$, $R$ and $R_D$ are simultaneous; in $S_A$, however, $R$ and $R_A$ are simultaneous. Due to this jump, the record of Twin S contains now a temporal gap for events on earth (highlighted interval). It is this “missing” time interval that overcompensates the slower running clocks on earth (as observed from $S_D$ and $S_A$) and makes Twin S conclude that Twin E ages faster (in agreement with the actual outcome of the experiment).

If you wonder what happened to the (missing) observations of events in the triangle $R_A R_R D$: there is a nice explanation in Schutz [4]. (The bottom line is that Twin S constructs a bad coordinate system by stopping the recording of events in system $S_D$ when he reaches $R$.)

- In the right diagram, we draw light signals (“pings”) of an earth-bound clock next to Twin E sent to Twin S. Twin S receives these signals and measures their period. This idealizes how Twin S sees (not observes!) the clocks ticking on earth (and, by proxy, how fast Twin E ages). It is important to understand the difference between this “seeing” and our operational definition of observing (using the contraption called an inertial system, as used in the left diagram). As demonstrated by the diagram, Twin S first sees the clock on earth ticking slower; but when he turns around at $R$, the clocks on earth (apparently) speed up significantly. In the end, this speedup overcompensates for the slowdown during the first part of the journey so that Twin S again arrives at the (correct) conclusion that Twin E ages faster. Note that the speedup of the earth-bound clock seen by Twin S during the second half of his journey does not contradict time-dilation.
because *seeing* is not *observing*. This is similar to the ↑ **Penrose-Terrell effect** in that a genuine relativistic effect (here: time-dilation) is distorted by an additional “imaging effect” due to the finite speed of light.

- In our careful derivation above, we not only showed that Twin S ages less than Twin E; we also showed that this conclusion is **independent** of the inertial observer! Thus we know that there will be no dispute about the different ages between different inertial observers.

- The Hafele-Keating experiment [46,47] and the muon decay experiments [44], mentioned previously in the context of time-dilation, are experimental confirmations of the twin “paradox.” So our theoretical prediction above (that Twin S ages less than Twin E) is experimentally confirmed. End of discussion.

- Our derivation of the accumulated proper time along trajectories in spacetime is both mathematically sound and experimentally confirmed. This qualifies **special relativity** as a successful theory of physics. **Operationally** there is nothing to complain about: the theory does its job to produce quantitative predictions of real phenomena. So why do so many people (physicists included) – despite the various efforts to visualize the phenomenon – have this nagging feeling of dissatisfaction that they cannot get rid of? The reason, so I would argue, is the human brain and its proclivity to inject concepts of absolute simultaneity into its model building. This qualifies the historical overemphasis of the twin “paradox” as a **meta problem**: The question to study is not how to “solve” the twin “paradox” (as we showed above, there is nothing to solve); the question to study is why so many people thought (and still think) that there is a problem in the first place. This **meta problem** is an actual problem to study; but it falls into the domain of cognitive science, and not physics!

7 | **Two lessons to be learned from this:**

You can live longer than your inertial-system-dwelling peers by changing inertial systems (= accelerating) at least once.

The mere fact that our universe **really** allows for this (at least in theory) makes it much more interesting than its boring alternative: a Galilean universe.

and

Phenomena like length contraction and the twin “paradox” are physically **real**. Their “paradoxical” flavor is a phenomenon of human cognition, not physics.

This is why we put “paradox” always in quotes in the context of **relativity**.
3. Mathematical Tools I: Tensor Calculus

In this chapter we introduce tensor calculus (↑ *Ricci calculus*) for general coordinate transformations \( \varphi \) (which will be useful both in *special relativity* and *general relativity*). The coordinate transformations \( \varphi \) relevant for *special relativity* are Lorentz transformations (and therefore linear) which simplifies expressions often significantly (→ Chapter 4). However, this special feature of coordinate transformations in *special relativity* is not crucial for the discussions in this chapter.

**Goal:** Construct Lorentz covariant (form invariant) equations
(for mechanics, electrodynamics, quantum mechanics)

**Question:** How to do this systematically?

Note that (we suspect that) Maxwell equations are Lorentz covariant. Clearly this is not obvious and requires some work to prove; we say that the Lorentz covariance is *not manifest*: it is there, but it is hard to see. Conversely, without additional tools that make Lorentz covariance more obvious, it is borderline impossible to *construct* Lorentz covariant equations from scratch (which we must do for mechanics and quantum mechanics!).

We are therefore looking for a “toolkit” that provides us with elementary “building blocks” and a set of rules that can be used to construct Lorentz covariant equations. This toolbox is known as tensor calculus or *Ricci calculus*; the “building blocks” are tensor fields and the rules for their combination are given by index contractions, covariant derivatives, etc. The rules are such that the expressions (equations) you can build with tensor fields are *guaranteed* to be Lorentz covariant. This implies in particular that if you can rewrite any given set of equations (like the Maxwell equations) in terms of these rules, you automatically show that the equations were Lorentz covariant all along. We then say that the Lorentz covariance is *manifest*: one glance at the equation is enough to check it.

Later, in *general relativity*, our goal will be to construct equations that are invariant under arbitrary (differentiable) coordinate transformations (not just global Lorentz transformations). Luckily, the formalism we introduce in this chapter is powerful enough to allow for the construction of such *general covariant* equations as well. This is why we keep the formalism in this chapter as general as possible, and specialize it to *special relativity* in the next Chapter 4. The discussion below is therefore already a preparation for *general relativity*; it is based on Schröder [1] and complemented by Carroll [52].

3.1. Manifolds, charts and coordinate transformations

1. \( D \)-dimensional Manifold
   
   = Topological space that *locally* “looks like” \( D \)-dimensional Euclidean space \( \mathbb{R}^D \):

\[ M \approx \mathbb{R}^D \]
• In relativity, the manifold of interest is the set of coincidence classes $\mathcal{E}$; it makes up the $D = 4$-dimensional manifold we call spacetime.

• A space that “locally looks like $\mathbb{R}^D$” is formalized as a topological space that is locally homeomorphic to Euclidean space $\mathbb{R}^D$. The structure defined in this way is then called a topological manifold.

2 Differentiable Manifolds:
We want to formalize this idea and introduce additional structure to the manifold so that we can differentiate functions on it:

i. Coordinate system / Chart $(U, u)$:

\[
\begin{align*}
    u & : U \subseteq M \rightarrow u(U) \subseteq \mathbb{R}^D \\
    u^{-1} & : u(U) \subseteq \mathbb{R}^D \rightarrow U \subseteq M
\end{align*}
\]

$U \subseteq M$: open subset of $M$; $u$ and $u^{-1}$ are continuous and $u \circ u^{-1} = \mathbb{1}$.

$U = M$ is allowed. This is the situation we assumed so far in special relativity: Our inertial coordinate systems cover all of spacetime $M = \mathcal{E}$.

ii. Two charts $(U, v)$ and $(V, v)$ and let $U \cap V \neq 0$:

\[
\begin{align*}
    \varphi := v \circ u^{-1} & : u(U \cap V) \rightarrow v(U \cap V) \\
    \varphi^{-1} := u \circ v^{-1} & : v(U \cap V) \rightarrow u(U \cap V)
\end{align*}
\]

$\varphi$: Coordinate transformation / Transition map

$U = M = V$ and $U \cap V = M$ is allowed. This is the situation we assume so far in special relativity where $(U = \mathcal{E}, u)$ and $(V = \mathcal{E}, v)$ correspond to the coordinate systems of two different inertial systems. The coordinate transformation $\varphi$ would then be a Lorentz transformation (defined on $U \cap V = \mathcal{E}$).

iii. Atlas := Family of charts $(U_i, u_i)_{i \in I}$ such that $M = \bigcup_{i \in I} U_i$

This definition of an atlas formalizes the notion of an atlas in real life (of the book variety): It contains many charts that, taken together, cover the complete manifold (typically earth). The different charts (on different pages of the book) all overlap on their edges such that you can draw any route on earth without gaps.

All $\varphi, \varphi^{-1}$ differentiable $\rightarrow M$: Differentiable Manifold
• \( \varphi \) and \( \varphi^{-1} \) are maps from \( \mathbb{R}^D \) to itself. It is therefore clear what “differentiable” means.

• In mathematics one is of course more precise about the degree of differentiability of the transition functions, and subsequently assigns this degree to the manifold. For example, if all coordinate transformations are infinitely often differentiable (= smooth), the manifold is called a smooth manifold. We are sloppy in this regard: For us all functions are differentiable as often as we need them to be.

In relativity we will only be concerned with differentiable manifolds.

3 | Example:

→ In general, a manifold cannot be covered by a single chart (Earth, mathematically \( S^2 \), needs at least two charts). In special relativity this is not a problem: There we assume that spacetime is a flat (pseudo-)Euclidean space \( \mathbb{E} \simeq \mathbb{R}^4 \) and the coordinates given by our inertial systems cover all of spacetime. Later, in general relativity, this will not necessarily be the case.

3.2. Scalars

4 | \( \star \) Scalar (field) := Function \( \phi : M \to \mathbb{R}/\mathbb{C} \)

• If \( \phi \) maps to \( \mathbb{R} \) (\( \mathbb{C} \)), we call \( \phi \) a real (complex) scalar field.

• \( \varphi! \) \( \phi \) is a geometric object because it only depends on the manifold itself. It does not rely on charts/coordinates and does not depend on a particular set of charts you might choose to parametrize the manifold. The notion of a mathematical object to be “geometric in nature” or “independent of the choice of coordinates” is absolutely crucial for the understanding of general relativity. The reason why these “geometric objects” are so important for physics is the following insight that took physicists (including Einstein) a long time to fully comprehend and implement mathematically:

Coordinates (charts) do not represent physical entities.
They are (useful) “mathematical auxiliary structures.”

• One reason why it is so hard for us to grasp and implement the “physical irrelevance” of coordinates is, so I believe, that the first (and often only) coordinates we encounter in school are Cartesian coordinates. They are particularly intuitive because they are simply the distances of a point to some coordinate axes. Distances are a geometric property and physically relevant.
(you can measure them with rods); they are not the invention of mathematicians. This makes students draw the (wrong) conclusion that coordinates in general have intrinsic physical meaning. The problem is that coordinates are inventions of mathematicians; they do not share the ontological status of physical quantities like lengths etc. To undo this misconception is key to understand general relativity (→ much later).

• Since both $M$ and $\mathbb{R}/\mathbb{C}$ are topological spaces, it makes sense to ask whether (or require that) $\phi$ is continuous. It does not make sense to ask whether $\phi$ is differentiable (and what is derivative is) because, in general, $M$ does neither come with a notion of “distance” between two points in $M$ nor can you add or subtract points ($M$ does not have to be a metric space and/or a linear space). So an expression like $\partial_p \phi(p)$ does not make sense (→ below)!

We just declared that coordinates are “not physical.” The problem is that without coordinates it is really hard (at least for physicists) to do actual calculations with the geometric objects we are interested in (for example: compute derivatives). In addition, comparing theoretical predictions with experimental observations typically requires some sort of coordinate representation. Our inertial systems, for example, are elaborate measurement devices that produce a specific coordinate representation of the observed events.

This is why we always assume in the following that we have one (or more) charts that allow us to parametrize a (part of the) manifold, and then express the geometric quantities as functions of these coordinates. This means for the scalar field:

< Two overlapping charts $u$ and $v$:

\[ \Phi(x) := \phi(u^{-1}(x)) \quad x \in u(U \cap V) \]  \hspace{1cm} (3.3a)

\[ \hat{\Phi}(\tilde{x}) := \phi(v^{-1}(\tilde{x})) \quad \tilde{x} \in v(U \cap V) \]  \hspace{1cm} (3.3b)

$\Phi$ and $\hat{\Phi}$ are functions on (subsets of) $\mathbb{R}^D$; in contrast to $\phi$ which is a function on the manifold $M$.

In an abuse of notation, some authors do not make this distinction and write $\phi$ and $\hat{\phi}$ instead.

\[ \hat{\Phi}(\tilde{x}) = \Phi(x) \quad \text{for} \quad \tilde{x} = \phi(x) \quad \text{with} \quad \varphi = v \circ u^{-1}. \]  \hspace{1cm} (3.4)

Note that $\Phi(\tilde{x}) \equiv \phi(p) \equiv \Phi(x)$ with $u^{-1}(x) = p = v^{-1}(\tilde{x})$.

• In relativity we typically work in a particular chart (coordinate system). Thus we write our fields as functions of coordinates (and not points on the manifold); e.g., when working with scalars, we typically work with $\Phi$ (and not $\phi$).

• ! The special transformation of a field Eq. (3.4) (given as function of coordinates) tells us that it actually encodes a geometric, chart-independent function $\phi$ (given as function of points on the manifold). This idea will be prevalent throughout this chapter and is the basis of our modern formulation of relativity: We work with functions that depend on specific coordinates (and therefore change when we transition to another chart); however, these functions satisfy certain transformation laws [like Eq. (3.4)] that guarantee that they actually encode geometric, chart-independent objects (which is what physics is about).

• As a function of coordinates, scalar fields are those fields the values of which do not change under coordinate transformations. A typical example would be the temperature as a function of position: When you move your coordinate system, the temperature of a particular point in space still is the same (only your coordinates of this particular point have changed!). This is exactly what Eq. (3.4) demands.
Note that being a scalar (field) does not simply mean “being a number.” The \( z \)-component of the electric field strength \( E_z(x) \), for example, assigns a number to every point \( x \); however, it does not transform like Eq. (3.4) under coordinate transformations. (Do you see why? What happens to \( E_z \) if you rotate your coordinate system?)

- In the literature, you will find the notation \( \hat{\Phi} = \Phi \) to characterize scalars. This does not mean \( \Phi(x) = \Phi(x) \) for all \( x \in \mathbb{R}^D \) (which characterizes form-invariance or functional equivalence), but rather \( \hat{\Phi}(\tilde{x}) = \Phi(x) \) (which characterizes scalar fields). Note that with \( x = \varphi^{-1}(\tilde{x}) \) it follows \( \hat{\Phi}(\tilde{x}) = \Phi(\varphi^{-1}(\tilde{x})) \) such that the function \( \hat{\Phi} \) is typically not functionally equivalent to \( \Phi \). This ambiguity is the price we have to pay if we want to express geometric objects in terms of coordinates.

- Since \( \hat{\Phi} : \mathbb{R}^D \rightarrow \mathbb{R} \), it is well-defined what “differentiability” of \( \Phi \) means. So expressions like \( \frac{\partial \hat{\Phi}(x)}{\partial x^k} \) make sense now (if \( \Phi \) is differentiable). One then defines that \( \phi \) is differentiable on \( M \) iff \( \Phi \) is differentiable for all charts of an atlas of \( M \).

### 3.3. Covariant and contravariant vector fields

Are scalar fields the only geometric objects that can be defined on a manifold? The answer is no, there are many more! And these objects are not just toys for mathematicians: they are necessary to represent physical quantities like the electromagnetic field. Unfortunately, the definition of these quantities is not so straightforward as for scalars. We will not be mathematically precise in our discussion; however, it is important to understand the conceptual ideas:

- **Tangent space \( T_p M \) at \( p \in M \)**
  
  = Vector space of directional derivative operators with evaluation at \( p \in M \) (=derivations)
  
  These operators can be applied to differentiable functions on the manifold (i.e., scalar fields).

- The tangent space \( T_p M \) is the mathematical formalization of the intuitive concept of the plane \( \mathbb{R}^2 \) that you can attach tangentially at any point \( p \) of a two-dimensional manifold. The problem with this picture is that it only works if you embed the manifold \( M \) into a higher-dimensional Euclidean space. Mathematically, such an approach is not satisfying because it presupposes additional structure to characterize the manifold (which, as it turns out, is not needed). Physically, the approach is also problematic: The manifold we are interested in is all of spacetime \( \mathcal{E} \). But \( \mathcal{E} \) is all there is, it is (to the best of our knowledge) not embedded into anything. It is therefore crucial that we can work with manifolds “stand alone”, without assuming any embedding into a higher-dimensional space. The price we have to pay is that tangent vectors must be defined, rather abstractly, as directional derivative operators.
• There is a different tangent space $T_p M$ at every point $p \in M$; these vector spaces all have the same dimension $D$ (like the manifold) and are therefore all isomorphic. However, without additional structure, there is no natural connection (isomorphism) between these different vector spaces at different points. The disjoint union of all tangent spaces is called the tangent bundle $TM$.

• Mathematically, the vectors in the tangent space can be defined as equivalence classes of smooth curves through $p$ with the same derivative (with respect to their parametrization) at $p$. This equivalence class corresponds to a particular directional derivative that one can apply to smooth functions on the manifold at $p$. We do not need this abstract “bootstrapping procedure” for $T_p M$ in the following.

Chart $(U, u)$ with coordinates $x = (x^0, x^1, \ldots, x^D)$

→ Coordinate basis $\{\partial_i \equiv \frac{\partial}{\partial x^i}\}$ for $T_p M$

Recall that partial derivatives are special kinds of directional derivatives (namely in the direction where you keep all but one coordinate fixed). You can therefore think of $\partial_i$ as the tangent vector at $p \in M$ that points into the $x^i$-direction mapped by $u^{-1}$ onto the manifold.
7 | Since $T_p M$ is a vector space for each point $p$ of the manifold $M$, we can define fields on $M$ that assign to each point $p$ a tangent vector:

**Vector field:** $A(p) = \sum_{i=1}^{D} A^i(x) \partial_i$ with $x = u(p)$

At every point $p \in M$ the vector field yields a tangent vector $A(p) = \sum_{i} A^i(u(p)) \partial_i \in T_p M$.

8 | $\varphi$ Coordinate transformation $\tilde{x} = \varphi(x) \iff x = \varphi^{-1}(\tilde{x})$

$\rightarrow$ Chain rule:

$$\frac{\partial}{\partial \tilde{x}^i} = \sum_{k=1}^{D} \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial}{\partial x^k} \quad (3.5)$$

$\rightarrow$ For $x = u(p)$ and $\tilde{x} = v(p)$ this is a basis change on the tangent space $T_p M$ from one coordinate basis $\{\partial_i\}$ to another coordinate basis $\{\tilde{\partial}_i\}$ via the (invertible) matrix $\frac{\partial x^k}{\partial \tilde{x}^i}$.

9 | $\varphi$ Vector field $A$ and expand it in different coordinate bases:

$$\sum_{i} A^i(x) \partial_i = A(p) = \sum_{i} \tilde{A}^i(\tilde{x}) \tilde{\partial}_i \quad (3.6)$$

with $x = u(p)$ and $\tilde{x} = v(p)$.

- The vector field $A$ is a geometric object, just as the scalar field $\phi$ was. That it does not depend on the chosen chart is the statement of this equation.

- You learned this (with different notation and without the $x/p$-dependency) in your first course on linear algebra: Given a vector space $V$, a vector $\tilde{v} \in V$, and a basis $\{e_i\}$ with $V = \text{span}\{e_i\}$, you can encode the vector in a basis-dependent set of numbers $v_i$ called components via linear combination: $\tilde{v} = \sum_i v_i e_i$. The same vector can be encoded by different components $v'_i$ in a different basis $\{\tilde{e}_i\}$: $\tilde{v} = \sum_i v'_i \tilde{e}_i$. In our terminology, the vector $\tilde{v}$ is a “geometric object” that does not depend on your choice of basis; only its components do. In this context, the gist of the story is that $\tilde{v}$ represents something physical (like the velocity of a particle). The components $v_i$ do so only indirectly because they depend on your choice of the basis $\{e_i\}$ – and this choice does not bear any physical meaning.

Eq. (3.6) $\rightarrow$

$$A = \sum_i A^i(x) \partial_i = \sum_i \tilde{A}^i(\tilde{x}) \tilde{\partial}_i \quad \text{Eq. (3.5) } \sum_{i} A^i(\tilde{x}) \tilde{\partial}_i = \sum_{k=1}^{D} \frac{\partial x^k}{\partial \tilde{x}^i} \tilde{A}^k(\tilde{x}) \tilde{\partial}_i = \sum_{k} \frac{\partial x^k}{\partial \tilde{x}^i} \tilde{A}^k(\tilde{x}) \tilde{\partial}_i \quad (3.7)$$
This motivates the following definition (we replace \( x \leftrightarrow \bar{x} \) and the indices \( i \leftrightarrow k \)):

\[
\begin{align*}
\text{\textbullet\ Contravariant vector field } \{ A^i(x) \} & \iff \tilde{A}^i(\bar{x}) = \sum_{k=1}^{D} \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \\
& \quad \quad \text{(3.8)}
\end{align*}
\]

**Contravariant vector (field) \( \rightarrow \) Superscript indices!**

This is a convention which relates syntax and semantics and is at the heart of tensor calculus. The idea is that whenever you are given a collection of fields \( A^i(x) \), you immediately know that they transform like Eq. (3.8) under coordinate transformations. (Unfortunately, there are exceptions to this rule, e.g., the Christoffel symbols.)

- ! Not every \( D \)-tuple of fields transforms as Eq. (3.8). To deserve the name “contravariant vector (field),” (and superscript indices) one has to check this transformation law explicitly!
- The rationale of Eq. (3.8) is the same as that of Eq. (3.4): Whenever we find a family of fields that transform under coordinate transformations as Eq. (3.8), we immediately know that together they encode a geometric, chart-independent object on the manifold that can be used to describe a physical quantity.

**11 | (Counter)Examples:**

- \( \triangleleft \) Only linear coordinate transformations: \( \bar{x} = \varphi(x) = \Lambda x \)
  - Coordinate functions \( X^i(x) := x^i \) as fields:
    \[
    \tilde{X}^i(\bar{x}) = \sum_{k=1}^{D} \Lambda^i_k X^k(x) = \sum_{k=1}^{D} \frac{\partial \bar{x}^i}{\partial x^k} X^k(x)
    \]
    \[
    \quad \quad \text{(3.9)}
    \]
  - \( \rightarrow \) Coordinate functions are contravariant vectors for linear transition maps.
    This is useful in special relativity because there we only consider global Lorentz transformations (which are linear).

- \( \triangleleft \) \( D \) scalar fields \( \Phi^i(x) \) (\( i = 1, \ldots, D \)):
  - For general \( \bar{x} = \varphi(x) \):
    \[
    \tilde{\Phi}^i(\bar{x}) = \Phi^i(x) \neq \sum_{k=1}^{D} \frac{\partial \bar{x}^i}{\partial x^k} \Phi^k(x)
    \]
    \[
    \quad \quad \text{(3.10)}
    \]
  - \( \rightarrow \{ \Phi^i(x) \} \) are not components of a contravariant vector field.
    - You see: not every collection of \( D \) fields is a vector!
    - ! \( \delta^i_k \) is the Kronecker symbol: \( \delta^i_k = 1 \) for \( i = k \) and \( \delta^i_k = 0 \) for \( i \neq k \). The notation \( \delta_{ik} \) is not used in tensor calculus (\( \rightarrow \) later).

**12 | Reminder: \( \rightarrow \) Dual spaces**
Remember: Linear algebra

Consider the vector space \( V = \mathbb{R}^D \) and a column vector \( \vec{v} = (v_1, \ldots, v_D)^T \in V \) (a \( 1 \times D \)-matrix). Let \( \vec{w}^T = (w_1, \ldots, w_D) \) be a row vector (a \( D \times 1 \)-matrix). We can then perform a matrix multiplication between the vectors and interpret it as a linear map \( \vec{w}^T \) acting on the vector \( \vec{v} \) and producing a number:

\[
\vec{w}^T : \vec{v} \in V \mapsto \vec{w}^T \cdot \vec{v} = (w_1 \ldots w_D) \begin{pmatrix} v_1 \\ \vdots \\ v_D \end{pmatrix} = \sum_i w_i v_i \in \mathbb{R}.
\] (3.11)

In mathematical parlance \( \vec{w}^T \) is a *linear functional* on the vector space \( V \). All linear functionals of this form make up another vector space \( V^* \) called the *dual space* of \( V \). You can think of \( V^* \) as the vector space of all \( D \)-dimensional row vectors and \( V \) as the vector space of all \( D \)-dimensional column vectors. The elements of the dual space are referred to as \( \downarrow \) *covectors*.

Remember: Quantum mechanics

In quantum mechanics, the state of a physical system is described by \( \downarrow \) *state vectors* in some Hilbert space \( \mathcal{H} \) (which is a special kind of vector space). Vectors in this space are written as \( \downarrow \) *kets*: \( |\Psi\rangle \in \mathcal{H} \). You can produce a \( \downarrow \) *bra* \( \langle \Psi| = |\Psi\rangle^\dagger \) by applying the complex transpose operator. As in the example above, the bra \( \langle \Psi| \) is a covector from the dual space \( \mathcal{H}^* \); indeed, it acts as a linear functional on state vectors via the inner product of the Hilbert space:

\[
\langle \Psi|\Phi \rangle := \langle \Psi|\Phi \rangle \in \mathbb{C}.
\] (3.12)

This is the gist of the famous \( \downarrow \) *Dirac bra-ket notation*.

Hopefully these examples convinced you that the dual space is just as important and useful as the vector space itself.

→ Dual space of the tangent space \( T_p M \)?

Given a coordinate basis \( \{\partial_i\} \in T_p M \) of a vector space, there is a standard way to define a basis of the dual space \( T^*_p M \):

\( \downarrow \) *Dual basis* \( \{dx^i\} \) with

\[
dx^i(\partial_j) := \delta^i_j = \frac{\partial x^i}{\partial x^j}
\] (3.13)

\( \downarrow \) \( \{dx^i\} \) is a basis of the \( \uparrow \) *Cotangent space* \( T^*_p M \)

\( T^*_p M \) is the dual space of \( T_p M \); it is common to write \( T^*_p M \) and not \( (T_p M)^* \).

\( \uparrow \) *Covector field* \( B(p) = \sum_{i=1}^{D} B_i(x) \, dx^i \) with \( x = u(p) \)

Just like the coordinate basis, the dual coordinate basis depends on the chart and changes under coordinate transformations:

\( \downarrow \) *Coordinate transformation* \( \bar{x} = \varphi(x) \):

\[
dx^i = \sum_{k=1}^{D} \frac{\partial \bar{x}^i}{\partial x^k} \, dx^k
\] (3.14)
Check that this is the correct transformation for the dual coordinate basis:
\[
dx^i(\partial_j) = \left[ \sum_k \frac{\partial \tilde{x}^i}{\partial x^k} \, dx^k \right] \left( \sum_l \frac{\partial x^l}{\partial \tilde{x}^j} \, \partial_l \right)
= \sum_k \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^k}{\partial \tilde{x}^j} \, dx^j = \delta^i_j \quad \otimes \quad (3.15)
\]

You might recognize Eq. (3.14): This is simply the rule to compute the total differential of the function \( \tilde{x} = \varphi(x) \). This is no coincidence and explains why we use the differential notation \( dx^i \) for the dual vectors: The objects \( dx^i \) that we physicists like to illustrate as "infinitesimal shifts" in \( x^i \) are actually linear functionals (↑ 1-forms).

Now we can play the same game on \( T_p^* M \) as before on \( T_p M \):

\(<\) Covector field \( B \) and expand it in different dual coordinate bases:
\[
\sum_i B_i(x) \, dx^i = B(p) = \sum_i \tilde{B}_i(\tilde{x}) \, d\tilde{x}^i \quad (3.16)
\]

with \( x = u(p) \) and \( \tilde{x} = v(p) \).

\( \ord \) The covector field \( B \) is another geometric object, just as the vector field \( A \) was. That it does not depend on the chosen chart is the statement of this equation.

Eq. (3.16) \( \Rightarrow \)
\[
B = \sum_i B_i(x) \, dx^i = \sum_i \tilde{B}_i(\tilde{x}) \, d\tilde{x}^i \quad \overset{\text{Eq. (3.14)}}{=} \quad \sum_k \left[ \sum_i \frac{\partial \tilde{x}^i}{\partial x^k} \tilde{B}_i(\tilde{x}) \right] \, dx^k \quad (3.17)
\]

This motivates the following definition (we replace \( x \leftrightarrow \tilde{x} \) and the indices \( i \leftrightarrow k \)):

\(<\) \( D \)-tuple \( \{ B_i(x) \} \) of fields (in some chart with coordinates \( x \)):

\[\star\star\text{ Covariant vector field} \{ B_i(x) \} : \quad \Leftrightarrow \quad \tilde{B}_i(\tilde{x}) = \sum_{k=1}^D \frac{\partial x^k}{\partial \tilde{x}^i} B_k(x) \quad (3.18)\]

Covariant vector (field) \( \rightarrow \) Subscript indices!

The rationale of Eq. (3.18) is the same as that of Eq. (3.8): Whenever we find a family of fields that transform under coordinate transformations as Eq. (3.18), we immediately know that together they encode a geometric, chart-independent object on the manifold that can be used to describe a physical quantity. To indicate that this object is a covariant vector field, we use subscript indices.

Example:

First, let us introduce an even shorter notation for partial derivatives: \( \Phi,_{i} \equiv \partial_i \Phi \)

Following our index convention, the lower index in these expressions is only warranted if the field transforms as a covariant vector field according to Eq. (3.18). Let us check this:

\[
\Phi,_{i}(\tilde{x}) = \partial_i \Phi(\tilde{x}) \quad \overset{\text{Eq. (3.4)}}{=} \quad \sum_{k=1}^D \frac{\partial x^k}{\partial \tilde{x}^i} \partial \Phi(x) = \sum_{k=1}^D \frac{\partial x^k}{\partial \tilde{x}^i} \Phi,_{k}(x) \quad (3.19)
\]
The gradient of a scalar is a covariant vector field.

What happens if we apply a covector field on a vector field at each point \( p \in M \)?

\[
\phi(p) := B(p)A(p) = \sum_{i,j} B_i(x)A^j_i(x) \underbrace{dx^i(\partial_j)}_{\delta_j^i} = \sum_i A^i_i(x)B_i(x) =: \Phi(x) \quad (3.20)
\]

\( \Phi(x) \) must be a scalar!

This is a good point to introduce a new (and very convenient) notation:

**Einstein sum convention:**

\[
\sum_{i=1}^{D} A^i_i(x)B_i(x) \equiv \underbrace{A^i_i(x)B_i(x)}_{\text{Einstein summation}} = A^i_i(x)B_i(x) \quad (3.21)
\]

The *Einstein sum convention* or *Einstein summation* is a syntactic convention according to which a sum is automatically implied (but not written) whenever two indices show up twice in an expression and one is up (contravariant) and one down (covariant). Note that such indices are "dummy indices" in the sense that you can rename them to whatever you want (as long as you do not use the same letter for other indices already!). The sum over one co- and one contravariant index is called a *contraction*.

With this new notation it is straightforward to check that \( \Phi \) transforms according to Eq. (3.4) by using the transformations Eq. (3.8) and Eq. (3.18):

\[
\tilde{\Phi}(\tilde{x}) = \tilde{A}^i(\tilde{x})\tilde{B}_i(\tilde{x}) = \underbrace{\frac{\partial \tilde{x}^i}{\partial x^k} A^k(x)}_{\text{Chain rule}} \underbrace{\frac{\partial x^l}{\partial \tilde{x}^i} B_l(x)}_{\delta_l^i} = \frac{\partial x^l}{\partial \tilde{x}^i} A^k(x)B_l(x) = A^l(x)B_l(x) = \Phi(x) \quad (3.22a)
\]

The intermediate expression contains three sums over the colored indices (which we don’t write)!

\( \rightarrow \) The contraction of a contra- and a covariant vector field yields a scalar field.

**Note on nomenclature:**

- If you compare Eq. (3.18) with Eq. (3.5) you find that the components \( B_i \) of a covector field transform like the basis vectors \( \partial_i \) of the tangent space. We say the components *covary* ("vary together") with the basis. This is why they are called *covariant*.

- A comparison of Eq. (3.8) and Eq. (3.14) shows that the components \( A^i \) of a vector field transform like the basis \( dx^i \) of the cotangent space – which is the inverse ("opposite") transformation as for the basis of the tangent space \( \partial_i \). Thus we say the components \( A^i \) *contravary* ("vary opposite to") the basis \( \partial_i \). This is why they are called *contravariant*.

### 3.4. Higher-rank tensors

You learned in your linear algebra course that two vector spaces \( V \) and \( W \) can be used to construct a new vector space \( V \otimes W \) called the *tensor product*. This allows us to generalize the notion of contra- and covariant vector fields to *tensor fields*, all of which are geometric, chart-independent objects defined on the manifold that are needed to describe physical quantities:
An \((p, q)\)-tensor \((T)\) of rank \(r = p + q\)

\[
T^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = T^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q}(x) \quad \text{or} \quad T^I_J = T^I_J(x),
\]

with \(\uparrow\) multi-indices \(I = (i_1 \ldots i_p)\) and \(J = (j_1 \ldots j_q)\),
transforms like the tensor product of \(p\) contravariant and \(q\) covariant vector fields:

\[
\begin{align*}
T^I_J(x) & = \frac{\partial x^i}{\partial \tilde{x}^I} \cdots \frac{\partial x^{i_p}}{\partial \tilde{x}^I} \frac{\partial x^{n_1}}{\partial \tilde{x}^J} \cdots \frac{\partial x^{n_q}}{\partial \tilde{x}^J} T^{m_1 \ldots m_p}_{n_1 \ldots n_q}(x) \\
& = \frac{\partial x^I_n}{\partial \tilde{x}^J} \ = T^I_J(x).
\end{align*}
\]

There are \(r = p + q\) sums in this transformation rule (Einstein summation!).

- \(\uparrow\) It is important that we do \textit{not} write contra- and covariant indices above each other like so: \(T_{ij}\) (at least not with additional knowledge about the tensor). This will become important below.
- Henceforth we always encode tensor fields by their chart-dependent \textit{components}. The actual tensor field is of course chart-independent and maps each point \(p \in M\) to an element of the tensor product

\[
T_p M \otimes \cdots \otimes T_p M \otimes T^*_p M \otimes \cdots \otimes T^*_p M.
\]

like so

\[
T(p) = \sum_{I,J} T^{i_1 \ldots i_p}_{j_1 \ldots j_q}(x) \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}.
\]

- Note that while tensors (more precisely: tensor components) are indicated by upper and lower indices (corresponding to their rank), not every object that is conventionally written with upper and lower indices does encode a tensor. For example, the transformation matrices \(\frac{\partial \tilde{x}^I}{\partial x^J}\), which describe a basis change on \(T^*_p M\), do not encode a tensor field.

\begin{itemize}
\item \(\downarrow\) Examples:
\begin{align*}
\text{Scalar } \Phi(x) & \rightarrow (0, 0)\text{-tensor} \\
\text{Contravariant vector } A^i(x) & \rightarrow (1, 0)\text{-tensor} \\
\text{Covariant vector } B_i(x) & \rightarrow (0, 1)\text{-tensor} \\
\text{Tensor product } T^i_j(x) := A^i(x) B_j(x) & \rightarrow (1, 1)\text{-tensor} \ (\text{Check this!})
\end{align*}
\end{itemize}

\begin{itemize}
\item \(\downarrow\) Properties:
\begin{itemize}
\item \textit{Equality}:
\[A = B \iff \forall i_1 \ldots i_p \forall j_1 \ldots j_q : A^{i_1 \ldots i_p}_{j_1 \ldots j_q} = B^{i_1 \ldots i_p}_{j_1 \ldots j_q}\]
\item \textit{Symmetry}:
\[T \text{ (anti-)symmetric in } k \text{ and } l \iff T^{i_1 \ldots i_{k-1} \ldots i_{l+1} \ldots i_l \ldots i_{k+1} \ldots}_{j_1 \ldots j_{k-1} \ldots j_{l+1} \ldots j_{k+1} \ldots} = (-) T^{i_1 \ldots i_{l-1} \ldots i_{k+1} \ldots i_l \ldots i_{k-1} \ldots}_{j_1 \ldots j_{l-1} \ldots j_{k+1} \ldots j_{l-1} \ldots}\]
\end{itemize}
\end{itemize}
Every contra- or covariant rank-2 tensor can be decomposed into a sum of symmetric and antisymmetric tensors:

\[ T_{ij} = \frac{1}{2} (T_{ij} + T_{ji}) + \frac{1}{2} (T_{ij} - T_{ji}) = T_{(ij)} + T_{[ij]}. \]  

(3.29)

23 | Constructing tensors:

New tensors can be constructed from known tensors as follows (Proofs: → Problemset 4):

- **Sum of** \((p, q)\)-tensors \(A\) and \(B\) yields \((p, q)\)-tensor \(C\):
  \[ C^{i_1\ldots i_p}_{j_1\ldots j_q} := A^{i_1\ldots i_p}_{j_1\ldots j_q} + B^{i_1\ldots i_p}_{j_1\ldots j_q} \]  
  or \[ C^I_J := A^I_J + B^I_J \]  
  (3.30a)

- **Product of** \((p, q)\)-tensor \(A\) and scalar \(\Phi\) yields \((p, q)\)-tensor \(C\):
  \[ C^I_J := \Phi A^I_J \]  
  (3.31)

- **Tensor product of** \((p, q)\)-tensor \(A\) and \((r, s)\)-tensor \(B\) yields \((p + r, q + s)\)-tensor \(C\):
  \[ C^{IK}_{JL} := A^I_J \cdot B^K_L \]  
  (3.32)

- **Contractions:**
  Summing over a pair of contra- and covariant indices yields a tensor of rank \((p - 1, q - 1)\):
  \[ A^{i_1\ldots i_p}_{j_1\ldots j_q} := A^{i_1\ldots k\ldots i_p}_{j_1\ldots k\ldots j_q} \]  
  (3.33)

The \(\bullet\) indicates that the index summed over on the right side is missing in the list.

Proof: → Problemset 4

A special case of a contraction (in combination with a tensor product) is the scalar obtained from a contra- and a covariant vector field above:

\[ \Phi = C^i_i = A^i B_i. \]  

(3.34)

- **Quotient theorem:**
  \[ \overrightarrow{AB} = C \text{ tensor for all tensors } B \implies A \text{ is tensor} \]  
  (3.35)

Here, \(\overrightarrow{AB}\) denotes (potentially multiple) contractions between indices of \(A\) and \(B\) (but not within \(A\) and \(B\)).

- As an example, rewrite an arbitrary contravariant vector \(A^i\) as \(A^i = \delta^i_j A^j\) with Kronecker symbol \(\delta^i_j\). The above theorem then implies that \(\delta^i_j\) transforms as a \((1, 1)\)-tensor (verify this using the definition!). Hence we actually should write \(\delta^i_j\) instead of \(\delta^i_j\). However, because the Kronecker symbol is symmetric in its indices, this simplified notation is allowed (→ later).
- Special case:
  \[ A_{ik} B^k = C_i \] covector for all vectors \( B^k \) \( \Rightarrow \) \( A_{ik} \) is \((0,2)\)-tensor (3.36)

Proof: \( \bullet \) Problemset 4

\(24\) Relative tensors:

Relative tensor are a generalization of the (absolute) tensors defined above. This generalization is useful because most of the rules for computing with tensors discussed so far carry over to relative tensors.

A \( \ast \) relative tensor of weight \( w \in \mathbb{Z} \) picks up an additional power \( w \) of the \( \downarrow \) Jacobian determinant under coordinate transformations:

\[
\tilde{R}^I_J (\tilde{x}) = \det \left( \frac{\partial x^I}{\partial x^J} \right)^w \frac{\partial \tilde{x}^I}{\partial x^M} \frac{\partial x^N}{\partial \tilde{x}^J} R_M^N (x) \quad \text{with weight } w \in \mathbb{Z}
\]

and Jacobian determinant

\[
\det \left( \frac{\partial x^i}{\partial \tilde{x}^j} \right) := \sum_{\sigma \in S_D} (-1)^\sigma \prod_{i=1}^D \frac{\partial x^{i_\sigma}}{\partial \tilde{x}^{o_j}}.
\]

Here \( S_D \) is the group of permutations \( \sigma \) on \( D \) elements.

Since \( \tilde{x} = \varphi(x) \) is invertible, \( x = \varphi^{-1}(\tilde{x}) \), it is \( \frac{\partial \tilde{x}}{\partial x} = \left( \frac{\partial x}{\partial \tilde{x}} \right)^{-1} \) and therefore \( \det \left( \frac{\partial \tilde{x}}{\partial x} \right) = \det \left( \frac{\partial x}{\partial \tilde{x}} \right)^{-1} \).

\( ii \) Examples:

- \( (\text{Absolute}) \) tensors \( \equiv \) Relative tensors of weight \( w = 0 \)
- \( \text{Volume form} \): Relative tensor of weight \( w = -1 \):

\[
d^D \tilde{x} = d^D x \det \left( \frac{\partial \tilde{x}}{\partial x} \right) = d^D x \det \left( \frac{\partial x}{\partial \tilde{x}} \right)^{-1}
\]

Remember the rule for integration by substitution with multiple variables!

- \( \ast \) \( \text{Tensor density} \ \mathcal{L}(x) := \) Relative tensor of weight \( w = +1 \) \( \rightarrow \)

\[
S = \int d^D x \mathcal{L}(x) = \int d^D \tilde{x} \tilde{\mathcal{L}}(\tilde{x})
\]

In this example, we assume that \( \mathcal{L}(x) \) is a \( \text{scalar} \) tensor density such that its integral is a \( \text{(absolute) scalar} \) quantity.

In \( \uparrow \) relativistic field theories (like electrodynamics), the \( \text{Lagrangian density} \ \mathcal{L}(x) \) is a scalar tensor density such that the \( \downarrow \) action \( S \) becomes a scalar.

- Let \( i_1, i_2, \ldots, i_D \in \{1, 2, \ldots, D\} \) and define the \( \ast \) \( \text{Levi-Civita symbol} \) as

\[
\varepsilon^I \equiv \varepsilon^{i_1 i_2 \ldots i_D} := \begin{cases} +1 & I \text{ even permutation of } 1, 2, \ldots, D \\ -1 & I \text{ odd permutation of } 1, 2, \ldots, D \\ 0 & \text{(at least) two indices equal} \end{cases}
\]
An even (odd) permutation of $1, 2, \ldots, D$ is constructed by an even (odd) number of transpositions (= exchanges of only two indices).

\[ \bar{\epsilon}^I = \epsilon^I \equiv \det \left( \frac{\partial x^I}{\partial \bar{x}^i} \right)^{+1} \frac{\partial x^I}{\partial \bar{x}^j} \bar{\epsilon}^J \]  

$\epsilon^I = \epsilon^{i_1 i_2 \ldots i_D}$ is a $(D, 0)$-tensor density

- $\bar{\epsilon}^I = \epsilon^I$ is true by definition: $\epsilon$ is a symbol defined by Eq. (3.41); this definition is independent of the coordinate system. In Eq. (3.42) we compare this trivial transformation with that of a (relative) tensor and conclude that it is equivalent to the statement that $\epsilon^I$ transforms as a $(D, 0)$-tensor density with weight $w = +1$. This knowledge is helpful in tensor calculus to construct covariant expressions that contain Levi-Civita symbols (→ below).

- To show this, note that the Levi-Civita symbol can be used to compute determinants:

\[ \det \left( \frac{\partial \bar{x}^i}{\partial x^j} \right) = \sum_{\sigma \in S_D} (-1)^\sigma \prod_{i=1}^{D} \frac{\partial \bar{x}^{i_\sigma}}{\partial x^{j_i}} = \frac{\partial \bar{x}^1}{\partial x^{j_1}} \cdots \frac{\partial \bar{x}^D}{\partial x^{j_D}} \epsilon^{11\ldots D}. \]  

(3.43)

Details: Problemset 4
3.5. The metric tensor

A differentiable manifold $M$ does not automatically allow us to measure the length of curves, the angles of intersecting lines, or the area/volume of subsets of the manifold; to do so, we need a metric on $M$ (which is an additional piece of information). While the continuity structure (an atlas) that comes with $M$ determines its topology, the metric determines its geometry (= shape). The same manifold $M$ can be equipped with different metrics; this corresponds to different geometries of the same topology (a potato and an egg both have the topology of a sphere, nonetheless they are geometrically distinct).

A differentiable manifold together with a (pseudo-)metric is called a (pseudo-)Riemannian manifold. In special relativity and general relativity, spacetime is modeled by such (pseudo-)Riemannian manifolds where the metric is used to represent spatial and temporal distances between events.

Motivation:

On linear spaces $V$, it is convenient to define an inner product (like in quantum mechanics where you consider Hilbert spaces and use their inner product to compute probabilities and transition amplitudes).

Recall the definition of a (real) inner product:

$$\langle \bullet | \bullet \rangle : V \times V \rightarrow \mathbb{R} \quad \text{with ...}$$

Symmetry:

$$\langle x | y \rangle = \langle y | x \rangle$$

(Bi)linearity:

$$\langle ax + by | z \rangle = a \langle x | z \rangle + b \langle y | z \rangle$$

Positive-definiteness:

$$x \neq 0 \Rightarrow \langle x | x \rangle > 0$$

Once you have an inner product, you get a norm, and subsequently a metric for free:

$$\mathcal{I}nter\text{m} \hspace{1em} \mathcal{N}orm \hspace{1em} \mathcal{M}etric$$

Thus an inner product is a rather versatile structure and nice to have!

Problem: We cannot define a inner product on the manifold directly because $M$ is not a linear space.

However: We can introduce an inner product on each of its tangent spaces $T_pM$ ! →

Riemannian (Pseudo-)Metric $ds^2 := $ Symmetric, non-degenerate $(0, 2)$-tensor field:

$$ds^2 : M \ni p \mapsto \left( ds^2_p : T_pM \times T_pM \rightarrow \mathbb{R} \right)$$

Bilinear & symmetric & non-degenerate

$$ds^2_p \text{ bilinear} \Rightarrow ds^2 \in T_p^*M \otimes T_p^*M$$

$$\Rightarrow ds^2_p = \sum_{i,j=1}^{D} g_{ij}(x) \, dx^i \otimes dx^j \equiv g_{ij}(x) \, dx^i dx^j$$

with $g_{ij} = g_{ji}$ (symmetry) and $g = \det(g_{ij}) \neq 0$ (non-degeneracy).
• The tensor product is non-commutative: \( dx^i \otimes dx^j \neq dx^j \otimes dx^i \). However, you can always decompose a tensor product as

\[
dx^i \otimes dx^j = \frac{1}{2} (dx^i \otimes dx^j + dx^j \otimes dx^i) + \frac{1}{2} (dx^i \otimes dx^j - dx^j \otimes dx^i)
\]

with the symmetrized tensor product \( dx^i \otimes dx^j \) and the anti-symmetrized tensor product \( dx^i \wedge dx^j \). (\( \wedge \) wedge product).

Since \( g_{ij} \) is assumed to be symmetric, only the symmetric component survives:

\[
g_{ij}(x) dx^i \otimes dx^j = g_{ij}(x) dx^i \otimes dx^j = g_{ij}(x) dx^i dx^j
\]

(3.48)

This means that when writing \( dx^i dx^j \) in the above formula, you can be sloppy and either mean \( dx^i \otimes dx^j \) or, equivalently, \( dx^i \otimes dx^j \). You will find both conventions in the literature. I will use \( dx^i dx^j \equiv dx^i \otimes dx^j \) so that \( dx^i dx^j = dx^j dx^i \).

• It would be more appropriate to write \( g = g_{ij} dx^i dx^j \) for the metric \( (0, 2) \)-tensor; it is conventional, however, to reserve \( g \) for the determinant \( \det(g_{ij}) \) so that we are stuck with \( ds^2 \) for the metric. Note that the \( d \) in \( ds^2 \) does not refer to an \( \d \) exterior derivative, it is purely symbolical.

• To define a proper \( \d \) inner product on \( T_p M \), we should demand \( \d \) positive-definiteness instead of non-degeneracy. This, however, is often (for example in RELATIVITY) too restrictive; as it turns out, non-degeneracy is all we need for an isomorphism between \( T_p M \) and \( T_p^* M \) (“pulling indices up and down”, \( \d \) below). This is why negative eigenvalues of \( g_{ij} \) are fine for many purposes, and motivates the concept of a \( \d \) signature:

\[
\text{Signature:}
\]

Since \( g_{ij}(x) = g_{ji}(x) \) and \( \det(g_{ij}(x)) \neq 0 \)

\[
\rightarrow g_{ij}(x) \text{ has } r \text{ positive and } s \text{ negative real eigenvalues for all } p \in M
\]

Since \( \det(g_{ij}(x)) \neq 0 \), these numbers must be the same for all \( p \in M \).

\[
\rightarrow (r, s): \d \text{ Signature of the metric } ds^2
\]

This classification does not depend on the coordinate basis (\( \d \) Sylvester’s law of inertia).

• \((r > 0, s = 0)\)
  \( \rightarrow ds^2: \text{Riemannian metric } \rightarrow (M, ds^2): \d \text{ Riemannian manifold} \)

I.e., \( g_{ij} \) has only positive eigenvalues for all \( p \in M \) and is therefore \( \d \) positive-definite. This produces a true, positive-definite inner product on \( T_p M \).

• \((r > 0, s > 0)\)
  \( \rightarrow ds^2: \text{pseudo-Riemannian metric } \rightarrow (M, ds^2): \d \text{ pseudo-Riemannian manifold} \)

I.e., \( g_{ij} \) has both positive and negative eigenvalues and is therefore \( \d \) indefinite.

- \((r > 0, s = 1) \text{ or } (r = 1, s > 0):\)
  \( \rightarrow ds^2: \text{Lorentzian metric } \rightarrow (M, ds^2): \d \text{ Lorentzian manifold} \)

In RELATIVITY we are only interested in metric tensors with one positive and three negative eigenvalues (equivalently: three positive and one negative eigenvalue). Mathematically speaking, spacetime is then a four-dimensional Lorentzian manifold and a special case of a pseudo-Riemannian manifold.
Example: (Details: Problemset 4)

i. \( \ast \) \( D = 2 \) Euclidean space \( E_2 \equiv (\mathbb{R}^2, ds_E^2) \)

The Euclidean metric in Cartesian coordinates \( x^1 = x \) and \( x^2 = y \) reads:

\[
ds_E^2 := dx^2 + dy^2 = g_{ij}(x)\,dx^i\,dx^j \quad \text{with} \quad (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.49)
\]

This is consistent with the notion of \( dx \) and \( dy \) as infinitesimal shifts in coordinates and \( ds^2 \) as the infinitesimal distance (squared) that corresponds to this shift:

![Diagram of Euclidean plane](image)

ii. We can now transition to a new chart, namely polar coordinates \( \bar{x}^1 = r \) and \( \bar{x}^2 = \theta \). The induced basis change on the cotangent space is given by the total differential of the coordinate functions Eq. (3.14):

\[
\varphi^{-1} : \begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases} \quad \Rightarrow \quad dx = \cos(\theta)\,dr - r \sin(\theta)\,d\theta \quad dy = \sin(\theta)\,dr + r \cos(\theta)\,d\theta
\]

\[\text{(3.50)}\]

iii. We find the components of the metric tensor field in the new basis \( \{d\bar{x}^1 = dr, d\bar{x}^2 = d\theta\} \):

\[
d\bar{s}^2 = dr^2 + r^2 d\theta^2 = \bar{g}_{ij}(\bar{x})\,d\bar{x}^i\,d\bar{x}^j \quad \text{with} \quad (\bar{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \quad (3.51)
\]

This expression is again compatible with infinitesimal shifts in the (new) coordinates \( r \) and \( \theta \):

![Diagram of polar coordinates](image)

- The Euclidean plane \( E_2 \) is therefore an example for a Riemannian manifold with metric signature \((2, 0)\); its distinctive feature is that it is flat.
• Note that here we compute the same infinitesimal length in different coordinates (with the same result)! We did not change the metric, only the coordinates and thereby the coordinate basis in which we express the metric tensor. This is flat Euclidean space in curvilinear coordinates. By contrast, later in general relativity we will study curved (non-flat, non-Euclidean) metric tensors, i.e., we will modify the geometry of space(time) itself.

Since the metric $ds^2$ is a $(0,2)$-tensor field:

$$\tilde{g}_{ij}(\tilde{x})d\tilde{x}^i d\tilde{x}^j = ds^2 = g_{ij}(x) dx^i dx^j$$

Eq. (3.14) $\rightarrow$

$$\tilde{g}_{ij}(\tilde{x}) = \frac{\partial x^i}{\partial \tilde{x}^m} \frac{\partial x^m}{\partial \tilde{x}^j} g_{lm}(x)$$

The metric (components) transforms as any other $(0,2)$ tensor. Nothing special!

Side note:
Let \( g := \det(g_{ij}) \) and $\tilde{g} := \det(\tilde{g}_{ij})$

$$\sqrt{|g|} = \left| \det \left( \frac{\partial x}{\partial \tilde{x}} \right) \right| \sqrt{|g|}$$

$\rightarrow \sqrt{|g|}$ is a pseudo scalar tensor density of weight $w = +1$. The “pseudo” indicates that the absolute value of the Jacobian determinant shows up, cf. Eq. (3.37).

$\sqrt{g} < 0 \rightarrow \int d^p x \sqrt{-g}$ is a scalar (\( \rightarrow \) later)!

30 | Length of curves on $M$:

One immediate benefit of having a Riemannian manifold is that we can now compute the length of curves $\gamma(t)$ on $M$ (parametrized by $t \in [a, b]$ and given in some chart):

$$L[\gamma] \equiv \int_{\gamma} ds := \int_a^b \sqrt{g_{ij}(\gamma(t)) \frac{d\gamma^i(t)}{dt} \frac{d\gamma^j(t)}{dt}} dt$$

Eq. (3.39)

$$= \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)} \, dt$$

$\mathfrak{I}$! If $ds^2$ is a true pseudo metric (i.e., $g_{ij}$ has at least one negative eigenvalue), one must make sure that the chosen curve $\gamma$ does not produce negative values under the square root. In relativity these will be $\uparrow$ time-like curves.

Example:
Let $\gamma$ be the circle with radius $R$ in the Euclidean plane $E_2$. A possible parametrization in Cartesian coordinates (with origin in the center of the circle) is $\tilde{\gamma}_{xy}(t) = (x(t), y(t)) = (R \cos(t), R \sin(t))$ with $0 \leq t < 2\pi$ so that one finds for the circumference:

$$L = \int_{\gamma} \sqrt{dx^2 + dy^2} = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2} \, dt \approx 2\pi R$$
The same length can of course be calculated with the parametrization \( \tilde{\gamma}_{r\theta}(t) = (r(t), \theta(t)) = (R, t) \) and \( 0 \leq t < 2\pi \) in polar coordinates:

\[
L = \int_0^{2\pi} \sqrt{r^2 + \dot{r}^2 + r^2 \dot{\theta}^2} \, dt = 2\pi R \quad (3.58)
\]

Details: Problemset 4

Besides computing lengths of curves (and other geometric quantities, → later), there is another benefit of having a metric tensor:

**Pulling indices down:**

\[
\tilde{T}_{i_1 \ldots i_p \ldots i_q} := g_{i_k} T_{i_1 \ldots k \ldots i_p \ldots i_q} \quad (3.59)
\]

→ \( \tilde{T} \) is a tensor of type \((p-1, q+1)\)

- In Eq. (3.59) we indicate “empty” slots for indices by \( \square \) to emphasize that in each index “column” an index can either be up (contravariant) or down (covariant). It is conventional to omit the \( \square \)-markers. Note that this explains why you never should write two indices directly above each other (except for special cases, → below).

Furthermore, since \( g \) is fixed, it makes sense to label \( \tilde{T} \) again by \( T \) (note that the difference between the original tensor and the new one is manifest in the different index patterns!):

\[
\tilde{T}_{i_1 \ldots i_p \ldots i_q} \quad \rightarrow \quad T_{i_1 \ldots \square \ldots i_p \ldots \square \ldots j_1 \ldots j_q} \quad (3.60)
\]

Example:

\[
A^i_{\ j \ k} := g_{jm} A^{imk}_l \quad (3.61)
\]

- This convention matches perfectly with the computation of an inner product (which is determined by the metric tensor \( g \)) of two contravariant vectors:

\[
\langle A, B \rangle \overset{\text{def}}{=} g_{ij} A^i B^j \overset{\text{def}}{=} \frac{1}{\text{Scalar}} A^i B_i \quad (3.62)
\]

**Pulling indices up:**

We would like to have a \((2,0)\)-tensor \( g^{ij} \) with the property

\[
g^{ij} T^j = T^k \overset{\text{def}}{=} g^{ki} T_i \overset{\text{def}}{=} g^{ki} g_{ij} T^j. \quad (3.63)
\]

\( g^{ij} \) allows us to revert the pulling-down of indices defined by the metric \( g_{ij} \). Note that \( g^{ij} \) is a different tensor than \( g_{ij} \), we could call it \( \tilde{g}^{ij} \); however, it is conventional to denote it with the same label due to the following close relationship with \( g \):

\[
g^{ki} g_{ij} \overset{1}{=} \delta^k_j \quad (3.64)
\]

This is an implicit equation for \( g^{ki} \)!
\( g^{ij} \) is the inverse matrix of \( g_{ij} \)

(Which always exists because \( d\Omega^2 \) is non-degenerate: \( \det(g_{ij}) \neq 0 \).)

→ In general:

\[
\tilde{T}^{i_1 \ldots i_p \square \ldots j \square \ldots i_q \square \ldots j_q} := g^{jk} T^{i_1 \ldots i_p \square \ldots j_1 \ldots \square \ldots j_k \square \ldots j_1 \ldots \square \ldots j_q}
\]

(3.65)

→ \( \tilde{T} \) is a tensor of type \((p + 1, q - 1)\)

• Again we relabel \( \tilde{T} \) to \( T \) and omit the \( \square \)-markers:

\[
\tilde{T}^{i_1 \ldots i_p \square \ldots j \square \ldots i_q \square \ldots j_q} \mapsto T^{i_1 \ldots i_p \ldots j_1 \ldots \ldots j_q}
\]

(3.66)

• Example:

\[
A^{ijkl} := g^{lm} A^{ijkm}
\]

(3.67)

• With these new definitions, we can now raise and lower contractions:

\[
A^i B_i = A^i \delta^i_i B_j = A^i g_{ik} g^{kj} B_j = A^i g_{ik} B^k = A_k B^k = A_i B^i
\]

(3.68)

• What happens if you pull the indices of the Kronecker symbol up or down?

\[
\delta^{ij} := g^{ik} \delta^i_k = g^{ij} \quad \text{and} \quad \delta_{ij} := g_{ik} \delta^k_j = g_{ij}
\]

(3.69)

\( \delta^{ij} \) and \( \delta_{ij} \) denote the metric and its inverse!

→ We never use the notation \( \delta^{ij} \) and \( \delta_{ij} \) to prevent confusion!

• Note that in general

\[
g^{ik} T^i_k \neq T^{ij} = g^{ik} T^i_k.
\]

(3.70)

This means that the “column” in which the index is located is important, and notations like \( T^{ij}_k \) are ill defined (if you pull \( k \) up by \( g^{ik} \), do you get \( T^{ij} \) or \( T^{ji} \)?) However, if the tensor is symmetric, \( T^{ij} = T^{ji} \), this does not matter and you can get away with the sloppy notation \( T^i_k \).

This explains why writing \( \delta^i_k \) for the Kronecker symbol is fine: \( g^{ij} = g^{ik} \delta^i_k \) is symmetric.

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Mathematical side note:

“Pulling indices up and down” is mathematically the application of an isomorphism between \( T_p M \) and \( T^*_p M \):

\[
g(\bullet, \bullet) : T_p M \times A \rightarrow g(A, \bullet) \in T^*_p M
\]

(3.71)

This has nothing to do with differential geometry or manifolds in particular; it is a general feature of non-degenerate bilinear forms on vector spaces. In differential geometry, this canonical isomorphism between the tangent bundle \( TM \) and the cotangent bundle \( T^* M \) is known as a musical isomorphism.

For example, you are using the same kind of isomorphism all the time in quantum mechanics, namely whenever you “dagger” a ket \( |\Psi\rangle \) to obtain a bra \( \langle \Psi| \):

\[
(\bullet)\dagger : \mathcal{H} \ni |\Psi\rangle \mapsto \langle \Psi| \in \mathcal{H}^* \text{ with } \langle \Psi|\Phi\rangle = \dagger (\langle \Psi|\Phi\rangle) \text{ for all } |\Phi\rangle \in \mathcal{H}.
\]

(3.72)
3.6. Differentiation of tensor fields

Remember: \( \partial_i \Phi \) is covariant vector if \( \Phi \) is scalar. However:

\(<\) Contravariant vector \( A^i \):

\[
A^i_{\, \, k} = \frac{\partial A^i}{\partial x^k} = \frac{\partial x^m}{\partial x^k} \frac{\partial}{\partial x^m} \left[ \frac{\partial x^i}{\partial x^l} A^l \right] = \frac{\partial^2 x^i}{\partial x^m \partial x^k} + \frac{\partial x^m}{\partial x^k} \frac{\partial x^i}{\partial x^m} \frac{\partial A^l}{\partial x^k} \frac{\partial A^l}{\partial x^m} \tag{3.73}
\]

Here we used the transformation of \( \tilde{A}^i \) [Eq. (3.8)] and \( \tilde{\partial}_k \) [Eq. (3.5)] and the product rule.

\(->\) In general: \( \frac{\partial A^i}{\partial x^k} \) is not a tensor!

\[35\]

How to define a derivative of tensor fields that again transforms as a tensor?

To solve this problem, we first need a new field:

\(->\) Christoffel symbols (of the second kind):

\[
\Gamma^i_{\, \, kl} := \frac{1}{2} g^{im} \left( g_{mk,l} + g_{ml,k} - g_{kl,m} \right) \tag{3.74}
\]

- The Christoffel symbols are symmetric in the lower two indices: \( \Gamma^i_{\, \, kl} = \Gamma^i_{\, \, lk} \)
- !\footnote{Despite the index notation, the Christoffel symbols are not tensors:}

\[
\tilde{\Gamma}^i_{\, \, kl} = \frac{\partial x^i}{\partial x^m} \frac{\partial x^m}{\partial x^k} \frac{\partial x^p}{\partial x^l} \Gamma^m_{\, \, np} - \frac{\partial x^m}{\partial x^k} \frac{\partial x^l}{\partial x^p} \frac{\partial^2 x^i}{\partial x^m \partial x^p} \tag{3.75}
\]

No tensor!
This is why they are called “symbols” and not “tensors”!

- There are also Christoffel symbols of the first kind:
  \[
  \Gamma^i_{k\ell} := g^{ij} \Gamma^j_{k\ell} = \frac{1}{2} \left( g_{i,k\ell} + g_{i\ell,k} - g_{k\ell,i} \right)
  \] (3.76)

- Mathematically, the Christoffel symbols are the coefficients (in some basis) of the Levi-Civita connection which is determined by the metric tensor \( g^{ij} \) (\( \rightarrow \) later).

\[ \Gamma^i_{k\ell} A^l = \frac{\partial x^i}{\partial \xi^m} \frac{\partial x^n}{\partial \xi^k} \Gamma^m_{n\ell} A^l = \frac{\partial x^i}{\partial \xi^k} \frac{\partial^2 x^i}{\partial \xi^m \partial x^n} \left[ \frac{\partial x^p}{\partial \xi^l} A^l \right] - \frac{\partial x^n}{\partial \xi^k} \frac{\partial x^m}{\partial x^n} \frac{\partial x^p}{\partial \xi^l} A^l \] (3.77)

Ideas: Add Eq. (3.73) and Eq. (3.77) to cancel the problematic term:

\[ A^l_{i;k} + \Gamma^i_{k\ell} A^l = \frac{\partial x^m}{\partial \xi^k} \frac{\partial x^n}{\partial \xi^l} \left[ A^l_{m,i} + \Gamma^l_{m\ell} A^l \right] \] (3.78)

This motivates the definition of the **Covariant derivative**:

- With this definition, \( A^l_{i;k} \) is a (1, 1)-tensor and \( B_{i;k} \) is a (0, 2)-tensor!

- With this definition, the product rule is valid for the covariant derivative:

\[ (A^l B_i)_{;k} = (A^l B_i)_{,k} \equiv A^l_{;k} B_i + A^l B_{i;k} \] (3.80)

- The construction of higher-rank tensors by tensoring contra- and covariant vectors Eq. (3.32) and the definitions of the covariant derivative above Eq. (3.79) can be used to construct covariant derivatives of arbitrary tensor fields. For example:

\[ T^i_{k;l} := T^i_{k;l} + \Gamma^i_{m\ell} T^m_{k;l} - \Gamma^m_{k\ell} T^i_{m;l} \] (3.81)

- With this generalization, we can apply the covariant derivative multiple times. For example:

\[ A^l_{i;k;l} \equiv \left( A^l_{i;k} \right)_{;l} \] (3.82)

- The covariant derivative is **not commutative** in general:

\[ A^l_{i;k;l} = A^l_{i;l;k} \neq 0 \] (3.83)

\( \rightarrow \) **Riemann curvature tensor** \( \rightarrow \) **GENERAL RELATIVITY** (\( \rightarrow \) later)

(This is not the case for the “normal” derivative: \( A^l_{i,l;k} = A^l_{i,l;k} \) )
Conclusion:
If you can formulate an equation that describes a physical theory in terms of tensors, it can always be brought into the form

$$ T^I_J(x) = 0. \quad (3.84) $$

(This equation is meant to hold for all values of indices $I$ and $J$ and all coordinate values $x$.)

Here is an example:

The (inhomogeneous) Maxwell equations on an arbitrary (potentially curved) spacetime read:

$$ F_{\mu\nu} + \frac{4\pi}{c} J^\mu = 0 \quad (3.85) $$

with current density $J^\mu$ and field strength tensor $F_{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} (A_{\rho\sigma} - A_{\rho\pi})$.

How does Eq. (3.84) look like in any other coordinate system $\tilde{x} = \varphi(x)$?

Easy:

$$ \tilde{T}^I_J(\tilde{x}) = \frac{\partial \tilde{x}^I}{\partial x^M} \frac{\partial x^N}{\partial \tilde{x}^J} T^M_N(x) = 0 \quad \Leftrightarrow \quad \tilde{T}^I_J(\tilde{x}) = 0. \quad (3.86) $$

This means:

Tensor equations are automatically form-invariant under arbitrary coordinate transformations; we say they exhibit \( \star \) (manifest) general covariance.

The “manifest” means that checking general covariance is just a matter of checking whether the equation “looks right”, i.e., whether it is built from tensors following the rules discussed in this chapter. If a property of an equation is manifest, you don’t have to do calculations to verify it!

In the next chapter, we take a step back and specialize the allowed coordinate transformations to the Lorentz transformations of special relativity. We can then use the form-invariance of equations built from “Lorentz tensors” to construct Lorentz covariant equations from scratch – which was our original goal!
4. Formulation on Minkowski Space

In this section we briefly reformulate what we already learned about SPECIAL RELATIVITY in terms of tensor calculus. We use this notation in subsequent chapters to make classical and quantum mechanics relativistic, and reformulate electrodynamics in a form where its Lorentz covariance is manifest. It also allows a smooth transition into GENERAL RELATIVITY.

The formulation of SPECIAL RELATIVITY on a unified, four-dimensional spacetime manifold goes back to Hermann Minkowski, Albert Einstein's former professors of mathematics at ETH. Minkowski writes in the notes of his lecture “Raum und Zeit” delivered 1908 in Cologne [53]:


Einstein, a physicist all through, didn’t appreciate this mathematical reformulation of his theory at first. According to Sommerfeld, he (Einstein) commented:

Seit die Mathematiker über die Relativitätstheorie hergefallen sind, verstehe ich sie selbst nicht mehr.

Einstein later changed his views and acknowledged that without Minkowski’s introduction of spacetime as a four-dimensional manifold, the development of GENERAL RELATIVITY would have been impossible.

For a historical account on the role of Minkowski, and his relationship (or absence thereof) to Einstein, see Ref. [54].

4.1. Minkowski space

<table>
<thead>
<tr>
<th>Manifold:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = \langle \text{Spacetime of events / coincidence classes } \mathcal{E} \rangle \simeq \mathbb{R}^4$ (4.1)</td>
</tr>
</tbody>
</table>

It is a well founded, but nonetheless empirical assumption that the spacetime manifold of events has the topology of $\mathbb{R}^4$. Note that at this point we do not impose restrictions on the geometry of spacetime, e.g., whether it is flat or curved; this follows below when we settle on a metric.

<table>
<thead>
<tr>
<th>Charts:</th>
</tr>
</thead>
<tbody>
<tr>
<td>In SPECIAL RELATIVITY, we restrict the coordinate systems to the ones that correspond to inertial observers / inertial coordinate systems:</td>
</tr>
<tr>
<td>$(\mathcal{E}, K) \leftrightarrow \text{Inertial (coordinate) systems } K \in J$ (4.2)</td>
</tr>
</tbody>
</table>
The coordinates are the ones obtained by an \textit{inertial observer}:

\begin{align}
K : \mathcal{E} \ni E &\mapsto K(E) := [E]_K = x \\
\text{with} \quad x^\mu &= (x^0, x^1, x^2, x^3)^T = (ct, x, y, z)^T
\end{align}

\begin{itemize}
\item Henceforth, \textit{Greek} indices \(\mu, \nu, \ldots\) run over 0, 1, 2, 3 where \(\mu = 0\) denotes the time component and \(\mu = 1, 2, 3\) denote the spatial components. \textit{Roman} indices \(i, j, \ldots\) run only over the spatial components 1, 2, 3.
\item We multiply the time \(t\) with the speed of light to measure times and distances in the same units.
\item Since we assumed that our inertial systems cover all of spacetime, the domains on which the coordinate functions are defined are the complete manifold.
\item The notation above is very suggestive: You can think of our inertial systems, namely the calibrated latticework of clocks and rods, as physical manifestations of the coordinate map of the corresponding chart. That is, an inertial system is a measurement device, or function, which assigns to every event \(E \in \mathcal{E}\) the coordinate tuple \(x = K(E) = (ct, \vec{x})_K \in E\).
\end{itemize}

3 | Transition maps:

\begin{itemize}
\item We worked hard in Section 1.4 to derive and select the correct coordinate transformations between different inertial systems. The most general ones have the form of…
\end{itemize}

\begin{align}
\text{Inhomogenous Lorentz transformations} \quad &\text{Poincaré transformations} \\
\begin{dcases}
\tilde{x} = \varphi(x) = \Lambda x + a
\end{dcases}
\end{align}

with \(a \in \mathbb{R}^4\) arbitrary and \(\Lambda \in \mathbb{R}^{4 \times 4}\) a \textit{Lorentz transformation}.

For the special case \(a = 0 \in \mathbb{R}^4\) we found:

\begin{align}
\text{Homogeneous Lorentz transformations:} \quad \tilde{x} = \varphi(x) = \Lambda x
\end{align}

\begin{itemize}
\item Since these transformations are affine, we find immediately:

\begin{align}
\frac{\partial \tilde{x}^\mu}{\partial x^\nu} = \Lambda_{\nu}^\mu \quad \text{and} \quad \frac{\partial x^\mu}{\partial \tilde{x}^\nu} = (\Lambda^{-1})_{\mu}^\nu \equiv \Lambda_{\nu}^\mu
\end{align}

Recall that the derivative of a linear (affine) map is simply the matrix which defines the map.
\end{itemize}

\begin{itemize}
\item We use the tensor-inspired notation \(\Lambda_{\nu}^\mu\) for the matrix elements of \(\Lambda\) to allow for well-defined contractions with the metric (\textit{\(\rightarrow\) later}). In \(\Lambda_{\nu}^\mu\), the upper index \(\mu\) denotes the \textit{rows}, the lower index \(\nu\) the \textit{columns} of the matrix. The notation \(\Lambda_{\nu}^\mu\) for the components of the inverse transformation matrix \(\Lambda^{-1}\) is purely conventional at this point; it will turn out to be consistent with pulling indices up and down with the Minkowski metric (\textit{\(\rightarrow\) below}).
\end{itemize}

This allows us to rewrite the coordinate transformation Eq. (4.5) in tensor notation:

\begin{align}
\tilde{x}^\mu = \Lambda_{\nu}^\mu x^\nu + a^\mu
\end{align}
The matrix-vector product $\Lambda x$ is now given by the Einstein summation (index contraction) highlighted blue. We will stick to this notation whenever possible. Since we are now in the world of tensor calculus, it is strongly discouraged to think of and write rank-2 tensors as “matrices” and contractions as matrix-vector products $\Lambda x$ (even though $\Lambda$ does not represent the components of a tensor). It is less error-prone (and simpler) to perform computations using the index notation introduced in Chapter 3.

Writing down the most general homogeneous Lorentz transformation is very complicated (and unnecessary). Here we provide the two special Lorentz transformations (boosts) discussed earlier in the new matrix notation, and an example for a spatial rotation about the $\hat{z}$-axis:

- Lorentz boost in $x$-direction $K \overset{v_x}{\rightarrow} \tilde{K} (\beta_x = \frac{v_x}{c})$:

$$\Lambda^\mu_\nu = [\Lambda_{v_x}]^\mu_\nu = \left( \begin{array}{cccc} \gamma & -\beta_x \gamma & 0 & 0 \\ -\beta_x \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)_{\mu\nu} \tag{4.9}$$

- Lorentz boost in $\tilde{v}$-direction $K \overset{v}{\rightarrow} \tilde{K} (v = |\tilde{v}|$ and $\tilde{\gamma} := \gamma - 1$):

$$\Lambda^\mu_\nu = [\Lambda_{\tilde{v}}]^\mu_\nu = \left( \begin{array}{cccc} \gamma & -\beta_x \gamma & 0 & 0 \\ -\beta_x \gamma & 1 + \tilde{\gamma} v_x^2 / v^2 & \tilde{\gamma} v_x v_y / v^2 & \tilde{\gamma} v_x v_z / v^2 \\ -\beta_y \gamma & \tilde{\gamma} v_x v_y / v^2 & \gamma & 0 \\ -\beta_z \gamma & \tilde{\gamma} v_x v_z / v^2 & 0 & \gamma \end{array} \right)_{\mu\nu} \tag{4.10}$$

- Spatial rotation $K \overset{R_z(\theta),\hat{\theta}}{\rightarrow} \tilde{K}$ by $\theta$ in $xy$-plane:

$$\Lambda^\mu_\nu = [R_z(\theta)]^\mu_\nu = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)_{\mu\nu} \tag{4.11}$$

Metric tensor:

We elevate the spacetime manifold $M$ to a pseudo-Riemannian (and Lorentzian) manifold by
introducing the following pseudo-Riemannian metric tensor (given in inertial coordinates):

\[
\mathbf{ds}^2 = (c dt)^2 - (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2
\]

with metric components

\[
\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}
\]

\[\text{Signature} (1, 3) = (+, -, -, -)\] (4.12a)

- The components \(\eta_{\mu\nu}\) of this metric tensor in Eq. (4.12b) are the same for all inertial coordinate systems [\(\rightarrow\) Eq. (4.21) below].
- Recall that \(\eta^{\mu\nu}\) is the matrix inverse of \(\eta_{\mu\nu}\).

\(\rightarrow\) We call the spacetime manifold equipped with this metric …

\[\mathbb{M} \text{inkowski space: } \mathbb{R}^{1,3} \equiv (\mathcal{E} \cong \mathbb{R}^4, \mathbf{ds}^2)\] (4.13)

- We will always use \(\eta_{\mu\nu}\) to denote the components of the Minkowski metric (in an inertial coordinate chart) to distinguish it from a generic metric \(g_{ij}\).
- Note that, informally speaking, \(\mathbf{ds}^2\) this is the infinitesimal form of the \(\leftrightarrow\) invariant spacetime interval Eq. (1.83) we introduced earlier (\(\rightarrow\) below).
- Minkowski space is therefore an example of a \(\leftrightarrow\) Lorentzian manifold. By fixing a metric, we fixed the geometry of spacetime. As we will see in our discussion of \(\leftrightarrow\) general relativity, the distinctive feature of Minkowski space is that it is flat (it has no curvature). It will turn out that, in reality, this assumption is only valid to some degree: The tenet of \(\leftrightarrow\) general relativity is that the deviations of spacetime from flat Minkowski space are what we experience as gravity!

\[\begin{align*}
\text{i)} &\text{ With the metric we can measure “lengths” of trajectories on spacetime:} \\
&\text{\hspace{1cm} Time-like trajectory } \gamma : s \mapsto x^\mu(s) \text{ for } s \in [s_a, s_b] \text{ in } \mathbb{R}^{1,3} \rightarrow
\\
&L[\gamma] \overset{\text{(4.14a)}}{=} \int_{s_a}^{s_b} \sqrt{\eta_{\mu\nu} \frac{dx^\mu(s)}{ds} \frac{dx^\nu(s)}{ds}} \, ds \\
&\overset{\text{(4.14b)}}{=} \int_{s_a}^{s_b} \sqrt{\dot{x}^0(s)^2 - \dot{x}^1(s)^2 - \dot{x}^2(s)^2 - \dot{x}^3(s)^2} \, ds \\
&\text{Choose parametrization } s := x^0/c \equiv t \quad \text{(4.14c)}
\\
&= \int_{t_a}^{t_b} \sqrt{c^2 - \vec{v}^2(t)} \, dt \\
&\overset{\text{(4.14d)}}{=} \Delta t[\gamma] \quad \text{(4.25)}
\end{align*}\]

Thus the “length” \(L[\gamma]\) of time-like curves in \(\mathbb{R}^{1,3}\) is the \(\leftrightarrow\) proper time \(\Delta t[\gamma]\) along the curve defined in Eq. (2.25) (multiplied by \(c\)); this explains why the Minkowski metric \(\mathbf{ds}^2\) is the right choice for \(\leftrightarrow\) special relativity.
4.2. Four vectors and tensors

Tensors are defined as in Chapter 3, with the restriction to $D = 4$ and that only homogeneous Lorentz transformations Eq. (4.7) are considered as transition maps. To emphasize this, we introduce a new nomenclature:

<table>
<thead>
<tr>
<th>Tensor calculus</th>
<th>SPECIAL RELATIVITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Contravariant vector $A^i$</td>
<td>Contravariant Lorentz vector / 4-vector $A^\mu$</td>
</tr>
<tr>
<td>Covariant vector $B_i$</td>
<td>Covariant Lorentz vector / 4-vector $B_\mu$</td>
</tr>
<tr>
<td>(Mixed) tensor $T^i_j$</td>
<td>(Mixed) Lorentz tensor / 4-tensor $T_{\mu \nu}$</td>
</tr>
<tr>
<td>Scalar $\Phi$</td>
<td>Lorentz scalar $\Phi$</td>
</tr>
</tbody>
</table>

Then a generic $(p, q)$ tensor transforms under the coordinate transformation Eq. (4.7) as:

$$T^{\mu_1 \cdots \mu_p}_{\nu_1 \cdots \nu_q} (\tilde{x}) = \left[ \Lambda^{\mu_1 \rho_1} \cdots \Lambda^{\mu_p \rho_p} \right] \left[ \Lambda^{-\nu_1 \pi_1} \cdots \Lambda^{-\nu_q \pi_q} \right] T^{\rho_1 \cdots \rho_p}_{\pi_1 \cdots \pi_q} (x)$$  \hspace{1cm} (4.15)

With the Minkowski metric, we can reformulate our classification for 4-vectors [recall Eq. (1.85)]:

- $X^\mu$ time-like \( X^2 = (X^0)^2 - (\vec{X})^2 > 0 \)
- $X^\mu$ light-like \( X^2 = (X^0)^2 - (\vec{X})^2 = 0 \)
- $X^\mu$ space-like \( X^2 = (X^0)^2 - (\vec{X})^2 < 0 \)

A light-like 4-vector is also called \( \leftrightarrow \) null.

! We use this classification scheme also for generic Lorentz vectors that are not coordinate differences between a pair of events (\( \rightarrow \) below). Since the pseudo-norm $X^\mu X_\mu = X^2$ is a Lorentz scalar, this classification is independent of the inertial system.

7 | Coordinate functions:

It is a particular feature of linear coordinate transformations (here: homogeneous Lorentz transformations) that the coordinate functions themselves transform as contravariant vector fields:

\(<\) Coordinate field $X^\mu (x) := x^\mu \rightarrow$

$$\tilde{X}^\mu (\tilde{x}) = \Lambda^\mu_\nu X^\nu (x) = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} X^\nu (x)$$  \hspace{1cm} (4.17)

We make the identification $X^\mu (x) = x^\mu$ and don’t write $X^\mu (x)$ henceforth.

Consequently, we can construct \( \leftrightarrow \) covariant coordinates (a covariant vector field) via the metric by pulling the index down:

$$x_{\mu} := \eta_{\mu \nu} X^\nu = (x^0, -x^1, -x^2, -x^3) = (ct, \vec{x})$$  \hspace{1cm} (4.18)

! To pull the index of a contravariant vector down, you multiply the spatial components by $-1$. 

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Coordinates of two events $x_A^\mu$ and $x_B^\mu \rightarrow \Delta x^\mu := x_B^\mu - x_A^\mu$ Lorentz vector

\[ \Delta x^2 \equiv \Delta x^\mu \Delta x_\mu \]

\[ \overset{\text{def}}{=} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \]

\[ = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 \]

\[ \overset{\text{def}}{=} \Delta s^2 \]

\[ (4.19a) \]

\[ (4.19b) \]

\[ (4.19c) \]

\[ (4.19d) \]

Remember [Eq. (1.84)]: $\Delta s^2 = \Delta s^2$ for arbitrary Lorentz transformations

\[
\frac{\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu}{\Delta s^2} = \frac{\Delta \tilde{x}_{\rho} \Delta \tilde{x}_{\pi}}{\Delta \tilde{s}^2} = \left[ \eta_{\rho\pi} \Lambda^\rho_{\phantom{\rho}\mu} \Lambda^\pi_{\phantom{\pi}\nu} \right] \Delta x^\mu \Delta x^\nu
\]

\[ (4.20) \]

Since this is true for all events $\Delta x^\mu \overset{\circ}{\rightarrow}$

\[
\Lambda^\rho_{\phantom{\rho}\mu} \Lambda^\pi_{\phantom{\pi}\nu} \eta_{\rho\pi} = \eta_{\mu\nu}
\]

\[ (4.21) \]

Concluding Eq. (4.21) from Eq. (4.20) is non-trivial because we consider “norms” $\eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$ and not “inner products” $\eta_{\mu\nu} \Delta x^\mu \Delta y^\nu$. However, for symmetric, real matrices $A$ and $B$, it is true that if $\tilde{x}^T A \tilde{x} = \tilde{x}^T B \tilde{x}$ for all real vectors $\tilde{x}$, then $A = B$. This is so because $A - B$ is a symmetric matrix that can be diagonalized by an orthogonal matrix and $\tilde{x}^T (A - B) \tilde{x} = 0$. The last condition implies that all eigenvalues of $A - B$ are zero and therefore $A - B = 0$. Alternatively, you can use the $\downarrow$ polarization identity to show that the invariance of the Minkowski (pseudo) norm implies the invariance of the Minkowski (pseudo) inner product.

We say:

\[
\text{Lorentz transformations are } \downarrow \text{ isometries of Minkowski space.}
\]

\[ (4.22) \]

With $\det(\eta_{\mu\nu}) \neq 0$, a corollary of Eq. (4.21) is:

\[
\det \left( \Lambda^\mu_{\phantom{\mu}\nu} \right) = \pm 1
\]

\[ (4.23) \]

If you want to write Eq. (4.21) in the old matrix notation, make the identifications $\Lambda^\mu_{\phantom{\mu}\nu} = A_{\mu\nu}$ and $\eta_{\mu\nu} = \eta_{\mu\nu}$. Here, subscripts of bold symbols denote the entries of matrices as usual (first index: row; second index: column). Equations that contain matrices (bold symbols) do not comply with the syntax of tensor calculus (which is why you should avoid them!).

Eq. (4.21) then reads in matrix notation:

\[
\Lambda_{\mu\rho}^T \eta_{\rho\pi} \Lambda_{\pi\nu} = \eta_{\mu\nu} \iff \Lambda^T \eta \Lambda = \eta
\]

\[ (4.24) \]

Here we defined the transposed matrix as $\Lambda^T_{\mu\rho} := \Lambda_{\rho\mu}$, i.e., the matrix where rows and columns are swapped. Eq. (4.24) immediately implies $\det(\Lambda^T) \det(\eta) = \det(\Lambda) = \det(\eta)$; using $\det(\eta) \neq 0$ and $\det(\Lambda^T) = \det(\Lambda)$, we find $\det(\Lambda) = \pm 1$. 
We can therefore conclude that:

\[ \Lambda_\rho^\sigma := \eta_{\rho\sigma} \Lambda_{\nu}^\mu = (\Lambda^{-1})^\sigma_\rho \]  

(4.26)

Note that this is consistent with our definition in Eq. (4.7).

In the literature (e.g. Schröder [1]) the concept of a “transposed” transformation is introduced. We refer to it as “pseudo-adjoint” transformation instead and label it by \( \ast \). It is defined analogous to proper adjoints on proper inner product spaces:

\[ \eta_{\mu\nu} A^\nu_\rho x^\nu y^\mu \stackrel{\text{def}}{=} (y, \Lambda x) \triangleq (\Lambda^* y, x) \stackrel{\text{def}}{=} \eta_{\mu\nu} (\Lambda^*)^\mu_\rho x^\nu y^\rho . \]  

(4.27)

This yields as reasonable definition for the pseudo-adjoint:

\[ (\Lambda^*)^\mu_\nu := \Lambda_{\nu\mu} \Rightarrow (\Lambda^*)^\mu_\nu = \Lambda^\nu_\mu \stackrel{\text{Eq. (4.26)}}{=} (\Lambda^{-1})^\mu_\nu . \]  

(4.28)

One can then define a corresponding matrix \( \Lambda^* \) such that \( (\Lambda^*)^\mu_\nu = \Lambda^\nu_\mu \) and use \( (\Lambda^{-1})^\mu_\nu = \Lambda^{-1}_{\mu\nu} \) to rewrite the above equation as

\[ \Lambda^* = \Lambda^{-1} . \]  

(4.29)

Recall that the pseudo-adjoint is implicitly defined via the inner product. At no point did we claim that the pseudo-adjoint matrix is given by the transpose matrix \( \Lambda^T \) (which is defined by swapping rows and columns)! To find a relation to the latter, we can rewrite Eq. (4.26) in matrix language:

\[ \Lambda^{-1}_{\sigma\rho} = \eta_{\rho\sigma} \Lambda_{\nu\nu} \eta^{-1}_{\nu\sigma} = (\eta \Lambda \eta)_{\rho\sigma} = (\eta \Lambda^T \eta)_{\sigma\rho} . \]  

(4.30)

Here we used that \( \eta^{-1} = \eta = \eta^T \) and that \( M^T_{ab} := M_{ba} \) for any matrix \( M \). So finally:

\[ \Lambda^* = \Lambda^{-1} = \eta \Lambda^T \eta . \]  

(4.31)

The take home message is that the transpose of a Lorentz transformation (given by swapping columns and rows) is not its inverse (there are additional minuses sprinkled in by the metric)! By contrast, the pseudo-adjoint (defined via the pseudo-inner product) is identical to the inverse.

**Warning**: In the literature you will find the notation \( T \) instead of \( \ast \) (e.g. Schröder [1]). Then one finds the (correct) relation \( (\Lambda^T)^\mu_\nu = \Lambda^\nu_\mu = (\Lambda^{-1})^\mu_\nu \). The problem is that this notation suggests that \( (\Lambda^T)^\mu_\nu \stackrel{\text{def}}{=} \Lambda^\mu_\nu \) and therefore \( \Lambda^{-1} \neq \Lambda^T \). As shown above, both equations are wrong!

**Covariant derivative:**

Since in inertial coordinate systems the Minkowski metric is given by \( \eta_{\mu\nu} \), it follows immediately for the Christoffel symbols Eq. (3.74):

\[ \Gamma^i_{kl} = \frac{1}{2} \eta^{im} \left( \eta_{mk,l} + \eta_{ml,k} - \eta_{kl,m} \right) = 0 \]  

(4.32)

If you would transform into *curvilinear* (non-inertial) coordinates, the Christoffel symbols would not vanish – even on flat Minkowski space (Problemset 5). That simple partial
derivatives produce Lorentz tensors is therefore a special feature of Minkowski space in inertial coordinates.

Eq. (3.79)

\[
\begin{align*}
\text{Lorentz Scalar:} & \quad \Phi_{,\mu} := \Phi_{,\mu} = \partial_\mu \Phi \\
\text{Contravariant Lorentz vector:} & \quad A_{\mu,\nu} := A_{\mu,\nu} = \partial_\nu A_\mu \\
\text{Covariant Lorentz vector:} & \quad B_{\mu,\nu} := B_{\mu,\nu} = \partial_\nu B_\mu
\end{align*}
\]

\[\text{(4.33a)}\]

\[\text{(4.33b)}\]

\[\text{(4.33c)}\]

ii | 4-Gradient:

This allows us to think of the differential operator \( \partial_\mu \) itself as a covariant Lorentz vector and motivates the introduction of its contravariant components:

\[
\begin{align*}
\partial_\mu & = \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \partial_t, + \vec{\nabla} \right)^T \\
\partial^\mu & := \eta^{\mu\nu} \partial_\nu = \frac{\partial}{\partial x^\mu} = \left( \frac{1}{c} \partial_t, - \vec{\nabla} \right)
\end{align*}
\]

\[\text{(4.34a)}\]

\[\text{(4.34b)}\]

Using Eq. (3.5), the transformation laws match that of co- and contravariant Lorentz vectors, respectively:

\[
\begin{align*}
\tilde{\partial}_\mu & = \frac{\partial}{\partial x^\mu} = \Lambda_\mu^\nu \frac{\partial}{\partial x^\nu} = \Lambda_\mu^\nu \partial_\nu \\
\tilde{\partial}^\mu & = \frac{\partial}{\partial x^\mu} = \Lambda^\mu_\nu \frac{\partial}{\partial x^\nu} = \Lambda^\mu_\nu \partial^\nu
\end{align*}
\]

\[\text{(4.35a)}\]

\[\text{(4.35b)}\]

The covariant 4-gradient (index down) is the partial derivative wrt. the contravariant coordinates (index up) and vice versa.

iii | These transformation properties immediately suggest two Lorentz scalars that can be constructed from 4-gradients \((A^\mu = (A^0, \vec{A}))\):

\[
\begin{align*}
\bullet \text{4-divergence:} & \quad \partial A := \partial_\mu A^\mu = \partial^\mu A_\mu = \frac{1}{c} \partial_t A^0 + \vec{\nabla} \cdot \vec{A} \\
\bullet \text{4-Laplacian:} & \quad \Box \equiv \partial^2 := \partial_\mu \partial^\mu = \left( \frac{1}{c} \partial_t \right)^2 - \vec{\nabla}^2
\end{align*}
\]

\[\text{(4.36a)}\]

\[\text{(4.36b)}\]

The 4-Laplacian \(\Box\) is also known as \(\dagger d’)Alembert operator.\)

Examples:

- In electrodynamics (\(\rightarrow\) later) the gauge potential transforms as a contravariant Lorentz vector \(A^\mu = (\frac{1}{c} \partial_t, \vec{A})\).

The \(\dagger Lorentz gauge\) is defined as \(\partial_\mu A^\mu = 0\); it is Lorentz invariant since the 4-divergence is a Lorentz scalar: \(\partial_\mu A^\mu(x) = \partial_\mu A^\mu(\tilde{x})\).

Note: The Lorenz gauge is named after \(\dagger Ludvig Lorenz\); by contrast, the Lorentz transformation is named after \(\dagger Hendrik Lorentz\). Thus: The Lorenz gauge (no “it”) is Lorentz invariant.
• In vacuum (and in Lorenz gauge), the gauge field of electrodynamics satisfies the wave equation
\[ \partial^2 A^\mu = \left( \frac{1}{c} \partial_t \right)^2 - \nabla^2 \right) A^\mu = 0 . \]  
(4.37)

Since \( \partial^2 \) is a Lorentz scalar and \( A^\mu \) a Lorentz vector, \( \partial^2 A^\mu \) transforms as a contravariant Lorentz vector and the equation is manifestly Lorentz covariant:
\[ \partial^2 A^\mu (x) = 0 \iff \tilde{\partial}^2 \tilde{A}^\mu (\tilde{x}) = 0 . \]  
(4.38)

• If we have a scalar field \( \Phi \), we can construct a manifestly Lorentz covariant wave equation:
\[ (\partial^2 + m^2) \Phi(x) = 0 \iff (\tilde{\partial}^2 + m^2) \tilde{\Phi}(\tilde{x}) = 0 . \]  
(4.39)

The parameter \( m \) is arbitrary and plays the role of a mass (spectral gap) of the excitations. This equation is known as \( \uparrow \) Klein-Gordon equation and describes, for example, the classical equation of motion of the Higgs field (without interactions).

11 | Relative tensors → Lorentz pseudo tensor:

Since \( \det(\Lambda) = \pm 1 \), the classification of tensors simplifies:

**Tensor:** \( \bar{T}^M_N (\bar{x}) = \Lambda^M_R \Lambda^P_N T^R_P (x) \)  
(4.40a)

**Pseudo tensor:** \( \bar{T}^M_N (\bar{x}) = \det(\Lambda) \Lambda^M_R \Lambda^P_N T^R_P (x) \)  
(4.40b)

Here we use again a multi-index notation: \( M = \mu_1, \ldots, \mu_p \) etc. Recall that \( \det(\Lambda) = \pm 1 \); pseudo tensors therefore pick up an additional minus sign under parity or time inversion (\( \rightarrow \) later).

\( \rightarrow \) Relative tensors of odd weight \( w \) are pseudo tensors under Lorentz transformations.

Example:

The Levi-Civita symbol is a Lorentz pseudo tensor [recall Eq. (3.42)]:
\[ \varepsilon^{\mu \nu \rho \pi} = \varepsilon^{\mu \nu \rho \pi} = \det(\Lambda) \Lambda^\mu_{\mu'} \Lambda^\nu_{\nu'} \Lambda^\rho_{\rho'} \Lambda^\pi_{\pi'} \varepsilon^{\mu' \nu' \rho' \pi'} . \]  
(4.41)

This means that if you contract a Levi-Civita symbol with an actual \( (0, 4) \) Lorentz tensor like \( F_{\mu \nu} F_{\rho \pi} \) (the tensor product of two electromagnetic field strength tensors), you obtain a pseudo (Lorentz) scalar:
\[ \Phi(\bar{x}) := \varepsilon^{\mu \nu \rho \pi} \bar{F}_{\mu \nu} \bar{F}_{\rho \pi} \overset{\approx}{=} \det(\Lambda) \varepsilon^{\mu \nu \rho \pi} F_{\mu \nu} F_{\rho \pi} = \det(\Lambda) \Phi(x) . \]  
(4.42)

Since this is a quadratic (pseudo) scalar quantity, you might try to add it to the Lagrangian of Maxwell theory (\( \theta \in \mathbb{R} \)):
\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu} F_{\mu \nu} + \theta \varepsilon^{\mu \nu \rho \pi} F_{\mu \nu} F_{\rho \pi} . \]  
(4.43)

(This Lagrangian is now only invariant under Lorentz transformations with \( \det(\Lambda) = +1 \).)

The new term is called \( \uparrow \theta \)-term. One can show that it is a total derivative and therefore does not affect the classical equations of motion (Maxwell’s equations). However, for non-abelian generalizations of electrodynamics like \( \uparrow \) quantum chromodynamics (\( \uparrow \) Yang-Mills theories), it does affect the theory (\( \uparrow \) Strong CP-problem [55]).

Note that we did not use the metric tensor \( g_{\mu \nu} \) to construct the term \( \varepsilon^{\mu \nu \rho \pi} F_{\mu \nu} F_{\rho \pi} \) (as compared to \( F_{\mu \nu} F_{\mu \nu} \), where we need it to pull two indices up); this makes the \( \theta \)-term an example of a so called \( \uparrow \) topological term (\( \uparrow \) topological field theory): the term doesn’t “see” the geometry of spacetime! In condensed matter physics, the term plays a role in the description of \( \uparrow \) topological insulators [56].
In the next chapter we want to construct a relativistic version of classical mechanics (using the framework of tensors calculus to make the equations Lorentz covariant). As a preparation, we can already define two 4-vectors with physical interpretation:

### 4-velocity:

**Question:** What is a reasonable definition for a relativistic (= Lorentz covariant) velocity?

- Particle trajectory $x^\mu(\lambda)$ parametrized by $\lambda$:

$$x^\mu(\lambda) = \left( c t(\lambda) \right) \xrightarrow{\text{d}x^\mu}{\text{d}\lambda} = \left( \frac{\text{d}t}{\text{d}\lambda} \right) \left( \frac{\text{d}x}{\text{d}\lambda} \right)$$  \hspace{1cm} (4.44)

First try: $\lambda = t$ (coordinate time) →

$$\frac{\text{d}x^\mu}{\text{d}t} = \left( \frac{\text{c}}{\frac{\text{d}x}{\text{d}t}} \right) = \left( \frac{\text{c}}{\tilde{v}(t)} \right)$$  \hspace{1cm} (4.45)

with coordinate velocity $\tilde{v}(t)$.

**Problem:**

$\frac{\text{d}x^\mu}{\text{d}t}$ is not a contravariant Lorentz vector because $\text{d}t \neq \text{d}\tilde{t}$ is not a Lorentz scalar. That is:

$$\frac{\text{d}x^\mu}{\text{d}t} \neq \Lambda^\mu_\nu \frac{\text{d}x^\nu}{\text{d}t}$$  \hspace{1cm} (4.46)

→ Eq. (4.45) is useless to construct Lorentz covariant equations!

**Idea:** The ↔ Proper time $\tau$ is a Lorentz scalar [Eq. (2.24)]: $\text{d}\tau = \text{d}\tilde{t}$

→ Set $\lambda = \tau$:

$$\star \star \star \text{ 4-velocity: } u^\mu := \frac{\text{d}x^\mu}{\text{d}\tau} = \left( \frac{\text{c}}{\frac{\text{d}x}{\text{d}\tau}} \right) = \gamma_\nu \left( \frac{\text{c}}{\tilde{v}} \right)$$  \hspace{1cm} (4.47)

Here we used $\frac{\text{d}x}{\text{d}\tau} = \gamma_\nu(t)$ [recall Eq. (2.23)].

By construction, the 4-velocity is a contravariant Lorentz vector: $\tilde{u}^\mu = \Lambda^\mu_\nu u^\nu$.

- Pseudo-norm:

$$u^2 = \eta_{\mu\nu} u^\mu u^\nu = (u^0)^2 - (\tilde{u})^2 \equiv c^2 > 0$$  \hspace{1cm} (4.48)

→ **Time-like 4-vector**

In Minkowski space, $u^\mu$ is the tangent at $x^\mu$ of the world line $x^\mu(\tau)$.

### 4-acceleration:

Following the same line of arguments above, the 4-acceleration is then defined as the derivative of the 4-velocity wrt. the proper time:

$$\star \star \star \text{ 4-acceleration: } b^\mu := \frac{\text{d}u^\mu}{\text{d}\tau} = \left( \frac{\text{c}}{\frac{\text{d}x(\mu)}{\text{d}\tau}} \right) \equiv \frac{\gamma_\nu}{\gamma_\nu} \left( \frac{\text{c}}{\tilde{v}} \right)$$  \hspace{1cm} (4.49)
Here $\ddot{a} := \frac{d^2a}{dt^2}$ is the coordinate acceleration or 3-acceleration.

It is now easy to show that $b^2 = b_\mu b^\mu < 0$ is a space-like Lorentz vector and that

$$
\frac{d(u^\mu u_\mu)}{dr} = \frac{d(e^2)}{dr} = 0 \Rightarrow u^\mu b_\mu = 0,
$$

i.e., the 4-acceleration is always “orthogonal” (in terms of the Minkowski metric) to the 4-velocity.

4.3. The complete Lorentz group

Details: ♻ Problemset 5

1 | The Lorentz group is a matrix group defined as the homogenous isometry group of the Minkowski metric $\eta$:

$$
\text{Lorentz group: } O(1, 3) := \left\{ \Lambda \in \mathbb{R}^{4 \times 4} \mid \Lambda^T \eta \Lambda = \eta \right\}
$$

(4.51)

with identification $\Lambda^\mu_\nu = \Lambda_{\mu\nu}$ and $\eta_{\mu\nu} = \eta_{\nu\mu}$.

- As shown previously [Eq. (4.21) and Eq. (4.24)], the matrix constraint in Eq. (4.51) is equivalent to the property

$$
\eta_{\mu\nu} x^\mu y^\nu \overset{\text{def}}{=} \eta(x, y) = \eta(\Lambda x, \Lambda y) \overset{\text{def}}{=} \left[ \eta_{\rho\sigma} \Lambda^\rho_\mu \Lambda^\sigma_\nu \right] x^\mu y^\nu
$$

(4.52)

for all 4-vectors $x, y$. Namely, the transformations $\Lambda$ do not change the inner product (and thereby length) of arbitrary vectors; maps with this feature are called ↑ isometries.

- If you replace the Minkowski metric $\eta_{\mu\nu} = \text{diag} (+1, -1, -1, -1)$ by the Euclidean metric $\delta_{\mu\nu} = \text{diag} (+1, +1, +1, +1)$, the homogeneous isometry constraint becomes $\Lambda^T \Lambda = 1$ since $\delta = 1$ is the identity matrix; this constraint characterizes orthogonal matrices. The homogeneous isometry group of a $D = 4$ Euclidean space is therefore O(4): the group of four-dimensional rotations and reflections.

2 | Continuous Lorentz transformations:

i | Mathematical fact: O(1, 3) is a ↑ Lie group (= a group that is also a differentiable manifold)

To be precise: O(1, 3) is a 6-dimensional (↑ below) ↑ non-compact ↓ non-abelian disconnected (↑ below) real matrix Lie group with components that are not ↑ simply connected.

→ In a neighborhood of $1$, elements of Lie groups can be written as exponentials:

$$
\Lambda = \exp(X) \quad \text{with} \quad X \in o(1, 3)
$$

(4.53)

where $o(1, 3)$ denotes the ↑ Lie algebra (= vector space with a Lie bracket):

$$
o(1, 3) = \left\{ X \in \mathbb{R}^{4 \times 4} \mid \exp(tX) \in O(1, 3) \quad \text{for all} \quad t \in \mathbb{R} \right\}.
$$

(4.54)
The isometry constraint on the group elements can be translated into the Lie algebra:

$$\Lambda^T \eta \Lambda = \eta \quad \text{Eq. (4.53)} \quad X^T = -\eta X \eta$$

 Most general form of $X$:

$$X = \begin{pmatrix} 0 & a & b & c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{pmatrix} \quad \text{with} \quad a, \ldots, f \in \mathbb{R} \quad (4.56)$$

Proof: \(\bigcirc\) Problemset 5

\(\Rightarrow\)

- \(\dim(o(1, 3)) = 6\)
  This is why \(O(1, 3)\) is a 6-dimensional Lie group.

- \(\text{Tr}[X] = 0 \Rightarrow \det \Lambda = \det[\exp(X)] = \exp(\text{Tr}[X]) = 1\)
  \(\rightarrow\) All Lorentz transformations connected to the identity have positive determinant. Recall that we found previously \(\det \Lambda = \pm 1\), so we should not expect to find all elements of \(O(1, 3)\) in this way.

Generators = Basis of \(o(1, 3)\) [57]:
We use the shorthand $+(-)$ for $+1(-1)$.

\[
L_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.57a)
\]

\[
K_x = \begin{pmatrix} 0 & + & 0 & 0 \\ + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 \end{pmatrix}, \quad K_z = \begin{pmatrix} 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (4.57b)
\]

Interpretation:

$$\exp(\varphi L_x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \varphi - \sin \varphi & \cos \varphi \\ 0 & 0 & \sin \varphi & \cos \varphi \end{pmatrix} = \Lambda_{R_x(\varphi)} \quad \rightarrow \text{Rotation around } x\text{-axis} \quad (4.58a)$$

$$\exp(-\theta K_x) = \begin{pmatrix} \cosh \theta - \sinh \theta & 0 & 0 \\ -\sinh \theta & \cosh \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Lambda_{v_x} \quad \rightarrow \text{Boost in } x\text{-direction} \quad (4.58b)$$

with \(\leftrightarrow\) \(\text{rapidity} \tan \theta = \frac{v_x}{c} \in (-1, 1)\) \(\bigcirc\) Problemset 3 and rotation angle \(\varphi \in [0, 2\pi)\).

$\left\{ L_x, L_y, L_z \right\}$: Generators of rotations

$\left\{ K_x, K_y, K_z \right\}$: Generators of boosts
An arbitrary element of O(1, 3) that is connected to the identity can then be written as
\[ \Lambda = \exp \left( \sum_i \phi_i L_i - \sum_i \theta_i K_i \right) \quad \text{with} \quad i \in \{ x, y, z \}. \]  
(4.60)

In particular [57]:

Pure boost:
\[ \Lambda_{\bar{\theta}} \equiv \Lambda_{\bar{\theta}} = \exp \left( -\bar{\theta} \cdot \vec{K} \right) \]  
(4.61a)

Pure rotation:
\[ \Lambda_{R_{\bar{\varphi}}} = \exp \left( \bar{\varphi} \cdot \vec{L} \right) \]  
(4.61b)

with rotation angle \( \varphi = |\bar{\varphi}| \), rotation axis \( \bar{\varphi} = \bar{\varphi}/|\bar{\varphi}| \), and rapidity vector
\[ \bar{\theta} = \bar{\theta}(\vec{u}) := \bar{\theta} \tanh^{-1} \left( \frac{\vec{u}}{c} \right). \]  
(4.62)

¡! The rapidity vector \( \bar{\theta} \) is not given by the rapidities \( \tanh^{-1} \frac{u_i}{c} \) of the components \( u_i \) of \( \vec{u} \).

iv | Lie algebra:

The Lie bracket (= commutator) on the Lie algebra determines the multiplicative structure of the Lie group via the Baker-Campbell-Hausdorff formula:
\[ \exp(X) \cdot \exp(Y) = \exp \left( X + Y + \frac{1}{2} [X, Y] + \ldots \right). \]  
(4.63)

→ The Lie algebra \( \mathfrak{o}(1, 3) \) determines the (local) group structure of O(1, 3):

Eq. (4.57) \( \overset{o}{\rightarrow} \)

\[
\begin{align*}
[L_i, L_j] & = \varepsilon^{ijk} L_k \\
[L_i, K_j] & = \varepsilon^{ijk} K_k \\
[K_i, K_j] & = -\varepsilon^{ijk} L_k
\end{align*}
\]  
(4.64a) (4.64b) (4.64c)

Some comments and implications:

• ¡! Because of Eq. (4.64) [and Eq. (4.63)], you cannot simply combine exponentials:
\[
\begin{align*}
\exp \left( -\bar{\theta} \cdot \vec{K} \right) \cdot \exp \left( \bar{\varphi} \cdot \vec{L} \right) & \neq \exp \left( \bar{\varphi} \cdot \vec{L} - \bar{\theta} \cdot \vec{K} \right), \\
\exp \left( -\bar{\theta} \cdot \vec{K} \right) \cdot \exp \left( -\bar{\theta}' \cdot \vec{K} \right) & \neq \exp \left( -\bar{\theta} + \bar{\theta}' \cdot \vec{K} \right), \\
\exp \left( \bar{\varphi} \cdot \vec{L} \right) \cdot \exp \left( \bar{\varphi}' \cdot \vec{L} \right) & \neq \exp \left( \bar{\varphi} + \bar{\varphi}' \cdot \vec{L} \right).
\end{align*}
\]  
(4.65a) (4.65b) (4.65c)

This is why the concatenation of Lorentz transformations is quite complicated in general.

• Eq. (4.64a) is written in physics often as \( [L_i, L_j] = i \hbar \varepsilon^{ijk} L_k \) with angular momentum operators \( L_k \). In this notation, they generate rotations \( U_{\bar{\varphi}} = \exp(\frac{\xi}{\hbar} \vec{L}) \). The additional phase \( i \) in the commutation relation matches a corresponding factor in an alternative definition of the generators \( \vec{L} \). (Recall that the \( L_j \) in Eq. (4.57) are anti-Hermitian whereas in physics we often prefer Hermitian operators.)

• Eq. (4.64a) shows that \( \mathfrak{o}(3) \) := span \{ \( L_x, L_y, L_z \) \} forms a subalgebra of \( \mathfrak{o}(1, 3) \). On the group level, this means that the group of spatial rotations SO(3) is a subgroup of the full Lorentz group O(1, 3).
By contrast, Eq. (4.64c) shows that the boost generators \( \{ K_x, K_y, K_z \} \) do not form a subalgebra, but mix with rotations. This implies that there is no “subgroup of pure boosts” in \( O(1, 3) \). In particular:

\[
\Lambda_\vec{u} \Lambda_{\vec{v}} = \Lambda_{\vec{u} \oplus \vec{v}} \Lambda_{R(\vec{u}, \vec{v})}
\]

(4.66)

with the \( \leftrightarrow \) Thomas-Wigner rotation \( R(\vec{u}, \vec{v}) \in SO(3) \) [recall Section 2.3].

- There is a more compact, 4-vector-inspired notation for the 6 generators in Eq. (4.57), namely [58]:

\[
\left( J^{\alpha \beta} \right)^\mu_v = \left( J^{\alpha \beta} \right)^\mu_v := \eta^{\mu \rho} \delta^\rho_\nu - \eta^{\beta \rho} \delta^\rho_\nu.
\]

(4.67)

Inspection shows that (\( \bullet \) Problemset 5)

\[
L_x = J^{23} = -J^{32}, \quad K_x = J^{01} = -J^{10},
\]

(4.68a)

\[
L_y = J^{31} = -J^{13}, \quad K_y = J^{02} = -J^{20},
\]

(4.68b)

\[
L_z = J^{12} = -J^{21}, \quad K_z = J^{03} = -J^{30}.
\]

(4.68c)

The three equations of the Lie algebra Eq. (4.64) can then be condensed into a single equation [58]:

\[
[J^{\mu \nu}, J^{\sigma \rho}] = \eta^{\rho \sigma} J^{\mu \nu} - \eta^{\mu \sigma} J^{\nu \rho} - \eta^{\nu \rho} J^{\mu \sigma} + \eta^{\mu \rho} J^{\nu \sigma}.
\]

(4.69)

This form is useful to construct other representations of the Lorentz group, especially in relativistic quantum mechanics (\( \rightarrow \) Dirac equation).

\( v \) | It is a useful mathematical fact that every continuous Lorentz transformation of the form Eq. (4.60) can be decomposed uniquely as follows:

\[
\Lambda = \Lambda_\vec{v} \Lambda_R = \Lambda_R \Lambda_\vec{w}
\]

(4.70a)

with parameters:

\[
\frac{v_i}{c} = -\frac{\Lambda_{i0}}{\Lambda_{00}}, \quad \frac{w_i}{c} = -\frac{\Lambda_{0i}}{\Lambda_{00}} \quad \text{and} \quad R_{ij} = \Lambda_{ij} - \frac{\Lambda_{i0} \Lambda_{0j}}{1 + \Lambda_{00}}
\]

(4.70b)

\( \Lambda_\vec{v} \) and \( \Lambda_R \) are defined in Eq. (4.61a) [or Eq. (1.75)] and Eq. (4.61b) [or Eq. (1.40)].

The proof can be found in Ref. [59]. This decomposition, sometimes referred to as \( \star \) rotation-boost decomposition, relates to the mathematical concept of \( \star \) Cartan decompositions [60].

If we use the multiplicative law \( \Lambda_R \Lambda_\vec{v} \Lambda_R^{-1} = \Lambda_{R\vec{u}} \) [recall Eq. (1.43a)] and choose \( R \) such that \( R\vec{v} = (v_x, 0, 0)^T \), we can also find a decomposition of the form

\[
\Lambda = \Lambda_R \Lambda_v \Lambda_{R_2}
\]

(4.71)

with appropriately chosen rotations \( R_1, R_2 \in SO(3) \) and a boost in \( x \)-direction by \( v_x \).

\[3\] Discrete generators:
It is easy to verify that the following two matrices also belong to $O(1, 3)$:

**Parity:**

\[ P : (t, \vec{x}) \mapsto (t, -\vec{x}) \quad \Rightarrow \quad P^\mu_\nu = P_{\mu\nu} := \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} \quad (4.72) \]

**Time reversal:**

\[ T : (t, \vec{x}) \mapsto (-t, \vec{x}) \quad \Rightarrow \quad T^\mu_\nu = T_{\mu\nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}_{\mu\nu} \quad (4.73) \]

In contrast to the continuous group elements above: $\det(P) = \det(T) = -1 \rightarrow P$ and $T$ are not generated by boosts or rotations!

4 | Structure of the Lorentz group:

Combining the discrete transformation $P$ and $T$ with the continuous transformations $\Lambda = \exp(X)$ yields the complete group $O(1, 3)$. Let us study its structure:

\[ O(1, 3) = L_+ \cup L_- \quad \text{det}(\Lambda) = \pm 1 \quad (4.74) \]

All Lorentz transformations that are continuously connected to $I$ are in $L_+$. One can transition between $L_+$ and $L_-$ by applying either $T$ or $P$.

\[ \text{i} \quad \text{In addition, we find:} \]

\[ 1 = \eta_{00} \overset{4.21}{=} \left( \Lambda^0_0 \right)^2 - \sum_{k=1}^{3} \left( \Lambda^k_0 \right)^2 \leq \left( \Lambda^0_0 \right)^2 \quad (4.75) \]

Thus $\Lambda^0_0 \neq 0$ and $\text{sign}(\Lambda^0_0) = \pm 1$ can be used to characterize Lorentz transformations. Note that $\text{sign}(P^0_0) = +1$ but $\text{sign}(T^0_0) = -1$ and $\text{sign}((PT)^0_0) = -1$.

\[ \text{ii} \quad \text{Neither det}(\Lambda) = \pm 1 \text{ nor } \text{sign}(\Lambda^0_0) = \pm 1 \text{ can be changed by continuously deforming a Lorentz transformation.} \]

\[ \rightarrow \text{Four disconnected components of } O(1, 3): \]

\[ L^\dagger_+ : \quad \text{det}(\Lambda) = +1 \quad \text{and} \quad \text{sign}(\Lambda^0_0) = +1 \quad (l \in L^\dagger_+ \quad (4.76a) \]

\[ L^\dagger_- : \quad \text{det}(\Lambda) = -1 \quad \text{and} \quad \text{sign}(\Lambda^0_0) = +1 \quad (P \in L^\dagger_- \quad (4.76b) \]

\[ L^\dagger_+ : \quad \text{det}(\Lambda) = +1 \quad \text{and} \quad \text{sign}(\Lambda^0_0) = -1 \quad (PT \in L^\dagger_+ \quad (4.76c) \]

\[ L^\dagger_- : \quad \text{det}(\Lambda) = -1 \quad \text{and} \quad \text{sign}(\Lambda^0_0) = -1 \quad (T \in L^\dagger_- \quad (4.76d) \]

\[ L^\dagger_+ : \quad \text{det}(\Lambda) = +1 \quad \text{and} \quad \text{sign}(\Lambda^0_0) = +1 \quad (l \in L^\dagger_+) \]

\[ L^\dagger_- : \quad \text{det}(\Lambda) = -1 \quad \text{and} \quad \text{sign}(\Lambda^0_0) = +1 \quad (P \in L^\dagger_-) \]

\[ L^\dagger_+ : \quad \text{det}(\Lambda) = +1 \quad \text{and} \quad \text{sign}(\Lambda^0_0) = -1 \quad (PT \in L^\dagger_+) \]

\[ L^\dagger_- : \quad \text{det}(\Lambda) = -1 \quad \text{and} \quad \text{sign}(\Lambda^0_0) = -1 \quad (T \in L^\dagger_-) \]
Graphically:

- **proper orthochronous Lorentz Group** (restricted LG)
  \( L_+ = \text{SO}^+ (1, 3) \)

- **orthochronous LG**
  \( L^+ = \text{O}^+ (1, 3) \)

- **proper LG**
  \( L_+ = \text{SO} (1, 3) \)

\[ L^+ \begin{cases} \uparrow & \text{no time inversion (sign } \Lambda^0_0 = +1 \text{)} \\ \downarrow & \text{time inversion (sign } \Lambda^0_0 = -1 \text{)} \\ + & \det \Lambda = +1 \text{ (proper)} \\ - & \det \Lambda = -1 \text{ (improper)} \end{cases} \]

### Subgroups:

We can define the following four subgroups of \( O(1, 3) \):

- **Proper LG**:
  \( \text{SO}(1, 3) \equiv L_+ := L_+^+ \cup L_+^- \) (4.77a)

- **Orthochronous LG**:
  \( \text{O}^+ (1, 3) \equiv L^+ := L_+^+ \cup L_-^+ \) (4.77b)

- **Proper orthochronous LG**:
  \( \text{SO}^+ (1, 3) \equiv L_+^+ \) (4.77c)

- **Orthochorous LG**:
  \( L_0 := L_+^+ \cup L_-^+ \) (4.77d)

Note that subgroups must contain the identity \( I \)!

In Greek, “chrónoς” (χρόνος) means “time” and “chóros” (χώρος) means “space”.

According to modern physics, Einstein's principle of relativity \( \text{SR} \) reads formally:

**All fundamental theories of nature must be invariant under the proper orthochronous Lorentz group \( \text{SO}^+ (1, 3) \).**

- **This does not prevent specific theories to have additional symmetries.** Quantum electrodynamics (QED), for example, is invariant under the full Lorentz group \( O(1, 3) \). This means that phenomena of electromagnetism – and its interaction with charged particles – are also symmetric under time inversion \( T \) and parity \( P \).

  So far, observations suggest that, besides the electromagnetic force, also gravity and the strong force are symmetric under \( P \) and \( T \). (Interestingly, there is no formal reason why the strong force should not break \( P \) and \( T \); the fact that it does not violate these symmetries is called the **strong CP problem**.)

- **However, today we know that there are terms in the standard model of particle physics that violate both \( P \) and \( T \).** For example, the weak interaction (responsible for radioactive \( \beta \)-decay) violates parity \( P \) strongly (**Wu experiment**). This means that you can use...
experiments that depend on the weak interaction to tell the difference between our world and its mirror image (or a right-handed and a left-handed coordinate system). There are also weak terms (concerning quarks) that violate time reversal $T \ (\neq CP \ \text{violation}).$ As a consequence, the standard model as a whole is only invariant under the proper orthochronous Lorentz group $\text{SO}^+ (1, 3)$.

This explains why we can only require symmetry under $\text{SO}^+ (1, 3)$, and not the full Lorentz group $\text{O}(1, 3)$: We already know by experiments that the latter is not a fundamental symmetry of nature!

- The fact that there are processes that violate parity symmetry $P$ contradicts our everyday experience: If you run an experiment using equipment found in a school physics lab and put a mirror next to it, there is no way to decide whether you are watching the experiment directly or via the mirror (i.e., parity inverted). The reason is that the physics we experience in everyday life is governed by electrodynamics and gravity, both of which are invariant under $P$. To unveil that nature secretly violates $P$, you must perform an experiment that involves the weak interaction (that is: a particle physics experiment). This is what Chien-Shiung Wu did in her now famous $\uparrow$ Wu experiment. At the time, the result (that $P$ is not a symmetry of nature) was unexpected and groundbreaking.

So if you are surprised that $P$ is not a symmetry of nature, you are not alone. Here is how Wolfgang Pauli reacted to the result of the Wu experiment [61]:

\begin{quote}
At one point, Temmer found himself in the presence of eminence grise Wolfgang Pauli, who asked for the latest news from the United States. Temmer told him that parity was no longer to be assumed conserved. “That’s total nonsense” averred the great man. Temmer: “I assure you the experiment says it is not.” Pauli (curtly): “Then it must be repeated!”
\end{quote}

### 4.4. Why is spacetime 3+1 dimensional?

Given the discussions in Chapter 3 and Chapter 4 it is clear that the mathematical formalism allows for straightforward generalizations to higher- (or lower-) dimensional spacetime manifolds with arbitrary signatures; these suggest spacetimes with various numbers of spatial and temporal dimensions.

It is therefore natural to ask:

\textit{Is there anything special about our 3+1-dimensional world?}

What follows is not a proof that spacetime must be 3+1 dimensional. Our goal is to argue that all spacetimes, except ours with three space and one time dimension, face severe problems that, most likely, would not allow for complex life.

The following discussion is based on Tegmark [39, 62].

1. **Pseudo-Riemannian manifold of signature \((t, s)\) with metric**

\[
g_{ij} = \text{diag}(+1, \ldots, +1, -1, \ldots, -1)
\]

- This is the generalization of Minkowski space to a (flat) spacetime manifold with, naively, $t$ time and $s$ space-dimensions.
- Most of our discussions in this chapter can be transferred to this more general setting.
\[ (\partial^2 + m^2) \Phi = \sum_{i=1}^{t} \frac{\partial^2 \Phi}{\partial x_i^2} - \sum_{i=t+1}^{s+t} \frac{\partial^2 \Phi}{\partial x_i^2} + m^2 \Phi = 0 \] 

(4.79)

- Recall that \( \partial^2 = g^{ij} \partial_i \partial_j \) where \( g^{ij} \) is given by (the inverse of) Eq. (4.78).
- The Klein-Gordon equation (KGE) is the simplest covariant field equation. It describes the time evolution of a scalar field of mass \( m \). It is ubiquitous in relativistic physics (especially in quantum field theory).
- For example, the components of the electromagnetic field in vacuum are described by the KGE for \( m = 0 \) and \( (t, s) = (1, 3) \) (which is then referred to as wave equation):

\[ \partial^2 E_i = \frac{1}{c^2} \partial_t^2 E_i - \nabla^2 E_i = 0, \] 

(4.80a)

\[ \partial^2 B_i = \frac{1}{c^2} \partial_t^2 B_i - \nabla^2 B_i = 0. \] 

(4.80b)

This motivates in Eq. (4.79) the (tentative) identification of the coordinates with positive sign as “time coordinates”, and the ones with a negative sign as “space coordinates”:

The difference between time and space is just a sign!

In the following, we use the KGE as a proxy for more general relativistic field equations.

→ Possible combinations of \( t \) time and \( s \) space dimensions:

3 | Partial differential equations (PDE):

The general KGE in Eq. (4.79) is an example of a partial differential equation (PDE). The theory of PDEs has been thoroughly developed by mathematicians and a lot is known about their solvability. The problem of solving a PDE, given some boundary/initial conditions, is known as Cauchy problem:

- Well-posed (Cauchy) problem: Given some boundary/initial data, there exists a unique solution to the PDE that satisfies these conditions, and this solution is robust. Here “robust” means that if you slightly modify the boundary/initial conditions, the solution also changes only slightly. Put differently: The solutions are not chaotic and you can use them to extrapolate reliably from boundary/initial states with finite errorbars. This is a crucial feature to use PDEs for predictions in the real world.
- Ill-posed (Cauchy) problem: Given some boundary/initial data, there either exist multiple solutions to the PDE that satisfy these conditions, or the unique solution is not robust. In both cases, the PDE cannot be used for predictions in the real world.
Elliptic PDEs have well-posed boundary problems:

\[ \Delta u = 0 \]

This corresponds to spacetimes that are Riemanian manifolds.

\[ \rightarrow \text{One cannot use Eq. (4.79) to make predictions} \]
\[ \rightarrow \text{No coordinate in Eq. (4.79) qualifies as a “time coordinate”} \]

Ultrahyperbolic PDEs have well-posed initial value problems:

\[ \Delta u = -1 \]

This corresponds to spacetimes that are generic pseudo-Riemannian manifolds.

\[ \rightarrow \text{One cannot use Eq. (4.79) to make predictions} \]

Hyperbolic PDEs have well-posed initial value problems:

\[ \Delta u = 1 \]

This corresponds to spacetimes that are Lorentzian manifolds.
We can use Eq. (4.79) to make predictions.

4 | Stability:
   • $<\! Newtonian Gravity in $s \geq 4$ spatial dimensions:
     $\rightarrow$ Two-body problem has no stable orbits (only scattering and attraction solutions).
     $\rightarrow$ No stable planetary systems possible
   • $<\! Hydrogen atom in $s \geq 4$ spatial dimensions:
     $\rightarrow$ Schrödinger equation has no bound states.
     $\rightarrow$ No stable atoms possible

The opposite cases with $t \geq 4$ and $s = 1$ are equivalent if one interprets space as time and vice versa (which is necessary to use hyperbolic PDEs to predict “the future”, $\rightarrow$ below).

5 | Simplicity:
GENERAL RELATIVITY in $s \leq 2$ spatial dimensions $\rightarrow$ No gravity ($\rightarrow$ later)!
$\rightarrow$ No stars, no planets, no orbits

The opposite cases with $t \leq 2$ and $s = 1$ are equivalent if one interprets space as time and vice versa (which is necessary to use hyperbolic PDEs to predict “the future”, $\rightarrow$ below).
Tachyon world:

In the literature both Lorentzian signatures $(1, 3)$ and $(3, 1)$ are used to formulate **special relativity**. Formulations in signature $(3, 1)$ have nothing to do with the Tachyon sector discussed here since they compensate for the global minus in their equations. For example, the KGE in signature $(1, 3)$ reads $(-\partial^2 + m^2)\Phi = 0$ which is equivalent to the KGE $(\partial^2 + m^2)\Phi = 0$ in signature $(+, -, -, -)$. The point here is that we do **not** add this additional minus:

\[
\text{Eq. (4.79)} \quad (\frac{1}{1,3}) \leftrightarrow (\frac{3,1}{3,1}) \quad (-\partial^2 + m^2)\Phi = 0 \iff (\partial^2 - m^2)\Phi = 0 \quad (4.81)
\]

In more detail:

For $t = 3$ and $s = 1$ the KGE reads

\[
\frac{\partial^2 \Phi}{\partial (x^1)^2} + \frac{\partial^2 \Phi}{\partial (x^2)^2} + \frac{\partial^2 \Phi}{\partial (x^3)^2} - \frac{\partial^2 \Phi}{\partial (x^4)^2} + m^2 \Phi = 0 . \quad (4.82)
\]

But because this an hyperbolic PDE, the Cauchy problem is only well-posed with initial conditions on a hypersurface spanned by $\{x^1, x^2, x^3\}$. Put differently: The PDE allows predictions only in $x^4$-direction! Thus we should interpret $x^4$ as **time** and $\{x^1, x^2, x^3\}$ as **space**:

\[
\frac{\partial^2 \Phi}{\partial (x^1)^2} + \frac{\partial^2 \Phi}{\partial (x^2)^2} + \frac{\partial^2 \Phi}{\partial (x^3)^2} - \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} + m^2 \Phi = 0 \quad (4.83)
\]

with $ct = x^4$. But this KGE is equivalent to

\[
\left(1 - \frac{\partial^2}{c^2} - \nabla^2 - m^2\right)\Phi = (\partial^2 - m^2)\Phi = 0 . \quad (4.84)
\]

Thus the “transposed” situation ($t \geq 1, s = 1$) is equivalent to the situation ($t = 1, s \geq 1$) with **negative** square-masses in the equations. Fields with negative square-mass (equivalently: imaginary mass) are called **tachyonic fields** or **tachyons** for short.

→ **All massive particles are tachyons** [63]

† Tachyonic fields are not science fiction; they do exist († below) but, contrary to the features assigned to them in science fiction, do **not** allow for faster-than-light propagation of information.

→ **Tachyon fields herald vacuum instabilities** [64]

The spontaneous symmetry breaking of the **Higgs mechanism** is an example of this phenomenon: The Higgs field has a negative square-mass which is responsible for the “Mexican hat potential.”
The consequence is spontaneous symmetry breaking, which, in this context, can be reframed as “tachyon condensation.” On the new, symmetry broken vacuum, excitations are not tachyons with negative square-mass but Higgs bosons with positive square-mass.

These arguments support the following hypothesis:

**Only a spacetime with 1 time and 3 space dimensions supports observers like us.**

What does this line of arguments explain? Well, if you would randomly construct universes by dicing the number of space and time dimensions, only the ones with one time and three space dimensions have the chance to develop complex observers like us (who then wonder why their universe is 3 + 1-dimensional). Thus the arguments above are important for “ensemble interpretations” of reality, like certain multiverse hypotheses or superstring theories (which can predict a plethora of different spacetime dimensions) [39, 65].
5. Relativistic Mechanics

Equipped with the machinery of Chapter 4, we can finally construct a relativistic (Lorentz covariant) version of classical mechanics.

5.1. The relativistic point particle

1 | Point particle in $\mathbb{R}^{1,3}$ with trajectory $x^\mu(\tau)$:

2 | It is reasonable to define the relativistic momentum of a massive particle as follows:

   $$ p^\mu := m u^\mu = m \frac{dx^\mu}{d\tau} = \left( \frac{m \gamma v}{m \gamma v} \right) \Rightarrow \left( \frac{p^0}{p} \right) $$

   with (rest) mass $m$ and 3-momentum $\vec{p}$.

   *(5.1) (5.2)*

- $m$ is the good old (inertial) mass we would assign to the particle in classical mechanics; it is a measure of the particles resistance to changes in its state of motion. You can determine it by applying a (weak) force to the particle at rest and observing its initial acceleration: $m = F/a$. This mass is an intrinsic property of the particle and does not depend on velocity. It is sometimes called rest mass, but we will simply call it mass.

- Since the 4-velocity $u^\mu$ is a Lorentz vector, the 4-momentum is also a Lorentz vector; i.e., under a Lorentz transformation $\Lambda$ the 4-momentum transforms as $\bar{p}^\mu = \Lambda^\mu_\nu p^\nu$.

- We will later rederive the expression for the 4-momentum as the conserved Noether charge for translations in spacetime.

3 | The spatial part of the momentum (the 3-momentum $\vec{p}$) is related to the velocity as follows:

   $$ \vec{p} = m \gamma v = \frac{m \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow m \vec{v} $$

   (Relativity)

   Newtonian mechanics

   *(5.3)*
In special relativity the kinetic momentum is no longer proportional to the velocity. In particular for \( v \to c \) the momentum of a massive particle diverges.

The non-relativistic limit (\( v \ll c \Rightarrow \beta \ll 1 \Rightarrow \gamma_0 \approx 1 \)) is consistent with the Newtonian (non-relativistic) relation \( \vec{p} = mv \) for the kinetic momentum; the 3-momentum \( \vec{p} \) is therefore the proper relativistic version of the momentum in Newtonian mechanics.

This explains why the above definition for the 4-momentum is reasonable – and why the mass \( m \) must be identified with the mass used in Newtonian mechanics.

At this point it is unclear how to interpret the time-component \( p^0 = m\gamma_v c \) of \( p^\mu \) (→ below).

\[ p^2 = p^\mu p_\mu = (p^0)^2 - \vec{p}^2 = m^2 c^2 4.48 = m^2 c^2 > 0 \]  

The mass \( m \) is a Lorentz scalar: \( m^2 = p^2 / c^2 \)

- The 4-momentum is a time-like 4-vector for massive particles.
- This means that the mass \( m \) can be measured/computed in every inertial system by measuring/computing the 4-momentum \( p^\mu \) and its pseudo-norm \( p^2 \). The numerical result will always be the same, namely \( m^2 c^2 \).

Equation of motion (EOM):

- We want an EOM that …
  - …is manifestly Lorentz covariant \( \rightarrow \) Lorentz tensor equation
  - …reduces to Newton’s equation of motion

\[ m\ddot{\vec{a}} = \frac{dp}{dt} = \vec{F} \quad \text{with} \quad \vec{p} = m\vec{v} \]  

in the non-relativistic limit (correspondence principle).

Suggestion:

\[ mb^\mu = \frac{dp^\mu}{dr} = K^\mu = \left( \begin{array}{c} K^0 \\ K \end{array} \right) \quad \text{with} \quad \star \text{4-force } K^\mu . \]

Because this is a equation built from Lorentz vectors, it is form-invariant (Lorentz covariant) by construction:

\[ mb^\mu = K^\mu \leftrightarrow m \Lambda^v_\mu b^\mu = \Lambda^v_\mu K^\mu \leftrightarrow \bar{m}\bar{b}^\nu = \bar{K}^\nu \]  

This is of course only so if the 4-force transforms like a Lorentz vector.

Instantaneous rest frame (IRF) \( K_0 \):

- At any time there is an inertial coordinate system \( K_0 \) in which the (potentially accelerated) particle is at rest at this very moment (if the particle is accelerating, it is also accelerating in this frame).

\[ mb_0^\mu = \left( \begin{array}{c} 0 \\ m\ddot{a} \end{array} \right) = \left( \begin{array}{c} 0 \\ \ddot{F}_0 \end{array} \right) = K_0^\mu \]
This follows from the correspondence principle: In the IRF the particle is in the non-relativistic, Newtonian limit. Thus its coordinate acceleration $\tilde{a}_0$ must be given by Newton’s equation of motion: $\tilde{m}\tilde{a}_0 = \tilde{F}_0$.

with

- **Proper acceleration $\tilde{a}_0$**

  The proper acceleration is the coordinate acceleration (3-acceleration) that you can measure (e.g., with an accelerometer) in the IRF $K_0$ of the particle.

  It follows immediately that the norm of the proper acceleration is a Lorentz scalar:

  $$b^2 = b^\mu b_\mu = -|\tilde{a}_0|^2 < 0$$

(5.9)

- **Proper force $\tilde{F}_0$**

  The proper force is the Newtonian force (3-force) you can measure (e.g., with a spring balance) in the IRF $K_0$ of the particle.

  We demand that this equation is Lorentz covariant, i.e., that $b^\mu$ and $K_\mu^0$ transform as contravariant Lorentz 4-vectors. We can then use a Lorentz boost to transform back into the lab frame in which the particle has coordinate velocity $\tilde{v}$:

  Eq. (1.75)

  4-acceleration: $b^\mu = (\Lambda_{\mu\nu})^\mu_0 b_0^\nu \overset{1.75}{=} \left( \gamma v \frac{\tilde{a}_0 \tilde{v}}{c} + \gamma v - 1 \tilde{a}_0 \cdot \tilde{v} \right)$

  (5.10a)

  4-force: $K^\mu = (\Lambda_{\mu\nu})^\mu_0 K_0^\nu \overset{1.75}{=} \left( \gamma v \frac{\tilde{F}_0 \tilde{v}}{c} + \gamma v - 1 \tilde{F}_0 \cdot \tilde{v} \right)$

  (5.10b)

  We will use these expressions later!

  On the other hand, we can return to Eq. (5.6) and study the 4-force $K^\mu$ in more detail:

  - Spatial components of Eq. (5.6):
    $$\frac{d\tilde{p}}{dt} = \gamma v(t) \frac{d\tilde{p}}{d\tau} = \tilde{K} \leftrightarrow \frac{d\tilde{p}}{dt} = \frac{\tilde{K}}{\gamma v} = : \tilde{F} \leftrightarrow \tilde{K} = \gamma v \tilde{F}$$

  (5.11)

  with **3-force $\tilde{F}$**.

  Here $\frac{d\tilde{p}}{dt}$ denotes the change in momentum measured in coordinate time; it makes sense identify this quantity with the relativistic analog of the Newtonian force.

  What is the time component $K^0$ of the 4-force? 

  $$0 \overset{4.50}{=} mb^\mu u_\mu \overset{5.6}{=} K^\mu u_\mu = K^0 u^0 - \tilde{K} \cdot \tilde{u} \overset{4.47}{=} \gamma_v (K^0 c - \tilde{K} \cdot \tilde{v})$$

  (5.12)

  $$K^0 = \frac{\tilde{K} \cdot \tilde{v}}{c} \overset{5.11}{=} \frac{\gamma_v}{c} \tilde{F} \cdot \tilde{v}$$

  (5.13)
In summary, the 4-force in terms of the 3-force and the 3-velocity reads
\[
4\text{-force: } K^\mu = \begin{pmatrix}
\gamma v \frac{\vec{F} \cdot \vec{v}}{c} \\
\gamma v F_0
\end{pmatrix}
\] (5.14)

Example:
In our discussion of electrodynamics (→ Chapter 6) we will find the following expression for the 3-force acting on a charged particle in an electromagnetic field:
\[
\vec{F} = q \vec{E} + \frac{q}{c} \vec{v} \times \vec{B}
\] (5.15)

This is the conventional Lorentz force.
This example demonstrates that the 3-force \( \vec{F} \) is indeed the proper relativistic analog of Newtonian forces. Note, however, that it is only the component of the 4-force and thus does not transform nicely under Lorentz transformations.

The Newtonian equation \( \frac{d\vec{p}}{dt} = \vec{F} \) therefore remains valid in special relativity for the 3-force \( \vec{F} \) and the 3-momentum \( \vec{p} \). By contrast, \( \vec{p} = m\gamma v \vec{v} \) is different from the Newtonian relation \( \vec{p} = m\vec{v} \) between momentum and velocity.

The time component of the EOM Eq. (5.6) can therefore be written as:
\[
\vec{F} \cdot \vec{v} = \frac{dE}{dt} = \frac{d}{dt} (m\gamma v c^2) = (\text{Change in energy})
\] (5.18)

We will discuss the expression for the energy in Section 5.2 below.
Eq. (5.10b) & Eq. (5.14) →

3-force \( \vec{F} \) as function of proper force \( \vec{F}_0 \) and velocity \( \vec{v} \):

\[
\vec{F} = \frac{\vec{F}_0}{\gamma_v} + \left( 1 - \frac{1}{\gamma_v} \right) \frac{\vec{F}_0 \cdot \vec{v}}{v^2} \vec{v}
\] (5.19)

Recall that the proper force is the Newtonian force you would measure with a spring scale in the IRS of the particle. In contrast to Newtonian mechanics, the force \( \vec{F} \) measured from a frame in relative motion is different from \( \vec{F}_0 \). In the non-relativistic limit \( \gamma_v \approx 1 \) we find \( \vec{F} \approx \vec{F}_0 \) and this distinction becomes irrelevant (as assumed by Newtonian mechanics).

7 | A similar comparison yields a relation between the 3-acceleration in the IRF (the proper acceleration) and the 3-acceleration in the rest frame:

Eq. (5.10a) & Eq. (4.49) →

3-acceleration \( \vec{a} \) as function of proper acceleration \( \vec{a}_0 \) and velocity \( \vec{v} \):

\[
\vec{a} \equiv \frac{1}{\gamma_v^2} \left[ \vec{a}_0 - \left( 1 - \frac{1}{\gamma_v} \right) \frac{\vec{v} \cdot \vec{a}_0}{v^2} \vec{v} \right]
\] (5.20)

This is again in sharp contrast to Newtonian mechanics where, as a consequence of absolute time, acceleration does not depend on the velocity of the reference frame. In the non-relativistic limit for \( \gamma_v \approx 1 \) we find \( \vec{a} \approx \vec{a}_0 \), consistent with Newtonian mechanics.

8 | Sanity check:

If we integrate the equation of motion Eq. (5.16), we find:

\[
\int_0^T \vec{F} \, dt = -\frac{m\vec{v}_T}{\sqrt{1 - \frac{\vec{v}_T^2}{c^2}}} - \text{const}.
\] (5.21)

For a finite 3-force \( |\vec{F}| < \infty \) and finite time \( T < \infty \), and non-zero mass \( m \neq 0 \), it follows for the final velocity \( \vec{v}_T \):

\[
\frac{m|\vec{v}_T|}{\sqrt{1 - \frac{\vec{v}_T^2}{c^2}}} < \infty \quad \Rightarrow \quad |\vec{v}_T| < c.
\] (5.22)

Thus the dynamics does not allow massive particles to reach the speed of light, no matter how strong the force or how long the acceleration! This is in direct contradiction to Newtonian mechanics and by now experimentally well-confirmed (→ below).

5.2. Momentum, Energy, and Mass

9 | To summarize, the 4-momentum of a massive particle can be written as:

\[
p^\mu = m u^\mu = \left( p^0 \atop \vec{p} \right) = \left( \frac{E}{c} \atop \vec{p} \right) = \left( \gamma_v mc \atop \gamma_v m \vec{v} \right)
\] (5.23)
The relativistic energy of a massive particle is then (as a function of 3-velocity):

\[ E = c p^0 = \gamma v m c^2 = \frac{m c^2}{\sqrt{1 - v^2/c^2}} \]  \hspace{1cm} (5.24)

With

\[ m^2 c^2 = p^2 = (p^0)^2 - (\vec{p})^2 = E^2/c^2 - \vec{p}^2 \]  \hspace{1cm} (5.25)

we find the alternative expression as a function of 3-momentum:

\[ E = \sqrt{\vec{p}^2 c^2 + m^2 c^4} \]  \hspace{1cm} (5.26)

- This expression is also valid in the massless case \( m = 0 \) (→ below).
- Eq. (5.25) has actually two solutions: \( E = \pm \sqrt{p^2 c^2 + m^2 c^4} \). In relativistic mechanics (and relativistic single-particle quantum mechanics), we can ignore the negative energy solution and consider only time-like 4-momenta \( p^\mu \) that point into the future light-cone. In quantum field theory, where interacting particles can be destroyed and produced, these negative energy solutions necessitate the introduction of \( \uparrow \) antiparticles (like the positron).
- For fixed mass \( m \), Eq. (5.25) determines a 3-dimensional hypersurface in the 4-dimensional “energy-momentum space” spanned by 4-momenta \( p^\mu = (p^0, \vec{p}) \in \mathbb{R}^4 \). For \( m \neq 0 \) this hypersurface is a hyperboloid of two sheets \( E = \pm \sqrt{\vec{p}^2 c^2 + m^2 c^4} \) (for \( m = 0 \) it is a cone: \( E = \pm c |\vec{p}| \)). This hypersurface is called \( \uparrow \) mass shell. If a 4-momentum satisfies the energy-momentum relation (with either sign) we say that it is “on-shell”; if not, it is “off-shell”. In quantum field theory, real particles that can be measured are always on-shell; intermediate “virtual particles” in scattering processes can be off-shell.

Rest energy:

For these considerations, it does not matter whether the particle is accelerating and this is an IRF, or whether the particle is in inertial motion and has a fixed rest frame. Formally, since \( p^2 = m^2 c^2 > 0 \) is a time-like Lorentz vector, there is always an inertial frame in which \( p^0 \neq 0 \) and \( \vec{p} = 0 \).

\[ p^0 = \left( \begin{array}{c} p^0  \\ 0 \end{array} \right) = \left( \frac{E_0}{c}  \\ 0 \end{array} \right) \]  \hspace{1cm} (5.27)

For these considerations, it does not matter whether the particle is accelerating and this is an IRF, or whether the particle is in inertial motion and has a fixed rest frame. Formally, since \( p^2 = m^2 c^2 > 0 \) is a time-like Lorentz vector, there is always an inertial frame in which \( p^0 \neq 0 \) and \( \vec{p} = 0 \).

\[ E_0 = mc^2 \]  \hspace{1cm} (5.28)

This is Einstein’s famous principle of equivalence of (inertial) mass and (rest) energy.

- \( \uparrow \) The total energy \( E \) is the time-component of a 4-vector: \( p^\mu = (E/c, \vec{p})^T \); thus it makes sense to refer to the rest energy \( E_0 \) – which is the component of this 4-vector in the rest frame \( K_0 \), i.e., the particular frame where \( \vec{p} = 0 \).
• ! By contrast, the mass is a Lorentz scalar, namely $p^2 = m^2 c^2$; hence it is the same in all inertial systems and it does not make sense to refer to the rest mass $m_0$ as this term suggests that there is a “non-rest mass” (which there isn’t).

• Einstein first derived the mass-energy equivalence in his Annus Mirabilis paper *Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?* [10]. In the paper, the equation is not given verbatim but encoded in the following statement:

  \begin{align*}
  \text{Gibt ein Körper die Energie $L$ in Form von Strahlung ab, so verkleinert sich seine Masse um $L/V^2$.}
  \end{align*}

Einstein concludes:

  \begin{align*}
  \text{Die Masse eines Körpers ist ein Maß für dessen Energieinhalt; […] Es ist nicht ausgeschlossen, daß bei Körnern, deren Energie in hohem Maße veränderlich ist (z.B. bei den Radiumsalzen), eine Prüfung der Theorie gelingen wird.}
  \end{align*}

Einstein further elaborates on the relativistic energy relation and its implications in [66]. He provides self-contained step-by-step derivation in Ref. [67]. Additional insight was provided over the years with alternative derivations by various authors [68–70].

The derivation by Feigenbaum and Mermin in [70] is particularly insightful as it follows Einstein’s original derivation in [10] closely without invoking electrodynamics. They also demonstrate that the heart of relativistic mechanics is actually Eq. (5.24) (where $mc^2$ appears as a coefficient), and not Eq. (5.28) (which is conventional).

\begin{note}
\textbf{Note 5.1: Some comments on $E_0 = mc^2$}
\end{note}

Eq. (5.28) is arguably the most famous equation in physics. The popularization of scientific concepts is often accompanied by simplifications and distortions. This is also the case for $E_0 = mc^2$:

• $E_0 = mc^2$ is often written as $E = mc^2$. This is either wrong or misleading (depending on the interpretation of the symbols); in any case, it is not consistent with modern conventions in relativity (→ below).

• $E_0 = mc^2$ is by no means Einstein’s most important equation. This is why it is not referred to as “Einstein equation;” this honor goes to

  \begin{align*}
  R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}
  \end{align*}

which are also known as the → Einstein field equations; these form the basis of general relativity and are empirically of much greater value than Eq. (5.28).

Luckily, the Einstein field equations look daunting and are not nearly as accessible as $E_0 = mc^2$; hence they weren’t seized (and mutilated) by pop culture like $E_0 = mc^2$ was.

• How statements are phrased determines our conceptualization of the world. The often heard phrase

  \begin{quote}
  “$E_0 = mc^2$ says that mass can be converted into energy”
  \end{quote}

makes me think of “mass” as a sort of coal that can be lighted and then produces energy (maybe in form of light and heat or an atomic explosion). I am quite convinced that there are many who got “conceptually derailed” by statements like this, and hence think of Einstein’s revelation as modern-day equivalent of an early human realizing, perhaps by witnessing a lightning strike, that wood can be kindled to produce heat. This is completely off the mark.
$E_0 = mc^2$ says that rest energy and inertial mass are equivalent; not that they can be “converted” into each other. It means that the Lorentz symmetry of spacetime necessitates that our concepts of “energy” (as a quantity that can make things change in time) and “inertial mass” (as a quantity that measures how hard it is to make the state of motion of an object change in time) are like two sides of the same coin. Note that we did not arrive at the equation by studying the microscopic dynamics and interactions of matter (like we do in quantum mechanics, and especially quantum field theory); the equivalence of rest energy and mass is a consequence of the symmetries of spacetime alone. One can take $E_0 = mc^2$ thus as a hint at the unanswered questions “What is time?” and “What is inertia?” because energy is the generator of time translations (think of the time-evolution operator in quantum mechanics) and mass quantifies the phenomenon of inertia.

To drive the point home, here a few examples:

- An atom in an excited electronic state is heavier than the same atom in the ground state.
- A battery gets lighter when being discharged.
- A chunk of metal is heavier when it is hot.
- If you put an atomic bomb into an opaque, completely sealed “super box” that survives the explosion, the weight of the box does not change when the bomb goes off. This makes it clear that mass is not “converted” into energy.
- If the box is made out of “super glass” that lets only photons escape, the box gets lighter by $E_{\text{phot}}/c^2$ if the photons carry away the energy $E_{\text{phot}}$.

- For these reasons, $E_0 = mc^2$ is not a magical blueprint to build atomic bombs. The equation is only relevant in this context because it provides a nice “shortcut” to compute the energies that the fission (splitting) of isotopes can yield (or cost, depending on the isotopes). Because one could measure the rest masses of isotopes rather easily (using mass spectrometry [71]) – but had almost no clue how to describe the inner workings (and therefore binding energies) of said nuclei – the equation allowed for a straightforward survey of the periodic table to identify suitable isotopes that would yield energy under fission. $E_0 = mc^2$ is not the reason why atomic weapons work, and these weapons are not so powerful “because they convert mass into energy.” This is pure nonsense. If you discharge the battery of your phone, it also loses mass – because rest energy and mass are equivalent: $E_0 = mc^2$! And yes, this mass difference is much smaller than the mass difference accompanied by a nuclear explosion. But this is not the reason; the reason is that the strength of electromagnetic interactions – which govern chemical processes (like discharging your battery) – is dwarfed by the strength of the strong interaction (and its residual, the nuclear force) – which governs nuclear reactions.

In a nutshell:

When studying reaction processes (of any sort), the change of restmass predicted by $E_0 = mc^2$ is an epiphenomenon. The mass change is not causal; it cannot be, because it is a consequence of the symmetries of spacetime, and not of the inner workings of matter.

Unfortunately, the notation and interpretation of special relativity has changed since its inception. In former times it was conventional to introduce the concept of a

\[ \text{Relativistic mass: } m_r := \gamma m = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}} \] (5.30)
which depends on velocity. With this definition, the relativistic relation between 3-velocity and 3-momentum reads \( \vec{p} = m_r \vec{v} \) and parallels the Newtonian relation \( \vec{p} = m \vec{v} \). The relativistic energy relation then reads \( E = m_r c^2 \).

The concept of a velocity-dependent, relativistic mass is avoided in most modern treatments of relativativity (and in this script). While this is mostly a matter of concepts and semantics, there are good reasons why the concept of a velocity dependent mass is less useful than it might seem (→ below).

Here a few comments on various notations that you might encounter:

\[
E_0 = mc^2 \quad \text{Correct} \quad \odot \\
E = \tilde{m}c^2 \quad \text{Only makes sense if } m = m_r \text{ (which we don’t use)}.
\]

\[
E_0 = m_0 c^2 \quad \text{Why } m_0 \text{? There is only } m! \\
E = \tilde{m}_0 c^2 \quad \text{Energy is frame-dependent. Do you mean } E_0 \text{? Otherwise: Wrong!}
\]

For more details and explanations see Refs. [72–74].

iii | → Take home message:

There is only one mass: the rest mass \( m \) (which we call mass). Thus mass does not depend on velocity.

This convention is used by almost all modern textbooks on relativativity.

Unfortunately the old conventions (using relativistic, velocity-dependent masses) are still used by school books and popular science books.

iv | Aside: Why introducing velocity depended masses leads nowhere.

If you are still inclined to think in terms of a velocity-dependent, relativistic mass \( m_r \), here is a compelling argument why this is a useless and artificial concept that needs to die:

The 3-component of the relativistic equation of motion Eq. (5.16) reads

\[
\vec{F} = \frac{d}{dt} (m \gamma_v \vec{v}) = m \gamma_v \vec{a} + m \gamma_v^3 \frac{\vec{v} \cdot \vec{a}}{c^2} \vec{v}
\]

(5.31)

with two extreme cases:

\[
\vec{v} \parallel \vec{a} \quad \Rightarrow \quad \vec{F} = m \gamma_v \vec{a}
\]

(5.32a)

\[
\vec{v} \perp \vec{a} \quad \Rightarrow \quad \vec{F} = m \gamma_v \vec{a}
\]

(5.32b)

If you insist on introducing a “mass” as the proportionality factor between 3-force and 3-acceleration to quantify the inertial response of an object at finite velocity, you are not only forced (\( \odot \)) to make this mass velocity dependent, you also need two masses:

“Longitudinal mass”: \( m_\parallel := m \gamma_v^3 \)

(5.33)

“Transverse mass”: \( m_\perp := m \gamma_v \)

(5.34)

The above result demonstrates that the concept of a mass as a measure for inertia is not very useful in special relativity. More precisely, the result shows that the quantities \( m_\parallel \) and \( m_\perp \) are relational properties between an object and an observer (they depend on the state of motion of the observer); they are not intrinsic properties of the object itself. Only the restmass \( m \) qualifies as such an intrinsic property. The velocity dependence of \( m_\parallel \) and \( m_\perp \) is not an intrinsic feature of matter, it is a feature of spacetime.
This is why in modern textbooks there is only one mass $m$ (the rest mass) which does not depend on $v$, and one has to accept that the Newtonian relation $\rho = m\tilde{v}$ is no longer valid. The concepts of “longitudinal mass” and “transverse mass” (and velocity dependent mass, for that matter) are therefore no longer used in modern literature.
12 | \( \ll \) Non-relativistic limit:

\[
E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \approx mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(\beta^4) \tag{5.35}
\]

This shows again that the correspondence principle is satisfied: For small velocities compared to \(c\), the kinetic energy of Newtonian mechanics is (up to a constant shift given by the rest energy) a good proxy for the true energy of the particle.

13 | The kinetic energy is: \( E_{\text{kin}} = E - E_0 = E - mc^2 \)

\( \rightarrow \) The velocity of a relativistic particle as a function of its kinetic energy is:

\[
\beta^2 = \left(\frac{v}{c}\right)^2 = 1 - \left[\frac{mc^2}{E_{\text{kin}} + mc^2}\right]^2 \beta \ll 1 \rightarrow \frac{2E_{\text{kin}}}{mc^2} \tag{5.36}
\]

Note that in the non-relativistic limit it is \( E_{\text{kin}} \ll mc^2 \).

This velocity dependence has been confirmed experimentally to high precision; for example with accelerated electrons [33] (see Refs. [33,34] for more technical details):

\( \rightarrow \) The relativistic energy relation Eq. (5.24) is correct ☺️

14 | Massless particles:

So far we considered only particles with non-vanishing mass \( m \neq 0 \). The definition of the momentum Eq. (5.1) and the relativistic energy Eq. (5.24) cannot be directly applied to particles without mass. However:

\( i \) \( \ll \) Eq. (5.26) with \( m \rightarrow 0 \):

\[
E = |\vec{p}|c \quad \text{(linear dispersion)} \tag{5.37}
\]

\( \ll \) Eq. (5.4) with \( m \rightarrow 0 \):

\[
p^2 = 0 \quad \text{(light-like)} \Rightarrow p^\mu = \left(\frac{|\vec{p}|}{\vec{p}}\right) \tag{5.38}
\]

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We take this as the definition of the 4-momentum for massless particles (it is the only definition that is consistent with $p^\mu = mu^\mu$ in the limit of vanishing mass). Note that there is no finite 4-velocity $u^\mu$ associated to massless particles.

The fact that $p^\mu$ becomes light-like for massless particles already suggests that they move with the speed of light. We can verify this:

$$E = \gamma_0 mc^2, \quad \vec{p} = \gamma_0 m \vec{v}$$

$$\Rightarrow \quad E = |\vec{p}| \frac{c^2}{v} \quad \text{Eq. (5.37)}$$

This limit is only consistent if $v \to c$ for $m \to 0$:

All particles with vanishing mass move with the speed of light. 

- Examples: Photons, Gravitons (if they exist)
- Massless particles do not have a rest frame.

You would need a boost with $v = c$ to reach such a frame; but such boosts are not defined (because the Lorentz factor diverges in this limit).

- The relativistic energy $E = \gamma_0 mc^2$ holds only for massive particles. For massless particles it does not follow $E = 0$ but rather $E = |\vec{p}|c \neq 0$. So photons do have energy and momentum, but no mass (neither rest- nor any other type of mass). You are also not allowed to use the “forbidden” equation $E = mc^2$ and declare $m_r = E/c^2 = |\vec{p}|c$ as the “dynamic mass” of the photon because (1) we argued above that this concept is not as useful as it sounds, and (2) you only renamed momentum, so what’s the point. And if you are afraid that later – in general relativity – our photons will not be deflected by stars or sucked into black holes because they “have no mass”: I assure you, they will; they have energy and momentum, that’s enough.

- This demonstrates why the “speed of light” is sort of a misnomer in this context, and we should have stuck to our $v_{max}$ (but then all our equations would look different from the literature). Then it would be conceptually clear that every particle with vanishing rest mass “runs into” the universal speed limit $v_{max}$.

### 5.3. Action principle and conserved quantities

In this section we study a more formal (and more versatile) approach to describe the dynamics of relativistic systems, namely in terms of the Lagrangian and the action. We do this for the free particle (no force!) and consider electromagnetic forces in the next Chapter 6.

- Action of free massive particle:

1. Trajectory $\gamma$ parametrized by $x^\mu = x^\mu(\lambda)$ with $\lambda \in [\lambda_a, \lambda_b]$ and $x^\mu(\lambda_a) = a^\mu, x^\mu(\lambda_b) = b^\mu$

Remember the characteristic property of the trajectory of a free particle (Section 2.4): The proper time ($= $ Minkowski distance) is maximized along the trajectory!

$$\rightarrow \text{Action: } S[\gamma] := \alpha \int_{\gamma} dx = \alpha \int_{\lambda_a}^{\lambda_b} \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \, d\lambda \quad (5.41)$$
with $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$.

The prefactor $\alpha$ is undetermined so far ($\rightarrow$ next step).

¡! The parameter $\lambda$ has no physical interpretation in this formulation as this action is reparametrization invariant ($\rightarrow$ Section 5.4).

### ii Correspondence principle $\rightarrow \alpha = -mc$

To determine the parameter $\alpha$, consider the non-relativistic limit of the Lagrangian in coordinate time parametrization $\lambda = t$:

$$
\tilde{L} = \alpha \sqrt{c^2 - \dot{x}^2} = \alpha c \sqrt{1 - \frac{\dot{x}^2}{c^2}} \quad \beta \ll 1 \quad \Rightarrow \quad L \approx \frac{\alpha v^2}{2c} \quad \frac{1}{2} mc^2.
$$

(5.42)

The non-relativistic limit yields – up to a constant that doesn’t change the equations of motion – the Lagrangian with Newtonian kinetic energy if we set $\alpha = -mc$.

### iii Lagrangian:

$$
L(x^\mu, \dot{x}^\mu) = -mc \sqrt{\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} = -mc \sqrt{\dot{x}_\mu \dot{x}^\mu}
$$

(5.43)

¡! This Lagrangian is only valid for massive particles.

• The Lagrangian Eq. (5.43) is fully specified as is; there is no need to fix a specific parametrization. In this form, the Lagrangian [more precisely: the action Eq. (5.41)] has a gauge symmetry: the parametrization $\lambda$ is arbitrary ($\rightarrow$ Section 5.4).

• On the contrary, if you fix a parametrization (= fix a gauge), e.g., by identifying $\lambda$ with the coordinate time $\lambda = t \equiv x^0/c$ (“static gauge”) or the proper time $\lambda = \tau$ (“proper time gauge”), you obtain different (but physically equivalent) Lagrangians which have no longer a gauge symmetry:

$$
\lambda \equiv t \quad \Leftrightarrow \quad c \lambda \equiv x^0 \quad \Rightarrow \quad \tilde{L}_t(\dot{x}, \ddot{x}) = -mc^2 \sqrt{1 - \dot{x}^2/c^2},
$$

$$
\lambda \equiv \tau \quad \Leftrightarrow \quad \dot{x}^\mu \dot{x}_\mu = c^2 \quad \Rightarrow \quad \tilde{L}_\tau(x^\mu, \dot{x}^\mu) = -mc^2.
$$

(5.44a)

(5.44b)

We denote gauge-fixed Lagrangians by $\tilde{L}$, and the gauge-invariant Lagrangian Eq. (5.43) by $L$. In the following we often work with the latter and choose specific parametrizations at the end of our calculations to express results in known quantities.

### 2 Euler-Lagrange equations:

$$
\delta S \equiv 0 \quad \Rightarrow \quad \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\sigma} - \frac{\partial L}{\partial x^\sigma} \bigg|_{\dot{x}^\sigma = 0} = 0 \quad \Rightarrow \quad \frac{d}{d\lambda} \frac{-mc \dot{x}_\sigma}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} = 0
$$

(5.45)

These are 4 differential equations ($\sigma = 0, 1, 2, 3$)!

→ Equations of motion in the “proper time gauge” $\lambda = \tau$ [where $\dot{x}_\mu \dot{x}^\mu = u^2 = c^2$]:

$$
\frac{m}{d\tau} \frac{d\dot{x}^\mu}{d\tau} = \frac{d\rho^\mu}{d\tau} = 0
$$

(5.46)

This is Eq. (5.6) for vanishing 4-force $\otimes$
3 | Action in “static gauge” \( \lambda = t = x^0/c \):

\[
S[\gamma] \equiv S_t[\vec{x}(t)] = \int_{t_a}^{t_b} L_t(\vec{x}, \dot{\vec{x}}) \, dt = -mc^2 \int_{t_a}^{t_b} \sqrt{1 - \frac{\dot{x}^2}{c^2}} \, dt \tag{5.47}
\]

i | Canonical momenta \((\vec{p} = \dot{\vec{x}})\):

\[
\vec{p} = \frac{\partial L_t}{\partial \dot{\vec{x}}} = \frac{m \vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} \tag{5.48}
\]

This is the expression for the relativistic 3-momentum Eq. (5.3) we found before, now derived as the canonical momentum of a Lagrangian.

ii | Hamiltonian:

\[
\tilde{H}_t = \vec{p} \cdot \vec{v} - \vec{L}_t = \frac{mc^2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}} = cp^0 \tag{5.26} = c \sqrt{\vec{p}^2 + mc^2} \tag{5.49}
\]

This is just the relativistic energy Eq. (5.24) we found before, now derived from a Lagrangian.

- Non-relativistic limit:

\[
\tilde{H}_t = mc^2 \sqrt{1 + \frac{\vec{p}^2}{m^2c^2}} \approx mc^2 + \frac{\vec{p}^2}{2m} \tag{5.50}
\]

- Contrary to the action Eq. (5.47), this Hamiltonian also makes sense for massless particles:

\[
\tilde{H}_t \overset{m = 0}{=} |\vec{p}| \tag{5.51}
\]

4 | Noether’s (first) theorem:

Details: Problemset 6

\(x^\mu\) cyclic \(\rightarrow\) Spacetime translations \(x^\mu + \delta x^\mu\) are continuous symmetries of \(S\)

These transformations correspond to the inhomogeneous part of Poincaré transformations: \(\delta x^\mu = x^\mu + a^\mu\). Every relativistic theory must have this symmetry; for field theories one obtains then four conserved currents: \(\rightarrow\) Energy momentum tensor.

\(\rightarrow\) Noether’s theorem \(\rightarrow\) Conserved Noether charges \(Q_\mu\): (set \(\lambda = t\) as the coordinate time)

\[
Q_\mu \equiv \begin{cases} 
\text{Time translation} & \Rightarrow \text{Energy } E/c \\
\text{Space translations} & \Rightarrow \text{Momentum } \vec{p} 
\end{cases} \tag{5.52}
\]

\[
= -\frac{\partial L}{\partial \dot{x}^\mu} = \frac{mc \dot{x}^\mu}{\sqrt{c^2 - \vec{v}^2}} = \left( \frac{1}{c} \frac{mc^2}{\sqrt{1 - \beta^2}} \right) = p_\mu \tag{5.53}
\]
Because \( x^\mu \) are cyclic coordinates, we can obtain the Noether charges directly from the Lagrangian as \( \frac{\partial L}{\partial p_\mu} \); the additional minus is conventional to connect to our definition of the 4-momentum.

\[ \frac{\partial L}{\partial p_\mu} = - \delta_\mu^\nu \text{ with additional minus.} \]

This shows that our definition of the 4-momentum is consistent, and the identification of its time-component \( p^0 \) as the total energy was correct: By definition, energy is the Noether charge that corresponds to translation invariance in time. Similarly, momentum is the charge for translation invariance in space.

5 | Noether charges for homogeneous Lorentz transformations?

Any relativistic theory is also invariant under (proper orthochronous) Lorentz transformations, \( \tilde{x}^\mu = \Lambda^\mu_{\nu} x^\nu \); for these there must exist additional conserved Noether charges:

Infinitesimal Lorentz transformations \( x^\mu + \delta x^\mu = \Lambda^\mu_{\nu} x^\nu \) are continuous symmetries of \( S \)

The infinitesimal transformation is antisymmetric: \( \delta x^\mu = -\delta x^\mu \), \( \triangleright \) Problemset 5.

\( \triangleright \) Conserved Noether charges:

\[ L^{\mu \nu} = x^\mu p^\nu - x^\nu p^\mu \]  
\[ (5.54) \]

This is an example of an antisymmetric \((2, 0)\) Lorentz tensor.

Proof: \( \triangleright \) Problemset 6

\( \triangleright \) Spatial components:

\[ L^{23} = x^2 p^3 - x^3 p^2 = l_1 \]
\[ L^{31} = x^3 p^1 - x^1 p^3 = l_2 \]
\[ L^{12} = x^1 p^2 - x^2 p^1 = l_3 \]

with 3-angular momentum \( \vec{l} = \vec{x} \times \vec{p} \). \( (5.55) \)

\( \triangleright \) 3-angular momentum \( \vec{l} \) is not (part of a) Lorentz vector but of a \((2, 0)\) tensor!

It is not surprising that invariance under spatial rotations \( \text{SO}(3) \subset \text{O}(1, 3) \) implies angular momentum conservation.

\( \triangleright \) Mixed components:

\[ L^{10} = x^1 \gamma_0 mc - ct p^1 = c n_1 \]
\[ L^{20} = x^2 \gamma_0 mc - ct p^2 = c n_2 \]
\[ L^{30} = x^3 \gamma_0 mc - ct p^3 = c n_3 \]  
\[ (5.56) \]

with \( \triangleright \) dynamic mass moment

\[ \vec{n} := m \gamma_0 (\vec{x} - t \vec{v}) = \frac{E}{c^2} \vec{x} - t \vec{p} = \text{const.} \]
\[ (5.57) \]

This is the relativistic version of the \( \triangleright \) center-of-mass theorem.

The center of mass (COM) is now the center of energy (COE). Since \( \vec{n} \) (and \( E \)) is conserved, we can set \( t = 0 \) to find \( \vec{n} = E/c^2 \vec{x}_0 \), which is the initial center of energy of the system (times \( E/c^2 \)).

For many particles this is slightly less trivial: One finds analogously the conserved quantity

\[ \vec{N} = \sum_i \vec{n}_i = \sum_i \left( \frac{E_i}{c^2} \vec{x}_i - t \vec{p}_i \right) = \text{const.} \]
\[ (5.58) \]
Division by the total (also conserved) energy $E = \sum_i E_i$ yields

$$\bar{X}_{\text{COE}}(t) := \frac{\sum_i E_i \bar{x}_i}{\sum_i E_i} = t \frac{\bar{P}}{E} + \text{const} = t \vec{V}_{\text{COE}} + \text{const} \quad (5.59)$$

with the total 3-momentum $\bar{P} = \sum_i \bar{p}_i$. Thus the center of energy $X_{\text{COE}}$ moves in a straight line with constant velocity $\vec{V}_{\text{COE}}$. Note that the center of energy becomes the Newtonian center of mass in the non-relativistic limit where $E_i \approx E_{i,0} = m_i c^2$.

6 | Multiple particles (covariantly coupled by fields):

The above arguments can be directly generalized to many (non-interacting) particles. This immediately yields the sum of the 4-momenta of these particles as conserved quantity. Interactions between the particles must be covariantly mediated by fields – which also carry 4-momentum (→ Chapter 6):

**Conserved Noether charge:**

\[ \text{Total 4-momentum:} \quad P^\mu := \sum_i p_i^\mu + p_{\text{Fields}}^\mu \quad (5.60) \]

with

- $p_i^\mu$ the 4-momentum of particle $i$, and
- $p_{\text{Fields}}^\mu$ the total 4-momentum of the fields mediating the interactions.

7 | Scattering process:

Long before and after the interactions play a role we can approximate the system by non-interacting particles and set $p_{\text{Fields}}^\mu = 0 →$

$$\sum_i p_{\text{in},i}^\mu = \sum_j p_{\text{out},j}^\mu \quad (5.61)$$

→ Conservation of energy ($\mu = 0$) and momentum ($\mu = 1, 2, 3$)

- In relativity, conservation of total energy and total momentum is combined into the conservation of 4-momentum.
- We will denote the 4-momenta of massive particles (solid lines) with $p^\mu$ and the 4-momenta of massless particles with $q^\mu$ (wiggly lines).
Examples:

i | Particle decay: \(<\) Radioactive Nucleus → Nucleus 1 & Nucleus 2

→ Energy-momentum conservation:

\[
p^\mu_{in} = p_1^\mu + p_2^\mu
\]  

(5.62)

\(<\) Center-of-mass frame where \(\vec{p} = \vec{p}_1 + \vec{p}_2 = 0\)

\[
m c^2 = m_1 c^2 + E_{\text{kin},1} + m_2 c^2 + E_{\text{kin},2}
\]  

(5.63)

→ Decay only possible if

\[
m \geq m_1 + m_2
\]  

(5.64)

If \(E_{\text{kin},1} \neq 0\) or \(E_{\text{kin},2} \neq 0\), it is \(m \neq m_1 + m_2\).

→ The rest mass of composite objects is not additive.

Composite objects also contain binding energy (potential energy) which contributes to the rest mass of the object.

\[
E_{\text{kin},1} = \frac{(m - m_1)c^2 - m_2c^2}{2m}
\]  

(5.65)

In the COM frame, the kinetic energy of the two decay products is constant and depends only on the masses of the particles. So if you find a non-trivial energy distribution for the products of a decay process, there must at least three particles be produced (of which you might not be able to detect all). This is how the neutrino was predicted by Pauli from the decay of the neutron: \(n \rightarrow p + e^- + \bar{\nu}_e\).

ii | Particle creation:

Note that a single massless (light-like) particle (like a photon) cannot decay into two massive (time-like) particles because \((p_1 + p_2)^2 = q^2 = 0\) cannot be solved if \(p_1^2 = m_1^2 c^2 > 0\).

Indeed (we set \(c = 1\)): With the \(\&\) Cauchy-Schwarz inequality we find

\[
m_1 m_2 + \vec{p}_1 \cdot \vec{p}_2 \leq \sqrt{m_1^2 + p_1^2} \sqrt{m_2^2 + p_2^2} = p_1^0 p_2^0
\]  

(5.66a)

\[0 < m_1 m_2 \leq p_1 \cdot p_2
\]  

(5.66b)

so that for arbitrary \(m_1\) and \(m_2\) (particle creation: \(q^\mu = p_1^\mu + p_2^\mu\))

\[(p_1 + p_2)^2 = m_1^2 + m_2^2 + 2 p_1 \cdot p_2 > 0 \quad \Rightarrow \quad \text{Time-like}
\]  

(5.67)

Furthermore, for \(m_1 = m_2\) (scattering: \(p_1^\mu - p_2^\mu = q^\mu\)):

\[(p_1 - p_2)^2 = m_1^2 + m_2^2 - 2 p_1 \cdot p_2
\]  

(5.68a)

\[\leq m_1^2 + m_2^2 - 2 m_1 m_2 \frac{m_1^2 + m_2^2}{2m_1 m_2} \overset{m_1 = m_2}{=} 0 \quad \overset{p_1 \neq p_2}{\Rightarrow} \text{Space-like}
\]  

(5.68b)

(For the Cauchy-Schwarz inequality, equality holds iff the two vectors are linearly dependent; for \(m_1 = m_2\) this is only possible if \(p_1 = p_2\), i.e., in the trivial case of no scattering.)
Eq. (5.67) shows that two particles (of arbitrary masses) can never annihilate into a single photon, and, vice versa, a single photon can never create a pair of massive particles. This is reason why we need an additional (heavy) nucleus for the creation of a particle & antiparticle pair from a photon.

By contrast, Eq. (5.68) tells us that a single massive particle cannot emit or absorb a single photon if it cannot change its mass (i.e., has no different energy states). This is true for free elementary particles like electrons (an electron cannot be excited, it always has the same mass). Thus a free electron cannot emit a single photon. If the massive particle in question has different internal energy states (and therefore the two masses $m_1$ and $m_2$ can be different), this argument does not hold. This is why atoms can spontaneously emit or absorb single photons.

\[ \text{Photon (+Nucleus)} \rightarrow \text{Electron & Positron (+Nucleus)} \]

\[ \rightarrow \text{Energy-momentum conservation:} \]

\[ P_{\text{in}}^\mu = q^\mu + p_{\text{out}}^\mu = p_1^\mu + p_2^\mu + \vec{p}^\mu = P_{\text{out}}^\mu \]

\[ \text{(5.69)} \]

With the mass $M$ of the nucleus and the momentum/energy $|q| = E/\gamma c$ of the incoming photon, we find

\[ \left( \frac{E_e + M c^2}{c} \right)^2 - \left( \frac{E_e}{c} \right)^2 = p_{\text{in}}^2 = \frac{1}{c} p_{\text{out}}^2 = \left( \frac{E_{\text{Nuc}} + E_{\gamma^-} + E_{\gamma^+}}{c} \right)^2 \]

\[ \text{(5.70)} \]

where the right hand side was evaluated in the COM frame with $\vec{P}_{\text{out}} = 0$ and the left hand side in the rest frame of the nucleus (which is allowed since $P^2 = P_{\mu} P^\mu$ is a Lorentz scalar).

Please appreciate the subtlety of this evaluation: The 4-momentum conservation Eq. (5.69) is Lorentz covariant. Therefore you cannot evaluate the left hand side $P_{\text{in}}^\mu$ in one inertial system and the right hand side $P_{\text{out}}^\mu$ in another. However, in any inertial system Eq. (5.69) implies $P_{\text{in}}^2 = P_{\text{out}}^2$ where left and right hand side are now Lorentz invariant; hence you can evaluate the two sides in different inertial systems.

\[ \rightarrow \text{Threshold for particle creation:} \]

\[ E_{\gamma,\text{min}} = 2m_e^2 c^2 \left( 1 + \frac{m_e}{M} \right) > 2m_e c^2 \]

\[ \text{(5.71)} \]

The threshold follows for vanishing kinetic energy of the products in the COM frame.

The threshold energy is larger than twice the rest energy of the electron $2m_e c^2$ (the positron has the same mass as the electron) because the scattering products necessarily acquire kinetic energy in the initial rest frame of the nucleus (to carry the momentum of the photon).
Energy-momentum conservation:

\[ P_{\text{in}}^\mu = p_1^\mu + p_2^\mu = q_1^\mu + q_2^\mu = P_{\text{out}}^\mu \]  \hspace{1cm} (5.72)

< COM frame:

\[ p_{\text{in}}^\mu = \left( \frac{E_e^-/c}{\bar{p}} \right) + \left( \frac{E_e^+/c}{-\bar{p}} \right) = \left( \frac{|\vec{q}|}{\bar{q}} \right) + \left( \frac{|\vec{q}|}{-\bar{q}} \right) = p_{\text{out}}^\mu \]  \hspace{1cm} (5.73)

Using that electron and positron have the same mass \( m_e \), we find for the energy of the emitted photons:

\[ E_{\nu} = c \sqrt{\vec{p}^2 + m_e^2 c^2} \]  \hspace{1cm} (5.74)

Note that the individual rest masses of particles in scattering processes are not conserved: \( p_1^2 = p_2^2 = m_e^2 c^2 > 0 \) for the incoming electron and the positron, but \( q_1^2 = q_2^2 = 0 \) for the outgoing photons. The rest mass of the composite system remains the same, though. In particular, the two photons together have the same rest mass as the electron-positron system before: \( p_{\text{out}}^2 = P_{\text{in}}^2 = 4(\vec{p}^2 + m_e^2 c^2) > 0 \).

→ The rest masses of individual particles are not conserved.

Compton scattering: < Electon & Photon → Electron & Photon

Details: 

Compton scattering is an example of elastic scattering where the total kinetic energy is conserved and the rest energies of in- and outgoing particles remains the same.

→ Energy-momentum conservation:

\[ q_{\text{in}}^\mu + p_{\text{in}}^\mu = q_{\text{out}}^\mu + p_{\text{out}}^\mu \]  \hspace{1cm} (5.75)

With \( q_1^2 = q_2^2 = 0 \) and \( p_1^2 = p_2^2 = m_e^2 c^2 \) one finds:

\[ \frac{E_1 E_2/c^2(1 - \cos \theta)}{\text{Rest frame of } e^-} = \frac{q_1 \cdot q_2}{\text{Lorentz invariant}} = \frac{p \cdot (q_1 - q_2)}{\text{Rest frame of } e^-} = \frac{m_e c (E_1/c - E_2/c)}{m_e c^2(1 - \cos \theta)} \]  \hspace{1cm} (5.76a)

Here the left and right hand sides are evaluated in the rest frame of the electron: \( p_1^\mu = (m_e c, 0)^T \); \( \theta \) is the angle between incoming and outgoing photon (scattering angle):
With the photon energy $E_i = h c / \lambda_i$ we find the change in wavelength due to scattering:

$$\Delta \lambda = \lambda_2 - \lambda_1 = \frac{h}{m_e c} \left( 1 - \cos \theta \right)$$

(5.77)

with the Compton wavelength $\lambda_c$ of the electron.

- With Compton scattering one can measure the Compton wavelength of the electron and thereby determine the Planck constant $h$.
- Because the Compton wavelength is the natural length scale associated to a massive quantum particle, it appears in many field equations of relativistic quantum mechanics (Klein-Gordon equation, Dirac equation, ...).

### 5.4. Reparametrization invariance

The action of the free relativistic particle Eq. (5.41) has the peculiar property of “reparametrization invariance”, a feature that plays an important role in general relativity, and is also crucial for the quantization of the relativistic string in string theory (+ Nambu-Goto action).

1. Trajectory $\gamma$ parametrized by $x^\mu (\lambda)$ for $\lambda \in [\lambda_a, \lambda_b]$.

2. Diffeomorphism $\varphi : [\lambda_a, \lambda_b] \to [\lambda_a, \lambda_b]$ with $\lambda_{a/b} = \varphi(\lambda_{a/b})$ and write $\tilde{\lambda} = \varphi(\lambda)$.

Diffeomorphism = Bijective map where both the map and its inverse are continuously differentiable.

3. Define new trajectory $\tilde{\gamma}$ via $\tilde{x}^\mu (\tilde{\lambda}) := x^\mu (\varphi^{-1}(\tilde{\lambda})) = x^\mu (\lambda)$ with $\tilde{\lambda} \in [\lambda_a, \lambda_b]$.

$\tilde{x}^\mu (\tilde{\lambda})$ is a reparametrization of $x^\mu (\lambda)$: $\tilde{x}^\mu$ and $x^\mu$ are different functions on $[\lambda_a, \lambda_b]$ that parametrize the same trajectory in spacetime $\mathbb{R}^{1,3}$.
→ Action of new trajectory:

\[ S[\tilde{\lambda}] \overset{\text{def}}{=} -mc \int_{\lambda_a}^{\lambda_b} \sqrt{\dot{x}_\mu(\lambda) \dot{x}^\mu(\lambda)} \, d\lambda \]  

(5.78a)

Rename the dummy variable: \( \lambda \to \tilde{\lambda} \)

\[ = -mc \int_{\lambda_a}^{\lambda_b} \sqrt{\dot{x}_\mu(\tilde{\lambda}) \dot{x}^\mu(\tilde{\lambda})} \, d\tilde{\lambda} \]  

(5.78b)

Use \( \ddot{x}^\mu(\tilde{\lambda}) = x^\mu(\lambda) \) and the chain rule

\[ = -mc \int_{\lambda_a}^{\lambda_b} \sqrt{\frac{d\lambda}{d\tilde{\lambda}} \dot{x}^\mu(\lambda) \frac{d\lambda}{d\tilde{\lambda}}} \, d\tilde{\lambda} \]  

(5.78c)

Substitution in the integral: \( \tilde{\lambda} = \varphi(\lambda) \)

\[ = -mc \int_{\lambda_a}^{\lambda_b} \sqrt{\dot{x}_\mu(\lambda) \dot{x}^\mu(\lambda)} \, d\lambda \]  

(5.78d)

\[ \overset{\text{def}}{=} S[\varphi] \]  

(5.78e)

→ \( S \) is invariant under diffeomorphisms on parameter space.

→ *Reparametrization invariance (RI)*

2 | Infinitesimal generators:

i | Consider infinitesimal deformations \( \varepsilon(\lambda) \) of the parametrization (i.e., |\( \varepsilon(\lambda) | \ll 1 \) for all \( \lambda \)):

\[ \tilde{\lambda} = \varphi(\lambda) \equiv \lambda + \varepsilon(\lambda) \]  

(5.79)

With this we find:

\[ x^\mu(\lambda) \overset{\text{def}}{=} \tilde{x}^\mu(\tilde{\lambda}) = \tilde{x}^\mu(\lambda + \varepsilon(\lambda)) = \tilde{x}^\mu(\lambda) + \varepsilon(\lambda) \partial_\lambda \tilde{x}^\mu(\lambda) + \mathcal{O}(\varepsilon^2) \]  

(5.80)

ii | The infinitesimal variation of the trajectory is:

\[ \delta_\varepsilon x^\mu := \tilde{x}^\mu(\lambda) - x^\mu(\lambda) \]  

(5.81a)

\[ = -\varepsilon(\lambda) \partial_\lambda x^\mu(\lambda) + \mathcal{O}(\varepsilon^2) \]  

(5.81b)

\[ \equiv G_\varepsilon x^\mu + \mathcal{O}(\varepsilon^2) \]  

(5.81c)

Note that we can replace \( \ddot{x}^\mu \) by \( x^\mu \) in linear order of \( \varepsilon \).

→ *Generators* of one-dimensional diffeomorphisms:

\[ G_\varepsilon = -\varepsilon(\lambda) \partial_\lambda \quad \text{for arbitrary (infinitesimal) } \varepsilon(\lambda). \]  

(5.82)

iii | We can expand \( \varepsilon(\lambda) \) into a Taylor series \( \varepsilon(\lambda) = \sum_n \frac{\varepsilon_n}{n!} \lambda^n \) to write

\[ G_\varepsilon = \sum_n \frac{\varepsilon_n}{n!} ( -\lambda^n \partial_\lambda ) \equiv \sum_n \frac{\varepsilon_n}{n!} G_n. \]  

(5.83)

→ Basis of generators that generate infinitesimal reparametrizations is given by

\[ G_n = -\lambda^n \partial_\lambda \quad \text{for } n \in \mathbb{N}_0. \]  

(5.84)
→ RI = Infinite-dimensional continuous symmetry group

Note that in particular \( \epsilon(\lambda) \) can be chosen such that it is non-zero only for a compact subinterval of \([\lambda_a, \lambda_b] \), i.e., reparametrization invariance is a local symmetry (local in parameter space).

→ RI is a gauge symmetry

3. Conserved quantities:

You know from your course on classical mechanics that Noether’s theorem assigns a conserved quantity to each continuous symmetry of an action. What are these quantities for the infinitely many symmetry transformations \( G_\epsilon \) associated to RI?

i. Variation of the Lagrangian \( L = -mc \sqrt{\dot{x}_\mu \dot{x}^\mu} \) under \( G_\epsilon \):

\[
\delta_\epsilon L = \frac{\partial L}{\partial \dot{x}^\mu} \delta_\epsilon \dot{x}^\mu
\]

Use \( \delta_\epsilon \dot{x}^\mu := \dot{x}^\mu - \dot{x}^\mu = \partial_\lambda (\delta_\epsilon x^\mu) \):\

\[\delta_\epsilon \dot{x}^\mu = \frac{mc \dot{x}_\mu}{\sqrt{\dot{x}_\sigma \dot{x}^\sigma}} \partial_\lambda [\epsilon(\lambda) \dot{x}^\mu] = \frac{mc}{\sqrt{\dot{x}_\sigma \dot{x}^\sigma}} \left[ \dot{x}_\mu \epsilon(\lambda) + \dot{x}^\mu \epsilon(\lambda) \dot{x}^\mu \right] = mc \sqrt{\dot{x}_\mu \dot{x}^\mu} \epsilon(\lambda) + mc \epsilon(\lambda) \partial_\lambda \sqrt{\dot{x}_\sigma \dot{x}^\sigma}\]

\[= \frac{d}{d\lambda} \left[ mc \epsilon(\lambda) \sqrt{\dot{x}_\mu \dot{x}^\mu} \right] = \frac{d}{d\lambda} K_{\epsilon, \lambda} \]

→ \( \delta_\epsilon L \) is a total derivative → \( G_\epsilon \) is a continuous symmetry of \( S \)

Note that in Eq. (5.78) we assumed \( \lambda_{a/b} = \varphi(\lambda_{a/b}) \) which corresponds to \( \epsilon(\lambda_{a/b}) = 0 = K_\epsilon(\lambda_{a/b}, \dot{x}^\mu) \) such the boundary terms vanish and the action is completely invariant.

ii. Noether’s (first) theorem:

For each continuous symmetry \( \delta_\epsilon x^\mu = G_\epsilon x^\mu \) there is a conserved Noether charge:

\[Q_\epsilon := \delta_\epsilon x^\mu \frac{\partial L}{\partial \dot{x}^\mu} - K_\epsilon \varepsilon(\lambda) mc \sqrt{\dot{x}_\sigma \dot{x}^\sigma} - \epsilon(\lambda) mc \sqrt{\dot{x}_\mu \dot{x}^\mu} = 0 \]

→ The Noether charge corresponding to \( G_\epsilon \) vanishes identically!

“Vanishing identically” means that \( Q_\epsilon(\lambda, x^\mu, \dot{x}^\mu) \equiv 0 \) for all functions \( x^\mu(\lambda) \), and not just those that satisfy the equations of motion.

• Naively, we expected infinitely many conserved quantities from the infinitely many symmetry generators \( G_\epsilon \). We found them, but quite surprisingly, they turned out to be trivially zero. This is a general feature of local or gauge symmetries; here we use the reparametrization invariance of the relativistic free particle only as an example.

• So while the conserved charges of local symmetries are trivial, such symmetries have other non-trivial implications: they enforce constraints on the equations of motion, so that they are no longer independent. Mathematically, this is described by ↑ Noether’s second theorem.

4. We can illustrate the implications of Noether’s second theorem for the relativistic free particle:
The Lagrangian

\[ L = -mc \sqrt{x_\mu x^\mu} \]  

(5.87)

leads to the conjugate momenta

\[ p_\sigma = \frac{\partial L}{\partial \dot{x}^\sigma} = -\frac{mc \dot{x}_\sigma}{\sqrt{x_\mu x^\mu}} \]  

(5.88)

which satisfy the identity

\[ p^2 = p^\mu p_\mu = m^2 c^2 \]  

(5.89)

- Eq. (5.89) is an identity, i.e., it holds for arbitrary trajectories \( x^\mu (\lambda) \). In particular, \( x^\mu (\lambda) \) does not need to satisfy the equations of motion for Eq. (5.89) to be valid. In Hamiltonian mechanics, such constraints are called primary constraints. So our four canonical momenta \( p^\mu \) are not independent!

- Eq. (5.89) is equivalent to:

\[ \frac{d p^2}{d\lambda} = 0 \iff \left( \frac{d p^\mu}{d\lambda} \right) p_\mu = 0 \]  

(5.90)

\[ \text{ii} \quad \text{Euler-Lagrange equations:} \]

\[ \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\sigma} - \frac{\partial L}{\partial x^\sigma} = \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\sigma} = \frac{d p_\sigma}{d\lambda} = 0 \]  

(5.91)

\[ \rightarrow \text{Four differential equations (} \sigma = 0, 1, 2, 3 \) for four undetermined functions \( x^\mu (\lambda) \).

However: Eq. (5.91) not independent:

\[ p^\mu \frac{d}{d\lambda} \frac{\partial L}{\partial \dot{x}^\mu} = p_\mu \frac{d p^\mu}{d\lambda} \stackrel{5.90}{=} 0 \]  

(5.92)

- Eq. (5.92) is again an identity, i.e., valid for all functions \( x^\mu \), and not only those that satisfy the equations of motion.

- As a consequence, the system of equations of motion Eq. (5.91) effectively looses one of the four equations, and is therefore underdetermined.

Put differently, if you specify a spacetime position \( x^\mu (\lambda = 0) \) and its first derivative \( \dot{x}^\mu (\lambda = 0) \) (note that the Euler-Lagrange equations are second-order differential equations), the equations of motion do not determine a unique solution \( x^\mu (\lambda) \). Mathematically speaking, the initial value problem is ill-posed. This is the characteristic property of a gauge theory.

- This makes sense in the light of reparametrization invariance: If \( x^\mu (\lambda) \) solves the equations of motion, you can construct a new solution \( \tilde{x}^\mu (\lambda) = x^\mu (\psi(\lambda)) \) where \( \psi \) is some diffeomorphism that is the identity except for a compact subinterval somewhere in the interior of \([\lambda_a, \lambda_b]\). In particular, \( \tilde{x}^\mu (\lambda) = x^\mu (\lambda) \) in the neighborhood of \( \lambda_a \), such that the two solutions cannot be distinguished by their initial value and derivative.

Note how important the locality of the symmetry is for this argument to hold!

- This is a special case of \( \text{Noether’s second theorem} \ [75, 76] \).
The fact that our theory is a gauge theory has another, at first glance surprising, consequence:

\[ H = p_\mu \dot{x}^\mu - L = \frac{mc \dot{x}_\mu \dot{x}^\mu}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} + mc \sqrt{\dot{x}_\mu \dot{x}^\mu} = 0 \]  
(5.93)

\[ \rightarrow \] The (canonical) Hamiltonian vanishes identically

- This does not mean that there is no time-evolution in our system. The Hamiltonian Eq. (5.93) describes the “parameter evolution” in \( \lambda \) – which, as we have seen, can be modified arbitrarily by gauge transformations; \( \lambda \) has therefore no physical significance.

This phenomenon will become important for the interpretation of the Einstein field equations in general relativity.

- If one fixes a gauge, the Hamiltonian that describes evolution in this parameter is non-zero in general. E.g., for the “static gauge” \( \lambda = x^0/c \) one finds the Hamiltonian Eq. (5.49) which coincides with the relativistic energy of the particle.
6. Relativistic Field Theories I: Electrodynamics

6.1. A primer on classical field theories

We start with a general discussion of classical field theories on Minkowski space; Maxwell’s electrodynamics is the prime example for such theories (→ next section).

Details: Chapter 1 of my QFT script [19]

6.1.1. Remember: Classical mechanics of “points”

With “points” we mean a discrete set of degrees of freedom.

1) Degrees of freedom $q_i$ labeled by $i = 1, \ldots, N$

2) Lagrangian $L(q_i, \dot{q}_i, t) = T - V$
   
   We write $q$ for $\{q_1, \ldots, q_N\}$. $T$ is the kinetic, $V$ the potential energy.

3) Action $S[q] = \int dt \ L(q(t), \dot{q}(t), t) \in \mathbb{R}$
   
   This is a functional of trajectories $q = q(t)$.

4) Hamilton’s principle of least action:

   $$\frac{\delta S[q]}{\delta q} = 0 \quad \Leftrightarrow \quad \frac{\delta S}{\delta q} = \int dt \frac{1}{L} = 0 \quad (6.1)$$

   $\delta$ denotes functional derivatives/variations.

5) Euler-Lagrange equations ($i = 1, \ldots, N$):

   $$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \quad (6.2)$$

6.1.2. Analogous: Lagrangian Field Theory

Now we consider a continuous set of degrees of freedom:

6) One or more fields $\phi(x)$ on spacetime $x \in \mathbb{R}^{1,3}$ with derivatives $\partial_\mu \phi(x)$
   
   If there are multiple fields we label them by indices: $\phi_k(x)$.
   
   In the following we suppress these indices for the sake of simplicity.

7) Lagrangian density $\mathcal{L}(\phi, \partial \phi, x)$
   
   Most general form: $\mathcal{L}(\phi_k, \{\partial_\mu \phi_k\}, \{x^\mu\})$. (No explicit $x^\mu$-dependence in the following!)
   
   → Lagrangian $L = \int_{\text{Space}} d^3x \ \mathcal{L}(\phi, \partial \phi)$
   
   (We omit the “density” in the following.)
Action:

\[ S[\phi] = \int dt L = \int dt \, d^3x \, L(\phi, \partial_t \phi) = \frac{1}{c} \int_{\text{Spacetime}} d^4x \, L(\phi, \partial_t \phi) \]  

(6.3)

\( S[\phi] \) is a functional of \( \text{“field trajectories”} \) in \( \mathbb{R}^{1,3} \).

Action principle:

The classical field evolutions of the system extremize the action:

\[ \delta S[\phi] = 0 \]  

(6.4)

This variation can be evaluated along the same lines as for the classical mechanics of points:

\[
0 \overset{!}{=} \delta S[\phi] = \int d^4x \, \delta L \\
= \int d^4x \left\{ \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right\} \\
\text{Add zero and use } \delta (\partial_\mu \phi) = \partial_\mu (\delta \phi) \\
= \int d^4x \left\{ \frac{\partial L}{\partial \phi} \delta \phi - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \right\} \delta \phi + \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi \right) \\
\text{Gauss theorem} \\
= \int_{\text{Boundary}} d\sigma_\mu \frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi = 0 + \int d^4x \left\{ \frac{\partial L}{\partial \phi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) \right\} \delta \phi \\
\]

(6.5a-d)

- Note that \( \phi \) is fixed on the boundary and therefore \( \delta \phi = 0 \).
- The second term vanishes because the integral must vanish for arbitrary variations \( \delta \phi \).

Euler-Lagrange equations (one for each field):

\[ \frac{\partial L}{\partial \phi} - \partial_\mu \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \right) = 0 \]  

(6.6)

- Note the Einstein summation over repeated indices.
- These equations are manifestly Lorentz covariant if \( L \) is a Lorentz scalar; such field theories are called \( \ast \) relativistic field theories.
- If there are multiple fields \( \phi_k \), there is one Euler-Lagrange equation per field (it is straightforward to generalize the derivation above).

Hamiltonian formalism:

Just like for the mechanics of points, we can define:

\[ \pi := \frac{\partial L}{\partial \phi} \]  

\( \ast \) Momentum density conjugate to \( \phi \)  

(6.7)

Like \( \phi(x) \), the momentum is a field: \( \pi(x) \). Here it is \( \dot{\phi}_0 \).

\[ H(\pi, \phi, \nabla \phi) := \pi \dot{\phi} - L(\phi, \partial_\mu \phi) \]  

\( \ast \) Hamiltonian density  

(6.8)
Here $\phi$ is to be expressed as a function of the conjugate momentum via Eq. (6.7).

The argument $\partial \phi$ of $\mathcal{L}$ is short for $\{\partial_{\mu} \phi\}$ or $\{\nabla \phi, \phi\}$.

\[ H := \int d^3x \mathcal{H} \quad \textit{Hamiltonian} \quad (6.9) \]

For given fields $\pi(x)$ and $\phi(x)$, $H$ is a (potentially constant) function of time. By contrast, the Hamiltonian density $\mathcal{H}$ is a function of space $\vec{x}$ and time $t$.

### 6.2. Electrodynamics: Covariant formulation and Lagrange function

We now want to reformulate Maxwell’s electrodynamics in this formalism, i.e., we want to find a Lagrangian density (and an associated action) such that the Euler-Lagrange equations are the Maxwell equations.

1 | Remember:

   i | $\triangleright$ **Maxwell equations** (in cgs units):

   - Magnetic Gauss’s law: $\nabla \cdot \vec{B} = 0 \quad (6.10a)$
   - Maxwell-Faraday law: $\nabla \times \vec{E} + \frac{1}{c} \partial_t \vec{B} = 0 \quad (6.10b)$
   - Electric Gauss’s law: $\nabla \cdot \vec{E} = 4\pi \rho \quad (6.10c)$
   - Ampère’s law: $\nabla \times \vec{B} - \frac{1}{c} \partial_t \vec{E} = \frac{4\pi}{c} \vec{j} \quad (6.10d)$

   with charge density $\rho(x)$ and current density $\vec{j}(x)$ that satisfy the $\triangleright$ **continuity equation**

   \[ \partial_t \rho + \nabla \cdot \vec{j} = 0. \quad (6.11) \]

This follows from the two inhomogeneous Maxwell equations Eqs. (6.10c) and (6.10d). Note that here $\rho$ and $\vec{j}$ are external fields and not dynamic degrees of freedom. The statement is therefore that only for external fields that satisfy Eq. (6.11) the Maxwell equations yield solutions for $\vec{E}$ and $\vec{B}$.

ii | Homogeneous Maxwell equations (HME) Eq. (6.10a) & Eq. (6.10b)

   $\triangleright$ \exists “Scalar” potential $\varphi$ and “Vector” potential $\vec{A}$:

   \[ \vec{E} = -\nabla \varphi - \frac{1}{c} \partial_t \vec{A} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A} \quad (6.12) \]

   - Constraining the fields $\vec{E}$ and $\vec{B}$ to this form satisfies the **homogeneous Maxwell equations** Eqs. (6.10a) and (6.10b) automatically.

   - Because of the two homogeneous Maxwell equations, the six fields $\{E_x, E_y, E_z, B_x, B_y, B_z\}$ are not independent so that all degrees of freedom can be encoded in the **four fields** $\{\varphi, A_x, A_y, A_z\}$. This suggests a reformulation of Maxwell’s theory in terms of these “potentials”.
Gauge transformation:

\[ A' := A + \nabla \lambda \quad \text{and} \quad \varphi' := \varphi - \frac{1}{c} \partial_t \lambda \quad (6.13) \]

This transformation of fields is called a \textit{gauge transformation (→ below)}.

\( \rightarrow \) The potentials \( \varphi \) and \( A \) are not unique.

Inhomogeneous Maxwell equations (IME) Eqs. (6.10c) and (6.10d) in terms of the potentials:

\[ \begin{align*}
\text{Eq. (6.10c)} & \iff \nabla^2 \varphi + \frac{1}{c} \partial_t (\nabla \cdot A) = -4\pi \rho \\
\text{Eq. (6.10d)} & \iff \nabla^2 A - \frac{1}{c^2} \partial_t^2 A = \frac{-4\pi}{c} j + \nabla \left( \nabla \cdot A + \frac{1}{c} \partial_t \varphi \right)
\end{align*} \quad (6.14a,b) \]

In this form, electrodynamics is a \textit{gauge theory} because it has a \textit{local} symmetry, namely the transformation Eq. (6.13). Indeed, it is straightforward to show that if \( (\varphi, A) \) is a solution of Eq. (6.14), then \( (\varphi', A') \) given by Eq. (6.13) is another solution. Since \( \lambda(x) \) is arbitrary, one can choose continuously differentiable \( \lambda(x) \) that vanish everywhere except for a compact region of spacetime. This makes Eq. (6.13) \textit{a local symmetry transformation} of the PDE system Eq. (6.14); such local symmetries are called \textit{gauge transformations}, and models that feature such symmetries are referred to as \textit{gauge theories}. The locality of the symmetry has profound implications:

\begin{center}
\[ \nabla \cdot \vec{A} + \frac{1}{c} \partial_t \varphi = 0 \quad \text{\textit{Lorenz gauge (LG)}} \quad (6.15) \]
\end{center}

Thus, if we want a deterministic theory (meaning: a theory with predictive power), we \textit{cannot} interpret the gauge fields \( (\varphi, \vec{A}) \) as physical (= observable) degrees of freedom. Our only choice (to save predictability) is to identify the equivalence classes \( [(\varphi, \vec{A})] \) of field configurations that are related by (local) gauge transformations as physical states; this is the defining property of a gauge theory. In a nutshell: local symmetries must be interpreted as gauge symmetries and fields related by such transformations are mathematically redundant descriptions of the \textit{same} physical state.

\[ \begin{align*}
\text{Eq. (6.14) Gauge theory} & \rightarrow \text{Fix a gauge:} \\
\nabla \cdot \vec{A} + \frac{1}{c} \partial_t \varphi & = 0 \quad \text{} \quad \textit{Lorenz gauge (LG)}
\end{align*} \quad (6.15) \]

It is straightforward to show that for any given \( (\varphi, \vec{A}) \) there is a gauge transformation \( \lambda \) such that \( (\varphi', \vec{A}') \) satisfies Eq. (6.15).
Expressed in potentials in the Lorenz gauge, the inhomogeneous Maxwell equations become a set of four decoupled wave equations.

We do not have to consider the homogeneous Maxwell equations in the gauge field representation because Eq. (6.12) ensures that Eqs. (6.10a) and (6.10b) are automatically satisfied.

2 | Observation: Charge \( dq = \rho \, d^3x \) in volume \( dV = d^3x \) independent of inertial system:

\[
\rho \, d^3x = \bar{\rho} \, d^3\bar{x} \quad \Rightarrow \quad \rho \, d^3x \frac{dx^\mu}{dt} = \rho \, d^3x \, dr \frac{dx^\mu}{dt} = \frac{1}{c} \, d^4x \rho \frac{dx^\mu}{dt} = \frac{1}{c} \, d^4x \frac{\rho \, dx^\mu}{dt} \quad \text{Eq. (4.23)}
\]

This suggests that charge and current density are actually components of a Lorentz 4-vector:

\[
j^\mu := \rho \frac{dx^\mu}{dt} = \left( \frac{c \rho}{\rho \, v} \right) = \left( \frac{c \rho}{j} \right) \quad \text{4-current (density)}
\]

with \( \star \) charge density \( \rho = \rho(x) \) and \( \star \) current density \( \vec{j} = \vec{j}(x) = \rho(x) \vec{v}(x) \).

- In the argument above, the trajectory \( \vec{x}(t) \) in \( x^\mu = (ct, \bar{x}(t)) \) parametrizes the movement of the infinitesimal volume \( dV = d^3x \) with charge \( dq = \rho dV \); the coordinate velocity \( \vec{v}(t) = \frac{d\vec{x}}{dt} \) is therefore the velocity of the charge distribution at position \( \vec{x}(t) \) at time \( t: \vec{v}(x) \). Thus, in general, the current density \( \vec{j}(x) = \rho(x) \vec{v}(x) \) depends on position and time via the charge density \( \rho(x) \) and the velocity field \( \vec{v}(x) \).

- That the charge density \( \rho \) is not a Lorentz scalar is intuitively clear as it is defined as charge per volume. Volumes, however, are clearly not Lorentz invariant because they are Lorentz contracted. Since the charge (not the charge density!) is Lorentz invariant (this is an observational fact), the ratio of charge by volume must change under boosts.
Eq. (6.18) and Eq. (6.16) suggest the compact notation

\[
\begin{align*}
\text{Eq. (6.16a)} & \quad \partial^2 A^\mu = \frac{4\pi}{c} j^\mu \quad \text{(IME in LG)} \\
\text{Eq. (6.16b)}
\end{align*}
\]

(6.19)

Remember that \( \partial^2 = \Box = \frac{1}{c^2} \partial_t^2 - \nabla^2 \).

with

\[
A^\mu := \left( \begin{array}{c} \phi \\ A \end{array} \right) \quad \text{4-potential}
\]

(6.20)

The covariant components of the gauge field are \( A_\mu = (\psi, -A) \).

The transformation of the 4-potential must be that of a Lorentz 4-vector:

\[
\begin{align*}
\tilde{\partial}^2 &= \partial^2 : \text{Scalar [Eq. (4.36b)]} \\
\tilde{j}^\mu &= \Lambda^\mu_\nu j^\nu : \text{4-vector [Eq. (6.18)]} \rightarrow \tilde{A}^\mu = \Lambda^\mu_\nu A^\nu : 4\text{-vector}
\end{align*}
\]

(6.21)

With this transformation, the Maxwell equations in their simple formulation Eq. (6.19) are manifestly Lorentz covariant:

\[
\tilde{\partial}^2 A^\mu = \frac{4\pi}{c} j^\mu \quad \text{(Continuity equation)}
\]

(6.22)

4. We can now rewrite our previous equations in tensor notation:

i. The Lorenz gauge condition can be compactly written as:

\[
\partial A \equiv \partial_\mu A^\mu = 0 \quad \text{(Lorenz gauge)}
\]

(6.23)

→ The Lorenz gauge is Lorentz invariant

*Note: The Lorenz gauge is named after Ludvig Lorenz; by contrast, the Lorentz transformation is named after Hendrik Lorentz. Thus: The Lorenz gauge (no “it”) is Lorentz invariant.*

ii. The continuity equation also becomes very simple (and Lorentz covariant):

\[
\partial j \equiv \partial_\mu j^\mu = 0 \quad \text{(Continuity equation)}
\]

(6.24)

iii. The gauge transformation can be written as follows:

\[
A'^\mu = A^\mu - \partial^\mu \lambda \quad \text{(Gauge transformation)}
\]

(6.25)

Recall that \( \partial^\mu = (\frac{1}{c} \partial_t, -\nabla) \).
Let us summarize our findings so far:

Maxwell equations: \( \partial^2 A^\mu = \frac{4\pi}{c} j^\mu \)
Lorenz gauge: \( \partial_\mu A^\mu = 0 \)
Continuity equation: \( \partial_\mu j^\mu = 0 \)

\[ \begin{cases} \partial^2 \tilde{A}^\mu = \frac{4\pi}{c} \tilde{j}^\mu \\ \tilde{\partial}_\mu \tilde{A} = 0 \\ \tilde{\partial}_\mu \tilde{j} = 0 \end{cases} \quad (6.26) \]

→ Electrodynamics satisfies Einstein’s principle of Special Relativity \( \text{SR} \)

- In contrast to Newtonian mechanics, electrodynamics was a relativistic theory all along and there was no need to modify it. It’s Lorentz covariance was simply not manifest and required a bit of work to unveil.

- The treatment above relies on (1) expressing the Maxwell equations in terms of the gauge fields and (2) choosing a particular gauge (the Lorenz gauge). While this is mathematically legit (and not restrictive), it would be nice to have manifestly Lorentz covariant expressions (1) without fixing a gauge and (2) in terms of the physically observable fields \( E \) and \( B \).

To achieve both goals, we first need a new tensorial quantity:

6 Field strength tensor:

i Motivation: We are looking for the simplest field that …
- …is gauge-invariant (i.e., has a physical interpretation).
- …is Lorentz covariant (i.e., can be used to construct Lorentz covariant equations).

ii Discretized spacetime on a (hypercubic) lattice (here we consider the \( xy \)-plane):
- The gauge field \( A^\mu \) lives on edges in \( \mu \)-direction.
- The gauge transformation \( \lambda \) lives on vertices of the lattice.

→ Discretized gauge transformation:

\[ A^\mu_{x,x+e_\mu} = A^\mu_{x,x} + \frac{1}{\partial_\mu \lambda} \left( \lambda_{x+e_\mu} - \lambda_x \right) \quad (6.27) \]

→ Sums along paths \( P \) transform non-trivially only at their “start site” \( s \) and “end site” \( e \):

\[ \sum_{e \in P} A^\mu_e = \sum_{e \in P} A^\mu_e + \frac{1}{2} \left( \lambda_e - \lambda_s \right) \quad (6.28) \]

Edges \( e \) are pairs of adjacent lattice sites, e.g., \( e = (x, x + e_\mu) \) with lattice vector \( |e_\mu| = \varepsilon \).

→ Sums \( \sum_{e \in L} A^\mu_e \) along closed loops \( L \) are gauge-invariant (because \( s = e \)!}
Smallest gauge-invariant loop (* loop around a single face $f = xy$):

\[ F_{yx} := A_{x,x} + e_x + A_{x,y} + e_y - A_{x,y,x} + e_x + e_y - A_{x,y,x} + e_y \]
\[ = \left( A_{x,x} + e_x - A_{x,y} + e_y - A_{x,y} + e_y + e_x \right) \]
\[ \to \partial_y A_x - \partial_x A_y \]

This motivates the definition:

\[ F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \]
\[ \Phi \text{ Field strength tensor (FST)} \]

\[ F_{\mu\nu} := \left( \begin{array}{cccc}
0 & E_x & E_y & E_z \\
-E_x & 0 & -B_z & B_y \\
-E_y & B_z & 0 & -B_x \\
E_z & -B_y & B_x & 0
\end{array} \right)_{\mu\nu} \]

Details:

- $F_{\mu\nu}$ is a $(0, 2)$ Lorentz tensor

  - The FST is gauge-invariant by construction. You can also check this by applying the gauge transformation Eq. (6.25).

  - It is easy to see that the FST has the following properties:

    - Antisymmetry: $F^{\mu\nu} = -F^{\nu\mu}$
    - Tracelessness: $F_{\mu}^{\mu} = g_{\mu\nu} F^{\mu\nu} = 0$

- $\Phi$ When we write “$E_x$”, we refer to the $x$-component of the original electric field $\vec{E}$ as it occurs in the Maxwell equations Eq. (6.10). In this context, an expression like $E_x$ does not make sense since $\vec{E}$ is not a 4-vector but the component of a rank-2 tensor.

Using that $\varepsilon^{\mu\nu\alpha\beta}$ is a Lorentz pseudo-tensor [recall Eq. (4.41)], we can define:

\[ \tilde{F}^{\mu\nu} := \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \]
\[ \Phi \text{ Dual field strength tensor (DFST)} \]

\[ \tilde{F}^{\mu\nu} := \left( \begin{array}{cccc}
0 & -B_x & -B_y & -B_z \\
B_x & 0 & E_z & -E_y \\
B_y & -E_z & 0 & E_x \\
B_z & E_y & -E_x & 0
\end{array} \right)_{\mu\nu} \]
Transformation of the electromagnetic field:

The field strength tensor Eq. (6.30) has the useful properties that (1) we know how it transforms under Lorentz transformations, and (2) we know how it relates to the observable fields $\vec{E}$ and $\vec{B}$. Hence we can use it to derive the transformation of the electromagnetic field when transitioning from one inertial system to another.

i) The (contravariant) FST transforms under a Lorentz transformation $\Lambda$ as follows:

$$\tilde{F}^{\mu\nu}(\tilde{x}) = \Lambda^{\mu}_\alpha \Lambda^{\nu}_\beta F^{\alpha\beta}(x)$$

Here it is $F^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} F_{\alpha\beta}$ as usual.

ii) Boost $\Lambda_\nu$ [Eq. (4.10)]:

$$\tilde{E}(\tilde{x}) = \gamma \left[ \tilde{E}(x) + \frac{v}{c} \tilde{\nu} \times \tilde{B}(x) \right] - (\gamma - 1) \frac{\tilde{v} \cdot \tilde{E}(x)}{v^2} \tilde{v}$$

$$\tilde{B}(\tilde{x}) = \gamma \left[ \tilde{B}(x) - \frac{v}{c} \tilde{\nu} \times \tilde{E}(x) \right] - (\gamma - 1) \frac{\tilde{v} \cdot \tilde{B}(x)}{v^2} \tilde{v}$$

with $x^\mu = (\Lambda_{-\nu})^\mu_\nu \tilde{x}^\nu$.

![Note that on the left-hand side the arguments are $\tilde{x}$ and on the right-hand side $x$!]

Electric and magnetic fields “mix” under boosts!

- Please appreciate what we showed: If you start from Maxwell Eq. (6.10) and perform an arbitrary Lorentz boost $\tilde{x}^\mu = \Lambda^\mu_\nu x^\nu$, transforming the derivatives as $\tilde{\partial}_\mu = \Lambda^\mu_\nu \partial_\nu$, you obtain a set of horribly looking PDEs. But if you recombine the equations appropriately, group the terms according to Eq. (6.34) and define the new fields $\tilde{E}(\tilde{x}), \tilde{B}(\tilde{x})$, the equations look again like Eq. (6.10), only with bars over coordinates and fields.

You could show this directly, without ever introducing the gauge field $A^\mu$ and without using the machinery of tensor calculus (this is what Einstein did for a boost in z-direction in his 1905 paper “Zur Elektrodynamik bewegter Körper” [9]); but hopefully you agree that our more advanced route (using the gauge field and tensor calculus) is a more elegant approach.

- Because of our motivation from Einstein’s principle of Special Relativity $\text{SR}$, we frame our discussion in the terminology of passive transformations (= coordinate transformation): The same electromagnetic field that looks like $E(x), B(x)$ in an inertial system $K$ looks like $\tilde{E}(\tilde{x}), \tilde{B}(\tilde{x})$ in another system $\tilde{K}$.

Because we showed that the Maxwell equations satisfy $\text{SR}$, they have exactly the same form in $\tilde{K}$ as in $K$. This, however, allows you to interpret the transformation actively: If you are given a solution of Maxwell equations $\tilde{E}(\tilde{x}), \tilde{B}(\tilde{x})$, then, for any $\tilde{v}$, the new functions $\tilde{E}(\tilde{x}), \tilde{B}(\tilde{x})$ defined by Eq. (6.34) and $x^\mu = (\Lambda_{-\nu})^\mu_\nu \tilde{x}^\nu$ are again solutions (in the same coordinates). This shows that the Lorentz group is (part of) the invariance group of the PDE system Eq. (6.10) we call Maxwell equations (just like the Galilei group was an invariance group of Newton’s equation, recall Section 1.2).
### Lecture 13 [23.01.24]

#### iii | Non-relativistic limit:

Eq. (6.34) \( \gamma \rightarrow 1 \)

\[
\begin{align*}
\vec{E}(\vec{x}) & \approx \vec{E}(x) + \frac{1}{c} \vec{v} \times \vec{B}(x) \\
\vec{B}(\vec{x}) & \approx \vec{B}(x) - \frac{1}{c} \vec{v} \times \vec{E}(x)
\end{align*}
\]

- The interconversion between magnetic and electric fields happens already in linear order of \( v/c \).
- The separation of the electromagnetic field into “electric” and “magnetic” components is observer dependent!
- Example: A charge at rest has a non-zero electric field, but a vanishing magnetic field. The same charge as seen from an inertial system in relative motion gives rise to a current that is accompanied by a non-vanishing magnetic field perpendicular to the direction of motion and the electric field. This is a direct consequence of Eq. (6.35):

\[
\vec{B}(\vec{x}) \approx -\frac{1}{c} \vec{v} \times \vec{E}(x) \neq 0.
\]

#### iv | Special case: Boost \( \Lambda_{\nu}{_\xi} \) in \( x \)-direction: Eq. (6.34) \( \vec{v} = (v_x, 0, 0) \)

\[
\begin{align*}
\vec{E}_x & = E_x, \quad \vec{E}_y & = \gamma \left[ E_y - \frac{v}{c} B_z \right], \quad \vec{E}_z & = \gamma \left[ E_z + \frac{v}{c} B_y \right] \\
\vec{B}_x & = B_x, \quad \vec{B}_y & = \gamma \left[ B_y + \frac{v}{c} E_z \right], \quad \vec{B}_z & = \gamma \left[ B_z - \frac{v}{c} E_y \right]
\end{align*}
\]

(Here the fields in \( \mathcal{K} \) on the left-hand side are functions of \( \vec{x} \) whereas the fields in \( \mathcal{K} \) on the right-hand side are functions of \( x \).)

- The field components parallel to the boost remain unchanged.
- The perpendicular components mix and get enhanced by a Lorentz factor \( \gamma > 1 \).
- Einstein derived this transformation directly (without using gauge fields and tensor notation) in his 1905 paper “Zur Elektrodynamik bewegter Körper” [9]; you follow this path in \( \mathcal{K} \) Problemset 7.

#### v | Lorentz scalars:

The electric and magnetic field components transform in a complicated way under Lorentz transformations. Is it possible to combine them into scalar quantities? Thanks to our knowledge of tensor calculus and the field strength tensor, this question is easy to answer:

- We can construct a scalar by contracting the FST with itself:

\[
F^{\mu\nu} F_{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} F_{\alpha\beta} F_{\mu\nu} \equiv 2(\vec{B}^2 - \vec{E}^2)
\]

\( \rightarrow \) If \( |\vec{E}| \lesssim |\vec{B}| \) is true in one IS, it is true in all IS.

- We can construct a pseudo scalar by contracting the FST with the DFST:

\[
\tilde{F}^{\mu\nu} F_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} F_{\mu\nu} \equiv -4(\vec{E} \cdot \vec{B})
\]

\( \rightarrow \) If \( \vec{E} \perp \vec{B} \) is true in one IS, it is true in all IS.
Some comments:

- Note that $\mathcal{F}_{\mu\nu} \equiv -F_{\mu\nu}$ (use contraction identities for Levi-Civita symbols to show this, Problemset 7); i.e., the two quantities above exhaust all elementary gauge-invariant scalar fields that we can construct ($A_{\mu} A_{\mu}$ is of course also a scalar, but not a gauge-invariant one).

- The combination of Eq. (6.37) and Eq. (6.38) can be used to infer whether inertial systems exist in which either the electric or magnetic field vanishes. For example: If $\mathcal{F}_{\mu\nu} = 0$ and $F_{\mu\nu} F_{\mu\nu} > 0$, it is possible to find an inertial system where $E = 0$ and $B \neq 0$ (but not the other way around). If $\mathcal{F}_{\mu\nu} \neq 0$ there is no inertial system in which one of the fields vanishes.

8 | Manifest covariant form of the Maxwell equations:

Using the FST and the DFST, we can write the Maxwell equations manifestly covariant without using the gauge field and/or fixing a gauge (cf. Eq. (6.19)):

i | The equations we look for must be …

- …manifestly covariant (→ tensor equations).
- …linear in the FST or the DFST (the ME are linear in $E^\mu$ and $B^\mu$).
- …use one 4-divergence $\partial_\mu$ (the ME are first-order PDEs).

→ (6.39a)

$$\partial_\mu \mathcal{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu
u\rho\sigma} \partial_\nu (\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \varepsilon^{\mu
u\rho\sigma} \partial_\nu A_\rho = 0$$

(6.39b)

$$\partial_\nu F^{\mu\nu} = \partial_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu (\partial A) - \partial^2 A^\mu$$

ii | The homogeneous ME Eqs. (6.10a) and (6.10b) must be identically true if the fields are given in terms of gauge fields. Eq. (6.39a) then suggests that the homogeneous ME are:

$$\partial_\nu \mathcal{F}^{\mu\nu} = 0 \quad \text{Homogeneous ME (??)}$$

(6.40)

To check this evaluate:

$$\partial_\nu \mathcal{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu
u\rho\sigma} \partial_\nu F_{\rho\sigma}$$

(6.41a)

$$= \frac{1}{6} \varepsilon^{\mu
u\rho\sigma} (\partial_\nu F_{\rho\sigma} + \partial_\sigma F_{\nu\rho} + \partial_\rho F_{\sigma\nu})$$

(6.41b)

$$= \frac{1}{2} \sum_{\nu<\rho<\sigma} \varepsilon^{\mu
u\rho\sigma} (\partial_\nu F_{\rho\sigma} + \partial_\sigma F_{\nu\rho} + \partial_\rho F_{\sigma\nu})$$

(6.41c)

Here we used that the Levi-Civita symbol is invariant under cyclic permutations of (subsets) of indices and that the FST (and the Levi-Civita symbol) is antisymmetric in its indices. Note that for every fixed $\mu$ there are $3! = 6$ non-vanishing assignments of indices $\nu \rho \sigma$. However, pairs of terms like $\varepsilon^{\mu
u\rho\sigma} \partial_\nu F_{\rho\sigma} = \varepsilon^{\mu
u\rho\sigma} \partial_\nu F_{\rho\sigma}$ are identical, so that only 3 distinct terms remain. These can be w.l.o.g. written like cyclic permutations as in Eq. (6.41c). Note that for a fixed index $\mu$, the sum contains only one non-vanishing summand.

→ (4 equations)

$$\forall \nu<\rho<\sigma : \partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0 \quad \Leftrightarrow \quad \forall \mu : \partial_\nu \mathcal{F}^{\mu\nu} = 0$$

(4 equations)

8 | Manifest covariant form of the Maxwell equations:

Using the FST and the DFST, we can write the Maxwell equations manifestly covariant without using the gauge field and/or fixing a gauge (cf. Eq. (6.19)):

i | The equations we look for must be …

- …manifestly covariant (→ tensor equations).
- …linear in the FST or the DFST (the ME are linear in $E^\mu$ and $B^\mu$).
- …use one 4-divergence $\partial_\mu$ (the ME are first-order PDEs).

→ (6.39a)

$$\partial_\nu \mathcal{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu
u\rho\sigma} \partial_\nu (\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \varepsilon^{\mu
u\rho\sigma} \partial_\nu A_\rho = 0$$

(6.39b)

$$\partial_\nu F^{\mu\nu} = \partial_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu (\partial A) - \partial^2 A^\mu$$

ii | The homogeneous ME Eqs. (6.10a) and (6.10b) must be identically true if the fields are given in terms of gauge fields. Eq. (6.39a) then suggests that the homogeneous ME are:

$$\partial_\nu \mathcal{F}^{\mu\nu} = 0 \quad \text{Homogeneous ME (??)}$$

(6.40)

To check this evaluate:

$$\partial_\nu \mathcal{F}^{\mu\nu} = \frac{1}{2} \varepsilon^{\mu
u\rho\sigma} \partial_\nu F_{\rho\sigma}$$

(6.41a)

$$= \frac{1}{6} \varepsilon^{\mu
u\rho\sigma} (\partial_\nu F_{\rho\sigma} + \partial_\sigma F_{\nu\rho} + \partial_\rho F_{\sigma\nu})$$

(6.41b)

$$= \frac{1}{2} \sum_{\nu<\rho<\sigma} \varepsilon^{\mu
u\rho\sigma} (\partial_\nu F_{\rho\sigma} + \partial_\sigma F_{\nu\rho} + \partial_\rho F_{\sigma\nu})$$

(6.41c)

Here we used that the Levi-Civita symbol is invariant under cyclic permutations of (subsets) of indices and that the FST (and the Levi-Civita symbol) is antisymmetric in its indices. Note that for every fixed $\mu$ there are $3! = 6$ non-vanishing assignments of indices $\nu \rho \sigma$. However, pairs of terms like $\varepsilon^{\mu
u\rho\sigma} \partial_\nu F_{\rho\sigma} = \varepsilon^{\mu
u\rho\sigma} \partial_\nu F_{\rho\sigma}$ are identical, so that only 3 distinct terms remain. These can be w.l.o.g. written like cyclic permutations as in Eq. (6.41c). Note that for a fixed index $\mu$, the sum contains only one non-vanishing summand.

→ (4 equations)

$$\forall \nu<\rho<\sigma : \partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0 \quad \Leftrightarrow \quad \forall \mu : \partial_\nu \mathcal{F}^{\mu\nu} = 0$$

(4 equations)
It is straightforward to check by hand, using Eq. (6.30), that the four Bianchi identities correspond to the four homogeneous Maxwell Eqs. (6.10a) and (6.10b). For example:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} \equiv -\nabla \cdot \vec{B} = 0 \quad \Leftrightarrow \quad \text{Eq. (6.10a)}$$  \hspace{1cm} (6.43)

Details: \(\bigcirc\) Problemset 7

- As shown in Eq. (6.39a), the homogeneous ME are \textit{identities} if the FST is expressed in terms of gauge fields.
- By contrast, if the FST is expressed in terms of physical fields \(\vec{E}\) and \(\vec{B}\) [as given in Eq. (6.30)], the equation \(\partial_\nu F^{\mu\nu} = 0\) becomes a non-trivial constraint on the field configurations.

iii \(\leftarrow\) Lorenz gauge Eq. (6.23) \rightarrow

Eq. (6.39b) \quad \Rightarrow \quad \partial_\nu F^{\mu\nu} = -\partial^2 A^\mu \quad \text{(6.44)}

Compare Eq. (6.19) (inhomogeneous ME in Lorenz gauge):

$$-\partial^2 A^\mu = \frac{4\pi}{c} j^\mu \quad \text{(6.45)}$$

This suggests that the inhomogeneous ME are:

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \quad \text{Inhomogeneous ME (?)} \quad \text{(6.46)}$$

It is straightforward to check by hand that these four equations are equivalent to the four inhomogeneous ME Eqs. (6.10c) and (6.10d) using Eq. (6.30). For example for \(\mu = 0\):

$$\partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} \equiv -\nabla \cdot \vec{E} = -\frac{4\pi}{c} j^0 = -4\pi \rho \quad \Leftrightarrow \quad \text{Eq. (6.10c)}$$  \hspace{1cm} (6.47)

Details: \(\bigcirc\) Problemset 7

- In this form, the continuity equation Eq. (6.24) follows trivially from the antisymmetry of the FST:

$$\partial_\mu j^\mu = -\frac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0 \quad \text{(6.48)}$$

- If you express the FST in terms of the gauge field, the inhomogeneous ME read (without fixing a gauge!):

$$\partial^2 A^\mu - \partial^\mu (\partial A) = \frac{4\pi}{c} j^\mu \quad \text{(6.49)}$$

This equation becomes Eq. (6.19) in the Lorenz gauge Eq. (6.23). It is easy to check that this equation is still gauge symmetric under the transformation Eq. (6.25).

iv \(\leftarrow\) In summary, the 8 (\(=1+3+1+3=4+4\)) Maxwell equations can be written in the covariant form:

Homogeneous ME: \(\partial_\nu \tilde{F}^{\mu\nu} = 0 \quad \text{(6.50a)}\)

Inhomogeneous ME: \(\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \quad \text{(6.50b)}\)
• Using Eqs. (6.30) and (6.32), these equations make sense without introducing the gauge field.

• Note that these equations show that under Lorentz transformations the four homogeneous (inhomogeneous) Maxwell equations mix among each other. You show this explicitly in Problemset 7 for a boost in $\chi$-direction.

• In particular, this means that the Maxwell equations written in their conventional form Eq. (6.10) (i.e., as two scalar and two vector equations) remain not invariant under Lorentz transformations for each equation separately, rather the magnetic Gauss law mixes with the Maxwell-Faraday law, and the electric Gauss law mixes with Ampère’s law. This explains why showing the Lorentz covariance of the PDE system Eq. (6.10) is quite messy and complicated without using the tensor formalism. This is why we say that its Lorentz covariance is not manifest. By contrast, the Lorentz covariance of the formulation Eq. (6.50) is manifest as these are tensor equations.

**Lagrangian formulation:**

Our final goal is to make a connection to the formalism introduced in Section 6.1 and obtain the Lorentz covariant Maxwell equations as the Euler-Lagrange equations of some action/Lagrangian:

1. It is convenient to construct the Lagrangian as a function of the gauge fields $A^\mu$ because in this formulation the HME are identically satisfied:

   $$\partial_\mu \tilde{F}^{\mu \nu} \equiv 0 \implies \mathcal{L} = \mathcal{L}(A, \partial A) \quad (6.51)$$

   → Only the inhomogeneous ME must follow as Euler-Lagrange equations

   Note that the counting matches: We have four fields $A^\mu$ and thus four Euler-Lagrange equations – just as we have four IME: $\partial_\nu F^{\nu \nu} = -\frac{\epsilon_\nu}{c} j^\nu$.

2. We have the following hints to construct a reasonable Lagrangian density:

   - The IME are Lorentz covariant. This can be ensured by a Lagrangian density that is a Lorentz (pseudo) scalar.
   - The Maxwell equations are linear (superposition principle!); thus the Lagrangian must be quadratic in the fields.
   - The IME are gauge invariant. This can be ensured by a Lagrangian density that is gauge invariant up to a total derivative (here: surface term) which does not affect the equations of motion.

   → Most general form:

   $$\mathcal{L}(A, \partial A) = a_1 F^{\mu \nu} F_{\mu \nu} + a_2 \frac{\tilde{F}^{\mu \nu} F_{\mu \nu}}{\epsilon_{\nu} F^{\mu \nu} F_{\mu \nu}} + a_3 \frac{\tilde{F}^{\mu \nu} \tilde{F}_{\mu \nu}}{\epsilon_{\nu} F^{\mu \nu} F_{\mu \nu}} + a_4 A_{\mu} j^{\mu} \quad (6.52)$$

   Surface term

   Gauge inv. up to surface term

   Details: Problemset 7

   - It is straightforward to check that

   $$\tilde{F}^{\mu \nu} F_{\mu \nu} \equiv -F^{\mu \nu} F_{\mu \nu} \quad (6.53)$$

   so that we can drop the $a_3$-term without loss of generality.
One can also check that
\[ F_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial_\rho A_\sigma - \partial_\sigma A_\rho) \]  
(6.54a)
\[ = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} (\partial_\mu A_\nu \partial_\rho A_\sigma - \partial_\nu A_\mu \partial_\rho A_\sigma - \partial_\nu A_\mu \partial_\sigma A_\rho - \partial_\mu A_\nu \partial_\sigma A_\rho) \]  
(6.54b)
\[ = 2 \varepsilon^{\mu\nu\rho\sigma} (\partial_\mu A_\nu) (\partial_\rho A_\sigma) \]  
(6.54c)
so that the \( a_2 \)-term has no effect on the equations of motion and we can drop it as well.

Note: The \( a_2 \)-term is known as the Chern-Simons 3-form and is special because it is topological (it does not “feel” the geometry of spacetime). This is easy to see: One does not need a metric tensor to construct it because the contravariant indices of the DFST stem from the Levi-Civita symbol! Despite being a surface term, such terms are important when one quantizes the theory and/or when the gauge theory is non-Abelian (like the SU(3) gauge theory of the strong interaction). Note also that this term is a pseudo scalar, i.e., it breaks parity symmetry (which we know electrodynamics does not).

The \( a_4 \)-term is not gauge invariant. However, the continuity equation ensures that it modifies the Lagrangian only by a surface term under gauge transformations:
\[ \tilde{A}_\mu j^\mu = (A_\mu - \partial_\mu \lambda) j^\mu = A_\mu j^\mu - (\partial_\mu \lambda) j^\mu = A_\mu j^\mu - \partial_\mu (\lambda j^\mu) \]  
(6.55)

(Here we used the continuity equation \( \partial_\mu j^\mu = 0 \).)

Consequently, the equations of motion must be gauge invariant despite the \( a_4 \)-term.

It is easy to check that the quadratic Lorentz scalar \( A_\mu A^\mu \) is not gauge invariant (not even up to a surface term); thus it is forbidden.

Note: Coincidentally, it is this term that would give the quantized excitations of the \( A \)-field a mass. Thus if you want massive gauge excitations (like the \( W^\pm \) and \( Z \)-bosons of the weak interaction), you must find a way to smuggle the term \( A_\mu A^\mu \) into your Lagrangian. This is what the Higgs mechanism achieves.

Thus we propose the Lagrangian density for Maxwell theory:
\[ \mathcal{L} \equiv \mathcal{L}_{\text{Maxwell}}(A, \partial A) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{e} A_\mu j^\mu \]  
(6.56)

The prefactors have been chosen such that the Euler-Lagrange equations match the IME (\( \rightarrow \) next step).

Euler-Lagrange equations:

Details: Problemset 7

There are four (\( \mu = 0, 1, 2, 3 \)) Euler-Lagrange equations:
\[ \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} \right) = 0 \]  
(6.57)
Straightforward calculations yield:

\[ \frac{\partial L}{\partial A_\mu} = -\frac{1}{c} j_\mu \quad \text{and} \quad \frac{\partial L}{\partial (\partial_\nu A_\mu)} = \frac{1}{4\pi} F^{\mu\nu} \]

(6.58)

Hence the Euler-Lagrange equations are exactly the inhomogeneous Maxwell equations:

\[ \partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \]

(6.59)

→ Eq. (6.56) is the correct Lagrangian density for Maxwell theory.

10 Coordinate-free notation:

Remember the coordinate-free concepts introduced in Chapter 3: All tensor fields \( T_{IJ} \) are the chart-dependent components of chart-independent objects \( T \) (the actual tensor fields). This formalism allows us to reformulate the Maxwell equations in the language of differential geometry, without using coordinates altogether:

i) First, write gauge field

\[ A := A_\mu dx^\mu \]

(6.60)

and the field strength coordinate-free:

\[ F := F_{\mu\nu} \ dx^\mu \otimes dx^\nu = \frac{1}{2} F_{\mu\nu} \left[ dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \right]. \]

(6.61)

We say that \( A \) is a 1-form and \( F \) is a 2-form.

ii) We can evaluate \( \text{exterior derivative} \) of the gauge field:

\[ dA \overset{\text{def}}{=} dA_\nu \wedge dx^\nu = \partial_\mu A_\nu \ dx^\mu \wedge dx^\nu = \frac{1}{2} F_{\mu\nu} \ dx^\mu \wedge dx^\nu = F \]

(6.62)

The exterior derivative \( d \) maps \( k \)-forms onto \( k+1 \)-forms.

iii) Now evaluate the exterior derivative of the field strength:

\[ dF \overset{\text{def}}{=} \frac{1}{2} \partial_\sigma F_{\mu\nu} \ dx^\sigma \wedge dx^\nu \wedge dx^\mu \]

(6.63a)

\[ = \frac{1}{6} \left( \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} \right) \ dx^\sigma \wedge dx^\nu \wedge dx^\mu \]

(6.63b)

\[ = \frac{1}{2} \sum_{\sigma=\nu<\mu} \left( \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} \right) \ dx^\sigma \wedge dx^\nu \wedge dx^\mu \]

(6.63c)

(Here we used the antisymmetry of the wedge product in all factors.)

Thus we find:

\[ dF = 0 \iff \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0 \quad \text{and} \quad \partial_\nu F^{\mu\nu} = 0 \]

(6.64)

If the field strength is expressed in terms of the gauge field, the homogeneous Maxwell equations \( \partial_\nu F^{\mu\nu} = 0 \) are identities. In the coordinate-free notation of differential geometry, this identity follows from the fact that applying an exterior derivative twice produces the zero field:

\[ dF = ddA = 0 \quad \text{since} \quad d^2 = 0 \]

(6.65)

The relation \( dF = 0 \) is known as a \( \text{Bianchi identity} \).
Define the linear Hodge star operator (here for a 4-dimensional Minkowski manifold):

\[ *(d\alpha^\mu) := \frac{1}{3!} \epsilon^\alpha_{\mu \nu \rho \sigma} (d\alpha^\nu \wedge d\alpha^\rho \wedge d\alpha^\sigma) \]  
(6.66a)

\[ *(d\alpha^\mu \wedge d\alpha^\nu) := \frac{1}{2!} \epsilon^{\mu \nu \rho} (d\alpha^\rho \wedge d\alpha^\sigma) \]  
(6.66b)

\[ *(d\alpha^\mu \wedge d\alpha^\nu \wedge d\alpha^\rho) := \frac{1}{1!} \epsilon^{\mu \nu \rho \sigma} (d\alpha^\sigma) \]  
(6.66c)

Note that the definition makes use of the metric tensor via pulling up/down indices of the Levi-Civita symbols. This implies in particular that any equation that uses the Hodge star depends on the geometry of spacetime (here flat Minkowski space).

The dual field-strength tensor (DFST) is the Hodge dual of the field-strength tensor (FST):

\[ *F = \frac{1}{2} F_{\mu \nu} \star (d\alpha^\mu \wedge d\alpha^\nu) \]  
(6.67a)

\[ = \frac{1}{4} F_{\mu \nu \rho \sigma} \epsilon^{\rho \sigma \alpha \beta} (d\alpha^\rho \wedge d\alpha^\sigma) \]  
(6.67b)

\[ = \frac{1}{2} \tilde{F}_{\rho \sigma} (d\alpha^\rho \wedge d\alpha^\sigma) \equiv \tilde{F} \]  
(6.67c)

Beware: The Hodge star \( \star \) is not a multiplication symbol (as the notation on the right-hand side might suggest) but a linear operator that acts on the differential form to the right.

The Hodge dual of the exterior derivative of the DFST yields:

\[ *(d \star F) = \frac{1}{4} \epsilon_{\mu \nu \rho \sigma} \partial_\pi F_{\mu \nu} \star (d\alpha^\pi \wedge d\alpha^\rho \wedge d\alpha^\sigma) \]  
(6.68a)

\[ = \frac{1}{4} \epsilon_{\mu \nu \rho \sigma} \epsilon^{\rho \sigma \alpha \beta} \partial_\alpha F_{\mu \nu} (d\alpha^\alpha) \]  
(6.68b)

\[ = \frac{1}{2} (\delta_\mu^\alpha \eta_{\nu \alpha} - \delta_\nu^\alpha \eta_{\mu \alpha}) \partial_\alpha F_{\mu \nu} (d\alpha^\alpha) \]  
(6.68c)

\[ = \eta_{\nu \alpha} \eta_{\mu \alpha} F_{\mu \nu} (d\alpha^\alpha) \]  
(6.68d)

\[ = \frac{4\pi}{c} j_\mu (d\alpha^\mu) \]  
(6.68e)

Here we used a contraction identity for Levi-Civita symbols (over the two red pairs of indices).

This motivates the definition of the coordinate-free current:

\[ J := \frac{4\pi}{c} j_\mu (d\alpha^\mu) \]  
(6.69)

In conclusion, the Maxwell equations can be written without using a coordinate system as:

**Homogeneous ME:** \( dF = 0 \)  
(6.70a)

**Inhomogeneous ME:** \( *d(*F) = J \)  
(6.70b)

- If one uses that \( (\star)^2 = +1 \cdot \mathds{1} \) on odd differential forms (\( d \star F \)) is a 3-form, Eq. (6.70b) can alternatively be written as \( d(\star F) \equiv \star J \). If one then defines the current not as a 1-form but as the dual 3-form, \( J := \frac{4\pi}{c} j_\mu \star d\alpha^\mu \), the inhomogeneous Maxwell equations take their simplest form: \( d(\star F) = J \).

- Eq. (6.70) is the most general and elegant formulation of the Maxwell equations. In this form, the equations remain valid even in **general relativity** on curved space times. Then the Minkowski metric used in the definition of the Hodge star \( \star \) (to pull the indices of the Levi-Civita symbols up/down) must be replaced by the dynamic, potentially curved metric of **general relativity**.
6.3. Noether theorem and the energy-momentum tensor

In the following, we consider first a generic (classical, relativistic) field theory, and specialize to electrodynamics later. This is to emphasize that most of the results in this chapter are not specific to electrodynamics.

Details: Chapter 1 of my QFT script [19]

1 | α General transformation of field $\phi \mapsto \phi'$:

\[ x \mapsto x' = x'(x) \quad \text{and} \quad \phi(x) \mapsto \phi'(x') = \mathcal{F}(\phi(x)) \tag{6.71} \]

Two effects: coordinates and (values of the) field transformed

These are active transformations that change physics. $x' = x'(x)$ is not a (passive) coordinate transformation; the frame of reference remains fixed in the following!

Example 6.1: Homogeneous Lorentz transformations

The (active) homogeneous Lorentz transformation of a vector field $A^\mu$ reads

\[ x^\mu \mapsto x'^\mu = \Lambda^\mu_v x^v \quad \text{and} \quad A^\mu_v(x) \mapsto A'^\mu_v(x') = \frac{\Lambda^\mu_v}{\mathcal{F}(A^\mu_v(x))} A^\mu_v(x) \tag{6.72} \]

whereas the Lorentz transformation of a scalar field $\phi$ reads

\[ x^\mu \mapsto x'^\mu = \Lambda^\mu_v x^v \quad \text{and} \quad \phi(x) \mapsto \phi'(x') = \frac{\phi(x)}{\mathcal{F}(\phi(x))} \tag{6.73} \]

2 | α Infinitesimal transformations (IT) ($|w_a| \ll 1$):

\[ x'^\mu = x^\mu + w_a \delta^a x^\mu(x) \quad \text{and} \quad \phi'(x') = \phi(x) + w_a \delta^a \phi(x) \tag{6.74} \]

Here, $w_a$ denotes infinitesimal parameters of the transformation (sum over $a$ implied!) and we label different transformations by the labels $a$.

Example 6.2: Homogeneous Lorentz transformations

Infinitesimal homogeneous Lorentz transformations take the form (α Problemset 4)

\[ \Lambda_w = \exp \left( -i \frac{1}{2} w_{a\beta} \eta^{a\beta} \right) \approx 1 - i \frac{1}{2} w_{a\beta} \eta^{a\beta} \tag{6.75} \]

(note that the $a = \alpha\beta$ are labels of generators that are not required to be tensor indices)

with generators

\[ (\eta^{a\beta})^\mu_v = i \left( \delta^{a\beta}_{\delta^\alpha_{\gamma\mu}} - \delta^{a\beta}_{\delta^\alpha_{\gamma\mu}} \right). \tag{6.76} \]

With this it follows for the coordinates

\[ w_{a\beta} \delta^{a\beta} x^\mu = x'^\mu - x^\mu = -\frac{i}{2} w_{a\beta} (\eta^{a\beta})^\mu_v x^v = w_{a\beta} \frac{1}{2} \left( \delta^{a\beta}_{\delta^\alpha_{\gamma\mu}} - \delta^{a\beta}_{\delta^\alpha_{\gamma\mu}} \right) x^v \tag{6.77} \]
so that
\[
\delta^\alpha_\beta x^\mu = \frac{1}{2} \left( \eta^\alpha_\beta x^\mu - \eta^\beta_\mu x^\alpha \right) .
\] (6.78)

Similar arguments yield \( \delta^\alpha_\beta A^\mu = \frac{1}{2} \left( \eta^\alpha_\beta A^\mu - \eta^\beta_\mu A^\alpha \right) \) for a vector field and \( \delta^\alpha_\beta \phi = 0 \) for a scalar field.

3 | Generator of IT:

\[
\delta_w \phi(x) := \phi'(x) - \phi(x) \equiv -i w^\alpha G_\alpha \phi(x)
\] (6.79)

With (omit first line and refer to previous equation)

\[
\phi'(x') = \phi(x) + w^\alpha \delta_\alpha \phi(x)
\] (6.80a)

\[
= \phi(x') - w^\alpha (\delta_\alpha x^\mu) \partial_\mu \phi(x') + w^\alpha \delta_\alpha \phi(x') + O(w^2)
\] (6.80b)

(Here we replaced \( x \) by \( x' \) in the last term because this is a \( O(w^2) \) modification.)

it follows (replace \( x' \) by \( x \); these are just labels!)

\[
i G_\alpha \phi = (\delta_\alpha x^\mu) \partial_\mu \phi - \delta_\alpha \phi
\] (6.81)

This function describes the infinitesimal change of the field at the same point.

Example 6.3: Translations

i | \( x'^\mu := x^\mu + w^\mu = x^\mu + w^\nu \delta_\nu x^\mu \) with \( \delta_\nu x^\mu = \delta_\nu^\mu \)

ii | \( \delta_\nu \phi = 0 \) (This is true for scalar and vector fields.)

iii | \( i G_\mu \phi = \delta_\mu^\nu \partial_\nu \phi - 0 \) and therefore

\[
G_\mu = -i \partial_\mu \equiv P_\mu
\] (6.82)

\( \rightarrow \) The “momentum operator” generates translations.

4 | So far the continuous transformations \( \phi \mapsto \phi' \) were arbitrary.

Continuous transformation [with infinitesimal form Eq. (6.74)] which is a

Symmetry of the action \( : \Rightarrow S[\phi] = S[\phi'] \) (6.83)

In principle, the action can vary by a surface term – equivalently, the Lagrangian density \( \mathcal{L} \) can vary by a 4-divergence \( \partial_\mu K^\mu(\phi, x) \) – under the symmetry transformation (because such modifications do not affect the equations of motion). Here we consider for simplicity only the case where no such terms exist and the action is strictly invariant.

Then one can prove (see Chapter 1 of my QFT script [19] or Refs. [1, 77]):
Noether's (first) theorem:

For solutions $\phi$ of the equations of motion, the $*$ (canonical) ($Noether$) currents

$$j^\mu_a = \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \frac{\partial L}{\partial \phi} \right\} \delta_a x^\nu - \frac{\partial L}{\partial (\partial_\mu \phi)} \delta_a \phi$$

(associated to the infinitesimal transformations of coordinates $\delta_a x^\nu$ and fields $\delta_a \phi$)

satisfy the continuity equations

$$\forall a: \quad \partial_\mu j^\mu_a = 0 \quad .$$

This means there is one conserved current $j^\mu_a$ for each generator $a$ of the continuous symmetry.

Conserved charge:

The currents Eq. (6.84) are called "conserved" because they describe the flow of a conserved...

$$Q_a := \int_{\text{Space}} d^{D-1}x \ j^0_a \quad * \ (Noether) \ charge$$

There is one conserved charge $Q_a$ for each generator $a$ of the continuous symmetry.

Indeed:

$$\frac{1}{c} \frac{dQ_a}{dt} = \int_{\text{Space}} d^{D-1}x \ \partial_\nu j^0_a \overset{6.85}{=} -\int_{\text{Space}} d^{D-1}x \ \partial_k j^k_a \overset{\text{Gauss}}{=} -\int_{\text{Surface}} d\sigma_k j^k_a = 0$$

Here we assume that $j^k_a \equiv 0$ on the spatial boundaries—typically at infinity, i.e., the universe is closed. $k = 1, 2, 3$ denotes the spatial coordinates.

\begin{itemize}
  \item \textbf{Note 6.1}
\end{itemize}

The current Eq. (6.84) is called \textit{canonical} current because it is not unique:

$$\tilde{j}^\mu_a := j^\mu_a + \partial_\nu B^{\mu\nu}_a \quad \text{with} \quad B^{\mu\nu}_a = -B^{\nu\mu}_a \quad \text{arbitrary} \quad \Rightarrow \quad \partial_\mu \tilde{j}^\mu_a = 0$$

This is particularly important for the energy-momentum tensor ($\rightarrow$ below).

6.3.1. Application: The Energy-Momentum Tensor (EMT)

Details: \textit{Problemset 7}

\begin{itemize}
  \item \textit{Infinitesimal} spacetime translations:
    \begin{align*}
    x'^\mu &= x'^\mu + u'^\mu \\
    \delta_\nu x'^\mu &= \delta_\nu^\mu \quad \text{and} \quad \delta_\nu \phi = 0
    \end{align*}

  \item Translation-invariant action: $S' = S$ (This includes translations in time!)
\end{itemize}
Conserved currents: Eq. (6.84) →

\[
\Theta^\mu := \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} \right\} \frac{\partial L}{\partial \phi} = \frac{\partial L}{\partial (\partial_\mu \phi)} \frac{\partial}{\partial \phi} - \delta^\mu_\rho \frac{\partial L}{\partial \phi} \tag{6.90}
\]

Note that the generator index \( \alpha \) is in this case a proper Lorentz index so that we can pull it up, \( \Theta^\mu = \eta^{\mu \nu} \Theta^\nu \), and obtain:

**Notation: (Canonical) Energy-Momentum Tensor:**

\[
\Theta^{\mu \nu} = \frac{\partial L}{\partial (\partial_\mu \phi)} \frac{\partial}{\partial \phi} - \eta^{\mu \nu} L \tag{6.91}
\]

with

\[
\partial_\mu \Theta^{\mu \nu} = 0 \quad \text{and four conserved charges} \quad P^\nu := \frac{1}{c} \int d^3x \Theta^{0 \nu}. \tag{6.92}
\]

- Note that these quantities are only conserved for solutions of the Euler-Lagrange equations.
- \( P^\nu \) is a 4-vector (show this!). Note that this is a non-trivial statement because \( d^3x \) is not a Lorentz scalar and \( \Theta^{0 \nu} \) not a 4-vector.
- The prefactor \( 1/c \) ensures that \( P^0 \) has the same dimension as a conventional 4-momentum with \( p^0 = E/c \); note that \( \Theta^{00} \) has the dimension of an energy density because \( L \) has this dimension.

**Interpretation:**

1. **Energy** (\( \nu = 0 \)):

\[
cP^0 = \int d^3x \Theta^{00} = \int d^3x \left\{ \frac{\partial L}{\partial \phi} - L \right\} = \int d^3x \mathcal{H}(\phi, \pi) = H \tag{6.93}
\]

→ The Hamiltonian is the component of a 4-vector and not Lorentz invariant!

By contrast, the Lagrangian is Lorentz invariant (for relativistic field theories).

2. **Kinetic momentum** (\( \nu = \lambda \)):

\[
P^\lambda = \int d^3x \Theta^{0 \lambda} = \int d^3x \frac{\partial L}{\partial \phi} (-\delta_{\lambda} \phi) = - \int d^3x \pi \delta_\lambda \phi \tag{6.94}
\]

\( \pi \) is the canonical momentum conjugate to the field \( \phi \).

**The canonical EMT of electrodynamics:**

1. **Free (\( j^\mu = 0 \)) electromagnetic field**: \( \mathcal{L}_{em} = -\frac{1}{16\pi} F_{\mu \nu} F^{\mu \nu} \)

→ Invariant under spacetime translations

Indeed, with \( x^\mu = x^\mu + u^\mu \) and the field transformation \( A'_\mu(x) := A_\mu(x - w) \) it is

\[
S_{em}[A'] = \int d^4x \mathcal{L}_{em}(A'(x), \partial A'(x)) = \int d^4x \mathcal{L}_{em}(A(x - w), \partial A(x - w)) \tag{6.95a}
\]

\[
= \int d^4y \mathcal{L}_{em}(A(y), \partial A(y)) = S_{em}[A] \tag{6.95b}
\]

where we integrate over the full Minkowski spacetime \( \mathbb{R}^{1,3} \), substituted \( y^\mu = x^\mu - w^\mu \) and used \( d^4x = d^4y \).
ii | → Canonical EMT conserved: \( \partial_\mu \Theta^\mu_\nu_{\text{em}} = 0 \) with

\[
\Theta^\mu_\nu_{\text{em}} = \frac{\partial \mathcal{L}_{\text{em}}}{\partial (\partial_\mu A_\sigma)} \partial^\nu A_\sigma - \eta^{\nu\lambda} \mathcal{L}_{\text{em}} \overset{6.58}{=} \frac{1}{4\pi} F^{\sigma\mu} \partial^\nu A_\sigma + \frac{\eta^{\mu\nu}}{16\pi} F^{\sigma\rho} F^{\sigma\rho}
\]  

(6.96)

Note that because the gauge field has multiple components \( A_\mu \), there is now an additional summation in the first term over these components (marked indices). This follows directly from a generalization of the proof of Noether’s theorem for fields with multiple components.

Details: \( \Box \) Problemset 7

iii | Problems:

The canonical EMT \( \Theta^\mu_\nu_{\text{em}} \) has two problematic properties:

- Because of the term \( \partial^\nu A_\sigma \), \( \Theta^\mu_\nu_{\text{em}} \) is gauge-dependent!

  This is problematic because it means that we cannot hope to identify physical quantities like the energy density or the momentum density of the electromagnetic field with (the components) of this tensor.

- The canonical EMT is non-symmetric: \( \Theta^\mu_\nu_{\text{em}} \neq \Theta^\nu_\mu_{\text{em}} \)!

In general relativity, we will find that the right-hand side of the → Einstein field equations (which determine how spacetime curves and evolves)

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}
\]

(6.97)

is given by the → Hilbert energy-momentum tensor

\[
T^{\mu\nu} = \frac{2}{\sqrt{g}} \delta(\mathcal{L}_{\text{Matter}}) / \delta g_{\mu\nu}
\]

(6.98)

where \( \mathcal{L}_{\text{Matter}} \) describes the Lagrangian density of all fields in the universe (except the metric tensor field). For example, \( \mathcal{L}_{\text{Matter}} \) contains the Maxwell Lagrangian \( \mathcal{L}_{\text{em}} \) (“matter” here includes every degree of freedom that has energy & momentum, i.e., also electromagnetic radiation).

Note that \( T^{\mu\nu} \) is symmetric because the metric \( g_{\mu\nu} \) is. Hence it cannot be identified with the canonical EMT \( \Theta^\mu_\nu_{\text{em}} \) in general (here for the example of Maxwell theory).

\( \Box \) These problems are not specific to electrodynamics but typically affect all theories that are gauge theories and/or include non-scalar fields.

→ How to solve these issues?

6.3.2. The Belinfante-Rosenfeld energy-momentum tensor (BRT)

We consider again first a generic field theory, and specialize to electrodynamics later.

Details: \( \Box \) Problemset 7

11 | Remember (Note 6.1) that the canonical EMT is not the only conserved EMT because

\[
\tilde{\Theta}^{\mu\nu} := \Theta^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} \quad \text{with} \quad K^{\rho\mu\nu} = -K^{\mu\rho\nu}
\]

(6.99)

yields another EMT \( \tilde{\Theta}^{\mu\nu} \) for any suitable tensor \( K^{\rho\mu\nu} \).

→ Idea: Find \( K^{\rho\mu\nu} \) such that \( \tilde{\Theta}^{\mu\nu} = \tilde{\Theta}^{\nu\mu} \) is symmetric (and hopefully gauge-invariant).
Let us assume that our theory is also invariant under homogeneous Lorentz transformations (in addition to the spacetime translations needed for the conservation of the EMT).

Generators of homogeneous LTs for coordinates:

\[ \delta^{a\beta} x^\mu = \frac{1}{2} \left( \eta^{a\mu} x^\beta - \eta^{\beta\mu} x^a \right) \]  

(6.100)

Assume that fields transform as \( \delta^{a\beta} \phi \).

For the following arguments, we do not need to fix whether our fields transform as scalar, vector, or even spinor fields.

Eq. (6.84) & Eq. (6.91) & Eq. (6.100) → 

Noether currents for homogeneous LTs:

\[ L^{\mu a\beta} = \frac{1}{2} \left( \omega^{a\mu} x^\beta - \omega^{\beta\mu} x^a \right) + \frac{1}{2} S^{\mu a\beta} \]  

(6.101)

with

\[ \text{Spin current: } S^{\mu a\beta} := -2 \frac{\partial L}{\partial (\partial_\mu \phi)} \delta^{a\beta} \phi \]  

(6.102)

which satisfies \( S^{a\beta} = -S^{\beta a} \).

(This follows because \( \delta^{a\beta} \phi = -\delta^{\beta a} \phi \) as the generators of homogeneous LTs are antisymmetric.)

The continuity equation reads

\[ \partial_\mu L^{\mu a\beta} = 0 \]  

(6.103)

Because homogeneous LTs describe rotations in space and time, the conserved current \( L^{\mu a\beta} \) can be identified as \( \text{† (canonical) angular momentum current}. \) The first part in Eq. (6.101) corresponds to the (canonical) orbital angular momentum while the second part \( S^{\mu a\beta} \) encodes the intrinsic angular momentum of the field (= its \( \text{spin} \)). This immediately explains why for a scalar field with \( \delta^{a\beta} \phi = 0 \), the spin current vanishes \( S^{a\beta} = 0 \).

Eq. (6.92) & Eq. (6.103) → 

\[ \partial_\mu S^{\mu a\beta} \equiv \Theta^{a\beta} - \Theta^{\beta a} \]  

(6.104)

This means that a non-vanishing divergence in the spin current is responsible for the “non-symmetry” of the canonical EMT!

Now define

\[ K^{\rho \mu \nu} := -\frac{1}{2} \left( S^{\mu \nu \rho} + S^{\nu \mu \rho} - S^{\rho \nu \mu} \right) \]  

(6.105)

\[ \rightarrow K^{\rho \mu \nu} = -K^{\mu \rho \nu} \]  

(This follows from \( S^{a\beta} = -S^{\beta a} \).)

With this we can finally define the …

\[ \text{Belinfante-Rosenfeld energy-momentum tensor (BRT):} \]

\[ T^{\mu \nu} \equiv \Theta^{\mu \nu} + \partial_\rho K^{\rho \mu \nu} := \Theta^{\mu \nu} - \frac{1}{2} \partial_\rho \left( S^{\mu \nu \rho} + S^{\nu \mu \rho} - S^{\rho \mu \nu} \right) \]  

(6.106)
It remains to be shown that $T^{\mu\nu}$ is always symmetric:

$$T^{\mu\nu} - T^{\nu\mu} \stackrel{6.104}{=} 0 \quad \odot$$

(6.107)

It can be rigorously shown that the BRT is identical to the Hilbert EMT that shows up in General Relativity as the source of gravity [78]. This is why the BRT gets its own symbol $T^{\mu\nu}$. 
16 | The BRT of electrodynamics:

Details: Problemset 7

i | Using \( \mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \) and the transformation of a vector field (\( = \text{spin-1} \))

\[
\delta^\alpha_\beta A_\mu = \frac{1}{2} \left( \delta^\alpha_\mu A_\beta - \delta^\beta_\mu A_\alpha \right)
\]

(6.108)

in Eq. (6.102) yields the spin current:

\[
S_{\mu\alpha\beta}^{\text{em}} = \frac{1}{4\pi} \left( F^{\mu\alpha} A_\beta - F^{\mu\beta} A_\alpha \right)
\]

(6.109)

ii | Eq. (6.96) & Eq. (6.106) & Eq. (6.109) →

\[
T_{\mu\nu}^{\text{em}} = \frac{1}{4\pi} F^{\mu}_\rho F^{\rho\nu} - \eta^{\mu\nu} \mathcal{L}_{\text{em}}
\]

(6.110a)

\[
= \frac{1}{4\pi} \left[ F^{\mu}_\rho F^{\rho\nu} + \frac{\eta^{\mu\nu}}{4} F^{\rho\sigma} F_{\rho\sigma} \right]
\]

(6.110b)

\[
= \left( \frac{\mathcal{E}}{c} \frac{c}{\bar{\Pi}} \begin{pmatrix} \Sigma \end{pmatrix} \right)_{\mu\nu}
\]

(6.110c)

To show this you have to use the Maxwell equations in vacuum: \( \partial_\nu F^{\nu}_\mu = 0 \).

Components:

- Energy density:
  \[
  \mathcal{E} = \frac{1}{8\pi} (\bar{E}^2 + \bar{B}^2)
  \]
  (6.111a)

- Momentum density:
  \[
  \bar{\Pi} = \frac{1}{4\pi c} (\bar{E} \times \bar{B})
  \]
  (6.111b)

- Maxwell stress tensor:
  \[
  \Sigma_{ij} = \frac{1}{4\pi} \left[ \frac{\delta_{ij}}{2} (\bar{E}^2 + \bar{B}^2) - E_i E_j - B_i B_j \right]
  \]
  (6.111c)

\(!\) Convince yourself that \( T_{\mu\nu}^{\text{em}} \) is symmetric and gauge invariant. Note that we did not construct it to be gauge invariant, only to be symmetric! We got this as a bonus.

iii | The conservation \( \partial_\mu T^{\mu\nu} = 0 \) of the BRT implies the following physical interpretations:

- \( \nu = 0 \):

\[
\partial_\mu T^{\mu\alpha} = \frac{1}{c} \frac{\partial \mathcal{E}}{\partial t} + c \nabla \cdot \bar{\Pi} = 0
\]

(6.112)

\[\rightarrow \downarrow \text{Poynting’s theorem} \text{ (in vacuum)}\]

\[
\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \bar{\mathcal{S}} = 0
\]

(6.113)
with

\[ Poynting \text{ vector: } \vec{S} = c^2 \vec{\Pi} = \frac{c}{4\pi} (\vec{E} \times \vec{B}) \]  \hspace{1cm} (6.114)

Eq. (6.113) → Poynting vector = Energy current density

This is simply the formal statement of energy conservation for the free electromagnetic field. As energy is the Noether charge for translations in time, it is of course no coincidence that the Poynting theorem follows from the time-component \( \nu = 0 \).

- \( \nu = i \):

\[ \partial_{\mu} T^{\mu i} = \frac{\partial \Pi_i}{\partial t} + \partial_k \Sigma_{ki} = 0 \]  \hspace{1cm} (6.115)

→ Conservation of momentum with ...
- \( \Pi_i \): \( i \)-momentum density
- \( \Sigma_{ki} \): \( i \)-momentum current density

→ Maxwell stress tensor = Momentum current density

Note that there are three momentum densities and corresponding current densities because there are three spatial momenta: \( i = x, y, z \).

Some final remarks:

- With the symmetric BRT one can define a gauge-invariant and conserved angular momentum tensor

\[ M^{\rho\mu\nu} := T^{\rho\mu} x^\nu - T^{\rho\nu} x^\mu \]  \hspace{1cm} (6.116)

with \( \partial_{\rho} M^{\rho\mu\nu} = 0 \) (show this!). The conserved Noether charges are

\[ J^{\mu\nu} := \frac{1}{c} \int d^3 x \ M^{0\mu\nu} = \frac{1}{c} \int d^3 x \left( T^{0\mu} x^\nu - T^{0\nu} x^\mu \right) \]  \hspace{1cm} (6.117)

which encodes the total angular momentum of the field. Indeed, for the spatial components one finds

\[ J_{ij} := \int d^3 x \left( \Pi_i x_j - \Pi_j x_i \right) \]  \hspace{1cm} (6.118)

Since \( \Pi_i \) is the momentum density, the three components \( J_x = J_32, J_y = J_13 \) and \( J_z = J_21 \) can be identified as the total angular momentum \( J \) of the field.

- If the electric current \( j^\mu \) does not vanish (i.e., the field is not in vacuum), the BRT derived above is no longer conserved. Rather one finds

\[ \partial_{\mu} T^{\mu e}_m = -c^{-1} F^{\nu\rho} j_\rho \]  \hspace{1cm} (6.119)

which can be identified as the Lorentz force density. This is perfectly reasonable as an external (non-dynamic) current \( j^\mu \) breaks the translation symmetry of the system in space and time on which the conservation of the BRT relies. Physically, the electromagnetic field is no longer a closed system because it can exchange momentum and energy with the charges described by \( j^\mu \). Only if one describes the charges as dynamic degrees of freedom (→ next section) and considers the total BRT

\[ T^{\mu\nu} = T^{\mu\nu}_e + T^{\mu\nu}_\text{charges} \]  \hspace{1cm} (6.120)

one would recover the conservation \( \partial_{\mu} T^{\mu\nu} = 0 \); this is then a statement about total energy and momentum conservation, including the energy and momentum of the charges.
6.4. Charged point particles in an electromagnetic field

1 | \(< N\) charged point particles with charge \(q_i\) and mass \(m_i\) in an EM field \(A_\mu\):

\[
S \left[ \{ x_k \}, A \right] = \int d^4x \left[ -\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c^2} A_\mu j^\mu \right] - \sum_{i=1}^{N} m_i c^2 \int dq_i \delta \left( x_i - x_i \right)
\]

Eq. (6.56) & Eq. (5.41) \rightarrow Relativistic action of the complete system:

Note that the Lagrangian is a Lorentz scalar! \(S \left[ \{ x_k \}, A \right]\) is short for \(S \left[ x_1, \ldots, x_N, A \right]\).

with current density

\[
j^\mu (x) = \sum_i \rho_i (x) \frac{dx_i^\mu}{dt} = \sum_i q_i \delta (\vec{x} - \vec{x}_i) \frac{dx_i^\mu}{dt}.
\]

2 | Coupling:

\[
S_c \left[ \{ x_k \}, A \right] = -\frac{1}{c^2} \int d^4x A_\mu (x) j^\mu (x) \pm \sum_i \left\{ \frac{q_i}{c} \int dx_i \frac{dx_i^\mu}{dt} \right\}
\]

Here we used \(\frac{dx_i^\mu}{dt} = dx_i^\mu\); the last integral is therefore a four-dimensional \(\text{line integral}\) of the 4-vectorfield \(A_\mu\) along the trajectory of particle \(i\).

3 | Hamilton’s principle:

\[
\delta S \left[ x_k, A \right] = 0 \iff \left\{ \begin{array}{l}
\frac{\delta S_{\text{em}} [A]}{\delta A} + \frac{\delta S_c \left[ \{ x_k \}, A \right]}{\delta A} = \frac{\delta S_j \left[ A \right]}{\delta A} = 0 \\
n_{i} : \frac{\delta S_c \left[ x_i, A \right]}{\delta x_i} + \frac{\delta S_{\text{p}} \left[ x_i \right]}{\delta x_i} = \frac{\delta S_A \left[ \{ x_k \} \right]}{\delta x_i} = 0
\end{array} \right. \]

(6.124)
4 | \(<\) Gauge field variations $\delta A$:
Here we don’t have to do anything because we already computed the Euler-Lagrange equations:

$$\frac{\delta S_A[A]}{\delta A} = 0 \iff \partial_\nu F_{\nu\mu} = \frac{4\pi}{c} \sum_i q_i \delta(\vec{x} - \vec{x}_i) \frac{dx_i}{dt}$$ (6.125)

These are the inhomogeneous Maxwell equations with the $N$ point particles as sources of the EM field. Note that this PDE system couples the particle coordinates $\{x_k^\mu\}$ to the EM field $A^\mu$.

5 | \(<\) Particle trajectory variations $\delta x_i$:

\(i\) | Eqs. (6.121) and (6.123) →

$$S_A[x_k] = -\sum_i \int \left[ m_i c \sqrt{\dot{x}_i^\mu \dot{x}_i^\mu} + \frac{q_i}{c} A_\mu(x_i) \dot{x}_i^\mu \right] d\lambda$$ (6.126)

Note that this action is again reparametrization invariant.

\(\Rightarrow\) Euler-Lagrange equation for particle $i$:

$$\frac{\delta S_A[x_k]}{\delta x_i} = 0 \iff \frac{d}{d\lambda} \left[ m_i c \dot{x}_i^\mu + \frac{q_i}{c} A_\mu(x_i) \dot{x}_i^\mu \right] + \frac{q_i}{c} \left[ \ddot{x}_i^\mu(x_i) - \dot{x}_i^\mu(\partial A_\mu(\dot{x}_i^\mu))/\partial x^\mu \right] = 0$$ (6.127)

\(\Rightarrow\) Choose proper-time parametrization $\lambda = \tau$:

$$m_i \frac{dx_i^\mu}{d\tau} + \frac{q_i}{c} \left( \frac{dA_\mu}{d\tau} - \frac{\partial A_\mu}{\partial x^\mu} \frac{dx^\nu}{d\tau} \right) = 0$$ (6.128)

Thus we find as the EOM for particle $i$:

$$m_i \frac{d\mu_\mu}{d\tau} = \frac{q_i}{c} \left( \frac{\partial A_\mu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) u_\nu$$ (6.129)

Or in the form discussed previously in Chapter 5 (we restore the particle index $i$):

$$\frac{dp_i^\mu}{d\tau} = \frac{q_i}{c} F_{\mu\nu}(x_i) u_i^\nu$$ (6.130)

with 4-momentum $p_i^\mu = m_i u_i^\mu$.

\(\Rightarrow\) The field strength tensor is evaluated at the position of the particle at a given time.

\(ii\) | Compare Eqs. (5.6) and (6.130) → 4-force:

$$K^\mu = \left( \begin{array}{c} \gamma_F \vec{E} \\ \gamma_F \vec{F} \end{array} \right) = \frac{q_i}{c} F_{\mu\nu} \frac{dx^\nu}{dt}$$ (6.131)
→ 3-force (we restore the particle index $i$):

\[
\vec{F}_i = q_i \vec{E}_i + q_i \left( \vec{v}_i \times \vec{B}_i \right) \quad \text{Lorentz force}
\]  

(6.132)

with $\vec{E}_i = \vec{E}(x_i)$, $\vec{B}_i = \vec{B}(x_i)$ and $\vec{v}_i = \frac{d\vec{x}_i}{dt}$.

- This result demonstrates that our concept of the relativistic 3-force introduced in Eq. (5.11) was reasonable: for a force due to an electromagnetic field, it exactly matches the Lorentz force.
- It also demonstrates that the common expression for the Lorentz force is already fully relativistic. However, note that the 3-force determines the change rate of the relativistic 3-momentum $\vec{p} = \gamma \mu \vec{v}$, recall Eq. (5.16).

6 | Comments:

- Eqs. (6.125) and (6.130) together are the equations of motion of the composite system, i.e., the EM field and the $N$ particles. Note that the system of differential equations is coupled: The dynamical positions of the particles determine the evolution of the EM field via Eq. (6.125), and the dynamical EM field determines the trajectories of the charged particles via Eq. (6.130).
- This model of $N$ charged particles interacting with and via an electromagnetic field is the culmination of our discussion of relativistic mechanics in Chapter 5 and electrodynamics in Chapter 6.
- The theory Eqs. (6.125) and (6.130) is fully relativistic as the EOMs are manifestly Lorentz covariant (they are tensor equations).
- Note that this model describes interactions between the $N$ particles not directly via forces (as one would in Newtonian mechanics), but via coupling to the dynamic EM field. Thus a particle can locally affect the EM field due to its motion, the EM field then can propagate with the speed of light through space and affect the trajectory of any other particle within the lightcone of the first. There is no instantaneous interaction between the particles!
- One can also consider the $\mu = 0$ component of Eq. (6.130). Then one finds with $p^0_i = E_i/c$:

\[
\frac{dE_i}{dt} \equiv q_i \vec{E}_i \cdot \vec{v}_i .
\]

(6.133)

This is just the statement that the change of energy for particle $i$ is given by the distance it travels collinear with the electric field per time. This is no surprise: The Lorentz force Eq. (6.132) tells us that the force due to the magnetic field is always perpendicular to the direction of motion and therefore cannot not perform work on the particle.

7 | Corollary: Single particle in a static electromagnetic field:

i | The action follows from Eq. (6.126) with $N = 1$ as:

\[
S_A[x] = \int d\lambda L(x^\mu, \dot{x}^\mu) = -\int \left[ mc \sqrt{x^\mu x_\mu} + q A_\mu(x) \dot{x}^\mu \right] d\lambda
\]

(6.134)

where $A_\mu$ is a fixed parameter (the static field configuration).

ii | Parametrization in coordinate time $\lambda = t$:

\[
L(\tilde{x}, \tilde{v}) = -mc^2 \sqrt{1 - \tilde{v}^2} + q A \cdot \tilde{v} - q \tilde{\phi}
\]

(6.135)

with $A_\mu = (\phi, -\vec{A})$ (covariant!) and $\tilde{x} = \tilde{v}$.
### Canonical momentum:

\[
\pi := \frac{\partial L}{\partial \dot{v}} \equiv m v_c \tilde{v} + \frac{q}{c} \tilde{A}
\]  

(6.136)

with mechanical momentum \( \tilde{p} = m v_c \tilde{v} \rightarrow \)

\[
\tilde{p} = \pi - \frac{q}{c} \tilde{A}
\]  

(6.137)

\( \tilde{p} \): Measurable momentum
\rightarrow Mechanical momentum \( \tilde{p} \) gauge-invariant
\rightarrow Canonical momentum \( \pi \) not gauge-invariant

### Hamiltonian:

\[
H = \pi \cdot \tilde{v} - L \equiv \sqrt{\frac{m c^2}{1 - \frac{v^2}{c^2}}} + q \varphi = \sqrt{\left(\pi - \frac{q}{c} \tilde{A}\right)^2 + m^2 c^2 + q \varphi}
\]  

(6.138)

so that

\[
E = H - q \varphi
\]  

(6.139)

\( E \) is gauge invariant \rightarrow \( H \) is not gauge invariant

### Summary:

Gauge invariant
\[
\left\{ \begin{array}{ll}
E = H - q \varphi & \Leftrightarrow E + q \varphi = H \\
\tilde{p} = \pi - \frac{q}{c} \tilde{A} & \Leftrightarrow \tilde{p} + \frac{q}{c} \tilde{A} = \pi
\end{array} \right.\]

\]  

(6.140)

For more details on the aspect of the gauge-(in)variance of certain quantities, see Ref. [79].
Note that these subtleties are not specific to a relativistic treatment, they already appear in
Newtonian mechanics (only the specific dependency of the Hamiltonian on the mechanical/-
canonical momentum and the functional form of the Lagrangian are relativistic).
6.5. Summary: The many faces of Maxwell’s equations

Here is a compact overview over the many (physically equivalent) forms of Maxwell’s equations that we encountered in this chapter:

<table>
<thead>
<tr>
<th>Magnetic Gauss</th>
<th>Maxwell-Faraday</th>
<th>Electric Gauss</th>
<th>Ampère</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{H}^1 ): ( \nabla \cdot \mathbf{B} = 0 )</td>
<td>( \mathbf{H}^2 ): ( \nabla \times \mathbf{E} + \frac{1}{c} \partial_t \mathbf{B} = 0 )</td>
<td>( \mathbf{I}^1 ): ( \nabla \cdot \mathbf{E} = \frac{4\pi}{c} \mathbf{j} )</td>
<td>( \mathbf{I}^2 ): ( \nabla \times \mathbf{B} - \frac{1}{c} \partial_t \mathbf{E} = \frac{4\pi}{c} \mathbf{j} )</td>
</tr>
</tbody>
</table>

Not manifest Lorentz covariant

Manifest Lorentz covariant

6.10

\[ \mathbf{H} = \mathbf{H}^1 + \mathbf{H}^2 \]

6.14

\[ \nabla^2 \varphi + \frac{1}{c} \partial_t (\nabla \cdot \mathbf{A}) = -4\pi \rho \]

\[ \nabla^2 \mathbf{A} - \frac{1}{c^2} \partial^2 \varphi \mathbf{A} = -\frac{4\pi}{c} \mathbf{j} \]

Introduce Gauge fields:
\( \mathbf{E} = -\nabla \varphi - \frac{1}{c} \partial_t \mathbf{A} \)
\( \mathbf{B} = \nabla \times \mathbf{A} \)

Check consistency
Derivation

6.50

\[ \partial_{\nu} F^\mu\nu = 0 \]

\[ \partial_{\nu} F^\mu\nu = -\frac{4\pi}{c} \mathbf{j}^\mu \]

Use
\( F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \)

Use
\( F = dA \)
and \( d^2 = 0 \)

6.60

\[ \mathbf{I} = \mathbf{I}^1 + \mathbf{I}^2 \]

6.49

\[ \partial^2 A^\mu - \partial^\mu (\partial A) = \frac{4\pi}{c} \mathbf{j}^\mu \]

Introduce (Dual) Field strength tensor:
\( F^\mu_{\nu} = \partial_{\nu} A^\mu - \partial_{\mu} A^\nu \)

Use
\( F^\mu_{\nu} = \frac{1}{2} \epsilon_{\nu\alpha\beta\gamma} F^\alpha_{\beta\gamma} \)

Identify
4-current:
\( j^\mu = (c\rho, \mathbf{j}) \)

4-potential:
\( A^\mu = (\varphi, \mathbf{A}) \)

Fix the Lorenz gauge:
\( \frac{1}{c} \partial_t \varphi + \nabla \cdot \mathbf{A} = 0 \)

6.16

\[ \left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) \varphi = \frac{4\pi}{c} \rho \]

\[ \left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) \mathbf{A} = \frac{4\pi}{c} \mathbf{j} \]
7. Relativistic Field Theories II: Relativistic Quantum Mechanics

Reminder

1 | The Schrödinger equation (SE)

\[ i \hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) = \hat{H} \psi(t, \vec{x}) \]  

(7.1)

is a linear field equation with Hamilton operator

\[ \hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{x}) = -\frac{\hbar^2}{2m} \Delta + V(\vec{x}) \]  

(7.2)

and the complex-valued field \( \psi : \mathbb{R}^{1,3} \rightarrow \mathbb{C} \).

It describes the time evolution of a single quantum particle with mass \( m \) in a potential \( V(\vec{x}) \) that is initially described by the wavefunction \( \psi_0(\vec{x}) = \psi(0, \vec{x}) \) at \( t = 0 \).

2 | The wavefunction has the interpretation

\[ |\psi(t, \vec{x})|^2 = \langle \text{Probability to find particle at time } t \text{ at position } \vec{x} \rangle \]  

(7.3)

which necessitates the normalization condition

\[ \forall t : \quad \| \psi(t) \|_2 := \int d^3x |\psi(t, \vec{x})|^2 = 1. \]  

(7.4)

Thus the wavefunction is an element of the Hilbert space \( \psi \in L^2 = L^2(\mathbb{R}^3, \mathbb{C}) \) of square-integrable functions.

The Hermiticity \( \hat{H} = \hat{H}^\dagger \) of the Hamiltonian implies a unitary time evolution and thereby guarantees a conserved norm:

\[ \frac{d}{dt} \| \psi(t) \|_2 = \int d^3x \left[ \psi^* \frac{\partial}{\partial t} \psi + \psi \frac{\partial}{\partial t} \psi^* \right] \stackrel{7.1}{=} \frac{1}{i\hbar} \int d^3x \left[ \psi^* (\hat{H} \psi) - \psi (\hat{H}^\dagger \psi)^* \right] \stackrel{7.6}{=} 0. \]  

(7.5)

where we used that for \( \psi, \phi \in L^2 \) and a Hermitian Hamiltonian

\[ \int d^3x \psi^* (\hat{H} \psi) \overset{\text{def}}{=} \langle \phi | \hat{H} \psi \rangle \overset{\text{def}}{=} \langle \hat{H}^\dagger \phi | \psi \rangle \overset{\text{def}}{=} \int d^3x (\hat{H}^\dagger \phi)^* \psi \overset{\hat{H} = \hat{H}^\dagger}{=} \int d^3x \psi (\hat{H} \phi)^* \].  

(7.6)

3 | Problem: The SE is Galilei covariant but not Lorentz covariant! (recall Problemset 1)

- The SE is of first order in time but of second order in the spatial derivatives. This asymmetry already suggests that the equation cannot be Lorentz covariant: Time is treated differently than space in (non-relativistic) quantum mechanics.
- We would like quantum mechanics to be described by a Lorentz covariant equation because we subscribed to Einstein’s principle of special relativity \( \text{SR} \) at the beginning of this course: All laws of physics must take the same form in all inertial systems (which are related by Lorentz transformations). This certainly includes quantum mechanics.

However, \( \text{SR} \) is just a (empirically motivated) principle, it is neither a law nor a theorem; there may be conceivable universes in which \( \text{SR} \) simply does not apply to the quantum realm – in which case the Schrödinger Eq. (7.1) would be a perfectly valid model.

As good physicists, we should seek for empirical evidence to settle the matter …
Evidence:

- First: The Schrödinger equation, published and studied by Erwin Schrödinger in a sequence of papers in 1926 [80–83] (so relativity was already known at the time), was (and is) a highly successful theory that describes a plethora of microscopic phenomena remarkably well. Examples are the ↓ double-slit experiment, ↓ quantum tunneling effects, and, of course, the ↓ spectrum of the hydrogen atom:

The Hamilton operator for the relative electron-proton system of the hydrogen atom is

\[ \hat{H} = -\frac{\hbar^2}{2\mu} \nabla^2 - \frac{e^2}{|\mathbf{r}|} \]  

(7.7)

with reduced mass \( \mu = m_e m_p / (m_e + m_p) \). The discrete part of the spectrum of the operator \( \hat{H} \) can be computed exactly (\( E_R \) is the ↓ Rydberg energy),

\[ E_n = -\frac{E_R}{n^2} \quad \text{with principal quantum number } n \in \{1, 2, \ldots\}, \]  

(7.8)

and determines the hydrogen spectrum:

The transitions between the levels of the hydrogen spectrum can be measured by spectroscopy (↓ Lyman series [84], ↓ Balmer series [85],…; these observations have been made around 1900). The explanation of these spectral lines by the non-relativistic Schrödinger equation is the crown jewel of quantum mechanics, and one of the most remarkable advances of 20th century physics.

- However, it’s not all sunshine and roses. It was already known at the end of the 19th century (due to advances in spectroscopy [86]) that the spectral lines of various atomic species (including hydrogen) had a ↓ fine-structure. Expressed in terms of the energy levels of the hydrogen atom, this means that some of the degenerate eigenstates of Eq. (7.7) are actually not exactly degenerate:
Note that this was known to Schrödinger when he published his equation in 1926; he writes in Ref. [83] (p. 132–133):

*Im Anschluß an die zuletzt erwähnten physikalischen Probleme, [...], möchte ich nun doch die vermutliche relativistisch-magnetische Verallgemeinerung der Grundgleichungen [...] hier ganz kurz mitteilen, wenn ich es auch vorerst nur für das Einelektronenproblem und nur mit der allergrößten Reserve tun kann. Letzteres aus zwei Gründen. Erstens beruht die Verallgemeinerung vorläufig auf rein formaler Analogie. Zweitens führt sie, wie schon in der ersten Mitteilung erwähnt wurde, im Falle des Keplerproblems zwar formal auf die Sommerfeldsche Feinstrukturformel und zwar mit „halbzahligen“ Azimutal- und Radialquant, was heute allgemein als korrekt angesehen wird; allein es fehlt noch die zur Herstellung numerisch richtiger Aufspaltungsbilder der Wasserstofflinien notwendige Ergänzung, die im Bohrschen Bilde durch den Goudsmit-Uhlenbeckschen Elektronendrall geliefert wird.*

Note that Schrödinger was very much aware that his equation lacked Lorentz covariance and viewed (and constructed) it as a *non-relativistic approximation* of a truly “relativistic quantum mechanics” (which he didn’t know how to formulate consistently).

He also makes this clear in the introduction of Ref. [82] (p. 439):

*Wesentlich größeres Interesse wird natürlich die (hier noch nicht darchgeführte) Anwendung auf den Zeemaneffekt bieten. Diese erscheint mir unlöslich geknüpft an eine korrekte Formulierung des relativistischen Problems in der Sprache der Wellenmechanik, weil bei vierdimensionaler Formalisierung das Vektorpotential von selbst dem skalaren ebenbürtig an die Seite tritt. Schon in der ersten Mitteilung wurde erwähnt, daß das relativistische Wasserstoffatom sich zwar ohne weiteres behandeln läßt, aber zu “halbzahligen” Azimutalquanten, also zu einem Widerspruch mit der Erfahrung führt. Es mußte also noch “etwas fehlen”. Seither habe ich [...] gelernt, was fehlt: in der Sprache der Elektronenbahnentheorie der Drehimpuls des Elektrons um seine Achse, der ihm ein magnetisches Moment verleiht.*

• We can also make a back-of-the-envelope calculation to estimate whether relativistic effects could be the root cause for the discrepancy between the non-relativistic Schrödinger equation and the observed fine-structure:

In a *classical* approximation, kinetic and potential energy are of the same order:

\[
\text{Kinetic energy} \quad \frac{1}{2}mv^2 \sim \frac{e^2}{r} \quad \text{Potential energy}.
\]

(7.9)

Because the system is quantum, momentum and position obey the *Heisenberg uncertainty relation* \(\Delta p \Delta r \sim \hbar\). In the energy eigenstates of an interacting quantum system (like an atom) we typically have \(\Delta p \sim p\) and \(\Delta r \sim r\), and in our semi-classical approximation it is
\[ p \sim mv, \text{ so that} \]
\[ v \sim \frac{e^2}{mvr} = \frac{e^2}{\hbar c} = \text{Fine-structure constant} \times c \approx \frac{c}{137}. \quad (7.10) \]

The semi-classical velocity of the electron \( v \) is therefore much smaller than the speed of light \( c \); this explains why the non-relativistic Schrödinger equation is so successful (and your course non non-relativistic quantum mechanics is no waste of time). However, the observed fine-structure splitting of spectral lines is indeed very small, so it is reasonable that relativistic effects can have small but measurable effects in atomic physics.

The situation is therefore similar to that of Newtonian mechanics before we made it relativistic: We have a very successful Galilei covariant theory that, however, shows signs of being the low-velocity/energy approximation of another, presumably relativistic theory.

(Note that historically the situation is very different, though: While Newtonian mechanics, born in the 17th century, had to wait more than 200 years to be “made relativistic”, the development of relativistic quantum mechanics was very fast: Non-relativistic quantum mechanics was established in 1925/26 – and just two years later, in 1928, Paul Dirac published the correct equation describing relativistic electrons: the \( \text{Dirac equation} [87] \).

Are there relativistic field equations which allow for a probabilistic interpretation?

### 7.1. The Klein-Gordon equation

The Klein-Gordon equation has been studied by Klein [88] and Gordon [89] in 1926 as a possible relativistic version of the Schrödinger equation. Schrödinger and Fock found the equation independently as well.

1. \(<\text{Complex scalar field: } \phi : \mathbb{R}^{1,3} \rightarrow \mathbb{C}\)

   \(\rightarrow\) Most general \textit{quadratic} (superposition principle!) and \textit{Lorentz covariant} Lagrangian density:

\[
\mathcal{L}_{\text{KG}}(\phi, \partial \phi) = (\partial \mu \phi)(\partial \mu \phi^*) - M^2 \phi \phi^* \quad (7.11)
\]

\(M = \frac{mc}{h} \in \mathbb{R}: \text{arbitrary parameter } (m \text{ will be the mass of the particle})\)

- Note that \( M = \frac{mc}{h} = \frac{2\pi}{\lambda} \) has the dimension of an inverse length; here \( \lambda = \frac{h}{mc} \) is the \(<\text{Compton wavelength} \> \text{Eq} \,(5.77)\).

- One can also derive the non-relativistic Schrödinger equation from a Lagrangian density \(\rightarrow\) below):

\[
\mathcal{L}_{\text{SE}}(\psi, \partial \psi) = i \hbar \psi^* \partial_t \psi - \frac{\hbar^2}{2m}(\nabla \psi^*)(\nabla \psi) - V(x) \psi^* \psi \quad (7.12)
\]

This is of course not a Lorentz scalar (you cannot write this combining only tensors).

2. \textbf{Euler-Lagrange equations:}

   \(\text{Trick: Consider } \phi \text{ and } \phi^* \text{ as independent fields; let } \phi^* \text{ be the complex conjugate of } \phi \text{ at the end.}\)

\[
\frac{\partial \mathcal{L}_{\text{KG}}}{\partial \phi^*} - \partial \mu \frac{\partial \mathcal{L}_{\text{KG}}}{\partial (\partial \mu \phi^*)} = 0 \quad \Rightarrow \quad -M^2 \phi - \partial \mu \partial \mu \phi = 0 \quad (7.13)
\]

The Euler-Lagrange equations for the field \( \phi \) yield the complex conjugate Klein-Gordon equation.
The Klein-Gordon equation (KGE) is the simplest relativistic wave equation. The non-relativistic Schrödinger equation follows along the same lines from Eq. (7.12):

$$\frac{\partial L_{SE}}{\partial \dot{\psi}^*} - \frac{\partial}{\partial (\partial_{\mu} \psi^*)} \frac{\partial L_{SE}}{\partial \psi} = 0 \Rightarrow i \hbar \partial_t \psi - V \psi + \frac{\hbar^2}{2m} \nabla^2 \psi = 0$$  \hspace{1cm} (7.15)

The Euler-Lagrange equations for \( \psi^* \) yield the complex conjugate of the Schrödinger equation.

3 | Lorentz symmetry of the KGE:
The KGE is manifest Lorentz covariant. However, it is instructive (and useful for our derivation of the Dirac equation \( \rightarrow \) later) to check its invariance manually. To this end, we view Lorentz transformations as active transformations, mapping solutions to different solutions. This is equivalent to the passive viewpoint where the coordinate system is transformed instead:

i | \(< \) Coordinate transformation: \( \bar{x} = \Lambda x \) & Field transformation: \( \bar{\phi}(\bar{x}) = \phi(x) \)
   We write \( \bar{x} = \Lambda x \) for \( \bar{x}^\mu = \Lambda_{\nu}^\mu x^\nu \).

ii | \(< \) \( \phi(x) \) with \((\partial^2 + M^2)\phi(x) = 0 \) for all \( x \)
   That is, \( \phi(x) \) is a solution of the KGE.

iii | \( \Theta \) \( \bar{\phi}(x) := \phi(\Lambda^{-1}x) \) is a new solution:
   Use the chain rule in the first step twice:
   \[
   (\eta^{\mu\nu}\partial_\mu \partial_\nu + M^2)\bar{\phi}(x) = [\eta^{\mu\nu}(\Lambda^{-1})^\rho_{\mu}\partial_{\rho}(\Lambda^{-1})^\sigma_{\nu}\partial_{\sigma} + M^2]\phi(\Lambda^{-1}x) \\
   \text{Use invariance of the metric Eq. (4.21)} \hspace{1cm} (7.16a) \\
   = (\eta^{\rho\sigma}\partial_\rho \partial_\sigma + M^2)\phi(\Lambda^{-1}x) \\
   \text{Solution} \hspace{1cm} (7.16b) \\
   = (\partial^2 + M^2)\phi(\Lambda^{-1}x) \phi \text{ solution} \hspace{1cm} (7.16c) \\
   \]  \hspace{1cm} (7.16d)

   Here \( \partial_\sigma \phi(\Lambda^{-1}x) \) must be read as \( \partial_\sigma \phi(y)|_{y=\Lambda^{-1}x} \), i.e., we compute the derivative of the function \( \phi \) with respect to its argument \( y \) and then plug in the value \( \Lambda^{-1}x \).

4 | Conserved current:
   i | \(< \) Global phase rotations:
   \[
   \phi'(x) = e^{i\alpha} \phi(x) \quad \text{for} \quad \alpha \in [0, 2\pi) \\
   \]  \hspace{1cm} (7.17)
   with infinitesimal generator \( |\alpha| = |w| \ll 1 \)
   \[
   \phi'(x) = \phi(x) + i \omega \phi(x) \equiv \phi(x) + w \delta \phi(x) \quad \Rightarrow \quad \delta \phi = i \phi \\
   \]  \hspace{1cm} (7.18)

   Note that this is an “internal symmetry” that has nothing to do with spacetime; thus \( \delta x = 0 \).
   For the complex conjugate field \( \phi^* \) one finds analogously \( \delta \phi^* = -i \phi^* \).

   \( \rightarrow \) Continuous symmetry:
   \[
   L_{KG}(\phi, \partial \phi) = L_{KG}(\phi', \partial \phi') \\
   \]  \hspace{1cm} (7.19)

   If the Lagrangian density is invariant, the action is trivially invariant!
Noether theorem Eq. (6.85) → Conserved Noether current density Eq. (6.84):

\[ j_{KG}^\mu = i (\partial^\mu \phi) \phi^* - i (\partial^\mu \phi^*) \phi \]  \hspace{2cm} (7.20)

Note that if one treats \( \phi \) and \( \phi^* \) independent fields, one has to sum over the two fields in the evaluation of the Noether current; this then yields the real-valued current density above.

→ Noether charge density:

\[ \rho_{KG}(x) := j_{KG}^0(x) = \frac{i}{c} (\dot{\phi} \phi^* - \dot{\phi}^* \phi) \quad \text{with} \quad \rho_{KG}(x) \in \mathbb{R} \] \hspace{2cm} (7.21)

→ Conserved Noether charge:

\[ Q = \int d^3x \rho_{KG}(x) = \frac{i}{c} \int d^3x (\dot{\phi} \phi^* - \dot{\phi}^* \phi) \] \hspace{2cm} (7.22)

Important: \( \rho_{KG}(x) \nless 0 \) is not positive-definite! →

\[ \rho_{KG}(x) \text{ cannot be interpreted as a probability density!} \] \hspace{2cm} (7.23)

• To sum up:
  - The inner product (= positive-definite, symmetric sesquilinear form) on \( L^2(\mathbb{R}^{1,3; \mathbb{C}}) \)

\[ \langle \phi | \psi \rangle_{L^2} := \int d^3x \phi^* \psi \] \hspace{2cm} (7.24)

is not conserved under the time-evolution of the KGE.
  - The indefinite symmetric sesquilinear form (which is not an inner product!)

\[ \langle \phi | \psi \rangle_{KG} := \frac{i \hbar}{2mc^2} \int d^3x (\phi^* \dot{\psi} - \dot{\phi}^* \psi) \] \hspace{2cm} (7.25)

is conserved under the time-evolution of the KGE. But because it is not positive- (semi)definite, we cannot interpret the induced “norm” as a probability.

The prefactor \( \frac{\hbar}{2mc^2} \) is chosen such that it has the dimension of a time (because \( \frac{\hbar}{mc} \propto \lambda \) has the dimension of a length). Then the square of the fields (= wavefunctions) has the dimension of one over a volume – which is the conventional dimension of wavefunctions. The factor \( \frac{1}{2} \) is chosen to simplify expressions later.

• Compare this to the conserved current for the same phase rotation symmetry that follows for the Schrödinger field Eq. (7.12) with \( \delta \psi = i \psi \) and \( \delta \psi^* = -i \psi^* \):

\[ j_{SE}^\mu = \begin{cases} \frac{\hbar c \rho_{SE}}{\hbar_{SE}} & \mu = 0 \\ \frac{i \hbar^2}{2mc} \left[ (\nabla \psi^*) \psi - (\nabla \psi) \psi^* \right] & \mu = i = 1, 2, 3 \end{cases} \] \hspace{2cm} (7.26)

(Recall that you must sum over the fields \( \psi \) and \( \psi^* \).)

This is the positive-definite probability density you already know from quantum mechanics,

\[ \rho_{SE}(x) = |\psi(x)|^2 \geq 0, \] \hspace{2cm} (7.27)
and the *probability current density*

\[ j_{\text{SE}} = \frac{i\hbar}{2m} \left[ (\nabla \psi^*) \psi - (\nabla \psi) \psi^* \right]. \tag{7.28} \]

In this context, Noether’s theorem ensures probability conservation:

\[ \partial_{\mu} j_\mu^\text{SE} = 0 \iff \partial_t \rho_{\text{SE}} + \nabla \cdot \vec{j}_{\text{SE}} = 0. \tag{7.29} \]

**5 | Solutions:** (for the free Klein-Gordon field)

i | The KG Eq. (7.14) is a wave equation:

\[ \left[ \frac{1}{c^2} \partial_t^2 - \nabla^2 + \frac{m^2}{\hbar^2} \right] \phi(t, \vec{x}) = 0 \tag{7.30} \]

\[ \rightarrow \text{Solution space spanned by plane waves:} \]

\[ \phi(t, \vec{x}) = e^{i \left( \vec{p} \cdot \vec{x} - Et \right)} \tag{7.31} \]

Plug this ansatz into Eq. (7.30) \( \rightarrow \) Dispersion relation:

\[ -\frac{E^2}{c^2 \hbar^2} + \frac{\vec{p}^2 \hbar^2}{\hbar^2} + \frac{m^2 c^2}{\hbar^2} = 0 \]

\[ E = \pm \sqrt{\frac{\vec{p}^2 c^2}{\hbar^2} + m^2 c^4} \tag{7.32} \]

- This is the relativistic energy-momentum relation Eq. (5.26).
- The KG Eq. (7.14) is a wave equation:

\[ \left[ \frac{1}{c^2} \partial_t^2 - \nabla^2 + \frac{m^2}{\hbar^2} \right] \phi(t, \vec{x}) = 0 \tag{7.30} \]

\[ \rightarrow \text{Solution space spanned by plane waves:} \]

\[ \phi(t, \vec{x}) = e^{i \left( \vec{p} \cdot \vec{x} - Et \right)} \tag{7.31} \]

Plug this ansatz into Eq. (7.30) \( \rightarrow \) Dispersion relation:

\[ -\frac{E^2}{c^2 \hbar^2} + \frac{\vec{p}^2 \hbar^2}{\hbar^2} + \frac{m^2 c^2}{\hbar^2} = 0 \]

\[ E = \pm \sqrt{\frac{\vec{p}^2 c^2}{\hbar^2} + m^2 c^4} \tag{7.32} \]

- This is the relativistic energy-momentum relation Eq. (5.26).
- There are two solutions for each 3-momentum \( \vec{p} \), one of which has negative energy \( E < 0 \) (if we interpret the prefactor of \( t \) as the energy as usual). This is a consequence of the quadratic nature of the KG (as compared to the SE), and therefore a direct consequence of its relativistic covariance.

- At the time of its inception, the negative energy solutions of the KG could not be interpreted properly. This (together with the fact that its conserved “norm” cannot be interpreted as a probability and it fails to predict the fine-structure of the hydrogen atom correctly, below) lead to its dismissal as a relativistic wave equation for quantum wave functions. It only became clear later that the negative energy solutions herald the existence of antiparticles. Only in modern relativistic quantum field theories [where the KG reappears as the equation of motion of (free scalar) quantum fields, see Chapter 2 of my script on QFT [19]] this “feature” can be cast into a consistent framework: The negative energy solutions are interpreted as eigenmodes of antiparticles with positive energies (and norms). If the particles are charged, their antiparticles have opposite charge; then the conserved Noether charge Eq. (7.22) is interpreted as charge conservation (and not probability conservation).

ii | As usual, one can “normalize” the plane wave solutions Eq. (7.31) if one considers a finite system with volume \( V = L^3 \). Then one finds the “orthonormal” solution basis of the KG:
\( \phi_{k}^{(\pm)} (t, \vec{x}) = N_{k} e^{i(k \cdot \vec{x} \mp \omega_{k} t)} \) with … \hfill (7.34a)

Dispersion: \( \omega_{k} = \sqrt{k^2 c^2 + m^2 c^4 / \hbar^2} \)

Momentum: \( \vec{p} = \hbar \vec{k} \in \hbar \frac{2\pi}{L} \mathbb{Z}^3 \) \hfill (7.34b)

Normalization: \( N_{k} = \sqrt{\frac{mc^2}{V \hbar \omega_{k}}} \)

- It is straightforward to check that these states are "orthonormal" with respect to the Klein-Gordon sesquilinear form Eq. (7.25):

  \( \langle \psi_{k}^{(\alpha)} | \psi_{k}^{(\beta)} \rangle_{KG} \equiv \alpha \delta_{\alpha, \beta} \delta_{k, \vec{k}'} \) with \( \alpha, \beta \in \{ \pm \} \). \hfill (7.35)

Note that the \((-\) states have negative "norm".

- The fact that there are "twice as many" linearly independent solutions (two for each momentum) means that you need "twice as many" parameters to specify a particular solution (i.e., a linear combination of the plane waves). This corresponds to the fact that the KGE is of second order in the time derivative, so that you need to provide both \( \phi(t = 0, \vec{x}) \) and \( \dot{\phi}(t = 0, \vec{x}) \) to specify a unique solution.
6 | Coupling to a static EM field:

The KGE can be coupled to the gauge field of electrodynamics. This is necessary to describe charged particles (in particular: the hydrogen atom). Note that in the following the gauge field is a parameter and not a dynamic degree of freedom.

i | Goal: Construct Lagrangian density that is …
   • … a Lorentz scalar.
   • … quadratic in $\phi$.
   • … gauge invariant under the gauge transformation $A'_\mu = A_\mu - \partial_\mu \lambda$.
   • … couples $\phi$ and $A_\mu$ in a non-trivial way.

Without additional tools, this is a tough job!

ii | \(<\text{Gauge transformation } A'_\mu = A_\mu - \partial_\mu \lambda(x) >\)

Let us assume that the KG field transforms under the gauge transformation as follows:

$$\phi'(x) := e^{iQ(x)} \phi(x) \quad \text{with the } U(1) \text{ charge } Q = \frac{q}{\hbar c} \in \mathbb{R}. \quad (7.36)$$

$q$: electric charge of the particle described by the wavefunction $\phi$

- It is reasonable to assume that the KG field must transform via phase factors because we already know [recall Eq. (7.19)] that the KG Lagrangian is invariant under global phase transformations $\lambda(x) = \text{const}$. Our hope is that we can “extend” this symmetry for arbitrary non-constant $\lambda(x)$.

- The charge $Q$ is a property of the field and quantifies how it transforms under gauge transformations; it essentially plays the role of the electric charge of the particle described by $\phi$; e.g., for an electron we would set $Q = -e < 0$.

The additional division by $\hbar c$ is necessary for dimensional reasons: $[\lambda] = L[\varphi]$ with $A^\mu = (\varphi, \vec{A})$; therefore $[\lambda q] = L[\varphi q] = L[E] = \frac{ML^3}{T^2}$ and it is $[h c] = \frac{ML^3}{T^2}$ as well.

In natural units (where $c = 1$), $Q = q$ is simply the electric charge.

- The term “$U(1)$ charge” highlights that the gauge transformation $e^{iQ(x)} \in U(1)$ is a $U(1)$ gauge transformation; the charge is the generator of this Lie group.

iii | Problem:

Derivatives transform complicated under gauge transformations:

$$\partial_\mu \phi'(x) = e^{iQ(x)} \left[ iQ \partial_\mu \lambda(x) \phi(x) + \partial_\mu \phi(x) \right] \quad (7.37)$$

→ It is hard to combine derivatives of fields to construct gauge-invariant terms!

Solution:

Define the …

\(<\text{(Gauge) Covariant derivative: } D_\mu := \partial_\mu + i Q A_\mu >\)

→ Lorentz vector (thus we can it use to construct Lorentz scalars!)
The covariant derivative has the following useful property:

$$D_0 \phi = \left[ \partial_0 + i Q \partial_0 \phi - i Q \partial_0 \phi \right] e^{iQ_0 \phi} \equiv e^{iQ_0 \phi} D_0 \phi$$  \hspace{1cm} (7.39)

→ $D_0 \phi$ transforms like $\phi$ under gauge transformations. [and not as ugly as Eq. (7.37)!]

This is useful because it allows us to combine derivatives into gauge-invariant terms.

iv) Using the covariant derivative, we can now construct the following general Lagrangian density that satisfies our four requirements above:

$$\mathcal{L}_A(\phi, \partial \phi) = (D^\mu \phi)(D_\mu \phi)^* - M^2 \phi \phi^*$$  \hspace{1cm} (7.40)

Please appreciate the ingenuity of the term $(D^\mu \phi)(D_\mu \phi)^*$: It is Lorentz invariant because we pair the indices correctly, and it is gauge invariant because we pair $(D_\mu \phi)$ with its complex conjugate $(D_\mu \phi)^*$ (which is sufficient because $D_\mu \phi$ gauge-transforms like $\phi$).

This Lagrangian density is gauge-invariant by construction in the sense that

$$\mathcal{L}_A(\phi, \partial \phi) = \mathcal{L}_A(\phi', \partial \phi') \quad \text{or} \quad \mathcal{L}(\phi, D \phi) = \mathcal{L}(\phi', D \phi') .$$  \hspace{1cm} (7.41)

• A comparison of the free Klein-Gordon Lagrangian Eq. (7.11) and the new one Eq. (7.40) reveals that we simply made the substitution $\partial_\mu \rightarrow D_\mu$, i.e., we replaced partial derivatives by covariant derivatives (which depend on the gauge field). This trick is not specific to the Klein-Gordon field and yields gauge-invariant theories in general. This procedure is called \textit{minimal coupling}.

• Note that the transformation Eq. (7.36) is a local phase rotation of the KG-field. In Eq. (7.17) we considered a global phase rotation and identified it as a continuous symmetry of the KG Lagrangian $\mathcal{L}_{KG}$. You can check that the new local transformation does not leave $\mathcal{L}_{KG}$ invariant, but it does leave $\mathcal{L}_A$ invariant if $A^\mu$ transforms together with $\phi$ as defined above. The transition from $\mathcal{L}_{KG}$ (with a global symmetry) to $\mathcal{L}_A$ (with a local version of the same symmetry) is called \textit{gauging the symmetry}. You can use this line of reasoning to “invent” the electromagnetic gauge field: If you start from a global continuous symmetry and demand that it becomes a local symmetry, you have to pay for it by introducing a new field: the gauge field.

v) Klein-Gordon equation in a static EM field:

The Euler-Lagrange equations of $\mathcal{L}_A$ yield: Eq. (6.6) \hspace{1cm} Eq. (7.40)

$$D^2 \phi(x) = 0$$  \hspace{1cm} (7.42)

with $D^2 = D_\mu D^\mu$ and $M = mc$.

In the form Eq. (7.42) both Lorentz covariance and gauge invariance are manifest (because we use the covariant derivative). If we expand everything, we loose these features but obtain a less abstract (but more complicated) form of the PDE:

$$\left[ \frac{1}{c^2} \left( \partial_t + i Q_\psi \right)^2 - \nabla \nabla - i Q \cdot A \right] \phi(t, \vec{x}) = 0$$  \hspace{1cm} (7.43)

Here we used $A_\mu = (\psi, -\vec{A})$ (covariant!).
Example: Hydrogen atom

Goal: Describe the electron of the hydrogen atom in the static EM field generated by the proton in terms of the KGE; i.e., we interpret the KG field naïvely as the wavefunction of the electron. Our hope is that the energy spectrum of this relativistic theory explains the observed fine-structure splitting.

\[ \text{Coulomb potential (of proton with charge } e > 0) \]

Choose a gauge where \( \mathbf{A} = 0 \) and \( \mathbf{E} \) is parallel to the axis:

\[ \frac{1}{c^2} \left( i \frac{\partial}{\partial t} + \frac{e^2}{\hbar |\mathbf{x}|} \right)^2 + \nabla^2 - \frac{m^2 c^2}{\hbar^2} \phi(t, \mathbf{x}) = 0 \quad (7.44) \]

With electron charge \( Q = \frac{e}{\hbar} < 0 \) one finds:

\[ \frac{c^2 \hbar^2 \Delta + \left( E + \frac{e^2}{|\mathbf{x}|} \right)^2 - m^2 c^4}{\hbar^2} \overline{\phi}(\mathbf{x}) = 0 \quad (7.45) \]

Ansatz \( \phi(t, \mathbf{x}) = \overline{\phi}(\mathbf{x}) e^{-i E t} \) → “Stationary” Klein-Gordon equation:

Note that this PDE is \textit{quadratic} in the energy \( E \) (and not linear, like the time-independent Schrödinger equation).

One can use a clever mapping to the non-relativistic Schrödinger equation to solve for \( \overline{\phi}(\mathbf{x}) \) and determine the energies \( E \) for which solutions exist:

\[ E_{n,l} = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2}{(\alpha - \delta_l)^2}}} \quad \text{with} \quad \delta_l = l + \frac{1}{2} - \sqrt{(l + \frac{1}{2})^2 - \alpha^2} \quad (7.47) \]

Here \( n = 1, 2, \ldots \) is the \textit{principal quantum number} and \( l = 0, 1, 2, \ldots \) is the \textit{orbital angular momentum quantum number}. \( \alpha = \frac{e^2}{\hbar} \approx \frac{1}{137} \) is the fine-structure constant.

Comments:

• The spectrum Eq. (7.47) predicts a splitting of the \( l \)-degeneracy; recall that this degeneracy is perfect in the non-relativistic hydrogen atom [cf. Eq. (7.8)]. Unfortunately, the spectrum Eq. (7.47) does not match observations! The reason is that the Klein-Gordon equation does not know about the electron spin. Schrödinger and his contemporaries were aware of this solution and its problems (this shines through in the quotes at the beginning of this chapter). This failure to predict the fine-structure correctly led to the dismissal of the Klein-Gordon equation and motivated Paul Dirac to search for another equation (→ next section).

• Today we know that the Klein-Gordon equation is \textit{not wrong}: It simply does not apply to particles with non-zero spin (and the electron in the hydrogen atom happens to have spin \( s = \frac{1}{2} \)). However, it \textit{does} apply to spin-0 particles like \textit{kaons} (K mesons, bound states of two quarks), \textit{pions} (pi mesons), and the \textit{Higgs boson} (the latter being the only \textit{elementary} particle with zero spin). But since we cannot build hydrogen atoms out of these particles, the significance of the above solution remains limited.

First-order formulation:

Here we consider again the free KGE (without EM field) for simplicity.

KGE = \textit{Second-order PDE in time}
Problem: $\phi(t = 0, \vec{x})$ does not specify the state of the system completely [unlike for the Schrödinger equation one also needs $i\hbar \partial_t \phi(t = 0, \vec{x})$ to pick out a unique solution $\phi(t, \vec{x})$].

Recall: Every higher-order differential equation can be recast as a first-order differential equation with multiple components.

Goal: Rewrite the KGE in the first-order form

\[ i\hbar \partial_t \Phi = \hat{H}_{KG} \Phi \quad \text{with} \quad \Phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix}. \tag{7.48} \]

Downside: In this form, the KGE is no longer manifest Lorentz covariant.

Define

\[ \dot{\phi}^\pm := \frac{1}{2} \left( \phi \pm \frac{i\hbar}{mc^2} \partial_t \phi \right) \tag{7.49} \]

so that

\[ \phi = \phi_+ + \phi_- \quad \text{and} \quad \frac{i\hbar}{mc^2} \partial_t \phi = \phi_+ - \phi_- . \tag{7.50} \]

Define the $2 \times 2$ differential operator

\[ \hat{H}_{KG} := \begin{pmatrix} \hat{H}_0 + mc^2 & \hat{H}_0 \\ -\hat{H}_0 & -\hat{H}_0 - mc^2 \end{pmatrix} = \hat{H}_0 \otimes (\sigma^z + i\sigma^x) + mc^2\sigma^z \tag{7.51} \]

with $\hat{H}_0 = -\frac{h^2}{2m} \nabla^2$ the free particle Hamiltonian and the Pauli matrices

\[ \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{7.52} \]

$\hat{H}_{KG}$ is a linear operator on the Hilbert space $L^2 \otimes \mathbb{C}^2$ of two-component square-integrable functions. Note that $\hat{H}_{KG}^2 = \hat{H}_0 \otimes (\sigma^z - i\sigma^x) + mc^2\sigma^z \neq \hat{H}_{KG}$ is non-Hermitian with respect to the conventional inner product on $L^2 \otimes \mathbb{C}^2$:

\[ \langle \Phi | \Psi \rangle_{L^2 \otimes \mathbb{C}^2} = \int d^3x \Phi^\dagger(x)\Psi(x) = \int d^3x \left( \phi^*_+ \psi_+ + \phi^*_- \psi_- \right) . \tag{7.53} \]

Check that the differential equation in first-order Schrödinger form

\[ i\hbar \partial_t \Phi = \hat{H}_{KG} \Phi \quad \Leftrightarrow \quad \begin{cases} i\hbar \partial_t \phi_+ = (\hat{H}_0 + mc^2)\phi_+ + \hat{H}_0 \phi_- \\ i\hbar \partial_t \phi_- = -\hat{H}_0 \phi_+ - (\hat{H}_0 + mc^2)\phi_- \end{cases} \tag{7.54} \]

is equivalent to the KGE:

Indeed, the difference of the two equations yields

\[ -\frac{h^2}{mc^2} \partial_\chi = (\hat{H}_0 + mc^2)\phi + \hat{H}_0 \phi \quad \Leftrightarrow \quad \frac{1}{c^2} \partial_t \chi - \nabla^2 \phi + \frac{m^2c^2}{h^2} \phi = 0 \tag{7.55} \]

where we defined $\phi := \phi_+ + \phi_-$ and $\chi := \frac{mc^2}{i\hbar}(\phi_+ - \phi_-)$.

By contrast, the sum of the two equation yields

\[ mc^2 \partial_t \phi = (\hat{H}_0 + mc^2)\chi - \hat{H}_0 \chi \quad \Leftrightarrow \quad \partial_t \phi = \chi . \tag{7.56} \]

Combining Eq. (7.55) and Eq. (7.56) returns the KGE:

\[ \frac{1}{c^2} \partial_t^2 \phi - \nabla^2 \phi + \frac{m^2c^2}{h^2} \phi = 0 . \tag{7.57} \]
If one defines the Klein-Gordon adjoint
\[ \Phi := \Phi^\dagger \sigma^z = (\phi^*_+ - \phi^*_-), \tag{7.58} \]

one can express the Klein-Gordon sesquilinear form Eq. (7.25) as
\[ \langle \Phi | \Psi \rangle_{KG} := \int d^3x \, \Phi(x) \Psi(x) = \frac{i\hbar}{2mc^2} \int d^3x \, (\phi^* \psi - \phi \psi^*) \tag{7.59} \]

Remember that this is not a proper inner product because it is not positive-definite.

If one defines additionally for an operator \( A \) on \( L^2 \otimes \mathbb{C}^2 \) the Klein-Gordon adjoint
\[ \tilde{A} := \sigma^z A^\dagger \sigma^z, \tag{7.60} \]

it follows \( \tilde{\Phi} = \Phi \tilde{A} \) and \( \tilde{A} = A \), and thereby
\[ \langle \Phi | A \Psi \rangle_{7.59} \equiv (\tilde{A} \Phi | \Psi \rangle. \tag{7.61} \]

It is easy to verify that the KG Hamiltonian is “Klein-Gordon Hermitian”, namely
\[ \hat{H}_{KG} = \hat{H}_{KG}. \tag{7.62} \]

because \( \sigma^z \sigma^\gamma \sigma^z = -\sigma^\gamma \).

With this machinery, we have now a new method to check that the time-evolution generated by the KGE leaves the KG sesquilinear form invariant:
\[ \frac{d}{dt} \langle \Phi | \Psi \rangle_{KG} \equiv \frac{d}{dt} \langle \Phi | \Psi \rangle_{KG} \tag{7.63a} \]
\[ = \langle \Phi | \dot{\Psi} \rangle_{KG} + \langle \dot{\Phi} | \Psi \rangle_{KG} \tag{7.63b} \]
\[ = \frac{1}{i\hbar} \langle \Phi | \hat{H}_{KG} \Psi \rangle_{KG} - \frac{1}{i\hbar} \langle \hat{H}_{KG} \Phi | \Psi \rangle_{KG} \tag{7.63c} \]
\[ = \frac{1}{i\hbar} \left( \langle \Phi | \hat{H}_{KG} \Psi \rangle_{KG} - \langle \Phi | \hat{H}_{KG} \Psi \rangle_{KG} \right) = 0 \tag{7.63d} \]

We already knew this from Noether’s theorem, but it is always nice to derive such statements in various ways.

Non-relativistic limit:

**Goal:** Derive a non-relativistic approximation of the Klein-Gordon equation
\[ \left[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{m^2 c^2}{\hbar^2} \right] \phi(t, \vec{x}) = 0. \tag{7.64} \]

**Kinetic energy:** \( E_{\text{kin}} = E - mc^2 = \sqrt{\vec{p}^2 c^2 + m^2 c^4} - mc^2 \approx \frac{1}{2} mu^2 + O(\beta^4) \)
(Not that both \( E_{\text{kin}} \) and \( E \) are non-negative!)

**Ansatz:**
\[ \phi_{\pm}(t, \vec{x}) = \tilde{\phi}_{\pm}(\vec{x}) e^{\mp \frac{i}{\hbar} E t} = \tilde{\phi}_{\pm}(\vec{x}) e^{\mp \frac{i}{\hbar} E_{\text{kin}} t} e^{\mp \frac{i}{\hbar} mc^2 t} \tag{7.65} \]
\[ = \hat{\phi}_{\pm}(t, \vec{x}) \]
\[ \hat{\phi}(t, \vec{x}) \]

\( c(t, \vec{x}) \) contains only the time evolution due to the kinetic energy, excluding the rest energy.
If we use that
\[ \partial_t^2 \phi_\pm = -\frac{E_{\text{kin}}^2}{\hbar^2} \phi_\pm, \]  
we can make the following approximation in the non-relativistic limit \( E_{\text{kin}} \ll mc^2 \):
\[ \partial_t^2 \phi_\pm = e^{\mp i mc^2 t} \left\{ \partial_t^2 \phi_\pm \mp \frac{2i mc^2}{\hbar} \partial_t \phi_\pm - \left( \frac{mc^2}{\hbar} \right)^2 \phi_\pm \right\} \]  
\[ = -e^{\mp i mc^2 t} \left\{ \pm \frac{2i mc^2}{\hbar} \partial_t \phi_\pm + \left( \frac{mc^2}{\hbar} \right)^2 \left[ 1 + \left( \frac{E_{\text{kin}}}{mc^2} \right)^2 \right] \phi_\pm \right\} \]  
\[ \approx -e^{\mp i mc^2 t} \left\{ \pm \frac{2i mc^2}{\hbar} \partial_t \phi_\pm + \left( \frac{mc^2}{\hbar} \right)^2 \phi_\pm \right\} \]  
\[ \text{Eq. (7.67c) in Eq. (7.64) yields:} \]
\[ e^{\mp i mc^2 t} \left[ \pm \frac{2i m}{\hbar} \partial_t + \frac{m^2 c^2}{\hbar^2} + \nabla^2 - \frac{m^2 c^2}{\hbar^2} \right] \phi_\pm (t, \vec{x}) = 0 \]

And finally:
\[ \pm i \hbar \partial_t \phi_\pm (t, \vec{x}) = -\frac{\hbar^2}{2m} \nabla^2 \phi_\pm (t, \vec{x}) \]
This is the Schrödinger equation for a free particle.
Note that the “negative energy solutions” \( \phi_- \) lead to the time-inverted Schrödinger equation.

### 7.2. The Dirac equation

The Dirac equation was published by Paul Dirac in [87], only two years after Schrödinger published the Schrödinger equation.

1 | **Goal:**

The Klein-Gordon equation has a few undesirable quirks:

- **It’s conserved** \( \text{U}(1) \) **current has no positive-definite density and therefore cannot be interpreted as a probability current.** Conversely, the conventional norm on \( L^2 \) is not conserved. In the first-order formulation, this corresponds to a non-Hermitian Hamiltonian.

  \( \rightarrow \) **Can we construct a relativistic field equation with a conserved positive-definite density that gives rise to a norm and a Hermitian Hamiltonian?**

- **In its manifest Lorentz covariant formulation, the KGE is of second order in time, so that we must provide both the wavefunction and its time derivative as initial data.**

  \( \rightarrow \) **Can we construct a relativistic field equation which is first order in time (just like the Schrödinger equation)?**

- **For each momentum there is are two solutions: one with positive and one with negative energy.**

  \( \rightarrow \) **Can we get rid of the negative energy solutions?**

The Dirac equation succeeds in solving the first two issues – but not the last one, i.e., there will still be negative energy solutions.
Observation:
To reach our goals we must equip our “toolbox” of tensor calculus with additional building blocks. As it turns out, there is another type of field (besides the tensor fields we introduced in Chapter 3) that plays an important role in quantum mechanics: \(\text{spinor fields}\).

Remember: Vector fields under rotations: \(\hat{\phi}'(\mathbf{x}) = R\hat{\phi}(R^{-1}\mathbf{x})\)

→ In general, a field \(\phi(x) \in \mathbb{C}^n\) can transform under homogeneous Lorentz transformations as

\[
\phi'_{\alpha}(x) = M_{ab}(\Lambda)\phi_{b}(\Lambda^{-1}x) \quad a = 1, \ldots, n
\]

where

\[
M(\Lambda')M(\Lambda)\phi(\Lambda^{-1}\Lambda'^{-1}x) = M(\Lambda'\Lambda)\phi((\Lambda'\Lambda)^{-1}x)
\]

is a \(n\)-dimensional representation of the (proper orthochronous) Lorentz group \(SO^+(1, 3)\).

- Regarding groups and their representations: \(\text{Problemset 1}\).
- More explicitly: The tensor fields (of various rank) we know so far allow only for the description of particles with \text{integer spin} \(S = 0; 1; 2; \ldots\) (spin = internal angular momentum). What we are missing are fields that can describe particles with \text{half-integer spin} \(S = \frac{1}{2}; \frac{3}{2}; \frac{5}{2}; \ldots\); these are the spinor fields.

The reason why this is crucial for relativistic quantum mechanics in particular has to do with the fact that multiplying wave functions by a global phase does not change the state. In mathematical parlance we are dealing with \(\text{projective Hilbert spaces}\) and \(\text{projective representations}\) of symmetries. Thus if you are interested what rotations \(SO(3)\) do to the quantum state of your system, you must study all \text{projective representations of SO(3)}. It turns out that these can be identified with the “conventional” (= linear) representations of another group: \(SU(2)\) (the so called \(\text{double cover of SO(3)}\)). And you know that the irreducible representations of \(SU(2)\) are labeled by “spin quantum numbers” \(s = 0; \frac{1}{2}; 1; \frac{3}{2}; 2; \ldots\). In general, the double covers of \(SO(n)\) are called \(\text{spin groups} Spin(n)\), and similarly, the double cover of the proper orthochronous Lorentz group \(SO^+(1, 3)\) is the group \(Spin(1, 3) \simeq SL(2, \mathbb{C})\) (the group of complex \(2 \times 2\) matrices with determinant one). It turns out that the irreducible representations of this group can be labeled by \(two\) numbers \((m, n)\) with \(m, n = 0; \frac{1}{2}; 1; \frac{3}{2}; \ldots\). The spinor representations we are interested in (the ones missing from our discussion of tensor fields) are the ones for which \(m + n\) is half-integer. Conversely, the \(\left(\frac{1}{2}; \frac{1}{2}\right)\) representation is our well-known 4-vector representation \(A^\mu\) and the \(\left(0; 0\right)\) representation is that of a scalar like \(\phi\).

We want a first-order relativistic field equation \(\rightarrow\) Ansatz:

\[
(\partial^\mu \partial_\mu + \text{const})\phi = 0 \quad \Rightarrow \quad (i \Box^\mu \partial_\mu + \text{const})\phi = 0
\]

We do not yet know what \(\Box\) is (only that it cannot be a derivative).

The \(i\) anticipates wave-like solutions for real \(\phi\).

A covariant equation of the form \(\partial_\mu \phi = 0\) or \(\partial_\mu A^\mu = 0\) would of course also be possible; their solutions, however, are either too simple or do not match observations.

Then (combine 2 & 3)

(a) Coordinate transformation \(x' = \Lambda x\) & Field transformation \(\phi'(x') = M(\Lambda)\phi(x)\)

That is, \(\phi(x)\) is a solution of the equation we want to construct.
When is $\phi'(x) = M(\Lambda)\phi(\Lambda^{-1}x)$ a new solution?

We want the equation to be Lorentz covariant; this means that the Lorentz group must be (part of) its invariance group: Lorentz transformations map solutions to new solutions.

\[(i \nabla^\mu \partial_\mu + \text{const})\phi'(x) = [i \nabla^\mu (\Lambda^{-1})^\nu_\mu \partial_\nu + \text{const}] M(\Lambda)\phi(\Lambda^{-1}x) \overset{!}{=} 0 \quad (7.73)\]

Multiply with $M^{-1}(\Lambda)$:

\[\Leftrightarrow [i M^{-1}(\Lambda) \nabla^\mu M(\Lambda) (\Lambda^{-1})^\nu_\mu \partial_\nu + \text{const}] \phi(\Lambda^{-1}x) \overset{!}{=} 0 \quad (7.74)\]

$\nabla^\mu \equiv \gamma^\mu$ must be $n \times n$-matrices with

\[M^{-1}(\Lambda)\gamma^\mu M(\Lambda) = \Lambda^\mu_\nu \gamma^\nu \quad (7.75)\]

The $\gamma$-matrices “translate” the “spinor”-representation $M(\Lambda)$ into the “vector”-representation $\Lambda$ and vice versa.

5 | **Question:** How to find appropriate $\gamma^\mu$ and $M(\Lambda)$ that satisfy Eq. (7.75)?

**Remember:** $SO^+(1, 3)$ is a Lie group (Recall $\bullet$ Problemset 4):

\[\Lambda = \exp \left[-\frac{i}{2} \omega_{\alpha\beta} J^{\alpha\beta}\right] \approx 1 - \frac{i}{2} \omega_{\alpha\beta} J^{\alpha\beta} \quad (7.76a)\]

\[M(\Lambda) = \exp \left[-\frac{i}{2} \omega_{\alpha\beta} S^{\alpha\beta}\right] \approx 1 - \frac{i}{2} \omega_{\alpha\beta} S^{\alpha\beta} \quad (7.76b)\]

$\omega_{\alpha\beta}$ antisymmetric tensor $\rightarrow$ 3 rotations (angles) + 3 boosts (rapidities)

It is $(J^{\alpha\beta})_{\mu\nu} = i (\delta^\alpha_{\mu} \delta^\beta_{\nu} - \delta^\alpha_{\nu} \delta^\beta_{\mu})$; these $4 \times 4$ matrices $J^{\alpha\beta}$ generate the 4-vector representation $(\frac{1}{2}, \frac{1}{2})$, i.e., the $4 \times 4$-matrices $\Lambda$. By contrast, the $n \times n$-matrices $S^{\alpha\beta}$ generate the spinor-representation $M(\Lambda)$ [we will find $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$]. The generators are antisymmetric in the spacetime indices.

- **Infinitesimal form of Eq. (7.75):**

\[\gamma^\mu, S^{\alpha\beta} \quad \overset{=}{} \quad (J^{\alpha\beta})_{\mu\nu} \gamma^\nu \overset{=}{} i (\eta^{\alpha\mu} \gamma^\beta - \eta^{\beta\mu} \gamma^\alpha) \quad (7.77)\]

- $\Leftrightarrow J^{\alpha\beta}$ (Problemset 4) $\rightarrow$ Lie-algebra of Lorentz group ($J = S, \gamma$):

\[\{J^{\mu\nu}, J^{\sigma\alpha}\} \overset{=}{} i (\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}) \quad (7.78)\]

The Lie algebra defines the structure of the Lie group by exponentiation and is therefore the same for all representations, recall Eq. (4.63).

6 | **Solution to Eq. (7.75) via Dirac’s trick [87]:** $\Leftrightarrow \gamma^\mu$ such that

\[\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} 1_{n \times n} \quad \text{Dirac algebra} \quad (7.79)\]

with the $\Leftrightarrow$ anticommutator $\{X, Y\} = XY + YX$. 
Matrices $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ that satisfy Eq. (7.79) are called Dirac matrices or Gamma matrices.

This is the 16-dimensional Clifford algebra $\mathbb{C}l_{1,3}$. Then

$$S^{\mu\nu} := \frac{i}{4} \left[ \gamma^\mu, \gamma^\nu \right]$$

(7.80)

satisfies the Lorentz algebra Eq. (7.78) and Eq. (7.77).

Check this by plugging Eq. (7.80) into Eq. (7.78) and Eq. (7.77) and using Eq. (7.79)!

Problem of solving Eq. (7.75) has been reduced to finding 4 matrices $\gamma^\mu$ that satisfy Eq. (7.79).

7 | Representations of Eq. (7.79):

- At least $n = 4$-dimensional (Think of the $\gamma^\mu$ as Majorana modes and construct ladder operators $\rightarrow$ 2 modes.)

- All 4-dimensional representations are unitarily equivalent (Actually, they constitute the unique irrep of the Dirac algebra which is 4-dimensional.)

- We use the Weyl representation (sometimes called chiral representation):

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3$$

(7.81)

- Recall the Pauli matrices Eq. (7.52):

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

(7.82)

- Other common choices are the Dirac representation and the Majorana representation.

- Henceforth: $\Lambda_{\frac{1}{2}} \equiv M(\Lambda)$

It turns out that these are two “copies” of a spin-$\frac{1}{2}$ projective representation: $\Lambda_{\frac{1}{2}}$ corresponds to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of $\text{SL}(2, \mathbb{C})$. Since $n + m = \frac{1}{2}$, this is a spinor representation, i.e., a projective representation of the Lorentz group $\text{SO}^+(1, 3)$. The fact that it is the sum of two such representations makes it reducible. The wavefunction $\Psi(x)$ has therefore $n = 4$ components and is a spinor field (and not a tensor field).

8 | Setting const $= -M = -\frac{mc}{h}$ (which has dimension of an inverse length), we find:

$$(i \gamma^\mu \delta_{\mu\nu} - M)\Psi = 0$$

(7.83)

Dirac equation

Here, $\Psi(x)$ is a (bi)spinor-field:

$$\Psi : \mathbb{R}^{1,3} \rightarrow \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2.$$ 

(7.84)

Introduce the Feynman slash notation: $\slashed{O} := \gamma^\mu O_{\mu}$

(Here, $O_{\mu}$ stands for any object with a 4-vector index.)
With the slash notation, the Dirac equation can be written as:

\[(i \partial - M)\Psi = 0\]  \hspace{1cm} (7.85)

The Dirac equation is engraved in a plaque on the floor of Westminster Abbey next to Isaac Newton’s tomb (they abbreviate \(\gamma \cdot \partial = \gamma^\mu \partial_\mu\) and are in natural units \(\hbar = 1 = c\) where \(M = m\)).

The components \(\Psi_a(x) (a = 1, 2, 3, 4)\) satisfy the KGE:

\[0 = (-i \gamma^\mu \partial_\mu - M)(i \gamma^\nu \partial_\nu - M)\Psi \overset{7.79}{=} (\partial^2 + M^2)\Psi\]  \hspace{1cm} (7.86)

On the right hand side of Eq. (7.86) there is an identity \(I_{4 \times 4}\) that we omit.

- The Dirac differential operator is the “square root” of the Klein-Gordon differential operator.
- \(\Psi\) has as many components as the EM gauge field \(A^\mu\), but we do not write these components as \(\Psi^\mu\), but either simply as \(\Psi\) (and think of it as a four-dimensional column vector), or as \(\Psi_a\) with spinor index \(a = 1, 2, 3, 4\). The purpose of this notational difference is to denote the different ways the fields transform under Lorentz transformations:

\[A^\mu = \Lambda^\mu_\nu A^\nu \quad \text{versus} \quad \Psi_a = (\Lambda_{\frac{1}{2}})_{ab} \Psi_b \quad \text{or simply} \quad \Psi' = \Lambda_{\frac{1}{2}} \Psi.\]  \hspace{1cm} (7.87)

Note that \(\Lambda_{\frac{1}{2}} = M(\Lambda)\) are not the same \(4 \times 4\) matrices!

**Dirac adjoint:**

We would like to find a Lagrangian density for the Dirac equation; since this must be a Lorentz scalar, we ask the question:

*How to form Lorentz scalars from Dirac spinors?*

**i** | First try: \(\Psi^{\dagger} \Psi\)

\[\Psi^{\dagger} \Psi' = \Psi^{\dagger} \Lambda_{\frac{1}{2}}^{\dagger} \Psi = \Psi^{\dagger} \Psi \neq 1\]  \hspace{1cm} (7.88)

\(\Lambda_{\frac{1}{2}}\) is not unitary because \(S^{\mu\nu}\) is not Hermitian for boosts \((\mu = 0\) and \(\nu = 1, 2, 3\)).

This is a consequence of the \(\dagger\) non-compactness of the Lorentz group due to boosts.

**ii** | Define

\[\bar{\Psi} := \Psi^{\dagger} \gamma^0 \quad \text{Dirac adjoint}\]  \hspace{1cm} (7.89)
ψ'ψ'' = and $\Psi \Lambda^{-1} \Lambda \frac{1}{2} \Psi = \bar{\Psi} \Psi \Rightarrow$ Lorentz scalar

Use Eq. (7.80) and Eq. (7.76b) and the Dirac algebra to show this!

11 | Lagrangian:

With these tools, it is reasonable to propose the following Lagrangian density:

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi} (i \gamma^{\mu} \partial_\mu - M) \Psi = \bar{\Psi} (i \not{\partial} - M) \Psi \quad (7.90)$$

→ Euler-Lagrange equations = Dirac equation

• Note that in explicit index notation, the Lagrangian density reads

$$\mathcal{L}_{\text{Dirac}} = i \bar{\Psi}_a \gamma^{\mu}_{ab} (\partial_\mu \Psi_b) - M \bar{\Psi}_a \Psi_a \quad (7.91)$$

where sums over pairs of spinor indices are implied.

The Euler-Lagrange equations follow again by treating $\Psi_a$ and $\bar{\Psi}_a$ as independent fields:

$$0 \overset{1}{=} \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial \partial_\mu \Psi_a} - \partial_\mu \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial (\partial_\mu \Psi_a)} = -M \bar{\Psi}_a - i (\bar{\partial}_a \bar{\Psi}_b) \gamma^b_{ba} \overset{1}{=} \left[ (i \not{\partial} - M) \Psi \right]_a \quad (7.92a)$$

$$0 \overset{1}{=} \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial \bar{\Psi}_a} - \partial_{\partial_\mu} \frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial (\bar{\partial}_\mu \bar{\Psi}_a)} = \left[ (i \not{\partial} - M) \Psi \right]_a \quad (7.92b)$$

Note that the two equations are Dirac adjoints of each other.

• Let us check that $\mathcal{L}_{\text{Dirac}}$ is a Lorentz scalar:

$$\mathcal{L}_1 = \bar{\Psi}' \left( i \gamma^{\mu} \partial_\mu - M \right) \Psi' = \bar{\Psi} \Lambda^{-1} \left( i \gamma^{\mu} \Lambda \frac{1}{2} \partial_\mu - M \right) \Lambda \frac{1}{2} \Psi \quad (7.93a)$$

$$= \bar{\Psi} \left( i \gamma^{\mu} \partial_\mu - M \right) \Psi \quad (7.93b)$$

$$= \bar{\Psi} \left( i \gamma^{\mu} \partial_\mu - M \right) \Psi \quad (7.93c)$$

$$= \bar{\Psi} \left( i \gamma^{\mu} \partial_\mu - M \right) \Psi \quad (7.93d)$$

$$= \bar{\Psi} \left( i \gamma^{\mu} \partial_\mu - M \right) \Psi \quad (7.93e)$$

Here we used the following fact:

The gamma matrices transform not like Lorentz vectors: $\gamma''^\mu = \gamma^\mu$. (7.94)

This is good because otherwise the Dirac equation would be different in different inertial systems.

This also means that slashed quantities (like $\not{\partial} = \gamma^{\mu} \partial_\mu$) are not Lorentz scalars. Think of it like this: they do not have a Lorentz index, but they do have two spinor indices (which we don’t write) because they are matrices. To get rid of these indices, you must pair them with the indices of spinor fields. That is, slashed quantities become Lorentz scalars if put between two Dirac spinors like in the Dirac Lagrangian: $\bar{\Psi} \not{\partial} \Psi$ is a scalar field.

12 | Conserved current:

Now that we have a Lagrangian, it is just a straightforward application of Noether’s theorem to obtain the conserved current associated to global phase rotations:
Global phase rotations:

Eq. (7.90) is clearly invariant under global phase rotations of the spinors:

$$\Psi'(x) = e^{i\alpha}\Psi(x) \quad \text{for} \quad \alpha \in [0, 2\pi) \quad (7.95)$$

with infinitesimal generator $|\alpha| = |w| \ll 1$

$$\Psi'(x) = \Psi(x) + i w \Psi(x) \equiv \Psi(x) + w \delta \Psi(x) \quad \Rightarrow \quad \delta \Psi = i \Psi \quad (7.96)$$

→ Continuous symmetry:

$$\mathcal{L}_{\text{Dirac}}(\Psi, \partial \Psi) = \mathcal{L}_{\text{Dirac}}(\Psi', \partial \Psi') \quad (7.97)$$

Noether theorem 6.85 → Conserved current density:

A straightforward calculation yields:

$$j^\mu_{\text{Dirac}} = \gamma^\mu \bar{\Psi} \Psi \quad \text{with} \quad \partial_\mu j^\mu_{\text{Dirac}} = 0 \quad (7.99)$$

Since the Lagrangian density $\mathcal{L}_{\text{Dirac}}$ is a Lorentz scalar, this Noether current must be a 4-vector. We can check this explicitly:

$$j^\mu_{\text{Dirac}} = \bar{\Psi} \gamma^\mu \Psi = \bar{\Psi} \Lambda^{-1}_x \gamma^\mu \Lambda_x^1 \Psi \quad (7.75)$$

→

Conserved Noether charge:

$$Q = \int d^3x \, \bar{\Psi}_0 \Psi = \int d^3x \, \bar{\Psi}^\dagger \Psi \geq 0 \quad (7.101)$$

Conserved norm on $L^2 \otimes \mathbb{C}^4$: $\|\Psi\|^2 := \int d^3x \, \bar{\Psi}^\dagger \Psi$

• $Q$! The positive-definite density $\bar{\Psi}^\dagger \Psi = \bar{\Psi}_0 \Psi$ is the time-component of a 4-vector and therefore not Lorentz invariant. However, the Noether charge $Q$ is a Lorentz scalar so that the norm is Lorentz invariant: $\|\Psi'\| = \|\Psi\|$.

Note that not all Noether charges are Lorentz scalars. The total field momentum Eq. (6.92), for example, is a 4-vector; similarly, the total field angular momentum Eq. (6.117) is a tensor of rank 2. However, it can be shown that the Noether charges of internal symmetries (like the $U(1)$ symmetry considered here) are necessarily Lorentz scalars (↑ Coleman-Mandula theorem [90]).

Let us prove $Q'^a = Q^a$ in the case where the Noether current $j^a_\mu$ has no other Lorentz index (and the internal group generators commute with the generators of Lorentz transformations):
We consider an infinitesimal Lorentz transformation. Coordinates transform according to Eq. (6.78),

\[ \delta \alpha^B x^\mu = \frac{1}{2} \left( \eta^{\alpha \mu} x^B - \eta^{\beta \mu} x^\alpha \right), \]  

and, as a 4-vector, the components of the current transform in the same way:

\[ \delta \alpha^B j^\mu_a = \frac{1}{2} \left( \eta^{\alpha \mu} j^\beta_a - \eta^{\beta \mu} j^\alpha_a \right) \equiv j^\nu_a \left( \partial_\nu \delta \alpha^B x^\mu \right). \]  

(The labels \( \alpha \) of the internal symmetry do not mix under this transformation because the internal symmetry is assumed to commute with Lorentz transformations.)

The generator of Lorentz transformations acts then according to Eq. (6.81) on the current field

\[ -i G_a \alpha^B j^\mu_a (x) = \delta \alpha^B j^\mu_a (x) - \left( \partial_\nu j^\mu_a \right) \delta \alpha^B x^\nu. \]  

In the following we suppress the indices \( \alpha \beta \) whenever possible.

It is easy to check that \( \partial_\nu \delta x^\nu = 0 \); furthermore, we know that \( \partial_\nu j^\mu_a = 0 \) from the Noether theorem. Together, this allows us to write the action of infinitesimal Lorentz transformations on the current as a 4-divergence:

\[ -i G^a \alpha^B j^\mu_a (x) = \left( \partial_\nu j^\nu_a \right) \delta x^\mu + j^\nu_a \left( \partial_\nu \delta x^\mu \right) - \left( \partial_\nu j^\mu_a \right) \delta x^\nu \]  

\[ = \partial_\nu \left( j^\nu_a \delta x^\mu - j^\mu_a \delta x^\nu \right). \]

Here we used that \( \delta j^\mu_a = j^\nu_a (\partial_\nu \delta x^\mu) \).

We finally obtain for the infinitesimal Lorentz transformation of the Noether charge:

\[ -i G Q_a = \int d^3x \left( -i G j^0_a \right) \]  

\[ = \int d^3x \partial_\nu \left( j^\nu_a \delta x^0 - j^0_a \delta x^\nu \right) \]  

\[ = \int d^3x \partial_i \left( j^i_a \delta x^0 - j^0_a \delta x^i \right) \]  

Gauss’s theorem

\[ = \int \partial_\nu \left( j^\nu_a \delta x^0 - j^0_a \delta x^\nu \right) = 0 \]

In the last step we used that on the surface \( \partial \) (typically spatial infinity) all fields vanish (for wavefunctions in \( L^2 \) this is clearly true).

Thus any Noether charge derived from internal symmetries transforms as a Lorentz scalar. In particular, the Dirac norm \( \| \Psi \| \) is invariant under Lorentz transformations of the bispinor fields \( \Psi (x) \).

Since the Dirac equation is first order in time, we can easily bring it into Schrödinger form and identify the Hamiltonian as the generator of time translations:

\[ \text{Eq. (7.83)} \quad \Leftrightarrow \quad \left[ i \hbar \gamma^0 \partial_t + i \hbar c \gamma^i \partial_i - mc^2 \right] \Psi = 0 \]  

Use \( (\gamma^0)^2 = 1 \rightarrow \)

\[ i \hbar \partial_t \Psi = \left[ -i \hbar c \gamma^0 \gamma^i \partial_i + \gamma^0 mc^2 \right] \Psi \]  

\[ (7.109) \]
Let us define the new matrices:

\[
\beta := \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_i := \gamma^0 \gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \quad i = 1, 2, 3 \tag{7.110}
\]

with \( \beta^2 = 1 = \alpha_i^2 \) and \( \{\alpha_i, \alpha_j\} = 0 = \{\alpha_i, \beta\} \) for \( i \neq j \), and in particular

\[
\beta^\dagger = \beta \quad \text{and} \quad \alpha_i^\dagger = \alpha_i. \tag{7.111}
\]

Note that the spatial gamma matrices are anti-Hermitian: \( (\gamma^i)^\dagger = -\gamma^i \).

With these matrices we can define the ... Dirac Hamiltonian:

\[
\hat{H}_{\text{Dirac}} = -i\hbar c \vec{\alpha} \cdot \nabla + \beta mc^2 = c \vec{\alpha} \cdot \vec{p} + \beta mc^2 \tag{7.112}
\]

with \( \vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \) and the \( \downarrow \) momentum operator \( \vec{p} = -i\hbar \nabla \).

The Dirac Hamiltonian is Hermitian:

(With respect to the standard inner product on \( L^2 \otimes \mathbb{C}^4 \):

\[
\hat{H}_{\text{Dirac}}^\dagger = c \vec{\alpha}^\dagger \cdot \vec{p}^\dagger + \beta^\dagger mc^2 = c \vec{\alpha} \cdot \vec{p} + \beta mc^2 = \hat{H}_{\text{Dirac}} \tag{7.113}
\]

Here we use that the momentum operator is self-adjoint (Hermitian) for (a dense subset of) functions in \( L^2(\mathbb{R}^3, \mathbb{C}) \):

\[
\langle \psi | \vec{p} \phi \rangle = \int d^3x \psi^*(-i\hbar \nabla \phi) = \int d^3x (-i\hbar \nabla \psi)^* \phi = \langle \vec{p} \psi | \phi \rangle. \tag{7.114}
\]

We used partial integration and \( \lim_{|x| \to \infty} \phi(x) = 0 = \lim_{|x| \to \infty} \psi(x) \) for admissible functions.

The Dirac equation then takes the Schrödinger form

\[
i \hbar \partial_t \Psi(x) = \hat{H}_{\text{Dirac}} \Psi(x) \tag{7.115}
\]

In this form its Lorentz covariance is no longer manifest.

Eq. (7.102) conserved \( \leftrightarrow \) Inner product on \( L^2 \otimes \mathbb{C}^4 \):

\[
\langle \Psi | \Phi \rangle := \int d^3x \Psi^\dagger(t, \vec{x}) \Phi(t, \vec{x}) \quad \text{with} \quad \|\Psi\| = \sqrt{\langle \Psi | \Psi \rangle} \tag{7.116}
\]

This inner product is constant under the evolution of the Dirac equation:

\[
\text{Eq. (7.113) \& Eq. (7.115)} \quad \Rightarrow \quad \frac{d}{dt} \langle \Psi | \Phi \rangle \triangleq 0 \tag{7.117}
\]

• This generalizes our previous finding in Eq. (7.102) about the conserved norm.
That the inner product is constant is straightforward to show:

\[
\frac{d}{dt} \langle \Psi | \Phi \rangle = \int d^3x \left[ \Psi^\dagger \Phi + \Psi^\dagger \Phi \right]
\]

We have

\[
\frac{1}{i \hbar} \int d^3x \left[ \Psi^\dagger \left( \hat{H}_{\text{Dirac}} \Phi \right) - \left( \hat{H}_{\text{Dirac}} \Psi \right)^\dagger \Phi \right] = 0
\]

\[
\frac{1}{i \hbar} \int d^3x \left[ \Psi^\dagger \left( \hat{H}_{\text{Dirac}} \Phi \right) - \Psi^\dagger \left( \hat{H}_{\text{Dirac}} \Phi \right) \right] = 0
\]

14 Conclusion:
Let us summarize our findings and compare them to the Klein-Gordon equation:

<table>
<thead>
<tr>
<th>Klein-Gordon Equation</th>
<th>Dirac Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\partial^2 + M^2)\phi = 0)</td>
<td>((i \partial - M)\Psi = 0)</td>
</tr>
<tr>
<td>Time derivative</td>
<td>second order</td>
</tr>
<tr>
<td>Function space</td>
<td>(L^2(\mathbb{R}^{1,3}, \mathbb{C}))</td>
</tr>
<tr>
<td>Wavefunction</td>
<td>Complex scalar field (\phi(x))</td>
</tr>
<tr>
<td>Conserved form</td>
<td>(i \int d^3x \left( \phi^\dagger \phi_2 - \phi^\dagger_1 \phi_2 \right))</td>
</tr>
<tr>
<td>Positive definite?</td>
<td>(x)</td>
</tr>
<tr>
<td>Hermitian?</td>
<td>(\checkmark)</td>
</tr>
<tr>
<td>Hamiltonian?</td>
<td>(x)</td>
</tr>
<tr>
<td></td>
<td>(\checkmark)</td>
</tr>
</tbody>
</table>

→ What about the eigenenergies and eigenstates of \(\hat{H}_{\text{Dirac}}\)?

7.2.1. Free-particle solutions of the Dirac equation

15 Eq. (7.86): Solutions of the Dirac equation satisfy the Klein-Gordon equation component-wise:

\[
\psi^\pm(x) = \psi^\pm(p)e^{\mp \frac{E}{\hbar} p x}
\]

with \(p^0 = \frac{E}{c} = \sqrt{p^2 + m^2 c^2} > 0\)

\[
(\pm y^\mu p_\mu - mc)\psi^\pm(p) = \begin{pmatrix} -mc & \pm p \sigma & \psi_L^\pm \\ \pm p \sigma & -mc & \psi_R^\pm \end{pmatrix} = 0
\]

with \(p \sigma = p_\mu \sigma^\mu\) and \(\sigma^\mu = (\mathbb{1}, \sigma^x, \sigma^y, \sigma^z)\) and \(\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^x, -\sigma^y, -\sigma^z)\).
Mathematical facts (check these!):

- \((p\sigma)(p\bar{\sigma}) = p^2 = m^2c^2\)
- Eigenvalues of \(p\sigma\) and \(p\bar{\sigma}\): \(p^0 \pm |\bar{p}| \rightarrow \) for \(p^0 > 0\) and \(m \neq 0\) positive spectrum
  \(\rightarrow p\sigma\) and \(p\bar{\sigma}\) are invertible and the positive square roots \(\sqrt{p\sigma}\) and \(\sqrt{p\bar{\sigma}}\) are Hermitian.

\(< \varphi^\pm_L = \sqrt{p\sigma} \xi^\pm \) with arbitrary, normalized \(\{(\xi^\pm)^\dagger \xi^\pm = 1\}\) spinor \(\xi^\pm \in \mathbb{C}^2\):

\[
\text{Eq. (7.121)} \quad \Rightarrow -m_c \sqrt{p\sigma} \xi^\pm \pm p\sigma \varphi^\pm_R = 0 \quad (7.122)
\]

Use \(\sqrt{p\sigma} \sqrt{p\bar{\sigma}} = m_c\):

\[
\varphi^\pm_R = \pm \frac{m_c}{\sqrt{p\sigma}} \xi^\pm = \pm \sqrt{p\sigma} \xi^\pm
\]

\(\rightarrow \varphi^\pm_L\) and \(\varphi^\pm_R\) are now parametrized by the spinor \(\xi^\pm \in \mathbb{C}^2\) (which is unconstrained!).

The second equation in Eq. (7.121) yields the same solution.

Solutions:

Let us adopt the more conventional notation

\[
\xi^+ \mapsto \xi \quad \text{and} \quad \psi^+ \mapsto u
\]

\[
\xi^- \mapsto \eta \quad \text{and} \quad \psi^- \mapsto v
\]

and choose the spinor basis \(\xi^s, \eta^s (s = \uparrow, \downarrow)\) with

\[
\xi^\uparrow, \eta^\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi^\downarrow, \eta^\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Then linearly independent solutions of the free Dirac equation can be written as:

\[
\Psi^+_{\vec{p},s}(x) = \begin{pmatrix} \sqrt{p\sigma} \xi^s \\ \sqrt{p\bar{\sigma}} \xi^s \end{pmatrix} e^{-\frac{i}{\hbar} \vec{p} \cdot \vec{x}} \quad \text{(positive energy solutions)}
\]

\[
\Psi^-_{\vec{p},s}(x) = \begin{pmatrix} \sqrt{p\bar{\sigma}} \eta^s \\ -\sqrt{p\sigma} \eta^s \end{pmatrix} e^{\frac{i}{\hbar} \vec{p} \cdot \vec{x}} \quad \text{(negative energy solutions)}
\]

with \(p^\mu = (p^0, \vec{p})\), \(p^0 = \sqrt{p^2 + m^2c^2} > 0\) and \(s = \uparrow, \downarrow\).

\(\rightarrow\) Four linearly independent solutions for each 3-momentum \(\vec{p}\) (\(\pm\) and \(s = 1, 2\)).

You can easily check that Eq. (7.126) form an orthogonal eigenbasis of the Dirac Hamiltonian:

\[
\hat{H}_{\text{Dirac}} \Psi^\pm_{\vec{p},s} = \pm \pm E_{\vec{p}} \Psi^\pm_{\vec{p},s} \quad \text{with spectrum} \quad E_{\vec{p}} = \sqrt{p^2c^2 + m^2c^4}.
\]

Their orthogonality follows with the identities

\[
[u^r(\vec{p})]^\dagger u^s(\vec{p}) \triangleq \frac{i}{\hbar} E_{\vec{p}} \delta^{rs}, \quad [v^r(\vec{p})]^\dagger v^s(\vec{p}) \triangleq \frac{i}{\hbar} E_{\vec{p}} \delta^{rs}, \quad [u^r(\vec{p})]^\dagger v^s(-\vec{p}) \triangleq 0.
\]

\(\rightarrow\) The Dirac equation still has \textit{negative-energy solutions}. \(\quad (7.129)\)
**Interpretation:**

- The negative energy solutions are not problematic as long as we consider a single particle (electron) without interactions (this is also why we can apply the Dirac equation to describe the hydrogen atom, → below). However, in reality the electron couples to a dynamic electromagnetic field and therefore could emit a photon (thereby lowering its energy). If the negative energy eigenstates really exist, there is no reason why this process should terminate; as a consequence, no stable electrons should exist.

Dirac writes in Ref. [91]:

> It is true that in the case of a steady electromagnetic field we can draw a distinction between those solutions [...] with $E$ positive and those with $E$ negative and may assert that only the former have a physical meaning (as was actually done when the theory was applied to the determination of the energy levels of the hydrogen atom), but if a perturbation is applied to the system it may cause transitions from one kind of state to the other. In the general case of an arbitrarily varying electromagnetic field we can make no hard-and-fast separation of the solutions of the wave equation into those referring to positive and those to negative kinetic energy. Further, in the accurate quantum theory in which the electromagnetic field also is subjected to quantum laws, transitions can take place in which the energy of the electron changes from a positive to a negative value even in the absence of any external field, the surplus energy [...] being spontaneously emitted in the form of radiation. [...] Thus we cannot ignore the negative-energy states without giving rise to ambiguity in the interpretation of the theory.

Dirac suggested a “fix” for this problem [91]: Because the electron is a fermion, it obeys the Pauli exclusion principle. Thus one can imagine that (for some reason) all the negative energy states are already occupied by electrons. The electrons we see can then only occupy the positive energy states and cannot decay to states of arbitrarily low energy. This construct is know as the ↑ hole theory because creating a “hole” in this ↑ Dirac sea of electrons with negative energy can be viewed as an excitation with positive energy. Dirac’s holes are of course a precursor to what we know today as ↑ antiparticles. (Dirac didn’t think of it this way, he conjectured that the holes in his sea of electrons are the protons!)

- However, Dirac’s interpretation is not how we deal with the negative-energy solutions today: Within the modern framework of ↑ relativistic quantum field theories, the four single-particle wave functions are associated (through “second” quantization of the Dirac field and the construction of a fermionic ↑ Fock space) to two particle types, both with positive energy and two internal spin-$\frac{1}{2}$ states:

<table>
<thead>
<tr>
<th>Type</th>
<th>Momentum</th>
<th>Spin</th>
<th>Energy</th>
<th>Charge</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Psi_{+\uparrow}^{\uparrow}$</td>
<td>fermion</td>
<td>$+\vec{p}$</td>
<td>$+\frac{1}{2}$</td>
<td>$+E_{\vec{p}}$</td>
</tr>
<tr>
<td>$\Psi_{+\downarrow}^{\downarrow}$</td>
<td>fermion</td>
<td>$+\vec{p}$</td>
<td>$-\frac{1}{2}$</td>
<td>$+E_{\vec{p}}$</td>
</tr>
<tr>
<td>$\Psi_{-\uparrow}^{\uparrow}$</td>
<td>antifermion</td>
<td>$+\vec{p}$</td>
<td>$-\frac{1}{2}$</td>
<td>$+E_{\vec{p}}$</td>
</tr>
<tr>
<td>$\Psi_{-\downarrow}^{\downarrow}$</td>
<td>antifermion</td>
<td>$+\vec{p}$</td>
<td>$+\frac{1}{2}$</td>
<td>$+E_{\vec{p}}$</td>
</tr>
</tbody>
</table>

(7.130)

Here “Spin” refers to the ↓ spin-polarization quantum number $m_z = \pm \frac{1}{2}$.

→ **Take home message:**

Relativistic quantum mechanics predicts spin and antiparticles. (7.131)
The negative energy solutions (and therefore the existence of antiparticles) are a necessary feature of relativistic quantum mechanics (more precisely: relativistic quantum field theories, via the *CPT*-theorem).

By contrast, the fact that particles can have an internal angular momentum (spin), and that this angular momentum can take half-integer values $S = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \ldots$ is not a relativistic feature *per se*: Spin enters quantum mechanics the moment one considers spatial rotations and its representations on the Hilbert space. Because these can be *projective*, one is forced to study the irreducible linear representations of $SU(2)$ – the double cover of the rotation group $SO(3)$ – which happen to be labeled by the “spin quantum numbers” $S = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \ldots$ Now, since the rotation group is a subgroup of the homogeneous Lorentz group, $SO(3) \subset SO^+(1, 3)$, the moment a quantum theory is relativistic [i.e., features a representation of $SO^+(1, 3)$], spin enters the stage automatically. However, you can describe quantum particles with spin *without* making quantum mechanics relativistic.

- The Dirac equation applies to all spin-$\frac{1}{2}$ fermions. The most prominent example is of course the electron $e^-$ and its associated antiparticle, the positron $e^+$. However, all other elementary fermions, namely leptons (like the muon/antimuon, the tau/antitau and the neutrinos) and the six quark/antiquark pairs, are described by the Dirac equation as well.

### 7.2.2. The relativistic hydrogen atom

**Dirac equation with a static EM field:**

To couple the Dirac field $\Psi$ in a gauge- and covariant way to a static EM field $A^\mu$, we use the same trick as for the Klein-Gordon equation:

$\leftarrow \text{Minimal coupling Eq. (7.38)} \rightarrow$

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + i Q A_\mu \quad \Rightarrow \quad \partial \rightarrow \mathcal{D} = \partial + i Q A = \gamma^\mu \partial_\mu + i Q \gamma^\mu A_\mu \quad (7.132)$$

For an electron it is $Q = -e$ with $e > 0$.

$$\quad \rightarrow \quad (i \mathcal{D} - M) \Psi = 0 \quad (7.133)$$

In this form, the Dirac equation is manifest Lorentz- and gauge invariant.

We can expand Eq. (7.133) to obtain a less abstract (but more convoluted) expression:

$$\begin{align*}
\left[ i \gamma^\mu \partial_\mu - Q \gamma^\mu A_\mu - M \right] \Psi &= 0 \\
\Leftrightarrow \left[ i \hbar y^0 \partial_t + i \hbar c y^i \partial_i - q y^0 \varphi + q y^i A_i - mc^2 \right] \Psi &= 0 \quad (7.134a) \\
\Leftrightarrow \left[ i \hbar \partial_t + i \hbar c \vec{\alpha} \cdot \nabla - q \varphi + q \vec{\alpha} \cdot \vec{A} - \beta mc^2 \right] \Psi &= 0 \quad (7.134b) \\
\end{align*}$$

Here we used $Q = \frac{e}{\hbar c}$, $M = \frac{mc^2}{\hbar}$, and $A_\mu = (\varphi, -\vec{A})$; $q$ is the charge of the particle.

In Schrödinger form the Dirac equation reads then:

$$\begin{align*}
i \hbar \partial_t \Psi &= \left[ -i \hbar c \vec{\alpha} \cdot \nabla + q \varphi - q \vec{\alpha} \cdot \vec{A} + \beta mc^2 \right] \Psi \quad (7.135a) \\
i \hbar \partial_t \Psi &= \left[ c \vec{\alpha} \cdot \left( \frac{\vec{p} - \vec{\mathcal{A}}}{\hbar} \right) + q \varphi + \beta mc^2 \right] \Psi \quad (7.135b)
\end{align*}$$
Choose the Coulomb potential (of the proton)
\[
\varphi(x) = \frac{e}{|x|} \quad \text{and} \quad \vec{A} = \vec{0}
\]
and set \( q = -e \) (charge of the electron) →
\[
i\hbar \partial_t \Psi = \left[ -i \hbar c \vec{a} \cdot \nabla - \frac{e^2}{|x|} + \beta mc^2 \right] \Psi
\]
(7.137)

With the ansatz \( \Psi(t, \vec{x}) = \psi(\vec{x}) e^{-i E t} \) one obtains the time-independent eigenvalue problem
\[
\left[ -i \hbar c \vec{a} \cdot \nabla - \frac{e^2}{|x|} + \beta mc^2 - E \right] \psi(\vec{x}) = 0 \quad \text{with} \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} : \mathbb{R}^3 \rightarrow \mathbb{C}^4.
\]
(7.138)

Note that \( \beta \) (unlike \( \alpha \)) is an off-diagonal block matrix that mixes the two spinors \( \psi_+ \) and \( \psi_- \); this complicates the solution. However, one can solve Eq. (7.138) exactly and compute the eigenvalues \( E \) and eigenstates \( \psi(\vec{x}) \).

\[\textbf{Solution:} \star \quad \text{Eigenenergies (including the rest energy of the electron):}\]

\[
E_{n,j} = mc^2 \left\{ 1 + \frac{\alpha^2}{n - j - \frac{1}{2} + \sqrt{(j + \frac{1}{2})^2 - \alpha^2}} \right\}^{-\frac{1}{2}}
\]
(7.139)

with
- \( \downarrow \textit{principal quantum number} \ n = 1, 2, \ldots \)
- \( \downarrow \textit{total angular momentum quantum number} \ j = \frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2} \)
- \( \downarrow \textit{fine-structure constant} \ \alpha \approx \frac{1}{137} \)

The principal quantum number \( n = 1, 2, \ldots \) constrains the allowed \textit{orbital} angular momentum to \( l = 0, 1, \ldots, n - 1 \). The allowed \textit{total} angular momentum is then given by the usual rules of angular momentum addition: \( |l - \frac{1}{2}| \leq j \leq |l + \frac{1}{2}| \) (in integer steps, \( s = \frac{1}{2} \) is the electron spin).

So for example \( n = 1 \) allows only for \( l = 0 \) and therefore \( j = \frac{1}{2} \); this is the \( 1S_{1/2} \) orbital and the ground state of the hydrogen atom. For \( n = 2 \) one finds again \( l = 0 \) with \( j = \frac{1}{2} \) (the \( 2S_{1/2} \) orbital) but also \( l = 1 \) with \( j = \frac{1}{2} \) and \( j = \frac{3}{2} \) (the \( 2P_{1/2} \) and \( 2P_{3/2} \) orbitals) – which are no longer degenerate because \( E_{2,1/2} \neq E_{2,3/2} \).

This result explains why in the hydrogen spectrum the degeneracy of the \( 2S_{1/2} \) and \( 2P_{3/2} \) orbitals is lifted whereas the \( 2S_{1/2} \) orbital remains degenerate with the \( 2P_{1/2} \) orbital (\( \star \) fine-structure).

\[\text{The Dirac equation explains the fine-structure of the hydrogen atom} \odot.\]
(7.140)

\textbf{Note:} You may have encountered the following Hamiltonian for the hydrogen atom with added relativistic corrections:
\[
\hat{H}_{\text{rel}} = \frac{\vec{p}^2}{2m} - \frac{e^2}{r} - \frac{1}{2mc^2} \left( \frac{\vec{p}^2}{2m} \right)^2 + \frac{e^2}{2m^2c^2} \frac{\vec{L} \cdot \vec{S}}{r^3} - \frac{e^2 \hbar^2}{8m^2c^2} \Delta \left( \frac{1}{r} \right) \quad \text{.}
\]
(7.141)

This Hamiltonian can reproduce the fine-structure as well. It has several drawbacks, though:
• It is only an approximation.
• It is hard to solve (perturbation theory!).
• The Schrödinger equation \( \text{i} \hbar \partial_t \psi = \hat{H}_{\text{rel}} \psi \) is not manifestly Lorentz covariant.
• The relativistic corrections are ad hoc and seemingly independent of each other.

Luckily, Eq. (7.141) does not have to appear out of thin air; one can show via a complicated derivation (Foldy-Wouthuysen transformation) that it is indeed the non-relativistic limit [with corrections in order \((v/c)^2\)] of the Dirac equation Eq. (7.138) in a Coulomb potential (without the rest energy \(mc^2\) of the electron).

7.2.3. The electron \(g\)-factor

Besides the fine structure, there is one other “mystery” that the relativistic treatment of the electron in terms of the Dirac equation finally explains: The non-classical ratio between the electrons internal magnetic moment and its spin.

24 | Dirac electron in homogeneous magnetic field \( \vec{B} = \nabla \times \vec{A} (\varphi = 0)\):

Eq. (7.135b) \[ \psi = \psi L e^{-\frac{\vec{p} \cdot \vec{A}}{2mc^2}} (\vec{p} + \frac{\vec{e}}{c} \vec{A}) + \beta mc^2 - E \] \[ \psi = 0 \] (7.142)

with bispinor

\[ \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} : \mathbb{R}^3 \to \mathbb{C}^4. \] (7.143)

Using Eq. (7.110) we can write this equation in terms of the two spinors:

\[ (-c \vec{\sigma} \cdot \vec{p} - E) \psi_L + mc^2 \psi_R = 0 \] (7.144a)
\[ (+c \vec{\sigma} \cdot \vec{p} - E) \psi_R + mc^2 \psi_L = 0 \] (7.144b)

Here we used \( \vec{\sigma} = \vec{p} + \frac{\vec{e}}{c} \vec{A} \) and introduced \( \vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z) \).

We can now use one of the two equations to decouple the system:

\[ (c \vec{\sigma} \cdot \vec{p} + E) (c \vec{\sigma} \cdot \vec{p} - E) \psi_R + (mc^2)^2 \psi_R = 0 \] (7.145a)
\[ \Leftrightarrow c^2 (\vec{\sigma} \cdot \vec{p})^2 \psi_R - [E^2 - (mc^2)^2] \psi_R = 0 \] (7.145b)

25 | Non-relativistic approximation:

We can use \( E^2 - (mc^2)^2 = (E - mc^2)(E + mc^2) \approx 2mc^2 \hat{E} \) with \( \hat{E} = E - mc^2 \) to find a non-relativistic approximation of Eq. (7.145b):

\[ \frac{1}{2m} (\vec{\sigma} \cdot \vec{p})^2 \psi_R = \hat{E} \psi_R \] (7.146)

Last, use the Pauli algebra \( \sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k \) and \( B_k = \epsilon_{ijk} (\partial_j A_i) \) to show that \( (\vec{\sigma} \cdot \vec{p})^2 \approx \vec{p}^2 + \frac{\hbar^2}{c^2} \vec{\sigma} \cdot \vec{B} \). We end up with the non-relativistic, time-independent Schrödinger equation of a charged particle in a magnetic field with a spin-dependent Zeeman term:

\[ \left[ \frac{1}{2m} \left( \vec{p} + \frac{\vec{e}}{c} \vec{A} \right)^2 + \frac{e\hbar}{2mc} \vec{\sigma} \cdot \vec{B} \right] \psi_R = \hat{E} \psi_R \] (7.147)
Potential energy of electron in magnetic field:

\[ E_{\text{mag}} \overset{\text{def}}{=} -\mu \cdot \vec{B} = \frac{\hbar}{2mc} \vec{\sigma} \cdot \vec{B} \quad (7.148) \]

Magnetic moment (operator) of the electron:

\[ \hat{\mu}_e = -\frac{\hbar}{2mc} \vec{\sigma} = g_e \frac{\mu_B}{\hbar} \vec{S} \quad (7.149) \]

with spin operator \( \vec{S} = \frac{\hbar}{2} \vec{\sigma} \) and Bohr magneton \( \mu_B = \frac{e\hbar}{2mc} \) and

\[ g_e = -2. \quad (7.150) \]

Comments:

- What makes Eq. (7.149) with \( g_e = -2 \) remarkable is that it is not what one would expect if the magnetic moment would be caused by a charge flying along a tiny orbit with angular momentum \( \vec{S} \). Indeed, a straightforward classical calculation yields for the relation between magnetic moment and (orbital) angular momentum \( \vec{L} \):

\[ \hat{\mu}_L = g_L \frac{\mu_B}{\hbar} \vec{L} \quad \text{with} \quad g_L = -1. \quad (7.151) \]

So, quite surprisingly, the Dirac equation predicts that the internal angular momentum (= spin) produces “twice as much” magnetic moment as one would naïvely expect.

That this really is the case can be easily measured: Just apply a magnetic field to hydrogen atoms and observe how strongly their spectral lines split as a function of the magnetic field strength (↑ anomalous Zeeman effect). This effect had already been experimentally observed at the end of the 19th century [92, 93]. Since it was unknown at the time that electrons had spin, certain line splittings could not be explained (therefore “anomalous”). The fact that the Dirac equation explains both – the electron spin and its “non-classical” \( g \)-factor – is therefore a remarkable feature of relativistic quantum mechanics.

- If one measures the electron \( g \)-factor really, really precisely, one finds [94]

\[ g_e = -2.00231930436118 \pm 0.000000000027. \quad (7.152) \]

You may notice that this is not exactly \(-2\) but a tiny bit off. One cannot explain this deviation with the Dirac equation because it stems from “virtual particles” that modify how the electron interacts with the EM field (and the Dirac equation is a single-particle wave equation). It is therefore remarkable that modern theories can explain this deviation perfectly (up to error bars), but for this one needs the machinery of ↑ relativistic quantum field theory.
Part II.

General Relativity
8. Limitations of SPECIAL RELATIVITY

8.1. Reminder: SPECIAL RELATIVITY

1 | SPECIAL RELATIVITY in a nutshell:
   - ← Inertial frames [Section 1.1]
     There exists a special class of infinitely extended reference frames (equipped with Cartesian coordinates) in which the law of inertia holds (\( \mathbf{IN} \) = the trajectories of free particles are straight lines that are traversed with constant velocity). All inertial frames move relative to each other with constant velocities:

   ![Inertial Frames Diagram]

   - ← Einstein’s principle of (special) relativity \( \mathbf{SR} \) [Section 1.3]
     The laws of physics (orange boxes in the sketch below) have the same form in all inertial systems. This extends Galilei’s principle of relativity which makes this claim only for the realm of mechanics. The modifier “special” emphasizes that the principle makes only claims about the special class of inertial systems:

   ![Einstein's Principle Diagram]

   We characterized \( \mathbf{SR} \) previously as follows: No experiment can distinguish between inertial frames. This description can be misleading, so let me prevent any misconceptions: What we mean is that there is no local physical experiment that you can perform in a sealed box (at rest in some inertial system) that allows you to figure out in which inertial system your box is at rest. In this way you probe the form of physical laws (e.g., whether there is an additional Coriolis term in your equation of motion or not) and thus probe the validity of \( \mathbf{SR} \) as formulated above.
The statement above does not mean that there is no operational way to label specific inertial systems. For example, we can define the (approximate) inertial frame in which the center of Earth is at rest and, for comparison, another inertial frame in which the cosmic microwave background (CMB) has no dipole structure (the latter has a velocity of roughly 360 km s\(^{-1}\) wrt. the former). Clearly there are experiments to decide whether you are in one or the other (measure the CMB dipole and/or the velocity of Earth relative to you). This does not violate SR though, because all phenomena you observe in these frames of reference are described correctly by the same equations (e.g. you can use the same Maxwell equations to describe the CMB radiation in both inertial frames). This is also why the existence of the global inertial frame labeled by a CMB without dipole is not in conflict with special relativity. There is a difference between physical states and physical laws; SR only makes claims about the latter.

**Lorentz transformations** [Section 1.5]

The coordinate transformations that map the record of physical events from one inertial system to another are given by Lorentz transformations (more generally: Poincaré transformations). (Proper orthochronous) Lorentz transformations form a group \(SO^+(1, 3)\) and are parametrized by a three-dimensional rotation and a three-dimensional boost velocity. They linearly map the spacetime coordinates \((t; \vec{x})\) of an event in one inertial system \(K\) to the spacetime coordinates \((t'; \vec{x}')\) of the same event in another inertial system \(K'\).

A pure boost in \(x\)-direction has the form:

\[
\Lambda(K, v_x \rightarrow K') : \begin{align*}
ct' &= \gamma (ct - \frac{v_x}{c^2}x) \\
x' &= \gamma (x - v_x t) \\
y' &= y \\
z' &= z
\end{align*}
\]

with Lorentz factor

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v_x^2}{c^2}}}
\]  

**Constancy of the speed of light** [Section 1.5]

Lorentz transformations are characterized (and differ from Galilei transformations) by the existence of a finite maximum velocity \(v_{\text{max}}\). Experience tells us to identify this velocity with the speed of light \(c\). Lorentz transformations then imply that this maximum velocity is the same for all initial observers (relativistic addition of velocities):

- Experiments \(v_{\text{max}} < \infty \Leftrightarrow \) Lorentz transformations
- \(v_{\text{max}} = \infty \Leftrightarrow \) Galilei transformations

**Tensor calculus** [Chapter 3 and Chapter 4]

Combining the principle of relativity with the assertion that Lorentz transformations translate between inertial systems implies that the laws of nature must be expressed as equations that are form-invariant under Lorentz transformations (Lorentz covariance). The Lorentz covariance of a theory can be quite tedious to show and even more tedious to ensure when constructing it from scratch. (Recall Maxwell equations in their conventional form!) This is why we prefer equations in which the Lorentz covariance is manifest. To achieve this, we developed tensor calculus as a “toolbox” to construct Lorentz covariant equations from Lorentz scalars, vectors, tensors, ….

For example, the equation of motion for a charged particle in an electromagnetic field reads...
and transforms as follows:

\[
\begin{align*}
\text{Inertial system } K & \quad \text{Inertial system } \tilde{K} \\
\frac{dp_i^\mu}{d\tau} = q_i c F^\mu_i u_i^\nu & \quad \tilde{p}_i^\mu = \Lambda^\mu_\nu p_i^\nu \\
\tilde{u}_i^\mu = \Lambda^\mu_\nu u_i^\nu & \quad \tilde{F}_i^\mu = \Lambda^\mu_\alpha \Lambda^\alpha_\beta F_i^\beta
\end{align*}
\]

(8.3)

2 | Problems:

Despite the undeniable success of \textit{special relativity}, it’s not just sunshine and roses:

- **What about gravitation?**
  
  In our discussion of \textit{special relativity} we explicitly avoided the phenomenon of gravitation (we will see below why). This makes \textit{special relativity} clearly incomplete (and \textit{special}) as a description of nature (which, on very large scales, is dominated by the gravitational force) and asks for a more \textit{general} theory.

- **Why are inertial coordinate systems special?**
  
  \textit{Special relativity} describes physics with respect to a particular class of reference frames (inertial frames) in a particular class of coordinates (Cartesian coordinates). Only in these coordinate frames the laws of nature take their “simplest” form and the Lorentz transformation only translates between these special coordinate systems. However, in our very general discussion of differential geometry (Chapter 3) we established the notion of “geometric objects” that are independent of coordinates. We also interpreted coordinates as mathematical auxiliary structures to label events, and denied their physical existence (“coordinates do not exist”). \textit{Special relativity} does not live up to this rather fundamental claim with its focus on inertial coordinate systems. Shouldn’t there be a formulation of physics in which coordinates play no role at all?

- **What is the origin of inertia?**
  
  Remember Newton’s bucket (p. 13)? It’s punchline was to argue for the existence of an entity (“absolute space”) which determines whether an object is accelerated or not. \textit{Special relativity}, of course, disposes of Newton’s absolute space w.r.t. to which position and velocity can be measured (no \textit{ether}!). The existence of such, however, was never implied by the bucket experiment anyway, which asks about the absolute notion of \textit{acceleration}. And \textit{special relativity} is silent about the origin of inertia and what determines whether the water in Newton’s bucket is concave or flat (we simply assumed that inertial frames exist, we neither asked where they come from nor what makes them inertial in the first place). This situation is clearly unsatisfactory.

3 | Non-problems:

Sometimes one hears that \textit{acceleration} is a problem for \textit{special relativity}. This is not so:

- **Accelerated motion ✓**
  
  \textit{Special relativity} of course describes accelerated objects perfectly well. Recall our concept of 4-acceleration in Section 5.1, the relativistic equation of motion in Eq. (5.6), and the validity of the proper time integral for arbitrary time-like trajectories in Eq. (2.25). Note that these equations are only valid in inertial frames, though.

- **Accelerated observers ✓**
  
  While our equations were given in inertial systems (where, according to Einstein’s principle of relativity, the laws of physics take the same \textit{and simplest} form), \textit{special relativity}
can describe the physics in accelerated non-inertial frames as well (e.g. using the concept of instantaneous rest frames). In such non-inertial coordinate systems the physical laws do not take their simplest forms and can look messy (in particular, one cannot Lorentz transform into these frames). This, however, does not mean that we cannot describe what happens in such systems. As an example recall the relativistic rocket of Problemset 6. (For details see Chapter 6 in Misner et al. [2] and also Einstein’s original work [95].)

8.2. The special role of gravity

Let us now focus on the problem of incorporating gravity into SPECIAL RELATIVITY. It is important to understand why the gravitational force poses a fundamental problem for the framework of SPECIAL RELATIVITY (and is not just a technical inconvenience).

Note on nomenclature:

In English, there are two terms with slightly different meaning (if we take *Merriam-Webster* as a reference):

- **Gravity**: the gravitational attraction of the mass of the Earth, the moon, or a planet for bodies at or near its surface
- **Gravitation**: a force manifested by acceleration toward each other of two free material particles or bodies or of radiant-energy quanta

This distinction has no counterpart in German as far as I can tell (perhaps “Schwerkraft” vs. “Gravitation”?). Given that even the English literature does not seem to be consistent, I will use these two terms interchangeably. Their context will suffice to establish semantic clarity.

4 | Recall Newton’s law of universal gravitation:

\[ \nabla^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x}) \rightarrow \text{Gravitational potential } \phi(\vec{x}) \]  \hspace{1cm} (8.4)

\[ G \approx 6.674 \times 10^{-11} \text{ m}^3 \text{ s}^{-2} / \text{kg}: \text{Gravitational constant} \]

Please appreciate the smallness of G (and therefore the weakness of gravity) as compared to the human-scale units in which it is given. Gravity is, if compared to the other three fundamental forces, by far (really really really far) the weakest force. It is a fundamental unsolved problem of physics why this is so.

→ Equation of motion of test mass (e.g. a satellite):

\[ m_1 \ddot{r} = -m_G \nabla \phi(\vec{r}) \]  \hspace{1cm} (8.5)

\[ m_1: \text{inertial mass} \]
\[ m_G: \text{gravitational mass} \]

We will discuss the relation of \( m_1 \) and \( m_G \) later (Section 9.1).

ii | Example: Static point mass \( M_G \gg m_G \) as source in origin (e.g. Earth):

\[ \phi(\vec{x}) = -\frac{G M_G}{|\vec{x}|} \Rightarrow m_1 \ddot{r} = F_G = -G \frac{m_G M_G}{r^2} \]  \hspace{1cm} (8.6)

If the source is dynamic as well, \( \vec{r} \) is the relative distance vector between the two masses and \( m_1 \) must be replaced by the reduced mass of the two bodies.
Observation: Equations [especially Eq. (8.4)] are not Lorentz covariant!

You can check that they are Galilei invariant, recall Eq. (1.18).

This is no surprise: We already know from our discussions in Section 6.4 that in relativity, classical forces can only act locally, and not at a distance. Interactions between distant objects must be mediated by dynamical degrees of freedom (a “field”) to obey the speed limit for information propagation imposed by Lorentz symmetry. But Newton’s gravitational potential \( \phi \) is static and not dynamic!

5 Problem: “Action at a distance” (Gravitational force acts instantaneously and has no dynamics.)

Isaac Newton writes in a letter to Bentley in 1692 [96]:

> It is inconceivable, that inanimate brute Matter should, without the Mediation of something else, which is not material, operate upon, and affect other Matter without mutual Contact, as it must be, if Gravitation in the Sense of Epicurus, be essential and inherent in it. And this is one Reason why I desired you should not ascribe innate Gravity to me. That Gravity should be innate, inherent and essential to Matter, so that one Body may act upon another at a Distance through a Vacuum, without the Mediation of any thing else, by and through which their Action and Force may be conveyed from one to another, is to me so great an Absurdity, that I believe no Man who has in philosophical Matters a competent Faculty of thinking, can ever fall into it. Gravity must be caused by an Agent acting constantly according to certain Laws; but whether this Agent be material or immaterial, I have left to the Confederation of my Readers.

Thus even Newton himself was not entirely satisfied with his law of universal gravitation (which describes an action at a distance) and anticipated some entity that mediates the force.

6 First try: Make gravitational potential a dynamic field:

\[
\text{Poisson equation Eq. (8.4)} \quad \rightarrow
\]

\[
\text{Wave equation:} \quad \partial^2 \phi(t, \vec{x}) = \left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) \phi = -4\pi G \rho(t, \vec{x}) \quad (8.7)
\]

\( \rightarrow \) Gravity propagates with the speed of light ☺

* Problems:

For a detailed study of a fully specified scalar theory of gravity: ☞ Problemset 1 (also Exercise 7.1 in Misner et al. [2]). See also Ref. [97].

- Electromagnetic field cannot couple to gravity \( \rightarrow \) No bending of light ☹

Today it is a well tested fact that light follows a curved trajectory in strong gravitational fields (\( \rightarrow \) later). Thus any theory that does not couple the EM field to gravity must be incorrect.

Here is a quick-and-dirty explanation why a theory of the form Eq. (8.7) fails to couple to the electromagnetic field in a relativistic setting:

Since \( \phi \) is assumed to be a scalar field, for Eq. (8.7) to be Lorentz covariant, \( \rho \) must be a scalar as well. In a relativistic theory, energy and (inertial) mass are equivalent (\( E_0 = mc^2 \)). If we assume that gravitational mass and inertial mass are equivalent (\( \rightarrow \) later), this implies that energy (density) must be a source of gravity. The problem is that the energy (density) (of any theory) is the 00-component of the \( \leftrightarrow \) energy-momentum tensor \( T^{00} \) (this is the charge density associated to the Noether current that comes from translation symmetry in time); in particular, the energy density is not a scalar and therefore cannot be used as a source on the right-hand side of Eq. (8.7). The only scalar we can construct from the energy-momentum tensor is the \( \leftrightarrow \) Laue scalar \( T = T^{\mu \nu} = \eta_{\mu \nu} T^{\mu \nu} \), i.e., the trace of the EMT. Thus a
simplistic but fully Lorentz covariant form of scalar gravity is

\[ \Box \phi = -\frac{4\pi G}{c^4} T. \quad (8.8) \]

(For a complete theory one also needs a Lorentz covariant analog of Eq. (8.5) which determines the motion of matter in dependence of the gravitational field \( \phi \). This equation is not relevant for the following argument.)

If the EM field couples to gravity, it must also be a source of gravity. This coupling is then described by the EMT of Maxwell theory Eq. (6.110) (in its symmetric, gauge-invariant form). The problem is that the trace of this particular EMT vanishes identically, \( T_{em} = (T_{em})^{\mu}_{\mu} = 0 \) (check this!), so that the scalar gravitational field and the EM field do not “feel” each other. In particular, there is no bending of light in the vicinity of massive bodies.

- **Wrong value for perihelion precession** (even with a wrong sign)

  The perihelion precession of Mercury deviates measurably from its Newtonian value (which is caused by perturbations by other planets). For Einstein, this anomaly served as a “litmus test” on his quest to generalize special relativity, and his first application of general relativity was the successful explanation of Mercury’s anomalous perihelion precession \([13]\) (→ later). Thus any theory that does not predict the correct value for the perihelion precession cannot be correct.

Historically, this first approach [Eq. (8.7)] to patch up Newton’s theory and make it consistent with special relativity goes back to the Finnish physicist Gunnar Nordström. He quickly dismissed Eq. (8.7) because of fundamental problems (especially its linearity, → below). He then proposed another (non-linear) scalar theory of gravity († Nordström’s theory of gravitation) which circumvented the most glaring issues but still failed to predict the bending of light (for the same fundamental reason sketched above) and produced the wrong value for the perihelion precession (even with a wrong sign!). Nonetheless, the theory merits consideration because it led Einstein and Adriaan Fokker to a groundbreaking realization \([98]\): Properly reformulated, the scalar field could be interpreted as a local “stretching” of the Minkowski metric. For the first time there was a clear formal link between a relativistic theory of gravity and a geometric deformation of spacetime, where the shape of the latter is determined by the distribution of mass and energy.

For a historical account of Nordström’s gravity and its role in the genesis of general relativity see Refs. [99,100].

7 **Second try: Make potential a vector field:**

Since scalar gravity fails to match observations, a natural next step would be to consider a vector field and treat gravity analogous to Maxwell theory. This is also reasonable insofar as the gravitational potential of a point mass in Newton’s theory and the Coulomb potential of a point charge in Maxwell’s theory share the same form. For example, we can take Eq. (6.121) as a blueprint and propose an analogous Lagrangian for a vector gravitational field:

Eq. (6.121) \( \rightarrow \llbracket \)

**Vector field \( \phi^\mu \) & particle with trajectory \( \gamma^\mu (\lambda) \):

\[
S_G[y, \phi] = \frac{1}{16\pi G} \int d^4x \ G_{\mu\nu} G^{\mu\nu} - mc \int \sqrt{-g} \gamma^\mu \gamma_\mu d\lambda - \frac{m}{c} \int \phi_\mu \gamma^\mu d\lambda \quad (8.9)
\]

with \( \gamma^\mu = \frac{dy^\mu}{d\lambda} \) and the “gravitational field strength tensor” \( G_{\mu\nu} := \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \).

- **Note the sign difference compared to Eq. (6.121)!**

  This ensures that equal charges (= masses) attract each other.
• The Lagrangian for the relativistic particle differs from the one given in Exercise 7.2 in Misner et al. [2]; the two are equivalent and lead to the same equations of motion.

→ Results:
For details see Exercise 7.2 in Misner et al. [2]; see also Ref. [97].
• No bending of light ☺
• Wrong perihelion precession ☺
• Gravitational waves have negative energy ☺

8 | Third try: Make potential a tensor field:
At that point, desperation starts to kick in. But since scalar and vector fields failed miserably, we have no other choice: add another index and consider a tensor field. Interestingly, this makes it rather straightforward to write down a Lorentz-covariant modification of Eq. (8.8) [or Eq. (8.4)] where we no longer must butcher the EMT by taking a trace:

< Symmetric tensor field $\phi^{\mu\nu} = \phi^{\nu\mu}$:

$$\nabla^2 \phi^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu} \quad (8.10)$$

The EMT on the right is the symmetric BRT of whatever matter occupies space (Section 6.3.2).

→ Results:
For details: ☺ Problemset 1 (also Exercise 7.3 in Misner et al. [2])
• Light is bent around gravitational potentials ☺
• Gravitational waves have positive energy ☺
• Describes perihelion precession not correctly ☺
• Theory not self-consistent ☺

Notes:
• Eq. (8.10) will describe the linearized version of the correct field equations of general relativity (the → Einstein field equations) with $\phi^{\mu\nu}$ essentially the (small) deviation of the metric tensor from flat Minkowski space.

• That the linear tensor theory of gravity Eq. (8.10) is not self-consistent follows if one completes the theory with dynamic matter (which is the source of the gravitational field, but also influenced by the latter). Then one can show that this system of differential equations has no solution.

• As we will discuss below, the deficiency of this theory is its linearity (in the gravitational field); this is the root cause for its inconsistency and wrong predictions. And here comes a fascinating insight: One can show [101] that if one systematically fixes the inconsistencies of this theory, it becomes inevitably non-linear and one eventually ends up with the correct equations of general relativity (which we will find much later via a different route)!

9 | So far, all our tentative theories of relativistic gravity failed (none of them describe observations correctly and they even suffer from intrinsic inconsistencies). There is a simple argument why this must be so, and why the correct theory must be more complicated:

i | The source (= charge) of gravity is, by definition, the gravitational mass $m_G$.

This is a physically vacuous statement.
A relativistic theory of gravity must be a field theory with a dynamical field. This is necessary so that gravity does not propagate faster than the speed of light.

Since the field is dynamical, it has a non-vanishing energy density. Recall that energy is the Noether charge of time translations and therefore generates the time evolution (think of the Hamiltonian in quantum mechanics).

As a relativistic theory it must obey the mass-energy equivalence: \( E_0 = m_I c^2 \). We write \( m_I \) to emphasize that Special Relativity only knows about inertial mass.

Experiments tell us that inertial and gravitational mass are the same: \( m_1 = m_G \). We will discuss this, and the closely related equivalence principle, in detail below.

Thus a gravitational field has a non-vanishing density of gravitational mass \( m_G \). Please appreciate how strange this is! If an analog statement were true for Maxwell theory (which it is not), electromagnetic waves would be electrically charged, and other electromagnetic waves could scatter off them!

Excitations of the gravitational field are sources of the gravitational field. This means that a relativistic theory of gravity must allow for self-interaction. In particular, it cannot feature a superposition principle and the field equations must be non-linear.

The field theory of gravity must be non-linear and allow for self-interactions.

- All of the above theories are linear in the gravitational field; hence they are bound to fail!
- This argument also clarifies the fundamental difference between relativistic theories of gravity and electrodynamics (both of which are classical field theories that mediate forces): The EM field is also dynamical and carries energy, hence, via the mass-energy equivalence and the equivalence of inertial and heavy mass, it is a source of gravity. But the mass/energy carried by the EM field is not the source of the EM field (electrical charge is). Thus Maxwell theory does not close the “vicious circle” from above and can be both relativistic and linear.
- If you want an even more boiled down version:
  Gravity is special in Relativity, because Special Relativity has something to say about (inertial) mass \( (E_0 = m_I c^2) \) and the latter is – via the → equivalence principle – the source of gravity.
10 | **Yet another problem:**

Besides the formal complications encountered above, there are also less formal yet fundamental lines of reasoning that suggest that the phenomenon of gravitation and the premises of [special relativity](#) are incompatible:

i | Experimental facts:

- Gravity cannot be shielded.
  Contrary to all other forces (which have negative charges), there is no negative mass.
- Gravity is typically *inhomogeneous*.
  In a gravitationally homogeneous universe there are no planets and we wouldn’t exist.
- In free fall, gravity is exactly countered by the inertial force.
  We will discuss this in more detail later (equivalence principle).

ii | For the machinery of special relativity to work, we need inertial frames. Can we find inertial frames in the presence of gravity?

**Thought experiment:**

a | ![Laboratory on the surface of Earth](#)

   → Not an inertial system 😞

   The problem is that we cannot simply shroud our lab by some magic material that shields the gravitational force. By contrast, this *can* be done for the electromagnetic field (Faraday cage, Mu-metal). Note that this is not a technical problem, it is a fundamental one!

b | ![Interior of orbital space station](#)

   → Approximate inertial system 😊

   The space station is equivalent to a free falling laboratory, where the gravitational force is canceled exactly by inertia. What makes a space station so convenient is that it also has orbital velocity so that it “falls around Earth” and therefore can be used much longer than a free falling lab that eventually crashes on the surface.
This is a situation where it actually makes sense to use the terms “gravity” and “gravitation” differently: In the space station, there is no gravity (the astronauts float), but there is a gravitational field! (The latter is just canceled by the inertial force due to free fall.)

This situation is different from being in a spaceship far away from Earth in interstellar space where no gravitational forces can be measured. (Although these two situations cannot be distinguished from within a small space station/spaceship, → later.)

c | < Very large orbital space station:

↑ Tidal forces → Not an inertial system 

When we extend the size of the space station, the inhomogeneity of the gravitational acceleration becomes noticeable and the “inertial test” \[ \text{IN} \] fails. Inhomogeneous gravitational fields therefore constrain the size (both in space and time) of reference systems that satisfy the properties of an inertial system. Hence our assumption that inertial systems cover all of spacetime (and therefore can describe arbitrary physical phenomena, not just local ones) is invalidated by the presence of inhomogeneous gravitational fields.

Note that there is another way to detect the inhomogeneous gravitational field and make the system non-inertial: Stay in the small spacecraft and wait longer. At some point you will notice that the two balls drift apart—even when they are only centimeters apart. This shows that the approximate inertial system really must be small in space and time.

d | < Two small orbital space stations:

→ Local inertial systems accelerated wrt. each other 

If you imagine that these small inertial systems overlap on their boundaries, you could ask how to transform the coordinates of an event in this overlap from one of these systems into the other. Because these systems are accelerated wrt. each other, this transformation cannot be linear, in particular it cannot be a Lorentz transformation! There seems to be something missing; what determines this transformation?
This thought experiment leads us to the following (troubling) conclusion:

Inertial systems can only exist locally in an inhomogeneous gravitational field. How to transform between these local inertial systems is unclear (→ later).

→ Extended phenomena cannot be described by SPECIAL RELATIVITY!

“Extended” here means “on the scale of gravitational inhomogeneities.”

In a nutshell:

SPECIAL RELATIVITY cannot …

• … describe the gravitational field itself.
• … describe physics in inhomogeneous gravitational fields.

How to fix this? → GENERAL RELATIVITY

8.3. ‡ The gravitational redshift and curved spacetime

Quite surprisingly, one can derive some predictions of GENERAL RELATIVITY without knowledge of the detailed theory. One is the → gravitational redshift of light, which, again without the usage of heavy math, implies that GENERAL RELATIVITY must describe a curved spacetime.

The following is based on Sections 7.2 and 7.3 of MISNER et al. [2] and Section 2.1 of CARROLL [102].

Gravitational redshift:

EINSTEIN already concluded in 1908 that light leaving a gravitational potential must be redshifted [95]. The following he showed in 1911 [103], that is, years before he finalized GENERAL RELATIVITY.

Particle of rest mass $m$ in (homogeneous) gravitational potential:

In the following, we assume that inertial and gravitational mass are equal: $m_I = m = m_G$. 
Step 1: Drop the particle by $\hbar$

New total energy:

$$E_{\text{tot}} = mc^2 + mGgh = m(c^2 + gh) \quad (8.11)$$

Step 2: Assume electron annihilates into a photon with energy:

$$E_{\uparrow} = E_{\text{tot}} \quad (8.12)$$

Step 3: Let the photon propagate upwards by $\hbar$ to New photon energy $E_{\downarrow}$:

- Possibility 1: Photon is not affected by gravity $E_{\uparrow} = E_{\downarrow} > mc^2 \times$
  This immediately leads to a violation of energy conservation because the photon can now be used to recreate the particle plus some kinetic energy that wasn’t there before.

- Possibility 2: Photon is redshifted by gravitational field such that $E_{\uparrow} = mc^2 \checkmark$
  This is the only possibility consistent with energy conservation, i.e., the photon must loose energy just as a particle would when climbing the potential.

Thus we find for the photon energies:

$$E_{\downarrow} = E_{\uparrow} \left(1 + \frac{gh}{c^2}\right) \quad (8.13)$$

Redshift parameter $z := \frac{\Delta \lambda}{\lambda} = \frac{\lambda_{\uparrow} - \lambda_{\downarrow}}{\lambda_{\downarrow}}$

The redshift $z$ measures the relative change in wavelength $\Delta \lambda$ wrt. a reference wavelength $\lambda$.

Gravitational redshift: (Use the photon energy $E = h\nu = hc/\lambda$)

$$1 + z = \frac{\lambda_{\uparrow}}{\lambda_{\downarrow}} = \frac{E_{\downarrow}}{E_{\uparrow}} = 1 + \frac{gh}{c^2} \quad (8.14)$$

Using the Mößbauer effect, Robert Pound and Glen Rebka verified this prediction in 1960 with their famous Pound-Rebka experiment [104, 105].

Schild’s argument:

The following reasoning goes back to Alfred Schild (see references in [2]) and demonstrates that a relativistic theory of gravity cannot be formulated on a flat Minkowski spacetime:

Assumptions:

- There exists an extended inertial frame $K$ attached to Earth’s center.
  (We relax our definition and allow particles to be accelerated near earth.)
- In this frame, proper time and lengths are given by the Minkowski metric.
- There is some gravitational field (of unspecified nature) that matches observations.
  (This implies the gravitational redshift derived above.)

Thought experiment:

- Two observers $O_{\downarrow, \uparrow}$ at height $z_{\downarrow, \uparrow}$ with $z_{\uparrow} = z_{\downarrow} + h$ at rest in $K$
- Observer $O_{\downarrow}$ emits a light signal with wavelength $\lambda_{\downarrow}$
  → Time for one wavelength: $\Delta t_{\downarrow} = \lambda_{\downarrow}/c$

Note that since both observers are at rest in $K$, their proper times and the coordinate time of $K$ coincide.
Observer $O_\uparrow$ receives the signal with wavelength $\lambda_\uparrow$

\[ \rightarrow \text{Time for one wavelength: } \Delta t_\uparrow = \lambda_\uparrow / c \]

Redshift $\rightarrow \lambda_\uparrow > \lambda_\downarrow \rightarrow \Delta t_\uparrow > \Delta t_\downarrow$

But in the Minkowski diagram of the (imagined) global inertial frame, the experiment looks as follows:

\[ \rightarrow \Delta t_\uparrow = \Delta t_\downarrow \]

Note that the only important aspect for the contradiction is that the two world lines of the start and end of one wavelength are congruent in Minkowski space. That is, we do not need to know how gravity affects the trajectory of light (maybe it is bent). The only important thing is that both trajectories are bent in the same way, which is to be expected in a static scenario where the gravitational field does not change.

Conclusion:

In the presence of gravity, the trajectories of light signals in spacetime must be congruent (if they are straight: parallel) — but at the same time their distance in time direction must change! This is impossible in the flat (pseudo-)Euclidean geometry of Minkowski space; but it is possible in a curved spacetime. As we will see later, the tendency of initially parallel “straight lines” (→ geodesics) to approach or recede from another is exactly what characterizes a curved space(time):

\[ \rightarrow \text{Spacetime must be curved!} \]
We can summarize:

A Lorentz covariant theory of gravity cannot be formulated on Minkowski space.

This already suggests that we will need the more general machinery of differential geometry, introduced in Chapter 3, to model spacetime not as flat Minkowski space, but as a more general, curved pseudo-Riemannian manifold.
9. Conceptual Foundations

9.1. Einstein’s equivalence principle

The wording of the equivalence principles are paraphrased from Carroll [102].

1 | Remember:

There are two concepts of mass in Newtonian physics:

\begin{align}
\text{Inertial mass } m_1 & : \quad \vec{F} = m_1 \vec{a} \tag{9.1a} \\
\text{Gravitational mass } m_G & : \quad \vec{F} = -m_G \nabla \phi \tag{9.1b}
\end{align}

Strictly speaking, two gravitational masses must be conceptually distinguished: The passive gravitational mass is the charge that couples to the gravitational field via Eq. (9.1b). The active gravitational mass is the source of the gravitational potential \( \phi = -\frac{GM_G}{r} \). However, given that Newton’s third law is valid (action equals reaction), the situation is completely symmetric and these two masses can be identified. Thus, in the following we only distinguish between inertial and gravitational mass.

\[ \rightarrow \text{Gravitational acceleration:} \]

\[ a_G = \frac{m_G \frac{GM_G}{r^2}}{m_1} \tag{9.2} \]

It has been long known (since Galileo Galilei) that the gravitational acceleration is independent of the material of the body (if one can ignore air resistance): All bodies fall at the same rate.

Experience:

\[ \frac{m_G}{m_1} = \text{const} \quad \rightarrow \quad \text{Choose units appropriately:} \quad \frac{m_G}{m_1} = 1 \tag{9.3} \]

In classical mechanics, this is just an observation; it is neither explained nor necessary for its consistency.

2 | The Eötvös experiment [106]: (See also the later publication [107].)

While there have been earlier experiments that quantitatively tested the equivalence of inertial and gravitational mass, the experiment by Hungarian physicist Roland Eötvös made huge improvements in precision. The experiment was used by Einstein as an argument for his \( \rightarrow \) equivalence principle.
Torsion balance with two different test bodies: \((\text{Details: Problemset 1})\)

\[ \tau \approx m_G a_x \left( \frac{m_1}{m_G} - \frac{m_1'}{m_G} \right) \]  
\[ \tau = 0 \Leftrightarrow \frac{m_1}{m_G} = \frac{m_1'}{m_G} = \text{const} \]  

Result by Eötvös [106]:
\[ \frac{\delta m}{m} = \frac{m_1 - m_G}{m_1} < 3 \times 10^{-9} \]  

The latest (2022) and most precise results testing the equivalence of inertial and gravitational mass come from the satellite-based MICROSCOPE experiment [108]; it improved the upper bound for a violation of the equivalence to \(\delta m/m < 10^{-15}\). Recent experiments also demonstrated the equivalence for antimatter [109].

Experimental fact:

Inertial mass and gravitational mass are proportional \((\text{w.l.o.g. } m_1 = m_G)\). \((9.6)\)

This trivial sounding assertion (when have you ever distinguished between these two masses?) has profound consequences: Recall that special relativity is concerned with the concept of inertia (e.g. by using inertial systems); in particular, the (rest) mass that shows up in the mass-energy equivalence \(E_0 = mc^2\) is the inertial mass \(m_1\) of the system. The equivalence above now links this mass to the gravitational mass, and therefore asserts the the inert bodies of special relativity must be affected by (and be sources of) gravity. But special relativity had nothing to say about gravity! Quite to the contrary: as discussed in Section 8.2, the theory cannot accomodate gravity in a consistent way.

Classical mechanics does not explain Eq. (9.6). However, if we take Eq. (9.6) for granted, a homogeneous gravitational field vanishes in an accelerated frame:

\[ m_k \frac{d^2 \vec{x}_k}{dt^2} = m_k \frac{\ddot{x}}{\ddot{u}} + \sum_{l \neq k} \vec{F}_{kl}(\vec{x}_k - \vec{x}_l) \quad \text{with } k = 1, \ldots, N. \]  

\[ (9.7) \]
Coordinate transformation into free-falling frame:

\[ t' = t \quad \text{and} \quad \vec{x}'_k = \vec{x}_i - \frac{1}{2}g t^2. \]  

This coordinate transformation is non-linear, in particular, it is not a Galilei transformation!

Equations of motion in the free-falling coordinate system:

\[ m_k \frac{d^2 \vec{x}'_k}{dt'^2} = \sum_{l \neq k} \tilde{F}_{kl}(\vec{x}'_k - \vec{x}'_l) \quad \text{with} \quad k = 1, \ldots, N. \]  

No gravity in the free-falling frame!

This matches our experience and can be illustrated with the following thought experiment:

\[ \text{Caveat:} \text{ Only true for homogeneous gravitational fields: } m_k \tilde{g}. \]

What about inhomogeneous gravitational fields?

Note that the inhomogeneity of gravity is essential for planets and stars to form; it is the root cause for complexity in the world that is necessary for life to exist. This is not a slight inconvenience we can sweep under the rug!

Small enough regions look homogeneous:

Gravity can be compensated locally in an accelerated frame.

! This implies that there is no transformation to a global free-falling reference frame in which inhomogeneous gravitational fields vanish. Thus acceleration and gravity are only equivalent locally; globally, they are physically distinct. In particular, this means that the phenomenon of gravity is not just “acceleration in disguise.” As mentioned previously, accelerated coordinate systems (and bodies) are something that special relativity can handle. If gravity and acceleration were equivalent globally, special relativity would be sufficient to describe gravity and there was no need for general relativity.
§ Given all these facts, it is reasonable to proclaim the following principle:

§ Postulate 1: **Weak equivalence principle** (Univesality of free fall)

In small enough regions of spacetime, the motion of freely-falling particles in a gravitational field and free particles in a uniformly accelerated frame are the same.

(This formulation formalizes the idea sketched in the upper panel of the sketch above.)

Equivalently:

For every event, there is a local reference frame, covering small enough regions of spacetime in its vicinity, such that gravity has no effect on the motion of arbitrary particles in this frame and the law of inertia holds.

(This formulation formalizes the idea sketched in the lower panel of the sketch above.)

6 | **Einstein’s generalization:**

The local equivalence of gravity and accelerated frames is true for all physical phenomena (and not only classical mechanics).

Einstein was aware of the Eötvös experiment and was convinced that the equivalence of inertial and gravitational mass hinted at a deep relationship between inertia (acceleration) and gravitation. He wrote in 1907 [95] (highlights are mine):

*Bisher haben wir das Prinzip der Relativität, [...], nur auf beschleunigungsfreie Bezugsstysteme angewendet. Ist es denkbar, daß das Prinzip der Relativität auch für Systeme gilt, welche relativ zueinander beschleunigt sind? [...]*

Wir betrachten zwei Bewegungsstysteme $\Sigma_1$ und $\Sigma_2$. $\Sigma_1$ sei in Richtung seiner $X$-Achse beschleunigt, und es sei $\gamma$ die (zeitlich konstante) Größe dieser Beschleunigung. $\Sigma_2$ sei ruhend; es befinde sich aber in einem homogenen Gravitationsfelde, das allen Gegenständen die Beschleunigung $–\gamma$ in Richtung der $X$-Achse erteilt.

*Soweit wir wissen, unterscheiden sich die physikalischen Gesetze in bezug auf $\Sigma_1$ nicht von denjenigen in bezug auf $\Sigma_2$; es liegt dies daran, daß alle Körper im Gravitationsfeld gleich beschleunigt werden. Wir haben daher bei dem gegenwärtigen Stande unserer Erfahrung keinen Anlaß zu der Annahme, daß sich die Systeme $\Sigma_1$ und $\Sigma_2$ in irgendeiner Beziehung voneinander unterscheiden, und wollen daher im folgenden die völlige physikalische Gleichwertigkeit von Gravitationsfeld und entsprechender Beschleunigung des Bezugsstystem al annnehmen.*
Pictorially, Einstein claims that any type of local experiment cannot distinguish gravity from acceleration (here, for example, some quantum mechanical scattering process):

We can formalize this as follows:

§ Postulate 2: Einstein’s equivalence principle (EEP)

In small enough regions of spacetime, the laws of physics reduce to those of special relativity: It is impossible to detect the existence of a gravitational field by means of local (non-gravitational) experiments.

Equivalently:

For every event, there is a local reference frame, covering small enough regions of spacetime in its vicinity, such that gravity has no effect on any (non-gravitational) experiment in this frame and the law of inertia holds.

• The EEP implies the WEP.
• Excluding non-gravitational experiments means that the intrinsic gravitational energy of our experiment does not contribute significantly to its mass (see SEP below). Note that we do not require that gravitational experiments (using large masses) can locally distinguish between gravity and acceleration; the WEP does simply not constraint such experiments.
• It is important to appreciate the profound implications of this principle for doing physics in a gravitational field: It asserts that as long as your laboratory is small (compared to the inhomogeneities of the gravitational field) and free-falling (e.g. a space station in orbit), special relativity is sufficient to describe all experiments that you can conduct in this lab. In particular, the fact that special relativity cannot describe gravity is not important because in the free-falling lab there is none. This means that everything we discussed last term remains valid—and therefore useful—locally. Thus gravity does not completely invalidate special relativity, it only restricts its domain of validity to local, free-falling inertial frames. I hope you are happy to hear that!
• More precisely, for every event (point in spacetime) there is an equivalence class of local inertial frames (related by boosts), equipped with inertial coordinate systems (related by translations and rotations), in all of which special relativity holds good. The coordinate transformations between these systems are given by Lorentz transformations. (You can check
the existence of such frames in our Newtonian calculation above by adding a term $\ddot{u}_{\dot{t}}$ to the transformation of the position coordinates.

- In our mathematical framework of differential geometry, the equivalence class of local inertial systems at a spacetime point will be identified with the tangent space of the spacetime manifold at that point.

8 | Concerning gravitational laws of physics:

In our definition of the EEP, we excluded experiments that depend on the gravitational interaction itself (e.g., use objects with considerable intrinsic gravitational energy). This exclusion follows Schröder [3], whereas other authors like Carroll [102] include the (unknown) gravitational laws of physics in the EEP.

For us, it makes then sense to define an extension of the EEP as follows:

§ Postulate 3: Strong equivalence principle SEP

The EEP is valid for all laws of physics, including the gravitational laws.

- **General relativity** satisfies the SEP (and thereby the EEP and the WEP).

- The reason to separate the EEP from the SEP is that alternatives to general relativity can satisfy the EEP (and the WEP) but violate the SEP. This alternatives can be metric theories like general relativity with additional fields; see Ref. [110] for details.

- In particular, the SEP requires that the universality of free fall (WEP) also holds for large bodies like planets (not just small test particles) with significant amounts of gravitational self-energy. More precisely, the SEP demands that the rest mass $E_{\text{grav}}/c^2$ that comes from the gravitational self-energy $E_{\text{grav}}$ accelerates just like any other rest mass in an external gravitational field.

Note that the validity of the SEP cannot be deduced from typical experiments that test the WEP because these experiments use small test masses with gravitational self-energy that is way too small to detect any violation of the SEP (because gravity is such a weak force). One needs to use planet-sized objects to draw conclusions about the SEP (→ next).

- Using reflectors on the moon (left by Apollo 11 in 1969), lunar laser ranging (LLR) can be used to experimentally test the SEP since the fractions of gravitational self-energy of moon and earth are different enough to modify moon’s orbit measurably if the SEP was violated. To date there is no evidence of such a violation to high precision [111, 112], hence we will assume that the SEP holds.

9 | If special relativity explains everything you can do in a local, free-falling laboratory, at which point does gravity enter the picture? Well, the hitch is that not all physical processes can be restricted to a single, local inertial frame:
Meteroid traversing the inhomogeneous gravitational field of earth:

How to model the trajectory?

Imagine you start in a local inertial system where you know the initial data (position, velocity) of the meteroid. Since special relativity is valid in this small patch, you can use the known equations of relativistic mechanics to compute the trajectory of the meteroid. However, at some point, the meteroid will leave the local inertial system and enter another one. To proceed with your application of relativistic mechanics, you need to know the coordinate transformation that maps the coordinates of the final position and velocity in the first inertial system to the coordinates of the next.

But a priori these inertial system are unrelated, in particular, they can be accelerated with respect to one another (recall the two small space stations in Section 8.2). To proceed with your application of relativistic mechanics, you need this coordinate transformation! And this is where gravity hides: The gravitational field (here generated by earth) and the pattern of local coordinate transformations are one and the same thing! This is what is meant by gravity becoming a geometric property of spacetime.

The gravitational field is the (dynamical) structure that determines which local frames of reference are inertial or, equivalently, how to transform from one local inertial frame to the next.
9.2. General relativity and covariance, background independence

The equivalence principle \( \text{EEP} \) is the foundation of general relativity; it motivates both the metrization of gravity (Section 9.4 and ??) and the minimal coupling of matter to gravity (Chapter 11). However, there are additional principles that are conceptually important to understand and were historically important for the genesis of general relativity as well:

10 | Motivation:

- No global inertial systems anymore \( \rightarrow \) More general coordinate charts needed!

As we have seen, the description of gravity forces us to give up the restriction to formulate physical models within the distinguished family of infinitely extended inertial coordinate systems. Hence we must formulate our physical theories in a way that is valid for arbitrary coordinate charts, and allows for arbitrary coordinate transformations between them.

Einstein was not satisfied with the distinguished role of inertial systems in special relativity. After all, relativity is all about the relativity of states of motion, i.e., only motion of systems with respect to one another are of physical significance – and no class of states of motion should be distinguished. Special relativity clearly does not live up to this rigorous form of relativity as it singles out inertial frames as special. Einstein’s ultimate goal was to make all states of motion (including accelerated motion) “equivalent.” General relativity does not achieve this goal! Even in general relativity, inertial motion is physically distinct from accelerated motion; the new thing is that mass and energy determine which states of motion are inertial.

We are therefore in the strange (and confusing) situation, that Einstein’s original motivation to seek out equations that have “the same form” in all coordinate systems does not achieve its goal, but nevertheless is the correct way forward (see \( \rightarrow \) next point). We will also see that “having the same form” means something different in special relativity than in general relativity because the former is formulated on a fixed background (Minkowski space) and the latter not (\( \rightarrow \) background independence) – and this changes what it means for two equations to have “the same form.” The situation is quite convoluted and we can disentangle it not until the end of this course.

- Chapter 3: Physics describes relations of geometric entities (‘Coordinates don’t exist.’) \( \rightarrow \) Coordinates should play no role in the formulation of physical models!

Recall our motivation in Chapter 3 for the introduction of tensor fields: We realized that coordinates are mathematical artifacts that we use to label events in spacetime. The essence of physical laws should clearly be independent of the labeling scheme we choose. Thus we should strive for a formulation of physical models (which, hopefully, capture physical laws) that is independent of coordinates, or at least makes it manifest that physical predictions do not depend on the choice of coordinates.

Note that this argument is very different from Einstein’s hope to extend the principle of special relativity \( \text{SR} \) by “equalizing” more states of motion. The argument is way more fundamental, has nothing to say about states of motion, and, in some sense, is almost tautological. It’s only physical content is the rather uncontroversial statement that “coordinates do not exist as physical entities.”
This motivates the following definition:

★ **Definition 1: General covariance (Coordinates don’t exist)**

An equation is said to be generally covariant if it is form invariant under arbitrary (differentiable) coordinate transformations.

→ Tensor equations are automatically generally covariant.

Generally covariant equations have an alternative coordinate-free formulation in terms of geometric objects on a manifold († differential forms):

**Examples:**

- Two vector fields \( A = A^\mu \partial_\mu \) and \( B = B^\mu \partial_\mu \):

\[
\begin{align*}
\phi_A &= \phi_B \\
\vec{A} &= \vec{B} \\
A^\mu &= B^\mu \\
\phi_B &= \ldots \\
\vec{B}^i &= \ldots
\end{align*}
\]

\[ \Leftrightarrow \quad \text{Manifestly (generally) covariant} \quad \text{Coordinate free} \] (9.10)

While mathematicians often prefer the coordinate-free notation, in physics, the coordinate-dependent, manifestly covariant notation is more widespread. This has to do with how physics is done: While coordinates do not exist a priori, physicists typically make them exist in their labs because measurements always use some form of reference system. The generally covariant equations are more useful in that regard because they can be specialized to any coordinate system most convenient for an experiment.

→ In this course we will only use the manifestly covariant notation.

- To decided whether an equation remains form invariant under arbitrary coordinate transformations, you must first know how the elementary fields of the equation transform. This is why the non-manifest notation is so cumbersome: In addition to the equation(s), you must figure out (or specify) how the different fields transform. (Recall the non-tensorial form of the Maxwell equations and how cumbersome it was to check their Lorentz covariance [see Eq. (6.34)]!)

This makes the benefit of the manifest notation clear: First, by convention, the tensor notation \( A^\mu \) implies that the transformation of the field is \( \vec{A}^\mu = \frac{\partial x^\lambda}{\partial x'^\mu} A^\nu \), and second, because of the rules of tensor calculus, checking the general covariance of a (valid) tensor equation is trivial.

- Inhomogeneous Maxwell equations on arbitrary spacetime (??):

\[
F^{\mu\nu} = - \frac{4\pi}{c} J^\mu 
\]

Manipulated generally covariant

\[ \Leftrightarrow \quad d(\ast F) = \ast J \] (9.11)

Remember that \( \ast \) denotes the covariant derivative Eq. (3.79) which implicitly depends on the metric of spacetime. In the coordinate-free notation, the metric is hidden in the definition of the Hodge star operator \( \ast \).
We can now use this definition to formulate our physical insight that the equations that describe physical laws must not single out specific coordinate systems:

In Einstein’s words [20]:

Die Gesetze der Physik müssen so beschaffen sein, daß sie in bezug auf beliebig bewegte Bezugsysteme gelten. (p. 772)

Die allgemeinen Naturgesetze sind durch Gleichungen auszudrücken, die für alle Koordinatensysteme gelten, d.h. die beliebigen Substitutionen gegenüber kovariant (allgemein kovariant) sind. (p. 776)

§ Postulate 4: General relativity \(GR\)

Models of laws of nature must take the same form in all coordinate systems; i.e., they must be expressed in terms of generally covariant equations.

Here the “must take the form” means that it must be possible to formulate them in a coordinate-independent way; if this were not the case, the theory (and its prediction) would implicitly depend on (and single out) a specific coordinate system. Note that there is nothing wrong in formulating such a theory in a way that is not generally covariant.

For example, Maxwell equations in their conventional (non-tensorial) form are not generally covariant, they are only Lorentz covariant. This is not a problem, though, because these equations are just a specialization of Eq. (9.11) to a particular class of coordinate systems (namely inertial systems). If you (naïvely) apply these specialized equations in a non-inertial frame (such as a laboratory on the surface of earth!), you can get incorrect results (Problemset 3 and Ref. [113, 114]).

What is the physical content of \(GR\)?

\(GR\), while being important for the formulation of physical models in general and being strictly satisfied in general relativity, is neither specific nor fundamental to and for general relativity. For example, the Maxwell equations in the manifestly covariant form of Eq. (9.11) satisfy the \(GR\) on the fixed background of Minkowski space and have nothing to do with general relativity.

The principle of general relativity \(GR\) has (almost) no physical content.

The “almost” refers to the fact that the principle asserts that there are no distinguished coordinate systems that exist as physically independent structures.

- The relativity postulate \(GR\), and its mathematical manifestation as general covariance \(GC\) have been criticized already in 1917 by Kretschmann [115]:

[Man] verengegenwärtigt sich, daß alle physikalischen Beobachtungen letzten Endes in der Feststellung rein topologischer Beziehungen ("Koinzidenzen") zwischen räumlich-zeitlichen Wahrnehmungsgegenständen besteht und daher durch sie unmittelbar kein Koordinatensystem vor irgend einem anderen bevorrechtigt ist, so wird man zu dem Schlüsse gezwungen, daß jede physikalische Theorie ohne Änderung ihres—beliebig—durch Beobachtungen prüfbaren Inhaltes mittels einer rein mathematischen und mit
• To drive the point home: One can also formulate good old non-relativistic Newtonian mechanics in a generally covariant form (it’s quite ugly, though)! See the original literature [116] and Misner et al. [2] (Box 12.4 and §12.5):

*Any physical theory originally written in a special coordinate system can be recast in geometric, coordinate-free language. Newtonian theory is a good example [...] Hence, as a sieve for separating viable theories from nonviable theories, the principle of general covariance is useless.*

• For a detailed account on the role general covariance plays in *general relativity* (and historically played in its inception), see Ref. [117].

We can summarize the relation of *SR, EEP, and GR* as follows:

- Both *SR* and *EEP* make claims about the equivalence (indistinguishability) of certain states of motion. These are physical claims about reality that can be assessed by experiments. Note that to check whether they are false or true you do not even know how to work with mathematical equations. It’s a simple matter of collecting the results of experiments (recall the Michelson-Morley experiment). It is this physical content that makes *SR* and *EEP* the foundations of *special relativity* and *general relativity*, respectively.

- By contrast, *GR* makes no such claims about reality. *GR* does not claim that all states of motion are indistinguishable (they are not, even in *general relativity* you can tell local inertial frames and accelerated frames apart); the principle only claims that all fundamental theories of physics should have a formulation that can be applied by all possible observers. *GR* is therefore more a statement about physical models than about reality.

- The sketch makes it clear that the *EEP* is actually more similar to the *SR* (in the role it plays for *general relativity*) than the *GR* is. In that sense “principle of general relativity” is kind of a misnomer.

There is another important concept that (contrary to *GC/ GR*) distinguishes *general relativity* from other theories and must itself be distinguished from *GC/GR*:
Definition 2: Background independence \(\textbf{BI} (\text{No prior geometry})\)

Physical models that do not contain the geometry of spacetime as an absolute element are called \(\textbf{BI} \) background independent. This implies that the geometry of spacetime emerges dynamically as solutions of the theory.

(Counter-)examples:

- \(\sqrt{\text{General Relativity}}\) is (and historically was the first example of) a background independent theory (\(\rightarrow\) below):
  \[
  S_{\text{Einstein–Hilbert}}[g] = \frac{c^4}{16\pi G} \int d^4x \sqrt{g} R
  \]
  Here \(R\) is the \(\rightarrow\) Ricci scalar that depends in a complicated way on the metric tensor field \(g_{\mu\nu}(x)\). \(\sqrt{g}\) is short for \(\sqrt{\det[g_{\mu\nu}(x)]}\) (Minkowski metric: \(\sqrt{\eta} = 1\)).

- \(\sqrt{\text{Maxwell theory}}\) is not background independent (recall Eq. (6.56)):
  \[
  S_{\text{Maxwell}}[A] = \int d^4x \sqrt{g} \left( -\frac{1}{16\pi} g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} F_{\mu\nu} \right)
  \]
  with field-strength tensor \(F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\).

- Note that you do not extremize this action wrt. the metric \(g\); the metric (e.g. Minkowski metric \(g = \eta\)) is given as a parameter (absolute element) of the theory. Hence it plays the role of a static background.

- Note also that the Maxwell equations (and their Lagrangian) are generally covariant: they are tensorial expressions that describe geometric objects on a manifold. That does not prevent them to have a tensor field (the metric) as an absolute element.

Beware:

**General Relativity** is most likely the first (and possibly last) generally covariant and background-independent theory that you will encounter in your university courses. Thus it is important to mention a peculiarity that, if ignored, can lead to lots of confusion when studying such theories:

- \(\sqrt{\text{Trajectory of particle in spacetime:}}\)

Because of general covariance (\(GC\)) there is always a coordinate system in which the (spatial) coordinates of an object are constant in time.

\(\rightarrow\) One cannot infer from coordinates whether an object is moving!
There is no absolute notion of “motion” in General Relativity; the only motion that makes sense is motion wrt. some reference object, next step.

Two objects at rest in some coordinate system:

In a background-independent theory (BI) the metric is a dynamic degree of freedom. Therefore two objects \( a \) and \( b \) can have constant coordinates in space while their distance varies over time! Note that the coordinates are completely independent of the metric in general.

→ One cannot infer from coordinates whether distances of objects change!

We can sum this up as follows:

Coordinates have no physical meaning in General Relativity; they are simply (arbitrary) labels of events.

• Please make sure you grasp this statement fully (we will see explicit examples later when we understand General Relativity better):

If I tell you in Special Relativity that (in some inertial frame) two test particles have constant spatial coordinates \( x^i \) and \( y^i \), you immediately know their relative velocity and distance:

\[
\vec{v}_{\text{rel}} = \hat{x} \cdot (x^i - y^i) = 0 \quad \text{and} \quad \Delta r^2 = -(x^0 - y^0)^2 = |x - y|^2.
\]

By contrast, if I tell you in General Relativity that two test particles have (in some coordinate system) constant spatial coordinates \( x^i \) and \( y^i \), this tells you nothing about their distance; not even whether it is constant or varies in time! This information is hidden in the values of the metric field, not in the coordinates.

• This is important, for example, when studying the effects of gravitational waves (→ later).

Summary:

• Every reasonable fundamental theory has a generally covariant formulation.
• A generally covariant theory does not need to be background independent.
• General Relativity is background independent and generally covariant.

• We will return to the question of general covariance, background independence (and diffeomorphism invariance) later when we know more about General Relativity. At this point it is only important that you know the conceptual difference between the terms general
covariance \( \text{GC} \) and background independence \( \text{BI} \) and the requirement of the principle of general relativity \( \text{GR} \).

- One sometimes hears that general covariance \( \text{GC} \) is a distinguishing feature of general relativity (as explained above, it is not). Sometimes even the very concepts of general covariance \( \text{GC} \) and background independence \( \text{BI} \) are confused. This confusion is partially rooted in history because Einstein himself didn’t separate the two concepts clearly. Misner et al. explain [2] (p. 431):

  > Mathematics was not sufficiently refined in 1917 to cleave apart the demands for “no prior geometry” and for a “geometric, coordinate-independent formulation of physics.” Einstein described both demands by a single phrase, “general covariance.” The “no-prior-geometry” demand actually fathered general relativity, but by doing so anonymously, disguised as “general covariance,” it also fathered half a century of confusion.

- For more details on the relation between the concepts of background independence, general covariance, and diffeomorphism invariance see Ref. [118] (and references therein).

9.3. Mach’s principle (an its failure in general relativity)

Mach’s principle is not a logical postulate of general relativity and mostly of historical (and perhaps philosophical) importance. However, it is also conceptually interesting because at a first glance one might conclude (as Einstein did), that general relativity actually satisfies the principle. It is rather subtle (and instructive) why this is not so:

18 | Recall \( \leftrightarrow \) Newton’s bucket:

\[ \text{Question: Rotation with respect to what determines the shape of the water surface?} \]

\[ \text{Newton: Absolute space!} \]

Note that the experiment already suggests that this “absolute space” must have certain symmetries since the experiment cannot distinguish specific points nor specific states of uniform motion. Special relativity tells us that the correct symmetry group of spacetime is the Poincaré group. So from our modern perspective, Newton’s answer must be read as follows: The experiment demonstrates the independent existence of an entity which determines the local inertial systems. We may call this entity “spacetime.”

19 | The austrian physicist \textbf{Ernst Mach} fervently disagreed with Newton [119]:

\[ \text{Der Versuch Newton’s mit dem rotirenden Wassergefäss lehrt nur, dass die Relativdrehung des Wassers gegen die Gefäßwände keine merklichen Centrifugalkräfte weckt, dass dieselben aber durch die Relativdrehung gegen die Masse der Erde und die übrigen Himmelskörper geweckt werden. Niemand kann sagen, wie der Versuch verlaufen würde, wenn die Gefäßwände immer dicker und massiger, zuletzt mehrere Meilen dick würden. Es liegt nur der eine Versuch} \]
Mach denied Newton’s notion of an independent entity responsible for inertia and proposed that the large-scale structure of matter in the cosmos determines the local inertial systems instead (relationalism):

\[\text{§ Principle: Mach’s principle} \]

Local inertial frames are determined by the cosmic motion and distribution of matter.

- **Mach** never formulated his principle precisely, which leaves room for interpretation and personal taste. This is why there are various readings of Mach’s principle in the literature, not all equivalent. The above phrasing is a rather strict version of the principle.

- Here is an alternative way to illustrate the point by Steven Weinberg [120] (p. 17):

  There is a simple experiment that anyone can perform on a starry night, to clarify the issues raised by Mach’s principle.

  First stand still, and let your arms hang loose at your sides. Observe that the stars are more or less unmoving, and that your arms hang more or less straight down. Then pirouette. The stars will seem to rotate around the zenith, and at the same time your arms will be drawn upward by centrifugal force. It would surely be a remarkable coincidence if the inertial frame, in which your arms hung freely, just happened to be the reference frame in which typical stars are at rest, unless there were some interaction between the stars and you that determined your inertial frame.

Put this way, the situation is quite puzzling indeed and Mach’s principle doesn’t seem far fetched at all.

- **Einstein** was responsible for coining the term “Mach’s principle” and was influenced by it during his construction of general relativity. At first, he believed that in his new theory of gravity the principle was indeed satisfied. He writes in a letter to Mach in 1913 [121]:


  Wenn ja, so erfahren Ihre genialen Untersuchungen über die Grundlagen der Mechanik – Planck’s ungerechtfertigter Kritik zum Trotz – eine glänzende Bestätigung. Denn es ergibt sich mit Notwendigkeit, dass die Trägheit in einer Art Wechselwirkung der Körper ihren Ursprung hat, ganz im Sinne Ihrer Überlegungen zum Newton’schen Eimer-Versuch.

  (You may wonder how Einstein could write this letter in 1913 when he finalized general relativity in November of 1915. Einstein refers to his paper with Marcel Grossmann published in 1913 [122] in which they established the “Entwurftheorie”, a precursor of general relativity that already included most of the pieces needed [but not yet the correct field equations].)

So here is the case Mach (& early Einstein) vs. Newton:
• **Newton:**

Space exists as an independent entity and determines locally which frames are inertial.

• **Mach:**

Space emerges from the relations between matter and does not exist independently. Hence the distribution of matter in the universe completely determines the local inertial systems.

Who is right according to *general relativity*?

21 | **Answer:** Both ... (in a sense, though Newton is more correct)

- **Newton**’s conclusion was correct: Because of locality (constancy of the speed of light) the matter distribution of the cosmos (the fixed stars) cannot immediately influence the local inertial frame; there must be a mediator, some “background” that is here right now. *General relativity* tells us what this is: the metric tensor field that determines the geometry of spacetime.

- **Mach** was right insofar as it is indeed not a coincidence that the local inertial frame on earth is at rest with respect to the fixed stars. There is a relation, although not a direct and immediate one. *General relativity* tells us that the large-scale distribution of matter (and energy) in the universe determines the (large-scale) metric of spacetime, which, in turn, determines the local inertial systems everywhere. But there is a hitch: the metric is not uniquely determined by the mass distribution. Thus the metric (and therefore spacetime) carries independent degrees of freedom. There is more than matter in the world, spacetime is a real entity!

**Notes:**

- Today we know that there are solutions of the Einstein field equations (e.g. the ↑ Gödel universe [123]) that violate Mach’s principle explicitly [124].

- **Mach**, in his critique of *Newton*’s bucket experiment, asked (rhetorically) what would happen if the walls of the bucket would become very thick and massive. His point was that it is not excluded that at some point the rotation of the bucket would influence the shape of the water. *General relativity* tells us that this is so indeed, because a very massive bucket affects the geometry of spacetime. This is known as the → Lense-Thirring effect [125,126] (also know as ↑ frame-dragging) and has been experimentally confirmed (not with a massive bucket, of course, but with earth) [127,128]. However, for the reasons explained above, this effect does not make *general relativity* comply with Mach’s principle in the strict sense.

- Because of the many different versions of *MP* floating around, for some the case is still not closed (at least for philosophers of science, it seems). For doing physics with *general relativity*, *MP* is irrelevant.

- **Mach** advocated a relational view of space(time): Only relations between the degrees of freedom of matter are observable. There is no independent meaning of, say, an electron being here now. It is interesting to realize that this relational view might very well be true (and in accordance with *general relativity*) if one accepts that the metric field is just another collection of degrees of freedom which can be in relations (coincide or interact) with other degrees of freedom. For example, an electron being here now might simply mean that an excitation of the field that describes the electron coincides/interacts with a particular degree of freedom of the metric field.

22 | **Conclusion:**

The controversy about the *MP* essentially boils down to the question whether spacetime has independent degrees of freedom (and therefore exists in a physical sense):
In its strict version, the **MP** denies spacetime this independent role. By contrast, **general relativity** grants spacetime independent degrees of freedom because the \( \rightarrow \) **Einstein field equations** only constrain the \( \rightarrow \) **Einstein tensor** but not the metric directly (\( \rightarrow \) **Gravitational waves**):

**general relativity** violates Mach’s principle **MP** because matter influences the geometry of spacetime but does not determine it uniquely.

This situation is exemplified by gravitational waves: When in 2015 the interferometers of LIGO detected a gravitational wave passing earth, the spacetime geometry in our vicinity changed by a very tiny bit. However, the mass distribution in the vicinity of earth didn’t change at all. So while the geometry of spacetime certainly is influenced by earth’s mass, it is not uniquely determined by it. LIGO therefore measured directly the dynamics of the degrees of freedom the existence of which **MP** denies.

### 9.4. Overview and Outline

Now that we know the conceptual starting point of **general relativity**, and argued that more general spacetimes than flat Minkowski space are needed to accommodate gravity, we can reveal the gist of **general relativity** and sketch the plan for the remainder of this course:

\( ! \) You are not required to fully grasp the **how** and **why** of the following statements. Understanding the details is the objective of this course. However, I think that it is useful to start off with a rough picture of what we want to accomplish because otherwise one is easily swamped by the details along the way.

**general relativity** **in a Nutshell**

- **Ontology:**

\[
\text{Spacetime} \equiv 4D \text{ differentiable manifold} \; M \\
\text{Gravitational field} \equiv \text{pseudo-Riemannian metric} \; g \text{ with signature } (1,3)
\]

\( \rightarrow \) Spacetime is a \( \leftrightarrow \) **4D Lorentzian manifold**

- Note that we only fix the dimensionality of \( M \) (and thereby its **local topology**) but not its **global topology** (i.e., whether it is simply \( \mathbb{R}^4 \), a sphere, a torus, or something even more fancy). Thus, for example, **general relativity** makes no a priori statement about the finiteness of the universe. (Asking about the local topology is like asking where space and time come from—and **general relativity** is silent about that. A reasonable theory of quantum gravity must address this question!)

- At this stage it is sufficient to interpret the points \( E \in M \) of the manifold as points in spacetime and therefore as (equivalence classes of) events. However, we will see that this interpretation is problematic (\( \rightarrow \) **Hole argument**) because of the diffeomorphism invariance of **general relativity**. It is thus questionable whether points of the manifold (and thereby the manifold itself) can be associated to any existing entity. An entity that certainly does exist, however, is the metric field.
• Equivalence principle:
  The EEP is built right into the mathematical framework of general relativity:

For every point with coordinates $y$, there exists a coordinate transformation $\varphi_y(x)$ such that:

$$\tilde{g}^{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu} g_{\alpha\beta}(x) \approx \eta^{\mu\nu}$$

and

$$\tilde{\partial}_\rho \tilde{g}^{\mu\nu}(x) \approx 0 \quad (9.14)$$

with Minkowski metric

$$\eta^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (9.15)$$

→ Locally inertial coordinates √(Problemset 2)

− ¡! In the presence of gravity, there is no coordinate transformation that brings the metric into Minkowski form everywhere on the spacetime manifold. Conversely, if this is possible, spacetime is flat Minkowski space and you were doing special relativity all along (perhaps in curvilinear coordinates).

− Note that metrization of gravity is not a mathematical corollary of the EEP (the latter is a physical principle, not a rigorous mathematical statement). However, the EEP is most naturally incorporated into a mathematical framework where gravity is described by a metric because the gist of the EEP is that all (local) physical phenomena are affected by gravity in the same way. This is exactly what happens if gravity is identified with the geometry of spacetime!

− At every point, the basis $\{\tilde{\partial}_\nu\}$ of the tangent space forms a so called local Lorentz frame.
  You can choose such a basis for all points of spacetime. However, in general there is no coordinate system that induces this basis everywhere; you have to use multiple charts to patch together spacetime.

• Important fields:
  All degrees of freedom (some gauge, some physical) of general relativity are stored in the metric tensor field. From the metric, one can then derive other fields that play important roles in the formulation of the theory:
Note that the Einstein tensor is non-linear in the metric and contains (up to) second-order derivatives.

- **Einstein field equations (EFE):** (here without cosmological constant)

  The centerpiece of **general relativity** is a tensorial partial differential equation that determines the metric tensor field in dependence of the energy momentum tensor of matter:

  \[
  G_{\mu\nu} = \kappa T_{\mu\nu} \rightarrow \text{Energy-momentum tensor}
  \]

  \( G_{\mu\nu} \) is the **Einstein tensor**
  \( T_{\mu\nu} \) is the **energy-momentum tensor**
  \( \kappa \) is a constant

  \( \text{“Geometry”} \) refers to the space-time geometry
  \( \text{“Matter”} \) refers to all degrees of freedom that carry energy and/or momentum. This includes bodies with rest mass but also electromagnetic radiation etc.

  \( \bowtie \) Eq. (9.17): Non-linear, second-order PDE for \( g_{\mu\nu} \):

  **General relativity** describes the geometry of space as a dynamical field that evolves “in time” according to a highly nontrivial PDE:

  \[
  \text{General relativity} = \bowtie \text{Geometrodynamics} \tag{9.18}
  \]

  The nonlinearity makes Eq. (9.17) hard to solve, even in vacuum were the right-hand side vanishes.

  - Spacetime geometry is dynamical \( \rightarrow \) Background independence \( \checkmark \)
  - Tensor equation \( \rightarrow \) General covariance (no preferred coordinate system) \( \checkmark \)
  - **Mass distribution** determines metric determines local inertial frames
    However: Fixing \( G_{\mu\nu} \) leaves some degrees of freedom of \( g_{\mu\nu} \) unconstrained!
    \( \rightarrow \) Boundary conditions required for unique solution
    \( \rightarrow \) Mach’s principle is not satisfied (but partially survives in spirit) \( \times / \checkmark \)
  - Recall Eq. (8.10) and our discussion that followed (also \( \checkmark \) Problemset 1). Eq. (9.17) is structurally similar but fixes the problem of linearity because the Einstein tensor is a non-linear function of the metric.

- **Physics with gravity:**

  Once gravity is described by the metric, one must generalize the other relativistic theories (mechanics, electrodynamics, ...) into a generally covariant form that couples to the metric. This generalization is a priori not unique because matter can couple in various ways to the fields derived from the metric.

  However, the EEP severely restricts the couplings that are allowed and leads to a “recipe” how the Lorentz covariant equations of **special relativity** must be rewritten to match the principles of **general relativity** (\( \rightarrow \) “Comma-Goes-to-Semicolon Rule”, Minimal coupling):

  The \( \text{GR} \) demands physical theories to be specified by tensor equations.
  The \( \text{EEP} \) restricts the possible couplings of matter and metric.

  \( \rightarrow \)

  - Mechanics: (with \( u^\mu = \frac{dx^\mu}{d\tau} \) the 4-velocity)
    \[
    m \frac{Du^\mu}{d\tau} = m \frac{d^2x^\mu}{d\tau^2} + m \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = K^\mu_{\text{4-force}} \tag{9.19}
    \]

  \( \text{Absolute derivative} \)
  \( \text{4-accel.} \)
  \( \text{New!} \)
Free particle: $K^\mu = 0 \rightarrow \text{Geodesic equation ("straight lines" in spacetime)}$

- Electrodynamics: Inhomogeneous Maxwell equations: (cf. Eq. (6.50))

$$F_{\mu\nu} = \frac{4\pi}{c} j^\mu \rightarrow \frac{4\pi}{c} J^\mu$$

The covariant derivative contains the connection $\Gamma$ and therefore the metric $g$. This is how the electromagnetic field is affected by the gravitational field (e.g., bent in the vicinity of heavy masses). Conversely, the EM field gives rise to the energy momentum tensor $T_{\mu\nu}$ [cf. Eq. (6.110)] and thereby contributes to the right-hand side of Eq. (9.17).

$\rightarrow$ Energy-momentum tensor $T_{\mu\nu}$ is dynamical

A generally relativistic theory of matter, interacting with and via gravity, is then described by a coupled, non-linear, higher-order system of partial differential equations (where $T_{\mu\nu}$ depends on the dynamical variables of the matter theory).

$\rightarrow$ Hard to solve in general $\rightarrow$ Approximations needed!

Outline of this course

Here is our approach for this course to study these various aspects of general relativity:

- **Step 1** (Section 9.4): How to describe non-Euclidean manifolds mathematically?

In Chapter 3 we introduced the concept of differentiable manifolds and introduced the concept of a (pseudo-)Riemannian metric to measure lengths of curves on the manifold. To formulate general relativity mathematically, we need to revisit and extend this toolbox of tensor calculus.

In particular, we will study two (at first independent) structures that can be put on a differentiable manifold:

- **Affine connection** $\rightarrow$ Determines parallel transport, straight lines, and curvature
- **Riemannian metric** $\rightarrow$ Determines lengths, shortest lines, and angles

As it turns out, when you are given a Riemannian metric, there is a unique way to construct an affine connection. This means that once you are given a spacetime manifold with a (pseudo-)Riemannian (Lorentzian) metric, all concepts in the list above are well-defined. This is then the framework we will use: The spacetime of general relativity is a Lorentzian manifold and the degrees of freedom (field) of the theory is the Lorentzian metric itself.

- **Step 2** (Chapter 11): How to formulate relativistic theories on non-Euclidean spacetimes?

In the first part of this course, we first established the tenets of special relativity (Lorentz symmetry) and then incorporated them successively in known theories of physics (point mechanics in Chapter 5, electrodynamics in Chapter 6, quantum mechanics in Chapter 7). Now that we established the tenets of general relativity (the spacetime metric is not necessarily the Minkowski metric but an arbitrary Lorentzian metric), we must again reformulate our theories to comply with this new insight. Recall p. 16 in the introduction:
This will lead us to generally covariant formulations of relativistic mechanics in \( \Phi \) and electrodynamics in \( \Phi \) (we will skip quantum mechanics this time, but this is also possible). Thanks to the combination of EEP and GR (and tensor calculus), the recipe to go from the equations of special relativity to that of general relativity will be very simple.

• **Step 3 (\( \Phi \)):** How to determine the geometry of spacetime dynamically?

Up until this point we simply declared that the metric of spacetime is an arbitrary Lorentzian metric and studied the effects on physics given such a metric. The core idea of general relativity (and maybe the most important insight of Albert Einstein) was that this metric was not part of the laws of nature but just another degree of freedom that had to be dynamically determined. This means that there is no a priori geometry of spacetime, a principle known as background independence. The equations that dynamically determine the metric are the Einstein field equations Eq. (9.17); they are the centerpiece of general relativity and determine the geometry of spacetime, given the distribution of mass and energy and some boundary conditions. We will derive these equations via an action principle from a Lagrangian.

Together with Step 2, this completes the framework of general relativity.

• **Step 4 (\( \Phi \)):** What does general relativity predict?

If we combine the results of Step 2 and Step 3 we obtain a self-contained, background independent framework to describe physics: Matter determines the geometry of spacetime (Step 3) and, conversely, this geometry determines how matter evolves (Step 2). This interplay makes for beautiful but mathematically complicated models. Thus, to study the predictions of general relativity, we typically resort to simplified approaches:

- Consider a static, inhomogeneous distribution of large masses (e.g. the sun). Using the Einstein field equations from Step 3 (and reasonable boundary conditions), calculate the geometry of spacetime induced by this distribution. Then use the results of Step 2 to determine the evolution of small test particles on this curved spacetime (without taking their backaction on spacetime into account). This approach leads to a variety of phenomena, e.g., the slowing down of clocks close to large masses (\( \Phi \)), the perihelion precession of Mercury (\( \Phi \)), the bending of light (\( \Phi \)), etc.

- Consider the Einstein field equations in vacuum, i.e., without any matter (or energy). Because the EFEs are non-linear (recall Section 8.2) and the geometry of spacetime is not uniquely determined by the distribution of mass and energy, this situation is not as boring and trivial as it sounds (its actually very complicated). But even in the weak field limit (where one drops the self-interactions) one finds something interesting: gravitational waves (\( \Phi \)).

- Consider an idealized universe that is homogeneously filled with matter and energy (and potentially dark matter and dark energy). If one calculates the solutions of the EFEs in such a scenario, one obtains the (approximate) spacetime geometry of the whole universe. This leads into the field of relativistic cosmology and to the current standard model of cosmology, known as \( \Lambda \)CDM. This is where one finds the possibility of an expanding universe and its origin, the Big Bang; this is also where the cosmological constant becomes important (\( \Phi \)).
10. Mathematical Tools II: Curvature

Here we continue our discussion of differential geometry in Chapter 3. We study two structures on a differentiable manifold that are particularly important for general relativity: \textit{connections} and the \textit{Riemannian metric} (the latter we already know). Since most of our results are not specific to general relativity, we mostly consider general $D$ dimensional manifolds, and only specialize to the case of $D = 3 + 1$ spacetime dimensions later.

- The mathematical framework of \textit{general relativity} is \textit{Riemannian geometry}, i.e., the field of differential geometry that studies differentiable manifolds equipped with a Riemannian (pseudo-)metric. The field was kickstared in 1854 by German mathematician Bernhard Riemann with his inaugural lecture in Göttingen titled “Über die Hypothesen, welche der Geometrie zu Grunde liegen” [129]. In the audience was Carl Friedrich Gauß, who had also picked the topic for Riemann’s habilitation (Gauß died one year later).

  \textbf{Fun fact:} In his 1854 lecture, Riemann speculated that the material bodies might determine the metric of space; many years before Einstein worked out \textit{general relativity} (see Part III, Paragraph 3 in Ref. [129], highlights are mine):

  Die Frage über die Gültigkeit der Voraussetzungen der Geometrie im Unendlichkleinen hängt zusammen mit der Frage nach dem innern Grunde der Maßverhältnisse des Raumes. Bei dieser Frage, welche wohl noch zur Lehre vom Raume gerechnet werden darf, kommt […] zur Anwendung, daß bei einer diskreten Mannigfaltigkeit das Prinzip der Maßverhältnisse schon in dem Begriffe dieser Mannigfaltigkeit enthalten ist, bei einer stetigen aber anders woher hinzukommen muß. Es muß also entweder das dem Raume zugrunde liegende Wirrliche eine diskrete Mannigfaltigkeit bilden, oder der Grund der Maßverhältnisse außerhalb, in darauf wirkenden bindenden Kräften gesucht werden.

  He continues …

  Die Entscheidung dieser Fragen kann nur gefunden werden, indem man von der bisherigen durch die Erfahrung bewährten Auffassung der Erscheinungen, wozu Newton den Grund gelegt, ausgeht und diese durch Tatsachen, die sich aus ihr nicht erklären lassen, getrieben allmählich umarbeitet; […]

  …and closes:

  Es führt dies hinüber in das Gebiet einer andern Wissenschaft, in das Gebiet der Physik, welches wohl die Natur der heutigen Veranlassung nicht zu betreten erlaubt.

  Not only did he sketch the route Einstein would take half a century later, he even seemed intrigued exploring it himself.

- The mathematical field of geometry was conceived in ancient times as a formalization of observable facts about physical space and culminated in the axiomatization of \textit{Euclidean geometry}. One of the facts/axioms of Euclidean geometry is the \textit{parallel postulate}:

  If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

  For two millenia (!) it was suspected that this axiom can be derived from the other four axioms of Euclidean geometry (so that it doesn’t deserve the title “axiom” after all). Finally, Gauß (and
contemporaries) recognized that the parallel postulate cannot be proven from the other four; it is an independent axiom that can be modified to define consistent geometries that differ from Euclid’s! The result is a non-Euclidean geometry which comes in two flavours, \( \text{elliptic geometry} \) and \( \text{hyperbolic geometry} \):

The realization by mathematicians that there are many consistent geometries opened a new question for physics: Are we sure that the geometry of space really is Euclidean? The answer of general relativity is: No, on large scales space is only approximately Euclidean, and it can be very non-Euclidean in regimes of strong gravitational fields.

### 10.1. Summary: What we know and what comes next

1 | Concepts we already know:

- \( \leftrightarrow \) **Differentiable manifolds** (Section 3.1):

  A \( D \)-dimensional manifold is locally homeomorphic (continuously isomorphic) to \( \mathbb{R}^D \) (it locally “looks like” Euclidean space). A continuous, invertible function that maps a region of the manifold to a subset of \( \mathbb{R}^D \) is called a (coordinate) chart. A collection of overlapping charts that covers the whole manifold is called an atlas. If the transition functions that map between different coordinates in regions where two charts overlap are all differentiable (smooth) on \( \mathbb{R}^D \), the manifold is a \( \leftrightarrow \) differentiable (smooth) manifold. On a differentiable manifold we can talk about the differentiation of functions defined on the manifold. In physics we consider almost exclusively such manifolds:

- \( \leftrightarrow \) **Tangent and cotangent spaces** (Section 3.3):

  Given a differentiable manifold (which is not a vector space in general!), there is a canonical way to associate a vector space to every point of the manifold: the \( \leftrightarrow \) tangent space \( T_pM \). Mathematically, it is the vector space of directional derivative operators that act on smooth functions on that point. Given a coordinate chart, the directional derivatives along the coordinates (evaluated at \( p \in M \)) induce a basis \( \{ \partial_i|_p \} \) of the tangent space \( T_pM \) (different
coordinates lead to different bases). In addition, to every vector space there is an associated
dual space spanned by the linear forms on the vector space; thus there is a dual tangent space:
the cotangent space $T^*_p M$. It is spanned by the dual basis $\{dx^i_p\}$ of differential forms:

\[
\text{Tangent space } T_p M \text{ at } p \in M = \text{span } \{\partial_i |_{p} \mid i = 1, \ldots, D\}
\]

(10.1)

With the dual basis (we often drop the subscript $p$)

\[
dx^i_p(\partial_j |_{p}) := \delta^i_j = \frac{\partial x^i}{\partial x^j} |_{p}
\]

(10.2)

we can define the

\[
\text{Cotangent space } T^*_p M \text{ at } p \in M = \text{span } \{dx^i_p \mid i = 1, \ldots, D\}
\]

(10.3)

• Tensor fields (Sections 3.2 to 3.4):

Since there are canonical vector and covector spaces associated to every point of the manifold,
we can consider (reasonably smooth) functions that map every point of the manifold to a
tensor product of $p$ vectors and $q$ covectors; we call such functions tensor fields of rank $(p, q)$. They are “geometric objects” in that they are independent of coordinate charts;
physical quantities (like the electromagnetic field) must be represented by such fields. Once
we have chosen a coordinate chart, we can encode these fields in terms of their components
wrt. the coordinate basis on tangent and cotangent space. The coordinate independence of
tensor fields translates then into a particular transformation law for their components:

\[
\begin{array}{c}
\text{Coordinate transformation } \tilde{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\tilde{x}) \\
\text{(p, q)-Tensor (field) } T \\
\end{array}
\]

\[
= \tilde{T}^{\text{ij}...\text{pq}}_{\text{jm}_{1}...\text{j}_{q}}(\tilde{x}) = \begin{bmatrix} \frac{\partial \tilde{x}^{m_1}}{\partial x^{j_1}} & \cdots & \frac{\partial \tilde{x}^{m_p}}{\partial x^{j_p}} \\ \frac{\partial x^{n_1}}{\partial \tilde{x}^{j_1}} & \cdots & \frac{\partial x^{n_q}}{\partial \tilde{x}^{j_q}} \end{bmatrix} \begin{bmatrix} T^{m_1...m_p}_{n_1...n_q} (x) \end{bmatrix} = T^{\text{ij}...\text{pq}}_{\text{jm}_{1}...\text{j}_{q}}(x)
\]

(10.4)

(Einstein sum convention = Sums over pairs of up- and down indices are implied.)
Examples:

(0, 0)-Tensor \equiv Scalar: \quad \Phi(\tilde{x}) = \Phi(x) \quad (10.5a)

(1, 0)-Tensor \equiv Contravariant vector: \quad \tilde{A}^i(\tilde{x}) = \frac{\partial x^i}{\partial x^k} A^k(x) \quad (10.5b)

(0, 1)-Tensor \equiv Covariant vector: \quad \tilde{B}_j(\tilde{x}) = \frac{\partial x^j}{\partial x^i} B^i(x) \quad (10.5c)

(1, 1)-Tensor \equiv (Mixed) Tensor: \quad \tilde{T}^i_j(\tilde{x}) = \frac{\partial x^i}{\partial x^k} \frac{\partial x^j}{\partial x^l} T^k_l(x) \quad (10.5d)

• ← Riemannian metric (Section 3.5):

A Riemannian metric is a (0, 2) tensor field with a few additional properties (symmetry and non-degeneracy) so that it defines a (pseudo-)inner product on the tangent space at every point of the manifold. A differentiable manifold equipped with such a metric is called a ← Riemannian manifold. On a Riemannian manifold we can measure angles between tangent vectors and lengths of curves:

Riemannian (pseudo-)metric \( ds^2 := \begin{cases} \text{Symmetric} \\ \text{non-degenerate} \end{cases} \) \( (0, 2) \)-tensor field \( (10.6) \)

More formally:

\[
\begin{align*}
\text{ds}^2 : M \ni p & \mapsto \left( \frac{\text{ds}^2}{p} : T_p M \times T_p M \to \mathbb{R} \right) \in T^*_p M \otimes T^*_p M \\
& \quad \text{Bilinear & symmetric & non-degenerate}
\end{align*}
\]

with coordinate representation

\[
\text{ds}^2 = \sum_{i,j=1}^D g_{ij}(x) \text{d}x^i \otimes \text{d}x^j = g_{ij}(x) \text{d}x^i \text{d}x^j \quad (10.8)
\]

where \( g_{ij} = g_{ji} \) (symmetry) and \( g = \det(g_{ij}) \neq 0 \) (non-degeneracy).

A Riemannian metric allows us to define the following geometric concepts:

- Angle between two vectors \( A = A^i \partial_i, B = B^i \partial_i \in T_p M \):

\[
\langle A, B \rangle \equiv \text{ds}^2_p(A, B) = g_{ij}(p) A^i B^j \equiv \|A\|_p \|B\|_p \cos \theta \quad (10.9)
\]

with the norm on \( T_p M \)

\[
\|A\|_p := \sqrt{\text{ds}^2_p(A, A)} = \sqrt{g_{ij}(p) A^i A^j} . \quad (10.10)
\]

- Length of curve \( \gamma : [a, b] \to M \):

\[
L[\gamma] := \int_a^b \sqrt{g_{ij}(\gamma(t)) \frac{dy^i(t)}{dt} \frac{dy^j(t)}{dt}} \text{d}t = \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)} \text{d}t . \quad (10.11)
\]

\( \dot{\gamma}(t) \) is the norm of the "velocity" vector.
Pulling indices up and down (Section 3.5):

A symmetric, non-degenerate bilinear form defines a canonical isomorphism between a vector space and its dual. A special case is a Riemannian metric which provides us with an isomorphism between tangent and cotangent spaces at every point of the manifold. In tensor calculus, this isomorphism is applied by “pulling indices up and down” with the metric:

\[
\text{Pulling down: } T^{i_1 \ldots i_p j_1 \ldots j_q} := g^{ik} T^{i_1 \ldots k \ldots i_p j_1 \ldots j_q}
\]

(10.12a)

\[
\text{Pulling up: } T^{i_1 \ldots i_p j_1 \ldots j_q} := g_{jk} T^{i_1 \ldots i_p k \ldots j} \cdot (10.12b)
\]

where \( g^{ij} \) is the inverse metric defined via \( g^{ik} g_{kj} = \delta^i_j \).

Christoffel symbols and covariant derivatives (Section 3.6):

We realized that the partial derivatives of tensor fields are not tensor fields themselves (this only works for scalars). This motivated the introduction of a “patched up derivative,” the so called \( \text{covariant derivative} \) that transforms again like a tensor. To define the covariant derivative, we needed a set of (non-tensorial) functions called \( \text{Christoffel symbols} \) that were defined by a given Riemannian metric:

\[
\Gamma^i_{\ k\ l} := \frac{1}{2} g^{im} \left( g_{mk,l} + g_{ml,k} - g_{kl,m} \right).
\]

(10.14)

¡! In the following two sections we will revisit, motivate and study the concept of a covariant derivative in more detail. We will also see where the Christoffel symbols come from and which role they play geometrically on the manifold.

If you were not satisfied with the way the covariant derivative and the Christoffel symbols appeared out of thin air in Section 3.6: Now comes the proper introduction!

Manifest covariance (Section 3.6):

The whole point of our endeavor was to find a mathematical toolbox that allows us to write down equations that are guaranteed to be form-invariant under arbitrary coordinate transformations. These equations describe relations between geometric objects on a manifold, such that their content is independent of the chosen coordinate chart. This toolbox is called \( \text{tensor calculus} \) and consists of rules how to combine/construct tensors (e.g. via multiplication, contraction of indices, covariant derivatives, …) to form generally covariant equations. The general covariance of tensorial equations is \( \text{manifest} \) because their mere structure guarantees general covariance:

\[
T^I_j(x) = 0 \quad \xrightarrow{\text{Coordinate trafo: } \tilde{x} = \varphi(x)} \quad \tilde{T}^I_j(\tilde{x}) = 0 \quad \xrightarrow{\text{manifest}} \quad \tilde{T}^I_j(\tilde{x}) = 0 \quad (10.15)
\]
2 | Plan:

- Section 10.2:
  Introduce and study connections and the concept of parallel transport and curvature.

- Section 10.3:
  Use a Riemannian metric to derive a special connection: the Levi-Civita connection. Study properties of this special connection: Riemannian curvature and geodesic curves.

10.2. Affine connections

- An affine connection is an additional structure on a differentiable manifold (no metric needed!) that allows for the definition of the following concepts:
  - Parallel transport
  - Covariant derivatives
  - Autoparallel curves
  - Curvature

  “Additional” means that it is not intrinsic or canonical to a manifold; you can add a connection to obtain more structure. It also implies that typically there are many connections to choose from.

- Terminology:
  In modern differential geometry, the term “connection” has a rather broad meaning. Generally speaking, a connection is a structure that allows one to “parallel transport” objects along curves on a manifold. The most straightforward objects to move around are vectors taken from the tangent spaces of the manifold; this type of connection is called an affine connection, and it is this variety we use in general relativity.

  However, you can also (artificially) attach other spaces to every point of a manifold (e.g., Lie groups like U(1)). Then you can ask how objects of these spaces are parallel transported around the manifold. This gives rise to other types of connections that are particularly important in modern formulations of gauge theories (gauge connections). The gauge field $A^\mu$ of electrodynamics is an example of a U(1) gauge connection; it transports U(1) phases around (not tangent vectors) and is therefore not an affine connection.

In the following we will often drop the “affine” and simply talk about “connections.” Keep in mind, however, that we only consider affine connections in this chapter (and this course).

1 | $\vec{\text{Differentiable}}$ $D$-dimensional manifold $M$; vector field $A = A^i \partial_i$; scalar field $\Phi$:

$$\partial_k \Phi \rightarrow \check{\text{covariant vector field}} \ (\text{cf. Eq. (3.19)})$$

$$\partial_k A^i \rightarrow \times \ no \ tensor \ field \ (\text{cf. Eq. (3.73)})$$

(10.16a) (10.16b)
This is a problem because we often need derivatives of tensors to formulate physical models; and since these equations must be generally covariant (GR!), we need them to transform as tensors!

2 | Problem:

i | Directional derivative of \( A = A^i \partial_i \) along a curve \( \gamma(\lambda) \) with \( \gamma(0) = p \in M \):

\[
\frac{dA(\gamma(\lambda))}{d\lambda}\Bigg|_{\lambda=0} = \lim_{\delta\lambda \to 0} \frac{A(\gamma(\delta\lambda)) - A(\gamma(0))}{\delta\lambda} = \lim_{\delta\lambda \to 0} \frac{A(q) - A(p)}{\delta\lambda}
\]

(10.17)

Note that \( A(q) \in T_qM \) and \( A(p) \in T_pM \), i.e., these values of the vector field belong to different vector spaces. Hence their difference is completely undefined!

ii | We can of course try to work with the components of the vector field wrt. a given chart instead. Since \( A^i \in \mathbb{R} \), the following expression is at least well-defined:

\[
\frac{dA^i(\gamma(\lambda))}{d\lambda}\Bigg|_{\lambda=0} = \lim_{\delta\lambda \to 0} \frac{A^i(q) - A^i(p)}{\delta\lambda}
\]

(10.18)

Unfortunately this does not solve the problem, because these components are given wrt. to different, coordinate-dependent bases on \( T_qM \) and \( T_pM \), respectively:

\[
A(q) = A^i(q) \partial_i|_q \quad \text{with} \quad \text{span}\{\partial_i|_q\} = T_qM
\]

(10.19)

\[
A(p) = A^i(p) \partial_i|_p \quad \text{with} \quad \text{span}\{\partial_i|_p\} = T_pM
\]

(10.20)

iii | To understand why this is a problem, imagine you fix the basis \( \{\partial_i|_p\} \) of \( T_pM \); this does not fix the basis \( \{\partial_i|_q\} \) of \( T_qM \) because choosing different (curvilinear) coordinates can be used to modify the induced basis \( \{\partial_i|_q\} \) without changing \( \{\partial_i|_p\} \):

As a consequence, the components \( A^i(q) \) can be modified arbitrarily without changing the vector field \( A \) itself. Thus the difference \( A^i(q) - A^i(p) \), and thereby the directional derivative above, do not encode a geometric, coordinate independent object! Mathematically, this is reflected in the non-tensorial transformation of the difference under arbitrary coordinate transformations

\[
\tilde{A}^i(q) - \tilde{A}^i(p) = \frac{\partial \tilde{x}^i(q)}{\partial x^k} A^k(q) - \frac{\partial \tilde{x}^i(p)}{\partial x^k} A^k(p) \neq \frac{\partial x^i(q)}{\partial x^k} \left[ A^k(q) - A^k(p) \right]
\]

(10.21)

This explains why partial derivatives of the form \( \partial_k A^i \) (which are simply directional derivatives along coordinate axes) fail to transform as tensors.
3 | Idea:

The problem is conceptually most transparent in Eq. (10.17) which is mathematically undefined. However, if we could make it well-defined, we would immediately obtain a geometric, coordinate-independent object. To make the difference between the two vectors well-defined, they must live in the same tangent space, though.

Our only way out is to assume that we are given some function \( \Gamma_{p\rightarrow q} : T_pM \rightarrow T_qM \) that establishes a correspondence between the two nearby tangent spaces by “parallel transporting” vectors between them. We then could “parallel transport” \( A(p) \) from \( T_pM \) to \( T_qM \) like so: \( \Gamma_{p\rightarrow q}(A(p)) \in T_qM \). With this new vector, the difference is mathematically well-defined:

\[
\frac{D A}{D \lambda} := \lim_{\delta \lambda \to 0} \frac{A(q) - \Gamma_{p\rightarrow q}(A(p))}{\delta \lambda}
\]

We use the capital letter \( D \) to indicate that the difference in the numerator of the difference quotient has been modified by (and depends on) \( \Gamma \).

\( \Gamma_{p\rightarrow q} \) is an \( \ast \) affine connection

(This is not yet very rigorous, we will specify our idea more formally below.)

- As already mentioned, the interpretation of an affine connection \( \Gamma \) is that it formalizes the notion of “parallel translating” or “parallel transporting” tangent vectors along curves on the manifold from one tangent space to another. It is important to realize that the notion of “parallel transport” is mathematically subtle and not trivial. It must be carefully defined and can lead to quite surprising results when considering curved manifolds:

Note that the (intuitive) parallel transport on the Euclidean plane (left) is independent of the path along which the vector is transported. By contrast, intuitively transporting vectors on a sphere (right) yields different results depending on the chosen path. The fact that there is no unique “parallel vector” to a given vector, but that the notion of parallelism depends on the path taken, is the hallmark of \( \rightarrow \) curvature.

- To be clear: the failure to produce a tensorial object from the directional derivative of a vector field is a fundamental and not a technical issue. We were neither too naïve when performing the derivative in Eq. (10.17), nor will our “solution” Eq. (10.22) render it magically tensorial. It is \( \text{impossible} \) to define a tensorial derivative on a manifold without specifying an additional structure (namely an affine connection \( \Gamma \)).

4 | Motivation:

To understand the properties of parallel transport (and thereby an affine connection \( \Gamma \)) better, we consider the simple example of parallel transporting a vector in the affine space \( M = E^2 = \mathbb{R}^2 \) (the Euclidean plane), described in curvilinear polar coordinates:
If you are given a differentiable manifold without additional structure, it does not make sense to ask whether a vector field “is constant.” For example, if we consider \( M = \mathbb{E}^2 \) as a manifold (forgetting about its Euclidean metric and affine structure), it does not make sense to call the vector field \( A \) “constant”; its components are constant wrt to the basis induced by a specific coordinate system. However, a coordinate-independent statement like \( A(p) = A(q) \) for all \( p, q \in M \) is nonsensical because \( A(p) \in T_p M \) and \( A(q) \in T_q M \), and there is no canonical isomorphism connecting \( T_p M \) and \( T_q M \); without an affine connection \( \Gamma \), these are completely unrelated vector spaces and we do not know how to compare vectors at different points on the manifold (there is no concept of “parallel” vectors).

\[ \begin{align*}
\text{i} & \quad \text{with Cartesian coordinates } \tilde{x} = (x, y) \text{ and Cartesian basis } \{ \partial_x, \partial_y \}: \\
\text{“Constant” vector field } & \quad A = A^x \partial_x + A^y \partial_y \quad \text{with } A^x = \text{const}, A^y = \text{const}:
\end{align*} \]

\[ A(r, \theta) = A^r \partial_r + A^\theta \partial_\theta \quad \text{(10.26)} \]

\[ A^r(r, \theta) = A^x \cos \theta + A^y \sin \theta \quad \text{(10.27a)} \]

\[ A^\theta(r, \theta) = \frac{1}{r} (A^y \cos \theta - A^x \sin \theta) \quad \text{(10.27b)} \]

\[ \begin{align*}
\text{ii} & \quad \text{Coordinate transformation } (x, y) = \varphi^{-1}(r, \theta) \text{ to polar coordinates:} \\
& \quad x = r \cos \theta \quad \text{(10.24a)} \\
& \quad y = r \sin \theta \quad \text{(10.24b)} \\
& \quad \text{Induced basis change on tangent spaces (\( \ast \text{ Eq. (3.5)} \))}:
\end{align*} \]

\[ \begin{align*}
\partial_r &= \cos \theta \partial_x + \sin \theta \partial_y \\
\partial_\theta &= -r \sin \theta \partial_x + r \cos \theta \partial_y
\end{align*} \]

\[ \text{Components of vector field:} \]

\[ A = A^x \partial_x + A^y \partial_y \]

\[ \text{with (no longer constant!)} \]

\[ \begin{align*}
A^r(r, \theta) &= A^x \cos \theta + A^y \sin \theta \\
A^\theta(r, \theta) &= \frac{1}{r} (A^y \cos \theta - A^x \sin \theta)
\end{align*} \]

\[ \begin{align*}
\text{iii} & \quad \text{Two infinitesimally separated points } p, q \in \mathbb{E}^2 \text{ with coordinates} \\
& \quad u(p) = (r, \theta) \quad \text{and} \quad u(q) = (r + \delta r, \theta + \delta \theta)
\end{align*} \]

\[ \begin{align*}
u(p) = (r, \theta) \quad \text{and} \quad u(q) = (r + \delta r, \theta + \delta \theta)
\end{align*} \]

\[ \text{(10.28)} \]
and associated vectors \((A^r = A^r(p) \text{ and } A^\theta = A^\theta(p))\)

\[
\begin{align*}
A(p) &= A^r \partial_r + A^\theta \partial_\theta, \\
A(q) &= [A^r + \delta A^r] \partial_r + [A^\theta + \delta A^\theta] \partial_\theta.
\end{align*}
\] (10.29a)

By Eq. (10.27) \(\circ\) (via Taylor expansion)

\[
\begin{align*}
\delta A^r &= r A^\theta \delta \theta, \\
\delta A^\theta &= -\frac{1}{r} (A^\theta \delta r + A^r \delta \theta).
\end{align*}
\] (10.30a, b)

If we now declare the vector field \(A\) to be constant, the variations Eq. (10.30) must be “fake” in the sense that they are caused by our choice of curvilinear coordinates rather than an “intrinsic” variation of the vector field itself.

This choice specifies an → affine connection.

Now that we specified which changes of the components of vector fields (in our coordinate system) are considered to be “fake”, i.e., artifacts of the coordinates, we can define the “real” changes of arbitrary vector fields (which then can be non-constant wrt. our specific notion of parallel vectors) as their “complete” variation corrected by the “fake” variation \(\delta A^i\):

\(<\) Arbitrary (“non-constant”) vector field with \(B^i(p) = B^i(r, \theta)\)

\[
\begin{align*}
[B^r(q) - B^r(p)] - \delta B^r &= \frac{\partial B^r}{\partial r} \delta r + \left( \frac{\partial B^r}{\partial \theta} - r B^\theta \right) \delta \theta, \\
[B^\theta(q) - B^\theta(p)] - \delta B^\theta &= \left( \frac{\partial B^\theta}{\partial r} + \frac{1}{r} B^r \right) \delta r + \left( \frac{\partial B^\theta}{\partial \theta} + \frac{1}{r} B^r \right) \delta \theta.
\end{align*}
\] (10.31a, b)

The idea is to use such “corrected” differences in the numerator of a difference quotient like Eq. (10.22) to define a derivative of the vector field that transforms like a tensor.

That is, we define

\[
A^i(p \rightarrow q) = A^i(p) + \delta A^i(p).
\] (10.32)
5 | Generalization:

Drawing from the example and the form of the particular connection Eq. (10.30), we can select reasonable properties that an general affine connection should satisfy (in terms of components):

(i) \( A^i (p \rightarrow q) \) is linear in \( A^i (p) \).

(ii) The variation \( \delta A^i (p) \) is linear in the first-order variation \( \delta x^i \) of coordinates.

We can satisfy both conditions if the variation has the general form (the minus is convention)

\[
\delta A^i (p) = -\Gamma^i_{kl} (p) A^k (p) \delta x^l
\]  
(10.33)

\[
A^i (p \rightarrow q) = A^i (p) + \delta A^i (p) = \left[ \delta^i_k - \Gamma^i_{kl} (p) \delta x^l \right] A^k (p)
\]  
(10.34)

with some undetermined set of coefficients \( \Gamma^i_{kl} \) that completely specify the affine connection (in the particular coordinates chosen):

\[
\Gamma^i_{kl} (x) : \# (coefficients of the) affine connection \Gamma (in \ x)
\]

Example:

From Eq. (10.30) and Eq. (10.33) it follows for the coefficients of the affine connection of the Euclidean plane, expressed in polar coordinates (Problemset 2):

\[
\Gamma^r_{kl} (r, \theta) = \begin{pmatrix} 0 & 0 \\ 0 & -r \end{pmatrix}_{kl} \quad \text{and} \quad \Gamma^\theta_{kl} (r, \theta) = \begin{pmatrix} 0 & \frac{1}{r} \\ \frac{1}{r} & 0 \end{pmatrix}_{kl}.
\]  
(10.35)

6 | Interpretation:

The affine connection establishes a connection (hence the name) between tangent spaces at different points on the manifold by establishing a notion of parallelism:

\[
\begin{align*}
\Gamma^i_{kl} (x) & : \# (coefficients of the) affine connection \Gamma (in \ x) \\
\text{Infinitesimal parallel transport} & \\
A(p) & \rightarrow A(p) = A^i (p \rightarrow q) \partial_i |q \\
& = [A^i (p) + \delta A^i (p)] \partial_i |q \\
& = [\delta^i_k - \Gamma^i_{kl} \delta x^l] A^k (p) \partial_i |q
\end{align*}
\]  
(10.36)
We say: $\Gamma_{p \rightarrow q}(A(p))$ is the vector in $q$ that is parallel to $A(p)$ in $p$.

### Absolute derivative:

We can now express the absolute derivative using the connection:

\[
\frac{dA^i}{d\lambda} \overset{\text{10.22}}{=} \frac{d}{d\lambda} \left( A^i(y(\lambda + \delta \lambda)) - A^i(y(\lambda)) - \delta A^i \right) = \frac{dA^i}{d\lambda} + \Gamma^i_{kl} \frac{dx^l}{d\lambda} \tag{10.33}
\]

We want the absolute derivative to transform as a contravariant vector:

\[
\frac{d\tilde{A}^i}{d\lambda} = \frac{\partial \tilde{x}^i}{\partial x^k} \frac{dA^k}{d\lambda} \tag{10.38}
\]

A straightforward but cumbersome calculation shows [recall Section 3.6] that this is the case if and only if the connection coefficients transform as follows:

\[
\tilde{\Gamma}^i_{kl} = \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^m}{\partial x^k} \frac{\partial x^l}{\partial x^l} \frac{\partial A^m}{\partial x^o} \Gamma^m_{no} + \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial^2 x^p}{\partial x^k \partial x^l} \tag{10.39}
\]

$\rightarrow \Gamma^i_{kl}$ does not transform as a tensor!

- For a given manifold $M$, there are infinitely many choices for an affine connection $\Gamma$.
- The definition Eq. (10.37) makes sense for any contravariant vector $A^i$ that is defined (and differentiable) along the curve $y(\lambda)$ [for example, a particle trajectory $x^\mu(\lambda)$]. Although we considered a vector field $A^i$ in our discussion, it is not necessary for $A^i$ to be defined in the neighborhood of the trajectory $y(\lambda)$; i.e., partial derivatives $\partial_j A^i$ do not need to be defined for the definition of the absolute derivative Eq. (10.37). This is why we distinguish between the absolute derivative and the covariant derivative.

- The additional term that makes the transformation of the connection coefficients non-tensorial is needed to compensate for a corresponding non-tensorial term from the total (non-covariant) derivative $\frac{dA^i}{d\lambda}$.
- Every set of fields $\Gamma^i_{kl}$ that transforms according to Eq. (10.39) can be used to define a connection (and therefore a notion of what “parallel” means on a manifold). This definition allows for more solutions than the specific type of connection that we used for our motivation, namely connections derived from declaring a given vector field as “constant.” Interestingly, not all connections can be constructed in this way (the ones that can are actually quite boring because they do not have curvature), and in Section 10.3 we will find a recipe to construct a special connection from every Riemannian metric.

### Torsion:

In general, the connection coefficients are not symmetric in their lower two indices.

\[
\Gamma^i_{kl} = \frac{1}{2} \left( \Gamma^i_{kl} + \Gamma^i_{lk} \right) + \frac{1}{2} \left( \frac{S^i_{kl}}{\Gamma^i_{(kl)}} \right) \tag{10.40}
\]
Eq. (10.39) (Note that the non-tensorial part in Eq. (10.39) is symmetric in $k$ and $l$!)

\[
\tilde{S}^i_{kl} = \frac{\partial x^i}{\partial x^m} \frac{\partial x^m}{\partial x^k} \frac{\partial x^o}{\partial x^l} S^m_{no} \tag{10.41}
\]

→ Antisymmetric part $S^i_{kl}$ of connection is a tensor: *Torsion tensor*

- \*! This is not true for the symmetric part.
- **General Relativity** is based on the assumption that the affine connection of spacetime is *torsion-free*. Hence it is sufficient to focus on symmetric, torsion-free connections to formulate the theory.
- **Interpretation:** On a manifold with torsion, infinitesimal parallelograms do not close:

To see this, consider two infinitesimal vectors $\delta x^1_i$ and $\delta x^2_i$ at some point $p \in M$. Then parallel transport $\delta x^1_i$ along $\delta x^2_i$ to produce $\delta x^1_i$ and vice versa:

\[
\begin{align*}
\delta x^1_i &= \delta x^1_i - \Gamma^i_{kl} (\delta x^k_1)(\delta x^l_2), \tag{10.42a} \\
\delta x^2_i &= \delta x^2_i - \Gamma^i_{kl} (\delta x^k_2)(\delta x^l_1). \tag{10.42b}
\end{align*}
\]

The amount by which this infinitesimal parallelogram does not close is:

\[
\Delta^i := (\delta x^1_i + \delta x^2_i) - (\delta x^2_i + \delta x^1_i) = (\delta x^1_i - \delta x^2_i) - (\delta x^2_i - \delta x^1_i) \\
\stackrel{\text{by (10.42)}}{=} (\Gamma^i_{kl} - \Gamma^i_{lk}) (\delta x^k_1)(\delta x^l_2) \equiv S^i_{kl} (\delta x^k_1)(\delta x^l_2). \tag{10.43}
\]

Non-vanishing torsion therefore implies:

\[
\Delta^i = S^i_{kl} (\delta x^k_1)(\delta x^l_2) \neq 0 \iff S^i_{kl} (\delta x^k_1)(\delta x^l_2) \neq S^i_{kl} (\delta x^k_2)(\delta x^l_1). \tag{10.44}
\]

→ The direction of paths matters: First going along $\delta x^1_i$ and then parallel to $\delta x^2_i$ leads to a different point than doing the opposite. (Similar to the motion of a screw, which is different for clockwise and counterclockwise rotation.)

- It is possible to extend General Relativity by allowing the torsion of spacetime to be non-zero (and dynamic as well) \[130,131\]. In such theories, the *spin* of particles becomes the source of torsion, just as their *mass* is the source of curvature. Such theories can predict additional forces between spinful particles, see Ref. \[132\] for a review.
- Since torsion is “just another tensor field” (which is not true for the symmetric part of the connection), it is reasonable to keep a geometric theory of gravity slim and assume torsion to vanish. If the theory matches observations, we didn’t produce unnecessary clutter by dragging torsion along (\*Occam’s razor*); however, if there happen to be phenomena that cannot be explained, we can still “patch” the theory by adding new (tensor) fields (that might play the role of torsion). In any case, there is no experimental evidence to date that makes a torsion field necessary.
Henceforth we consider only torsion-free connections:

\[ \Gamma^i_{kl} = \Gamma^i_{lk} \]

9 | Locally geodesic coordinate systems:

Since we know how the coefficients of a connection transform, we can ask whether there are special coordinate systems in which the connection looks particularly simple:

Details: \( \mathcal{O} \) Problemset 2

i | Goal:

Show that for every point \( p \in M \) there is a coordinate system in which the connection coefficients in this point vanish:

\[ \forall p \in M \exists \text{ Chart } u \text{ with } u(p) = x_0 : \quad \Gamma^i_{kl}(x_0) = 0 \quad \forall_{ijk} \quad (10.45) \]

\( u: \mathcal{O} \) Locally geodesic coordinate system

First, show the alternative form of the transformation: (recall Eq. (3.75))

\[
\tilde{\Gamma}^i_{kl} = \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^m}{\partial x^k} \frac{\partial x^o}{\partial x^l} \Gamma^m_{no} - \frac{\partial x^m}{\partial x^k} \frac{\partial x^p}{\partial x^l} \left( \frac{\partial^2 \tilde{x}^i}{\partial x^p \partial x^m} \right) \quad (10.46)
\]

This follows from Eq. (10.39) by differentiating \( \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^m}{\partial x^k} = \delta^i_j \).

\( \langle \) Coordinates \( v \) with \( v(p) = 0 \in \mathbb{R}^D \) (in general it is \( \tilde{\Gamma}^i_{kl}(0) \neq 0 \) in this chart)

\( \rightarrow \) Coordinate transformation \( \tilde{x} = \varphi(x) = u \circ v^{-1}(x) \) in vicinity of \( p \in M \):

\[ \tilde{x}^i = x^i + \frac{1}{2} C^i_{kl}(0) x^k x^l + \ldots \quad (10.47) \]

with (w.l.o.g.) symmetric coefficients \( C^i_{kl} = C^i_{lk} \).

Partial derivatives at \( u(p) = 0 = v(p) \):

\[ \frac{\partial \tilde{x}^i}{\partial x^m} \bigg|_{x=0} = \delta^i_m \quad \text{and} \quad \frac{\partial^2 \tilde{x}^i}{\partial x^p \partial x^m} \bigg|_{x=0} = C^i_{pm}(0) \quad (10.48) \]

\( \rightarrow \) \( \tilde{\Gamma}^i_{kl} \triangleq \tilde{\Gamma}^i_{kl} - C^i_{kl} \)

\( \Gamma^i_{kl}(0) \neq 0 \) and \( \tilde{\Gamma}^i_{kl} = \tilde{\Gamma}^i_{lk} \) (torsion-free!) \( \rightarrow \) \( C^i_{kl}(0) := \Gamma^i_{kl}(0) \)

Notes:

- ![\!] Note that we only showed that the connection coefficients can be made zero \emph{in a single point}; in general one cannot find a coordinate system where the coefficients vanish everywhere. This also implies that in general the derivatives \( \partial_m \Gamma^i_{kl}(0) \) do \emph{not} vanish in \( p \).

- In locally geodesic coordinates, the absolute derivative Eq. (10.37) is simply the “normal” total derivative. As a consequence, in the context of Riemannian manifolds, the coordinate lines are local geodesics ("shortest paths", \( \rightarrow \) \emph{later} – hence the name.

- The above argument fails for connections with non-vanishing torsion \( S^i_{kl} \neq 0 \) since the latter transforms as a tensor and cannot be zeroed by a coordinate transformation (unless it vanishes in all coordinates).
• The fact that locally geodesic coordinates exist at every point will be the foundation for the implementation of Einstein’s equivalence principle (EEP) in the mathematical framework of general relativity. Physically, these coordinates will be identified with the free falling, local inertial frames.
10.2.1. Covariant derivatives

The definition Eq. (10.37) of the \( A^i(\lambda) \) did not require \( A^i(\lambda) \) to be defined in a neighborhood of the curve \( \gamma(\lambda) \). However, if \( A^i(\lambda) \equiv A^i(\gamma(\lambda)) \) is defined on the whole manifold (or at least in a neighborhood of the curve), we can define a more useful derivative:

\[
\frac{dA^i}{d\lambda} = \frac{\partial A^i}{\partial x^k} \frac{dx^k}{d\lambda} = \frac{DA^i}{D\lambda} = \left( \frac{\partial A^i}{\partial x^k} + \Gamma^i_{mk} A^m \right) \frac{dx^k}{d\lambda} = A^i;k \frac{dx^k}{d\lambda} \tag{10.49}
\]

→ **Covariant derivative of a contravariant vector:**

\[
\begin{cases}
D_k A^i \\
\nabla_k A^i \\
A^i;k
\end{cases}
:=
\begin{cases}
\frac{\partial A^i}{\partial x^k} \\
\partial_k A^i \\
A^i;_k
\end{cases} + \Gamma^i_{mk} A^m
\tag{10.50}
\]

\( A^i;k \) is \((1, 1)\)-tensor

Proof: Via the \(<\) quotient theorem or by straightforward calculation using Eq. (10.39) (\(<\) Section 3.6).

11 | Covariant derivative of a scalar:

\[ \Phi_{,k} := \Phi_{,k} \tag{10.51} \]

\( \Phi_{,k} \) is \((0, 1)\)-tensor [Proof: Eq. (3.19)]

That the partial derivatives of scalar fields encode geometric objects, and there is no need to use the additional structure of a connection, is a consequence of the fact that scalar fields map to \( \mathbb{R} \) and not \( T_p M \). Note that it makes sense to talk about a constant scalar field \( \phi(p) = \phi(q) \) for all \( p, q \in M \) without referring to a particular coordinate system or specifying an additional structure!

12 | One demands that the \( \rightarrow \) **Leibniz product rule** is valid for covariant derivatives:

\[
(A^i B_i)_;k \equiv A^i;k B_i + A^i B_i;k
\tag{10.52}
\]

→ Covariant derivative of **covariant vector**:

\[ B_{i;k} := B_{i;k} - \Gamma^m_{ik} B_m \tag{10.53} \]

Cf. Eq. (10.50): Different summation indices and different sign!

\( B_{i;k} \) is \((0, 2)\)-tensor
Proof. First we note that
\[ A^i \cdot B_{i;k} + A^i \cdot B_i;_{;k} \stackrel{10.52}{=} (A^i \cdot B_i)_{;k} \stackrel{10.51}{=} (A^i \cdot B_i)_{,k} = A^i_{,k} B_i + A^i B_i_{,k} \] (10.54)
since \( A^i \cdot B_i \) is a scalar. With the definition Eq. (10.50) it follows
\[ A^i \cdot B_i;_{;k} = A^i \left( B_i_{,k} - \Gamma^m_{i,kl} B_m \right) . \] (10.55)
Since this must be true for arbitrary \( A^i \), Eq. (10.53) follows. \[ \square \]

13 | Covariant derivatives of higher-rank tensors:
The above structure can be generalized to tensors of arbitrary rank:

\[
\mathcal{T}^{i_{r \ldots} k_{s \ldots} j} := T^{i_{r \ldots} k_{s \ldots} j} + \Gamma^i_{m l} T^m_{r \ldots} k_{s \ldots} j + \ldots - \Gamma^m_{r l} T^{i_{r \ldots} k_{s \ldots} j} + \ldots \]

\[ \forall \text{ upper indices} \quad \forall \text{ lower indices} \]
(10.56)

Example:

Covariant derivatives of rank-2 tensors:

\[
\begin{align*}
T^{i_{r \ldots} k_{s \ldots} j} & = T^{i_{r \ldots} k_{s \ldots} j} + \Gamma^i_{m l} T^m_{r \ldots} k_{s \ldots} j + \Gamma^k_{m l} T^{i_{r \ldots} m_{s \ldots} j} + \ldots \quad \rightarrow (2, 1)-tENSOR \quad (10.57a) \\
T^i_{k,l} & = T^i_{k,l} - \Gamma^m_{i l} T^m_{k} + \Gamma^m_{k l} T^i_{m} \quad \rightarrow (0, 3)-tENSOR \quad (10.57b) \\
T^i_{k,l} & = T^i_{k,l} + \Gamma^m_{i m} T^i_{k l} - \Gamma^m_{k l} T^i_{m} \quad \rightarrow (1, 2)-tENSOR \quad (10.57c)
\end{align*}
\]

For a proof, see Schröder [3] (p. 53).

10.2.2. Parallel vector fields and autoparallel curves

Vector field \( A^i = A^i \delta_i \) & curve \( \gamma \):

\[ A \text{ is a } \parallel \text{ parallel (vector field) along } \gamma \]
\[ \Leftrightarrow \quad \frac{DA^i}{D\lambda} = \frac{dA^i}{d\lambda} + \Gamma^i_{k l} A^k \frac{dx^l}{d\lambda} = 0 \] (10.58)

- Given a connection \( \Gamma \), Eq. (10.58) is a first-order differential equation for \( A^i \). By solving it for a given initial value of \( A^i (\lambda = 0) \), one can reconstruct a parallel vector field on the curve \( \gamma \).
• For higher-rank tensors, one defines parallelism along a curve analogously:

$$\frac{\partial^{\gamma}i_{...}}{\partial \lambda} = 0$$

(10.59)

15 Autoparallel curve: Generalization of a straight line in $\mathbb{R}^D$:

Straight line: Curve that "keeps its direction constant."

We cannot characterize a straight line as "the shortest curve between two points" because we do not have a metric, only a connection!

Curve $\gamma$ with parametrization $\gamma^{\mu}(\lambda)$ (in some chart)

$\gamma$ is autoparallel $\iff$ Tangent field $A = A^i \partial_i := \frac{dy^i}{d\lambda} \partial_i$ is $\leftrightarrow$ parallel along $\gamma$:

$$\frac{d^2 \gamma^i}{d\lambda^2} + \Gamma^i_{kl} \frac{dy^k}{d\lambda} \frac{dy^l}{d\lambda} = 0 \quad \Rightarrow \quad \gamma \text{ is autoparallel}$$

(10.60)

• If a parametrization of a curve satisfies the DGL Eq. (10.60), the curve is autoparallel and the given parametrization is called affine. Since Eq. (10.60) is not reparametrization invariant (→ below), there are other (non-affine) parametrizations of the same autoparallel curve that do not satisfy Eq. (10.60). Every autoparallel curve has such an affine parametrization (which is unique up to affine transformations).

• Once we have a metric and a compatible connection (→ Section 10.3), the autoparallel curves will be identical to the curves of shortest length (→ geodesics).

• Let us assume that an affine parametrization of an autoparallel curve satisfies Eq. (10.60). Now consider a reparametrization $\mu = f(\lambda)$ given by some strictly monotone function $f$.

The new parametrization is then $\tilde{\gamma}^i(\mu) = \gamma^i(f(\lambda))$ and satisfies the DGL

$$\frac{d^2 \tilde{\gamma}^i}{d\mu^2} + \Gamma^i_{kl} \frac{d\gamma^k}{d\mu} \frac{d\gamma^l}{d\mu} \equiv h(\mu) \frac{d\tilde{\gamma}^i}{d\mu} \frac{d\tilde{\gamma}^j}{d\mu} \frac{d\mu^2}{d\lambda^2} = 0$$

(10.61)

The definition of $h$ is equivalent to the DGL

$$\frac{d^2 \mu}{d\lambda^2} + h(\mu) \left( \frac{d\mu}{d\lambda} \right)^2 = 0$$

(10.62)
If \( \lambda \) is an affine parameter, the transformation \( f \) yields another affine parameter \( \mu \) if and only if \( h(\mu) = 0 \), i.e.,

\[
\frac{d^2 \mu}{d\lambda^2} = 0 .
\] (10.63)

which is solved by reparametrizations of the affine form \( \mu = f(\lambda) = a\lambda + b \). That is, affine parametrizations are unique up to affine reparametrizations.

- This problem does not affect the definition of a parallel vector field because Eq. (10.58) is reparametrization invariant.

### 10.2.3. The curvature tensor

Now that we have a formal concept of the parallel transport of vectors from one tangent space to another, we can ask whether the result of such a transport depends only on the final destination, or whether the path of the transport also plays a role. The answer will be that, for a generic connection, parallel transport indeed is path dependent, and that this path dependence is a manifestation of the intrinsic curvature of the manifold (more precisely: its connection).

16 | ≪ Parallel transport of vector \( A = A^j \partial_j \) from \( q \) to \( q' \) via different paths \( \gamma_1 \) and \( \gamma_2 \):

\[ \rightarrow \text{It is easier (and sufficient) to study an infinitesimal parallelogram.} \]

17 | ≪ Path \( p \xrightarrow{p_1} p' \):

The first parallel transport along \( \delta x_1 \) yields:

\[ A^i(p) \xrightarrow{\delta x_1} A^i(p_1) = A^i(p) + \delta_1 A^i(p) = A^i(p) - \Gamma^i_{kl}(p) A^k \delta x_1^l \] (10.64)

The subsequent parallel transport along \( \delta x_2 \) yields:

\[
A^i(p) \xrightarrow{\delta x_1} A^i(p_1) \xrightarrow{\delta x_2} A^i(p_1') = A^i(p) + \delta_1 A^i(p) + \delta_2 A^i(p_1) \delta x_1^1
\]

\[ = A^i + \delta_1 A^i - \Gamma^i_{nm}(p_1) \left[ A^n + \delta_1 A^n \right] \delta x_2^m \] (10.65a)

Our goal is to express everything in the initial point \( p \). →

\[ \Gamma^i_{nm}(p_1) \approx \Gamma^i_{nm}(p) + \partial I \Gamma^i_{nm}(p) \delta x_1^l \] (10.66)
(Since we consider an infinitesimal parallelogram, we only need linear variations of all quantities.)

With this expansion, we find for the parallel vector in \( p' \):

\[
A^i(p \xrightarrow{\delta x_1} p_1 \xrightarrow{\delta x_2} p') = \frac{A^i - \Gamma^i_{kl} A^k \delta x_1^l - \Gamma^i_{nm} A^n \delta x_2^m}{\delta_1 A^i(p)}
\]

\[+ \Gamma^i_{nm} \Gamma^n_{kl} A^k \delta x_1^l \delta x_2^m - \partial_l \Gamma^i_{nm} A^n \delta x_1^l \delta x_2^m + \mathcal{O}((\delta x)^3) \]

(10.67)

In this expression, all connection coefficients and fields are evaluated in \( p' \).

\[\langle \text{Path } p \xrightarrow{\delta x_2} p_2 \xrightarrow{\delta x_1} p' \rangle: \text{Same expression with } \delta x_1 \leftrightarrow \delta x_2: \]

\[
A^i(p \xrightarrow{\delta x_2} p_2 \xrightarrow{\delta x_1} p') = \frac{A^i - \Gamma^i_{kl} A^k \delta x_2^l - \Gamma^i_{nm} A^n \delta x_1^m}{\delta_2 A^i(p)}
\]

\[+ \Gamma^i_{nm} \Gamma^n_{kl} A^k \delta x_2^l \delta x_1^m - \partial_l \Gamma^i_{nm} A^n \delta x_2^l \delta x_1^m + \mathcal{O}((\delta x)^3) \]

(10.68)

18 | \[\rightarrow \text{Path dependence:} \]

\[\Delta A^i := A^i(p \xrightarrow{\delta x_1} p_1 \xrightarrow{\delta x_2} p') - A^i(p \xrightarrow{\delta x_2} p_2 \xrightarrow{\delta x_1} p') \]

\[= \left\{ \begin{array}{l}
\text{Change of } A^i \text{ after parallel transport along } p \\
\text{closed path } p \rightarrow p_1 \rightarrow p' \rightarrow p_2 \rightarrow p.
\end{array} \right. \]

Drop \( \mathcal{O}((\delta x)^3) \) terms.

\[\equiv R^i_{klm} A^k \delta x_1^m \delta x_2^l \]

(10.69)

with the \[\ast\] curvature tensor

\[
R^i_{klm} = \partial_l \Gamma^i_{km} - \partial_m \Gamma^i_{kl} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}.
\]

(10.70)

Although \( \Gamma^i_{kl} \) is no tensor, this particular combination is a \((1, 3)\)-tensor (Proof: \(\rightarrow\) next).

19 | \[\rightarrow \text{Path dependence:} \]

\[A^i[p \xrightarrow{\delta x_2} p_2 \xrightarrow{\delta x_1} p'] = A^i[p \xrightarrow{\delta x_1} p_1 \xrightarrow{\delta x_2} p'] = R^i_{klm} A^k \delta x_1^m \delta x_2^l \]

(10.71)

\[\rightarrow \text{Covariant derivatives of tensors are not commutative (in general)!} \]

[Eq. (10.71) is valid in this form only for torsion-free connections.]

\[A^i[p \xrightarrow{\delta x_2} p_2 \xrightarrow{\delta x_1} p'] = (0, 3)-\text{tensor} \leftarrow \text{Quotient theorem} \]

\[R^i_{klm} \text{ is } (1, 3)-\text{tensor} \checkmark \]

- Alternatively, you can prove the tensorial transformation of \( R^i_{klm} \) manually using the expression Eq. (10.70) and the transformation of the connection coefficients Eq. (10.39) and partial derivatives Eq. (3.5).
• Compare the non-commutativity of the covariant derivative of tensors with the commutativity of conventional partial derivatives:

\[ A_{k,[l,m]} = A_{k,l,m} - A_{k,m,l} = \partial_l \partial_m A_k - \partial_m \partial_l A_k = 0. \]  

(10.72)

Notes:

• The curvature tensor can be interpreted geometrically as follows:

Since curvature is the property that vectors parallel transported around infinitesimal loops change their direction, one can encode all features of curvature in an object that tells you how an arbitrary vector is transformed if transported around any infinitesimal parallelogram in the \( ml \)-plane. This object is the curvature tensor, and from this perspective it is clear that it must be of rank four (two indices to specify the plane, two for the transformation of the vector).

• (A manifold with) a connection is called flat iff the curvature tensor is identically zero everywhere: \( R^i_{\ klm}(p) = 0 \). In particular, this means (for a torsion-free connection) that in a neighborhood of every point on the manifold (and not just the point itself!) you can find a coordinate system in which the connection coefficients vanish identically (i.e., these neighborhoods behave like flat Euclidean space).

In summary, the following statements are equivalent:

- The curvature tensor vanishes identically.
- The manifold is flat.
- Parallel transport is path-independent.
- Covariant derivatives are commutative.

• Whether a space is curved or not is a property of its connection and not of its topology! For example, here are two topologically equivalent (homeomorphic) tori ("donuts"): 
The left one is defined by identifying opposite edges with each other and inherits the connection of the Euclidean plane. The right torus is embedded in 3D Euclidean space and inherits the metric of $\mathbb{R}^3$ and its induced connection. Both spaces are topological tori, but the left one is flat whereas the right one is not [as illustrated by the path(in)dependence of parallel transport].

So if someone asks you whether a torus is flat or curved, the correct answer is that this is an undefined question unless a particular connection is specified! (Interestingly, this is not true for the two-dimensional sphere $S^2$. While there are many connections you can assign to a 2D sphere, none of them is flat! This is a corollary of the ↑ Gauss-Bonnet theorem or, alternatively, the ↑ hairy ball theorem.)

### 10.3. Affine connections on Riemannian manifolds

We already know the benefits of a Riemannian manifold $(M, g)$, i.e., a manifold equipped with a (pseudo-)Riemannian metric $g$. In the previous section, we studied another type of structure that lives on a manifold: a connection $\Gamma$. In this section we bring both (a priori independent) concepts together by asking whether, among all possible connections, there are distinguished ones on a Riemannian manifold. This will lead us to a connection that can be constructed directly from the metric and plays a central role in general relativity.

#### 10.3.1. The Levi-Civita connection

1 | **Motivation:**

   In Euclidean space, the parallel transport of two vectors does not change their inner product (in particular, their norm/length remains constant):

$$\mathcal{E}^k = (\mathcal{M}, g, \Gamma)$$

2 | $\Leftrightarrow$ Riemannian manifold $(M, g)$ with (pseudo-)Riemannian metric $g_{ij}(x)$

A connection $\Gamma$ is called a **metric-compatible**

$$\Leftrightarrow \frac{d}{d\lambda} (A, B) \overset{\text{def}}{=} \frac{d}{d\lambda} (g_{ik} A^i B^k) \overset{10.51}{=} \frac{D}{D\lambda} (g_{ik} A^i B^k) \overset{\gamma}{=} 0$$

along any curve $\gamma(\lambda)$ for all parallel vector fields $A$ and $B$ along $\gamma$.

Recall that for a scalar the total and absolute derivative are identical.
A and B parallel vector fields: \( \frac{\partial A^j}{\partial \lambda} = 0 = \frac{\partial B^k}{\partial \lambda} \) → Eq. (10.73) \( \leftrightarrow \frac{\partial g_{ik}}{\partial \lambda} \equiv 0 \leftrightarrow \forall i,k,l: g_{ik;l} \equiv 0 \) (10.74)

Use the Leibniz product rule Eq. (10.52) to show this.

\( g_{ij}(x) \) is covariantly constant

\[ \partial_l g_{ik} - \Gamma^m_{il} g_{mk} - \Gamma^m_{kli} g_{im} = 0 \] (10.75)

Since Eq. (10.74) holds for arbitrary indices, we also have equations with cyclic permutations:

\[ \partial_k g_{li} - \Gamma^m_{lk} g_{mi} - \Gamma^m_{mil} g_{li} = 0, \] (10.76a)

\[ -\partial_l g_{kl} + \Gamma^m_{ki} g_{ml} + \Gamma^m_{mlk} g_{li} = 0. \] (10.76b)

Adding up the three equations yields

\[ \Gamma_{i(kl)} = \Gamma^m_{(kl)} g_{mi} = \frac{1}{2} (\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl}) + \frac{1}{2} (S^m_{li} g_{mk} + S^m_{kli} g_{ml}) \]

\[ = \frac{1}{2} (\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl}) + S_{(kl)i} \] (10.77)

with torsion \( S^m_{li} = \Gamma^m_{li} - \Gamma^m_{il} \) and the symmetrized coefficient \( \Gamma^m_{(kl)} := \frac{1}{2} (\Gamma^m_{kl} + \Gamma^m_{lk}) \) and torsion tensor \( S_{(kl)i} := \frac{1}{2} (S_{kli} + S_{lki}) \).

If we assume a torsion-free connection, it is \( \Gamma_{i(kl)} = \Gamma_{ikl} \) and \( S_{(kl)i} = 0 \) so that

\[ \Gamma_{ikl} = \frac{1}{2} (\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl}) . \] (10.78)

These are the connection coefficients of the unique Levi-Civita connection.

\[ \Gamma_{i(kl)} = \Gamma^i_{lk} \] (torsion-free!)

Use symmetry \( \Gamma^i_{lk} = \Gamma^i_{lk} \) and definition \( \Gamma_{ikl} := g_{im} \Gamma^m_{kl} \)

\[ \frac{\partial g_{ik}}{\partial \lambda} = \frac{1}{2} (\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl}) \] (10.79a)

\[ \frac{\partial g_{im}}{\partial \lambda} = \frac{1}{2} (\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}) \] (10.79b)

\[ \partial_l g_{mk} \neq \partial_l (g^{im} g_{mk}) = \partial_l \delta^i_k = 0. \]

This torsion-free, metric-compatible connection is unique and called the Levi-Civita connection.

\[ \Gamma^i_{kl} = \text{Connection coefficients of the Levi-Civita connection} \]

\[ \Gamma^i_{kl} = \text{Connection coefficients of the Levi-Civita connection} \]

\[ \Gamma^i_{kl} \]

\[ \text{In general relativity, we only work with the Levi-Civita connection; i.e., when we use the symbols } \Gamma^i_{kl}, \text{ we always refer to the Christoffel symbols Eq. (10.79) (and not to generic coefficients of a [metric-compatible] connection, \( \rightarrow \text{below} \)).} \]
For a given metric, there are many compatible connections (→ next). However, if we demand in addition that the connection is symmetric (= torsion-free), there is only one possible choice: the Levi-Civita connection (∩ Fundamental theorem of Riemannian geometry).

The Christoffel symbols are sometimes written as $^{[131,132]}$

$$\begin{align*}
\Gamma^i_{kl} &= \frac{1}{2} g^{im} \left( \partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk} \right), \quad (10.80)
\end{align*}$$

(Einstein used an “upside down” version of this notation in his original work on general relativity, e.g., in Ref. [11].)

Then it follows from Eq. (10.77) that a general metric-compatible connection can be written as

$$\Gamma^i_{kl} = \Gamma^i_{kl} + \Gamma^i_{[kl]} = \left\{ i \right\}_{kl} + \frac{1}{2} \left( \mathcal{S}^i_{kl} - \mathcal{S}^i_{jk} + \mathcal{S}^i_{kj} \right), \quad (10.81)$$

with $\Gamma^i_{[kl]} = \frac{1}{2} \mathcal{S}^i_{kl}$; the tensor $\mathcal{K}^i_{kl}$ is known as $\uparrow$ contorsion tensor (“Verdrehungstensor”).

The torsion-free Levi-Civita connection is the special case where

$$\Gamma^i_{kl} = \left\{ i \right\}_{kl}. \quad (10.82)$$

Because we use only the torsion-free Levi-Civita connection in general relativity, we don’t make use of this notation and only write $\Gamma^i_{kl}$.

5 | Interpretation:

For the special case of a 2D manifold embedded in 3D Euclidean space, the Levi-Civita connection can be geometrically interpreted as follows:

![Diagram](image)

¡! This illustration is based on an embedding of the manifold into an ambient Euclidean space (which induces a metric on the manifold). Note, however, that the Levi-Civita connection is intrinsically defined and does not require such an embedding.
6 | **Corollaries:**

- Working with a metric-compatible connection has the benefit that one can pull indices up and down within a covariant derivative:

\[
T_{i;k} = (g_{im} T^m)_{;k} = g_{im;k} T^m + g_{lm} T^m_{;k} \overset{10.74}{=} g_{im} T^m_{;k} \quad (10.83)
\]

- The inverse metric is also covariantly constant:

\[
g_{ik;l} = 0 \quad (10.84)
\]

To show this, note that \(\delta^i_{j;l} = 0 \) [Eq. (10.57b)] and use the Leibniz product rule:

\[
0 = \delta^i_{j;l} = (g^{ik} g_{kj})_{;l} = g^{ik} g_{lj} + g^{ik} g_{kj;l} \overset{10.74}{=} g^{ik} g_{kJ} \cdot \quad (10.85)
\]

7 | **Local inertial coordinates:** (Details: ★ Problemset 2)

i | **< Levi-Civita connection in < locally geodesic coordinates at** \(p \in M\):  
(For simplicity, we assume that the point \(p\) has the coordinates \(u(p) = 0\).)

\[
\Gamma^m_{ij}(0) = 0 \quad (10.75) \\
\left[ \Gamma^m_{ij}(0) g_{mk} + \Gamma^m_{kj}(0) g_{im} \right] = 0 
\]

\[\rightarrow \text{In these coordinates, the metric tensor is constant in linear order:} \]

\[
g_{ij}(x) = g_{ij}(0) + \frac{1}{2} \partial_\alpha \partial_\beta g_{ij} (0) x^\alpha x^\beta + \mathcal{O}(x^3) \quad (10.87)
\]

ii | **< Affine coordinate transformation:** \(\bar{x}^i = M^i_j x^j + b^i \) \overset{10.39}{=} \Gamma^i_{kl} = 0

Note that under affine/linear coordinate transformations, the connection coefficients transform like tensors! In particular, if the connection coefficients vanish in one (geodesic) coordinate system, they vanish in all coordinates that can be reached by affine transformations; i.e., geodesic coordinates are not unique!

\[\rightarrow \text{Use linear transformation to bring metric of signature} \ (r, s) \text{ into the form} \]

\[
\bar{g}_{ij}(0) = \text{diag}(+1, \ldots, +1, -1, \ldots, -1). \quad (10.88)
\]

That this is possible follows from ★ Sylvester’s law of inertia: First, use the symmetry of the metric to diagonalize the matrix \(g_{ij}(0)\) by an orthogonal transformation, then use another non-singular transformation to normalize the eigenvalues to \(\pm 1\).

iii | **< Special case \((r = 1, s = 3) = \text{Lorentzian manifold} \rightarrow\)**

\[
\text{Metric in ★ locally inertial coordinates:} \quad \tilde{g}_{\mu\nu}(\tilde{x}) \approx 0 \eta_{\mu\nu} + \frac{1}{2} \partial_\alpha \partial_\beta \tilde{g}_{\mu\nu}(0) \tilde{x}^\alpha \tilde{x}^\beta \quad (10.89)
\]
In words: For every point of a Lorentzian manifold there exist coordinate systems such that the metric in this point takes the Minkowski form $\eta_{\mu \nu}$ and is constant in linear order; we call such charts *locally inertial coordinates*.

Recall that Lorentz transformations are linear and leave the Minkowski metric invariant $\left[\rightarrow \text{Eq. (4.21)}\right]$. This implies that locally inertial coordinates are also not unique: You can use arbitrary Lorentz transformations without changing the structure of Eq. (10.89).

**Useful relations:**
Here we list a few identities that will be useful for many calculations in *general relativity*. You prove these relations in $\otimes$ Problemset 2.

- The trace of the Christoffel symbols simplifies to
  \[
  \Gamma^i_{\ k i} = \frac{1}{2} g^{im} g_{im,k} .
  \quad \text{(10.90)}
  \]

- With the determinant of the metric $g = \det(g_{im})$, the inverse metric can be written as
  \[
  g^{im} = \frac{1}{g} \frac{\partial g}{\partial g_{im}} .
  \quad \text{(10.91)}
  \]

- With Eqs. (10.90) and (10.91), the trace of the Christoffel symbols takes the simple form
  \[
  \Gamma^i_{\ k i} = \frac{1}{2g} g_{k,} = \left(\ln \sqrt{g}\right)_{,k} .
  \quad \text{(10.92)}
  \]
  such that $\pm g > 0$.

  *Note:* In *general relativity* it is $\det(g_{\mu \nu}) < 0$ (because of the Lorentzian signature) and we redefine $g := -\det(g_{\mu \nu}) > 0$ to simplify expressions.

- The other trace of the Christoffel symbols can also be written in a compact form:
  \[
  g^{kl} \Gamma^i_{\ kl} = \frac{1}{\sqrt{g}} \left(\sqrt{g} g^{im}\right)_{,m} .
  \quad \text{(10.93)}
  \]

- It is straightforward to show the following useful identity:
  \[
  g_{ik}(g^{kl})_{,m} = - (g_{ik})_{,m} g^{kl} .
  \quad \text{(10.94)}
  \]

- The *covariant divergence* of a contravariant vector field is defined as one would expect:
  \[
  A^i_{\ ;i} = \frac{1}{\sqrt{g}} A^i_{\ ,i} + A^i \left(\ln \sqrt{g}\right)_{,j} = \frac{1}{\sqrt{g}} \left(\sqrt{g} A^i\right)_{,j}
  \quad \text{(10.95)}
  \]

- For the covariant divergence of an *antisymmetric* $(2, 0)$-tensor there is a similar expression:
  \[
  A^{ik}_{\ ;k} = \frac{1}{\sqrt{g}} \left(\sqrt{g} A^{ik}\right)_{,k} \quad \text{with} \quad A^{ij} = - A^{ji} .
  \quad \text{(10.96)}
  \]

- Eq. (10.95) can be used to rewrite the covariant Laplacian (divergence of a gradient) of a scalar:
  \[
  \Delta \phi = \phi^{;i}_{\ ;i} = \frac{1}{\sqrt{g}} \left(\sqrt{g} g^{ik} \phi_{,k}\right)_{,i} .
  \quad \text{(10.97)}
  \]

  The differential operator $\Delta$ maps scalar functions onto scalar functions and is known as $\uparrow$ *Laplace-Beltrami operator*. 

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• Generalized divergence theorem:
  
  i | \( \varphi(\mathbf{x}) \) is a coordinate transformation \( \tilde{\mathbf{x}} = \varphi(\mathbf{x}) \)
  
  \( \rightarrow \) \( D \)-dimensional (oriented) volume element (more precisely: volume form) transforms as (Eq. (3.39))
  
  \[
  d^{D}\tilde{\mathbf{x}} = \det \left( \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{x}} \right) d^{D}\mathbf{x}
  \] (10.98)
  
  with \( \Delta \) Jacobian determinant \( \det \left( \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{x}} \right) \).
  
  ii | The determinant of the metric transforms in the opposite way (Eq. (3.54)):
  
  \[
  \sqrt{g} = \left| \det \left( \frac{\partial \mathbf{x}}{\partial \tilde{\mathbf{x}} \right)} \right| \sqrt{\tilde{g}}
  \] (10.99)
  
  (Note the absolute value of the Jacobian determinant!)
  
  iii | Hence the product of metric determinant and (oriented) volume element transforms like a pseudo scalar:
  
  \[
  \sqrt{g} d^{D}\tilde{\mathbf{x}} = \text{sign} \left[ \det \left( \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{x}} \right) \right] \sqrt{g} d^{D}\mathbf{x}
  \] (10.100)
  
  Here \( \text{sign} \left[ \det \left( \frac{\partial \tilde{\mathbf{x}}}{\partial \mathbf{x}} \right) \right] \) denotes the sign of the Jacobian determinant, which encodes whether the coordinate transformation is orientation preserving (+1) or not (−1). This makes \( \sqrt{g} d^{D}\mathbf{x} \) transform like a pseudo scalar.
  
  If we are only interested in non-oriented volume elements, or restrict ourselves to orientation-preserving coordinate transformations, Eq. (10.100) simplifies to a true scalar transformation:
  
  \[
  \sqrt{g} d^{D}\tilde{\mathbf{x}} = \sqrt{g} d^{D}\mathbf{x}
  \] (10.101)
  
  This subtlety will not be important in the following and we use Eq. (10.101) henceforth.
  
  iv | Eq. (10.101) is the reason why integrals over scalar quantities \( \phi(\tilde{\mathbf{x}}) = \phi(\mathbf{x}) \) are form-invariant under arbitrary coordinate transformations if we use the “modified” volume element \( \sqrt{g} d^{D}\mathbf{x} \) for integration:
  
  \[
  \int d^{N}\tilde{\mathbf{x}} \sqrt{g(\tilde{\mathbf{x}})} \phi(\tilde{\mathbf{x}}) \stackrel{\tilde{\mathbf{x}} = \varphi(\mathbf{x})}{=} \int d^{N}\mathbf{x} \sqrt{g(\mathbf{x})} \phi(\mathbf{x})
  \] (10.102)
  
  Using the covariant divergence Eq. (10.95) and the modified volume element Eq. (10.101), we find the generalized form of the divergence theorem
  
  \[
  \int_{V} d^{D}\mathbf{x} \sqrt{g} A^i \stackrel{10.95}{=} \int_{V} d^{D}\mathbf{x} \left( \sqrt{g} A^i \right) \stackrel{\text{Gauss}}{=} \oint_{\partial V} \text{d} \sigma_i \sqrt{g} A^i
  \] (10.103)
  
  where \( \partial V \) is the surface of \( V \) and \( \text{d} \sigma_i \) denotes the \( D-1 \)-dimensional surface element.
10.3.2. The Riemann curvature tensor

Now that we identified the special Levi-Civita connection (which can be computed from the metric), we can also express its curvature tensor (then called Riemann curvature tensor) in terms of the metric as well:

**Detailed calculations:** Problemset 3

9 | Locally geodesic coordinates LG:
\[
\{R_{iklm}\}_{LG}^{10.70} = g_{ia} R^{a}_{\, iklm} \equiv g_{ia} \left( \partial_{j} \Gamma^{a}_{\, klm} - \partial_{m} \Gamma^{a}_{\, kl} \right)
\] (10.104)

Recall that the connection coefficients – but not their derivatives – vanish in these coordinates!

10 | Now use the explicit form of the Levi-Civita connection to find an expression in terms of the metric:
\[
\{R_{iklm}\}_{LG}^{10.79} = \frac{1}{2} \left( g_{im,k,l} + g_{kl,i,m} - g_{li,k,m} - g_{km,i,l} \right)
\] (10.105)

- Recall that \(g_{ij,k} = 0\) in locally geodesic coordinates [\(\rightarrow\) Eq. (10.86)].
- This expression tells us that curvature prevents us from finding coordinates in which the second derivatives of the metric vanish.

11 | In general coordinates, the expression becomes more complicated:
\[
R_{iklm} = \frac{a}{\{R_{iklm}\}_{LG}^{10.79}} \left( g_{ab} \left( \Gamma^{a}_{\, kli} \Gamma^{b}_{\, im} - \Gamma^{a}_{\, kml} \Gamma^{b}_{\, il} \right) \right)
\] (10.106)

To show this, start from Eqs. (10.70) and (10.79) and use Eqs. (10.75) and (10.94).

12 | Algebraic identities:
- Eqs. (10.105) and (10.106) \(\rightarrow\)
\[
R_{kilm} = -R_{kilm}, \quad R_{iklm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}
\] (10.107)

In words: the Riemann tensor is antisymmetric in the first two and last two indices, but symmetric if both pairs of indices are swapped.

- **First/Algebraic Bianchi identity:**

  The cyclic sums of Riemann tensors vanish identically:
\[
R_{i(klm)} = R_{iklm} + R_{ilmk} + R_{imkl} \equiv 0
\] (10.108)

The same is true for the cyclic sums of arbitrary triples of indices.

The relations Eqs. (10.107) and (10.108) are identities, i.e., their validity follows directly from the definition of the Riemann curvature tensor, independent of the specific metric. This means that a Riemann tensor in \(D\)-dimensions has less independent components as the naïve count \(D^4\) suggests.

For example, on the \(D = 4\)-dimensional spacetime of General Relativity, at most 20 (and not \(4^4 = 265\)) numbers are needed to specify \(R_{iklm}\) in every point of the spacetime manifold (Problemset 3). [Beware: This does not mean that there are 20 physical degrees of freedom in General Relativity! \(R_{iklm}\) is still a tensor and can be modified by arbitrary coordinate transformations without changing its physical content. We will see \(\rightarrow\) later that General Relativity has a large gauge group (\(\rightarrow\) diffeomorphism invariance) so that there are way less physical degrees of freedom than the 20 alluded to above.]
**Second/Differential Bianchi identity:**

The cyclic sums of covariant derivatives of the Riemann tensor vanish identically:

\[
R^a_{k(l|m;n)} = R^a_{klm;n} + R^a_{kmn;l} + R^a_{knl;m} \equiv 0 \tag{10.109}
\]

**Proof.** A neat trick to prove tensor relations is to choose a coordinate system in which their derivation is simple, and then use the tensor character of the involved objects to infer the validity of the relation in general coordinates.

Both the Riemann tensor and covariant derivatives are particularly simple in locally geodesic coordinates:

\[
\begin{align*}
\{ R^a_{klm;n} \}^\text{LG} & \overset{10.70}{=} \Gamma^a_{km,l;n} - \Gamma^a_{kl,m;n} . \\
\{ R^a_{klm;n} \}^\text{LG} + \{ R^a_{kmn;l} \}^\text{LG} + \{ R^a_{knl;m} \}^\text{LG} & = 0 .
\end{align*}
\tag{10.110}
\]

Adding up the cyclic permutations of this expression yields:

\[
\begin{align*}
\{ R^a_{klm;n} \}^\text{LG} & = \Gamma^a_{km,l;n} - \Gamma^a_{kl,m;n} . \\
\{ R^a_{klm;n} \}^\text{LG} + \{ R^a_{kmn;l} \}^\text{LG} + \{ R^a_{knl;m} \}^\text{LG} & = 0 .
\end{align*}
\tag{10.111a}
\]

\[
\begin{align*}
\{ R^a_{klm;n} \}^\text{LG} & = \Gamma^a_{km,l;n} - \Gamma^a_{kl,m;n} + \Gamma^a_{kn,m,l} - \Gamma^a_{km,n,l} + \Gamma^a_{kl,n,m} - \Gamma^a_{kn,l,m} . \\
& = 0 .
\end{align*}
\tag{10.111b}
\]

Now, since \( R^a_{k(l|m;n)} \) is a tensor and vanishes in one coordinate system, it vanishes in all coordinate systems (because tensor components transform linearly under coordinate transformations); thus \( R^a_{k(l|m;n)} = 0 \) and we are done.

**Notes:**

- Remember that commutators \([A, B] = AB - BA\) satisfy the \(\ast\) Jacobi identity:

\[
\tag{10.112}
\]

But the \(\ast\) Ricci identity Eq. (10.71) relates the curvature tensor (not necessarily a Riemannian one, but the connection must be torsion-free) to the commutator of covariant derivatives:

\[
A_k[\ell,m] = A_k R^a_{k[l;m]} .
\tag{10.113}
\]

Using this, one can derive the second (and also the first) Bianchi identity from the Jacobi identity; see Nakahara [133] (p. 269).

### Derived tensors:

The following tensors can be derived from the Riemann tensor and will play an important role in the formulation of General Relativity:

i | The only non-trivial contraction of the Riemann tensor sums one index of the first pair with one index of the second pair (all other contractions vanish due to symmetries):

\[
\ast \text{Ricci tensor: } R_{kl} := R^a_{k[\ell;l]} = -R^a_{k[\ell;l]} \tag{10.114}
\]

ii | The Ricci tensor is symmetric:

\[
R_{kl} = R_{lk} \tag{10.115}
\]
To show this, contract the first Bianchi identity Eq. (10.108),
\[ R^a_{\ kla} + R^a_{\ lka} + R^a_{\ akl} = 0, \]  \hspace{1cm} (10.116)
and use \( R^a_{\ akl} = 0 \) due to the antisymmetry of the Riemann tensor.

iii | We can contract the Ricci tensor to obtain a curvature scalar:

\[ \text{RICCI scalar: } R := g^{ab} R_{ab} = R^a_a \]  \hspace{1cm} (10.117)

iv | **Contracted Bianchi identity:**

Ricci tensor and scalar obey an identity that derives from the second Bianchi identity:

\[ R^a_{\ n;la} = \frac{1}{2} R_{;na} \]  \hspace{1cm} (10.118)

**Proof.** To show this, contract the differential Bianchi identity Eq. (10.109) over \( a \) and \( m \):

\[ R_{k;n} - R_{kn;l} - R^a_{\ k n;la} = 0. \]  \hspace{1cm} (10.119)

Tracing out \( k \) and \( l \) (recall that our connection is metric-compatible, i.e., we are allowed to pull indices up/down inside covariant derivatives) yields:

\[ 0 = g^{kl} R_{k;n} - g^{kl} R_{kn;l} - g^{kl} R^a_{\ k n;la} \]  \hspace{1cm} (10.120a)
\[ = R_{n} - R_{n} - R^a_{\ n;la} \]  \hspace{1cm} (10.120b)
\[ = R_{n} - R_{n} - R^a_{\ n;la} \]  \hspace{1cm} (10.120c)
\[ = R_{n} - 2 R^a_{\ n;la} \]  \hspace{1cm} (10.120d)

v | As preparation for **GENERAL RELATIVITY**, we define another tensor using the Ricci tensor, Ricci scalar, and metric:

\[ \text{EINSTEIN tensor: } G_{ij} := R_{ij} - \frac{1}{2} g_{ij} R \]  \hspace{1cm} (10.121)

For \( D = 4 \) on a Lorentzian manifold, this tensor will be used as the left-hand side of the → **Einstein field equations**.

vi | The form of Eq. (10.121) is structurally similar to the contracted Bianchi identity. Indeed, Eq. (10.118) immediately implies:

\[ \text{Eq. (10.118)} \Rightarrow G^a_{\ 1;na} = 0 \]  \hspace{1cm} (10.122)

• Eq. (10.122) will be crucial for the consistency of the → **Einstein field equations** with energy momentum conservation.
• For $D = 4$ one can show that the Einstein tensor $G_{\mu\nu}$ (besides the metric tensor $g_{\mu\nu}$) is the only rank-2 tensor with vanishing (covariant) divergence that one can construct from the metric and its first and second derivatives [134, 135]. This result is known as Lovelock’s theorem and states under which conditions the field equations of General Relativity (including the cosmological constant) are unique ($\rightarrow$ later). The uniqueness of $G_{\mu\nu}$ and Lovelock’s theorem impose important constraints on possible extensions (or modifications) of General Relativity.

10.3.3. Geodesics

In Section 10.2 we defined “straight lines” as curves that keep their direction constant, and formalized this notion as autoparallel curves. Now that we have a metric at hand, we can also define “straight lines” as the shortest curves connecting two points. We will show now that these two concepts coincide for the metric-compatible, torsion-free Levi-Civita connection induced by the metric:

15 | $\bowtie$ Length of curve $\gamma$ connecting two points $P_2$ and $P_2$ ($\equiv$ Eq. (3.55)):

$$L[\gamma] = \int_{\gamma} ds = \int_{\lambda_1}^{\lambda_2} d\lambda \sqrt{g_{ij} \dot{x}^i \dot{x}^j} \quad (10.123)$$

Here, $x^i(\lambda_{1/2})$ are the coordinates of $P_{1/2}$ in some chart. The right expression is independent of both the parametrization $x^i(\lambda)$ of the curve and the coordinate system.

To see the latter, recall that for a coordinate transformation $\tilde{x} = \varphi(x)$ it is

$$\frac{d\tilde{x}^i}{d\lambda} = \frac{\partial \tilde{x}^i}{\partial x^m} \frac{dx^m}{d\lambda} \quad \text{and} \quad \tilde{g}_{ij} = \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j} g_{kl}. \quad (10.124)$$

Remember that the directional derivatives $\dot{x}^i \partial_i$ along a curve are vectors in the tangent space $T_p M$ and transform accordingly. Thus, in the expression Eq. (10.123), the total derivative wrt. $\lambda$ is important! In contrast to the special coordinate transformations of Special Relativity (Lorentz transformations), the coordinates $x^i$ themselves do not transform as tensors (they transform like $\tilde{x} = \varphi(x)$, which is non-linear in general).

16 | $\bowtie$ “Straight line” from $P_1$ to $P_2$ $\equiv$ Shortest curve $\gamma^*$ ($\Rightarrow$ Geodesics) from $P_1$ to $P_2$

Strictly speaking, we will not study globally shortest curves, but curves that locally extremize the length functional Eq. (10.123). For now, you can think of geodesics as “shortest curve” connecting two points, but keep in mind that this is not necessarily true ($\rightarrow$ comments below).

$\rightarrow$ Extremize length over curves starting at $P_1$ and terminating at $P_2$:

$$\delta L = \delta \int_{P_1}^{P_2} ds \overset{!}{=} 0 \quad (10.125)$$
17 | Strictly monotonic, differentiable function $\chi$ & Class of “Lagrangians”

$$\mathcal{L}_\chi(x, \dot{x}) := \chi(g_{kl}(x)\dot{x}^k \dot{x}^l)$$  \hspace{1cm} (10.126)

For example: $\chi(x) = \sqrt{\gamma}$ yields the integrand of Eq. (10.123) as Lagrangian.

→ More general variation principle:

$$\delta \int_{P_1}^{P_2} \, d\lambda \, \mathcal{L}_\chi(x, \dot{x}) = 0$$  \hspace{1cm} (10.127)

Depending on $\chi$, this “action” is no longer reparametrization invariant in general.

18 | → Euler-Lagrange equations:

$$\frac{d}{d\lambda} \left( \frac{\partial \mathcal{L}_\chi}{\partial \dot{x}^i} \right) - \frac{\partial \mathcal{L}_\chi}{\partial x^i} = 0 \quad \Leftrightarrow \quad \frac{d}{d\lambda} \left( \chi'(y)2g_{ik} \dot{x}^k \right) - \chi'(y) \frac{\partial g_{kl}}{\partial \dot{x}^l} \dot{x}^k \dot{x}^l = 0$$  \hspace{1cm} (10.128)

19 | Parametrization with $y = g_{ij}(x)\dot{x}^i \dot{x}^j \equiv \|\dot{x}\|^2 = 1 = \text{const}$

This choice fixes an affine parametrization $\lambda = s$ of the curve $\gamma$ where the “velocity” $\|\dot{x}\|_x$ is constant. Since we require $\|\dot{x}\|_x = 1$, the “time” $\lambda$ is equal to the length $s$ of the curve from the start to $x'(\lambda)$ (up to a constant offset).

Later, on the (pseudo-Riemannian) Lorentzian manifolds of general relativity, we will also consider space-like geodesics with $y < 0$; for such curves, you must add an additional minus in the square root of Eq. (10.123) and choose $y = -1 = \text{const}$ instead. The rest of the derivation is then completely analogous.

→ $\chi'(y) = \text{const} \neq 0$ (strict monotonic!)

$$\text{Eq. (10.128)} \quad \Leftrightarrow \quad g_{ik} \ddot{x}^k + g_{ik, l} \dot{x}^k \dot{x}^l - \frac{1}{2} g_{kl, i} \dot{x}^k \dot{x}^l = 0$$  \hspace{1cm} (10.129)

Note that this differential equation is independent of $\chi$!

$$\text{Eq. (10.129)} \quad \Leftrightarrow \quad g_{ik} \ddot{x}^k + \left( g_{il, k} + g_{ik, l} - g_{kl, i} \right) \dot{x}^k \dot{x}^l = 0.$$  \hspace{1cm} (10.130)

20 | Identify Christoffel symbols Eq. (10.79)

$$\frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} = 0 \quad \star \quad \text{Geodesic equation}$$  \hspace{1cm} (10.131)

Solutions of this DGL are called $\star$ (affinely parametrized) Geodesics.

21 | Notes:

• ! We derived the Geodesic equation by a variational principle extremizing the length between two points. This means that geodesics are not necessarily the shortest curves between two points. Ignoring the peculiarities of pseudo-Riemannian metrics for now (→ Section 11.1),
Geodesics are only *locally* the shortest connections between close by points, but not necessarily *globally*. Put differently: Every shortest path connecting two points is a geodesic, but not every geodesic connecting two points is a shortest path.

An example is a great circle on a sphere connecting two points (→ below), say the north pole and a point on the equator. The great circle satisfies the geodesic equation everywhere, and is therefore a geodesic. The shortest path connecting the two points is part of the great circle (and therefore also a geodesic). But the “long way around” is certainly not the shortest path (but still a geodesic, as it is also part of the great circle).

- With our derivation we showed that the curves (Geodesics) that solve the geodesic equation Eq. (10.131) not only extremize the length Eq. (10.123), but the more general class of “actions” defined by the “Lagrangian” Eq. (10.126). This will be useful → later when we study the classical mechanics of points on the Lorentzian manifolds of *general relativity*.

- As already discussed previously (in Section 10.2.2), Eq. (10.131) is not invariant under arbitrary but only affine reparametrizations \( \mu = a \lambda + b \). The geodesic equation therefore not only picks out the locally shortest (more precisely: extremal) curves on the manifold, but selects also a particular way to parametrize them (namely a parameter that is proportional to the length of the curve, i.e., an *affine parametrization*).

- The geodesic equation is a second-order differential equation. As such it has a unique solution \( x^i(\lambda) \) for any point \( p \) of the manifold and tangent vector in \( v_p = v_p^i \partial_i \in T_p M \); in coordinates:

\[
\begin{align*}
x^i(0) &:= x_p^i \\
\dot{x}^i(0) &:= v_p^i
\end{align*}
\]

Geodesic \( x^i(\lambda) \) through \( p \) in direction \( v_p \). \hspace{1cm} (10.132)

This is reminiscent of classical mechanics where, given some potential \( V(\vec{x}) \), Newton’s law determines a unique trajectory for every initial position \( \vec{x}_0 \) and initial velocity \( \vec{v}_0 \) of a test particle by solving the second-order differential equation

\[
m \ddot{\vec{x}} + \nabla V(\vec{x}) = 0. \hspace{1cm} (10.133)
\]

However, there is a subtle difference between Eq. (10.133) and Eq. (10.131): Solutions of Newton’s equation of motion are *not* invariant under affine reparametrizations in general. That is, if \( \vec{x}(t) \) is a solution of Eq. (10.133), the rescaled trajectory \( \vec{x}(\alpha t) := \vec{x}(\alpha) \) is no longer a solution (check this!). Note that the effect of the time rescaling \( \alpha \) is to scale the initial velocity: \( \vec{v}'(0) = \alpha \vec{v}(0) = \alpha \vec{v}_0 \). Physically, this makes sense: If you throw a ball in the same direction with different velocity, its trajectory will look different in a generic potential.

In conclusion, the solutions of Eq. (10.133) form a family of curves through every point, with many different curves going off in the same direction:

Compare this to the geodesic Eq. (10.131):
Given an (affinely parametrized) geodesic \( x^i(\lambda) \), which shoots off from \( x^i(0) \) in direction \( \dot{x}^i(0) = v_p^i \), the reparametrized curve \( y^i(\lambda) := x^i(\alpha \lambda) \) is again a solution (check this!). This new curve has again a rescaled tangent vector at \( p \) ("initial velocity"), namely \( \dot{y}^i(0) = \alpha \dot{x}^i(0) = \alpha v_p^i \). But the two curves \( x^i(\lambda) \) and \( y^i(\lambda) \) trace out the same curve on the manifold, only with a different parametrization ("speed").

The affine reparametrization symmetry of the geodesic equation therefore leads to a unique geodesic shooting off in every direction \( v_p \in T_p M \) at every point \( p \in M \). Rescaling \( v_p \) produces the same geodesic, only with a different parametrization (left sketch):

Note that geodesics emanating from a point can meet and cross each other at other points of the manifold (this depends on the curvature, and therefore the metric).

An example is the sphere (right sketch); its geodesics are great circles. At every point of the sphere there is a unique great circle for every direction. But two great circles shooting off in different directions eventually cross again at the antipode of the point where the started from.

- You may wonder: If we know all (unparametrized/projected) geodesics through all points in all directions, do we then know the metric of the manifold? This question is actually of physical significance in General Relativity, where the geodesics of spacetime correspond to the trajectories of free falling bodies (→ later). In the language of General Relativity, the question then asks whether one can reconstruct the metric of spacetime by observing enough free falling bodies (asteroids, stars, etc.).

In its strictest sense, the answer to the question is negative. This is easy to see: Consider \( \mathbb{R}^2 \) and equip this manifold with (1) the Euclidean metric \( \delta_{ij} \), and (2) the Minkowski metric \( \eta_{ij} \). Since both metrics are constant, their Christoffel symbols vanish identically and the solutions of the geodesic Eq. (10.131) are all straight lines for both metrics. On says that the two metrics are \( \uparrow \text{geodesically equivalent} \).

However, in general it turns out that this is a quite subtle question to answer, see Ref. [136]. Note that one must carefully distinguish between unparametrized geodesics (you know only the traces of geodesics on the manifold), and (affinely) parametrized geodesics (where you know also the lengths along the traces). Despite the example above, it turns out that generic metrics can be characterized by their geodesics (even unparametrized ones); i.e., two metrics being geodesically equivalent is not the norm but the exception.

- Imagine you are given a Riemannian manifold and a machine that, input two nearby points on the manifold, spits out the affinely parametrized geodesic through these points (i.e., a curve with "distance ticks" on it). Using this device, you can reconstruct the Levi-Civita connection on the manifold (i.e., you can use it to parallel transport tangent vectors) via a geometric construction known as \( \uparrow \text{Schild’s ladder} \) [Misner et al. [2] §10.2, pp. 248–249)].

Fun fact: There is also a science fiction novel called Schild’s Ladder [137] by the Australian mathematician and Hugo Award winning author Greg Egan. If you are a fan of hard, mind-bending science fiction à la Lem, Asimov and Heinlein (and not afraid to encounter
concepts from your physics courses in a work of fiction), you might give his novels a try.

- On a Riemannian manifold with a generic metric-compatible connection, that is not necessarily the torsion-free Levi-Civita connection, the coefficients $\Gamma^i_{kl}$ in Eq. (10.131) are still the Christoffel symbols (which no longer equal the connection). So the geodesic equation on such a manifold still reads (now with the alternative notation for Christoffel symbols to distinguish them from the connection coefficients):

$$\frac{d^2 x^i}{d\lambda^2} + \left[ \Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} \right] = 0.$$  \hspace{1cm} (10.134)

This equation determines the “shortest lines” (geodesics) on the manifold.

By contrast, the “straightest lines” (autoparallels) are determined by the autoparallel equation Eq. (10.60):

$$\frac{d^2 x^i}{d\lambda^2} + \Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} = 0.$$  \hspace{1cm} (10.135)

Here we used the general form of a metric-compatible connection Eq. (10.81) with the contorsion tensor $K^i_{kl}$. Introducing the symmetric part $K^i_{kl}$ of the contorsion tensor yields (for reference see e.g. [138])

$$\frac{d^2 x^i}{d\lambda^2} + \left[ \Gamma^i_{kl} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda} \right] = K^i_{(kl)} \frac{dx^k}{d\lambda} \frac{dx^l}{d\lambda}.$$  \hspace{1cm} (10.136)

The geodesic equation and the autoparallel equation are therefore equivalent if and only if the symmetric part $K^i_{(kl)}$ of the contorsion tensor vanishes (a sufficient, but not necessary, condition is that the torsion $S^i_{kl}$ vanishes).

In conclusion, knowing all the geodesics on a manifold only conveys information about the symmetric part of the connection; the geodesics know nothing about torsion (but autoparallels do, at least partially). Thus, for a generic metric-compatible connection, there is a difference between “shortest lines” (geodesics) and “straightest lines” (autoparallels).

- If the metric $g_{ij}(x)$ is independent of a coordinate $x^i$, Eq. (10.128) implies for the allowed choice $\chi(x) = x/2$

$$p_i := g_{ik} \dot{x}^k = \text{const}.$$  \hspace{1cm} (10.137)

This “constant of motion” corresponds to the cyclic variable $x^i$ and can be used to simplify the solution of the geodesic equation.

22 | Geodesic deviation:

Details: Problemset 3

- Continuous family of nearby (non-crossing) geodesics $\gamma^i_s(t)$:
Define two vector fields:

\[ T^i := \frac{\partial y^j_s(t)}{\partial t} \quad \text{("Velocity") and} \quad S^i := \frac{\partial y^j_s(t)}{\partial s} \quad \text{("Deviation")}. \quad \text{(10.138)} \]

Relative acceleration of nearby geodesics:

\[ A^i := \frac{D^2 S^i}{Dt^2} = 10.49 \left( T^n S^i_{j:m} \right)_m. \quad \text{(10.139)} \]

The covariant acceleration \( A^i \) measures whether two infinitesimally close geodesics “attract” or “repel” each other.

Using the ← geodesic equation and the ← Ricci identity, one finds:

Eqs. (10.71) and (10.131) →

\[ \frac{D^2 S^i}{Dt^2} = R^i_{jkl} T^j T^k S^l \quad \text{Geodesic deviation equation} \quad \text{(10.140)} \]

Proof: Problemset 3

(Note that the geodesic equation can be written as \( T^k T^j_{;k} = 0 \).)

Curvature makes parallel geodesics attract/repel each other!

But this looks very much like gravity (more precisely: the tidal effects of gravity):

(Note that this sketch is a projection of geodesics from spacetime to space.)

Reasonable approach to a geometric theory of gravity:

- Free-falling bodies follow geodesics in spacetime: \( \rightarrow \) Chapter 11
- Masses create curvature of spacetime: \( \rightarrow \) ??
11. Classical physics on curved spacetime

Our mathematical toolbox is now fully equipped to formulate general relativity. In this chapter, we start by assuming some spacetime metric as given, and study how relativistic mechanics and electrodynamics can be formulated on this (curved) spacetime. Where the metric actually comes from will be discussed in the next ??.

11.1. Spacetime

1 | Setting the stage:
Here are some facts:

• We live in 3 spatial and 1 time dimension.
  For an argument why $3 + 1$-dimensional spacetimes are special, recall Section 4.4.
• The EEP requires the existence of $\leftrightarrow$ locally inertial coordinates ($\leftrightarrow$ Section 10.3.1).
  Recall that in such coordinates the metric locally looks like the Minkowski metric.
→ Spacetime is a $\leftrightarrow 4D$ Lorentzian manifold:

| Spacetime | $\equiv$ 4D Lorentzian manifold $(M, g)$ with pseudo-Riemannian metric $g$ of signature $(1, 3)$ |

• Henceforth all manifolds are of this type. We indicate this by using Greek indices $\mu, \nu, \ldots = 0, 1, 2, 4$ for tensors; Latin indices $i, j, \ldots = 1, 2, 3$ are now reserved for the spatial components of tensors.
• With the metric $g$ we can measure lengths of curves on the spacetime manifold and norms of and angles between vectors in the tangent bundle. There is also a lot of bonus structure: The metric defines a Levi-Civita connection, which, in turn, defines concepts like parallel transport, covariant derivatives, and curvature.
• Note that the global topology of $M$ is not specified by general relativity, e.g., whether $M$ is compact in all or some dimensions. For example, the universe could be periodic in one or more spatial dimensions, i.e., it could be a torus. While currently there are no observations that indicate a non-trivial topology, such topologies are also not conclusively ruled out and subject to ongoing research [139]. (Note that even assuming a completely flat universe – which is consistent with observations – does not rule out non-trivial topologies; recall the flat torus in Section 10.2.3.)

2 | Geodesics on Lorentzian manifolds:
In Section 10.3.3 we considered generic (pseudo-)Riemannian manifolds. We are now interested in $D = 4$-dimensional Lorentzian manifolds of signature $(1, 3)$ (“Spacetime”). This comes along with a few peculiarities concerning geodesics on this spacetime:
### Null cones:

**Tangent space** $T_p M$ with basis $\{\partial_\mu\}$ induced by **locally inertial coordinates**:

![Null cone diagram]

$\rightarrow$ The **null cone** is the subset of tangent vectors $v = v^\mu \partial_\mu \in T_p M$ with

$$\|v\|^2_p = \Delta_p (v, v) = \eta_{\mu\nu} v^\mu v^\nu = 0.$$ 

(11.1)

- That the null vectors of the Minkowski metric $\eta$ form a cone was discussed in Section 1.6.
- Recall $[\leftarrow Eq. (4.16)]$ that all other vectors with strictly positive (negative) Minkowski norm are called $\leftarrow$ time-like ($\leftarrow$ space-like). We adopt this nomenclature for vectors in the tangent spaces of Lorentzian manifolds.
- We call the cone “null cone” and not “light cone” because the latter term is reserved for a similar but distinct structure on the manifold ($\rightarrow$ below).

$\rightarrow$ A Lorentzian metric induces a “null cone texture” on the manifold ($\rightarrow$ below). This means that you can think of a Lorentzian manifold as being covered with little null cones that vary smoothly from point to point (not only their orientation, but also their “opening angle” can vary!). The null cones live in the tangent spaces and indicate which directions on the manifold are time-like, light-like (null), or space-like.

---

### Classification of geodesics:

**Geodesic** $\gamma^\mu (t)$ in an arbitrary coordinate system and parametrization

We can use the null cone structure to classify geodesics on a Lorentzian manifold. To this end, consider the sign of the norm (squared) of the “velocity vector” of a geodesic:

$\leftarrow$ **Sign of norm of tangent at geodesics:**

$$\text{sign } \|\gamma'(t)\|^2_p = \text{sign } \left[ g_{\mu\nu}(\gamma(t)) \gamma'(t)^\mu \gamma'(t)^\nu \right]$$

(11.2)

$\rightarrow$ Eq. (11.2) is ...

- ... constant along the geodesic.

It is easy to check by straightforward calculation that the norm of the tangent vector is
constant along a geodesic:

\[
\frac{d\|\dot{\gamma}\|^2_\gamma}{d\lambda} = g_{\mu\nu,\sigma} \dot{\gamma}^\mu \dot{\gamma}^\nu \dot{\gamma}^\sigma + 2g_{\mu\nu} \ddot{\gamma}^\mu \dot{\gamma}^\nu \tag{11.3a}
\]

\[
= 2g_{\mu\nu,\sigma} \dot{\gamma}^\mu \dot{\gamma}^\nu \dot{\gamma}^\sigma - g_{\nu\sigma,\mu} \dot{\gamma}^\mu \dot{\gamma}^\nu \dot{\gamma}^\sigma + 2g_{\mu\nu} \ddot{\gamma}^\mu \dot{\gamma}^\nu \tag{11.3b}
\]

\[
= 2\ddot{\gamma}^\mu \left(g_{\mu\nu,\sigma} \dot{\gamma}^\nu \dot{\gamma}^\sigma - \frac{1}{2}g_{\nu\sigma,\mu} \dot{\gamma}^\nu \dot{\gamma}^\sigma + g_{\mu\nu} \dot{\gamma}^\nu \right) \tag{11.3c}
\]

\[
\|\dot{\gamma}\|^2_\gamma = 0 \tag{11.3d}
\]

\[
\rightarrow \|\dot{\gamma}\|^2_\gamma = \text{const along a geodesic } \gamma.
\]

This of course immediately follows from our observation that geodesics are autoparallel curves, together with the metric-compatibility of the Levi-Civita connection.

- … invariant under reparametrizations.

The independence of the sign on the parametrization of the curve is easy to show if one remembers that a reparametrization \( \gamma(t) = \gamma(t) \) is given by a strictly monotone function \( \tau = \tau(t) \):

\[
\text{sign} \left[ \frac{dy(t)_\mu}{dr} \frac{dy'(t)_\mu}{dr} \right] = \text{sign} \left[ \frac{d\dot{\gamma}(\tau)_\mu}{d\tau} \frac{d\dot{\gamma}(\tau)_\mu}{d\tau} \right] = \text{sign} \left[ \frac{d\dot{\gamma}(\tau)_\mu}{d\tau} \frac{d\dot{\gamma}(\tau)_\mu}{d\tau} \right].
\]

Note that the norm of the “velocity vector” itself (without the sign) does depend on the parametrization! This makes sense if you think of the parameter as time: Changing how you measure time of course changes how you measure velocity.

\[
\rightarrow \text{sign } \|\dot{\gamma}\|^2_\gamma \text{ characterizes geodesics:}
\]

\[
\gamma \text{ time-like } \quad \gamma \text{ light-like (or null)} \quad \gamma \text{ space-like}
\]

\[
: \quad \text{sign } \|\dot{\gamma}\|^2_\gamma = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \tag{11.5}
\]

Hence there are three types of geodesics on a Lorentzian manifold.

- We adopt the same nomenclature also for spacetime curves that are not geodesics. In this case, the claim that the sign is constant along the curve is not (necessarily) the consequence of some dynamical law, but simply a feature of a particular curve.

- On the \( D = 4 \)-dimensional spacetime of **general relativity**, the time-like geodesics correspond to possible trajectories of free-falling bodies (also: possible time axes). The light-like geodesics are the trajectories of, well, light rays. Space-like geodesics are the analog of “straight lines” in space.

- There is a subtlety regarding light-like/null geodesics: Since their “velocity” vanishes (by definition), their length Eq. (10.123) vanishes as well. As a consequence, we cannot use their length \( s \) as an affine parameter \( \lambda \). To see what goes wrong, note that for \( \chi(y) = \sqrt{\gamma} \) setting \( y = \|\dot{\gamma}\|^2_\gamma = 0 \) in Eq. (10.128) is undefined (division by zero).

Luckily, this is only a technical inconvenience. Recall that in our setting, the equations for autoparallel curves Eq. (10.60) and geodesics Eq. (10.131) are identical. While the norm \( \|\dot{\gamma}\|^2_\gamma \) of a null vector vanishes, the vector itself \( \dot{\gamma} \) is a perfectly normal vector in

\[
\]
the tangent space (courtesy of $g_{\mu\nu}$ being a pseudo-metric). We then can simply fall back to the autoparallel equation Eq. (10.60) to describe null geodesics. The only difference is then that the affine parameter of light-like solutions of Eq. (10.60) (or, equivalently, Eq. (10.131)) cannot be interpreted as the length along the geodesic anymore.

**Light cones:**

- **Point/Event** $E \in M$; Draw all null geodesics emanating from $E$

```
\rightarrow \text{Light cone of } E:
```

**Notes:**

- Null geodesics remain null everywhere, i.e., their tangent vectors at every point lie on the null cone of the corresponding tangent space. Since the metric is Lorentzian (but otherwise arbitrary) the null cones can point in “different directions” at different points, so that the light cone can be warped and deformed.

In summary: The null cones live in the tangent spaces attached to the manifold, the light cone lives on the manifold itself and warps according to the local null cones (and thereby the metric).

- Note how all null cones on the future light cone point “inward”, whereas all null cones on the past light cone point “outward”. They act like unidirectional “pores” in a membrane that allow time-like trajectories (not necessarily geodesics) to leave the past light cone and enter the future light cone (but not to other way around).

- All time-like geodesics through $E$ stay within its past- and future light cone. Conversely, all space-like geodesics remain outside of this light cone.

Note that because of curvature in the metric [← Eq. (10.140)], geodesics can “attract” each other; in particular, two time-like geodesics emanating from a common event might cross again at another event! (Example: Imagine two satellites orbiting earth on the same orbit in opposite directions. Both are falling freely and – according to general relativity – follow geodesics in spacetime. But they periodically meet each other, i.e., their geodesics cross in spacetime repeatedly.)

- The null cone texture (also called a ↑ cone field) induces a ← partial order of events, which encodes a ← causality structure on the spacetime manifold (recall Section 1.6 for the case of Minkowski space). Up to a local (conformal) deformation of time- and length scales, this structure is essentially equivalent to the Lorentzian metric [140]! This suggests
the intriguing possibility that the null cone texture (equivalently: the causal structure of events) might be the truly fundamental field of general relativity, and the Lorentzian metric is just a convenient tool to encode it.

(Note that a local “stretching” of the metric by a strictly positive scalar field, $\Omega(x)g_{\mu\nu}(x)$, does neither alter the null cone texture nor angles between tangent vectors, thus it is a conformal transformation. This is why one says that the null cone texture determines the conformal class of the Lorentzian metric.)

For more details on Lorentzian manifolds, null cones, light cones, and the causal structure of spacetime, see the monograph [141].

- By comparison, in flat Minkowski space all geodesics are straight lines and never cross twice:

\[ \text{null cone} \]

Note how the null cone (which lives in the tangent space) of the reference event coincides with its light cone (which lives on the manifold). Mathematically, Minkowski space $\mathbb{R}^{1,3}$ is not just a Riemannian manifold (with Minkowski metric $g$) but also an affine space; this allows for a natural embedding of its tangent spaces into the manifold itself. Minkowski space is therefore a rather “degenerate” case of a generic spacetime and is not well suited to carve out the essential features of general relativity.

- Remember that there are locally inertial coordinates for every point of the manifold where (1) the Christoffel symbols vanish and (2) the metric has the Minkowski form (Section 10.3.1). This concept can be generalized:

For every geodesic, there is a coordinate system, defined in a “tube” around the geodesic, in which the metric takes the Minkowski form and the Christoffel symbols vanish (and so do the first derivatives of the metric). Such coordinates are called Fermi normal coordinates [142] and are useful for a freely falling observer to describe physics along (and close to) its time-like geodesics (which is then the time-axis of these coordinates).

---

iv | Extremal properties of geodesics:

We defined geodesics by a variational principle Eq. (10.125). Hence they extremize their Riemannian length locally. Since null geodesics have vanishing length, we focus here on time-like and space-like geodesics.
Length of time/space-like geodesics $\gamma$:

Proper time:  
$$L_{\text{Time}}[\gamma] = \int_{\gamma} \sqrt{g_{\mu\nu} dx^\mu dx^\nu} \sim c^2 dr^2 - (dx^2 + dy^2 + dz^2) > 0$$  \hspace{1cm} (11.6a)

Proper distance:  
$$L_{\text{Space}}[\gamma] = \int_{\gamma} \sqrt{-g_{\mu\nu} dx^\mu dx^\nu} \sim (dx^2 + dy^2 + dz^2) - c^2 dr^2 > 0$$  \hspace{1cm} (11.6b)

Recall that the metric has signature $(1, 3) = (+, -, -, -)$. The expressions below the integrals are valid approximately in locally inertial coordinates.

Local variations (in locally inertial coordinates; geodesic w.l.o.g. along coordinate axis):

- Light-like geodesics are local maxima of proper time.
- Space-like geodesics are local saddle points of proper distance.
Part III.

Excursions
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