Special and General Relativity

Lecture Notes • Winter Term 2023/24

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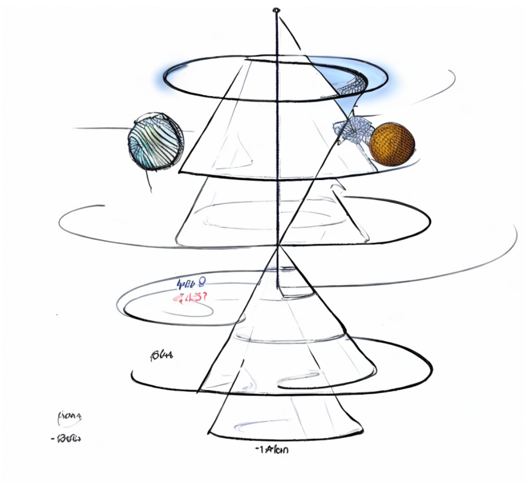

How *Stable Diffusion* imagines a light cone.

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Preliminaries

Important

This script is in development and continously updated. To download the latest version:

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If you spot mistakes or have suggestions, send me an email:

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Requirements for this course

We assume that students are familiar with the following concepts:

- Classical mechanics (Lagrangian and Hamiltonian formalism ...)
- Non-relativistic quantum mechanics (Schrödinger equation ...)
- Classical electrodynamics (Maxwell equations ...)
- Basics of algebra & linear algebra (groups, linear maps, ...)
- Second quantization and path integrals ★
  This is only required for the excursions on quantum gravity!

Literature recommendations

Special relativity

- Schröder: *Spezielle Relativitätstheorie* [1]
  ISBN 978-3-808-55653-5
  Compact, pedagogic, mathematically precise introduction (in German).
- Misner, Thorne, and Wheeler: *Gravitation* [2]
  ISBN 978-0-691-17779-3
  Extensive standard textbook on special and general relativity (in English).

General relativity

- Schröder: *Gravitation: Einführung in die Allgemeine Relativitatstheorie* [3]
  ISBN 978-3-817-11874-8
  Compact, pedagogic, mathematically precise introduction (in German).
- Misner, Thorne, and Wheeler: *Gravitation* [2]
  ISBN 978-0-691-17779-3
  Extensive standard textbook on special and general relativity (in English).

- Schutz: *A First Course in General Relativity* [4]
  `ISBN 978-1-108-49267-6`
  Extensive, pedagogic, mathematically precise introduction.
- Rovelli: *General Relativity: The Essentials* [5]
  `ISBN 978-1-009-01369-7`
  Very high level and compact overview with links to quantum gravity.

Quantum gravity

- Zwiebach: *A First Course in String Theory* [6]
  `ISBN 978-0-521-88032-9`
  Extensive, pedagogic introduction with many detailed calculations.
- Rovelli: *Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory* [7]
  `ISBN 978-1-108-81025-8`
  Compact, pedagogic introduction, omitting some technical details.

This course follows roughly the textbook *Spezielle Relativitätstheorie* by Ulrich Schröder [1] in the first part on special relativity (with admixtures from Schutz [4] and Straumann [8]). The second part on general relativity follows roughly the textbook *Gravitation* by Ulrich Schröder [3] (with admixtures from Schutz [4] and Rovelli [5]). The excursions on quantum gravity at the end draw from Barton Zwiebach's *A First Course in Sring Theory* [6] for the primer on bosonic string theory, and Carlo Rovelli's *Covariant Loop Quantum Gravity* [7] for the sneak peek at loop quantum gravity.

Original literature

- A. Einstein: *Zur Elektrodynamik bewegter Körper* [9]
  `Annalen der Physik, 17, p. 891-921, (1905)`
  Einstein bootstraps SPECIAL RELATIVITY (in German).
- A. Einstein: *Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?* [10]
  `Annalen der Physik, 18, p. 639-641, (1905)`
  Einstein derives the famous mass-energy equivalence (in German).
- A. Einstein: *Zur allgemeinen Relativitätstheorie* [11]
  `Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 778-786, 799-801, (1915)`
  A. Einstein: *Die Feldgleichungen der Gravitation* [12]
  `Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 844-847, (1915)`
  Einstein bootstraps the field equations of general relativity (in German).
- A. Einstein:
  Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie [13]
  `Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 831-839, (1915)`
  Einstein explains Mercury's apsidal precession (in German).
- A. Einstein: *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie* [14]
  `Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 142-152, (1917)`
  Einstein kickstarts relativistic cosmology and introduces the cosmological constant (in German).
- A. Einstein: *Über Gravitationswellen* [15]
  `Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 154-167, (1918)`
  Einstein predicts and studies gravitational waves (in German).

Goals of this course

The goal of this course is to gain a thorough understanding of relativity, our modern theory of space and time ("spacetime"). This includes both the *symmetries* and the *dynamics* of spacetime; the former being described by *special relativity*, the latter by *general relativity*. We close with an (optional) excursion into the quantization of gravity, and briefly discuss the two most prominent contenders: string theory and loop quantum gravity.

In particular ($\star$ optional):

Special relativity

- Conceptual foundations special relativity
- Galileian and Einsteinian relativity principles
- Lorentz transformations and the principle of invariance
- Kinematical consequences of Lorentz transformations
- Tensor calculus and the metric tensor
- Special relativity in Minkowski space
- Lorentz- and Poincaré group
- Relativistic mechanics
- Lagrange function and principle of least action
- Electrodynamics as a relativistic field theory
- Noether theorem and the energy momentum tensor
- Relativistic quantum mechanics (Klein-Gordon- and Dirac equation)
- Limitations of special relativity

General relativity

- Incompatibility of gravitation and special relativity
- Mathematical toolbox:
  Riemannian manifolds, metric tensor, Levi-Civita connection, curvature, ...
- Conceptual framework of general relativity:
  Metric field, general covariance vs. background independence, ...
- Classical mechanics in curved spacetime
- Electrodynamics in curved spacetime $\star$
- Dynamics of general relativity (Einstein field equations)
- Implications of the Einstein field equations:
  Newtonian limit, Gravitational time dilation, Apsidal precession, Light deflection ...
- Application: Gravitational waves (linearized Einstein equations)
- Application: Black holes (Schwarzschild solution)
- Application: The standard model of cosmology (FLRW metric, ΛCDM, ...)
- Limitations of general relativity:
  Einstein-Hilbert action, quantum field theory, (non-)renormalizability, ...

Quantum gravity (excursion)

- The bosonic string ★ :
  Quantization, Virasoro algebra, anomalies, Hilbert space, gravitons, tachyons, …
- Concepts of quantum loop gravity ★ :
  Discretized gravity, spin networks, vertex amplitude, transition amplitudes, …

Notes on this document

- This document is not an extension of the material covered in the lectures but the script that I use to prepare them.
- Please have a look at the given literature for more comprehensive coverage. References to primary and secondary resources are also given in the text.
- The content of this script is color-coded as follows:
  - Text in black is written to the blackboard.
  - Notes in red should be mentioned in the lecture to prevent misconceptions.
  - Notes in blue can be mentioned/noted in the lecture if there is enough time.
  - Notes in green are hints for the lecturer.
- One page of the script corresponds roughly to one covered panel of the blackboard.
- Enumerated lists are used for more or less rigorous chains of thought:
  1 | This leads to …
  2 | this. By the way:
    i | This leads to …
    ii | this leads to …
    iii | this.
  3 | Let's proceed …
- In the bibliography (p. 244 ff.) you can find links to download most papers referenced in this script (they look like this: Download). Because most of these papers are not freely available, you need a username & password to access them. These credentials are made available to students of my classes.
- This document has been composed in **`Vim`** on **`Arch Linux`** and is typeset by LuaLaTeX and BibTeX. Thanks to all contributors to free software!
- This document is typeset in **Equity**, **Concourse** and MathTimeProfessional.

Acknowledgements

- Several students and colleagues spotted typos in the script. Thanks!

Symbols & Scientific Abbreviations

The following abbreviations and glyphs are used in this document:

cf	confer ("compare")
dof	degree(s) of freedom
eg	exempli gratia ("for example")
etc	et cetera ("and so forth")
et al	et alii ("and others")
ie	id est ("that is")
viz	videlicet ("namely")
vs	versus ("against")
wlog	without loss of generality
wrt	with respect to
$\lhd$	"consider"
$\rightarrow$	"therefore"
¡!	"Beware!"
$\overset{\circ}{=}$	non-obvious equality that may require lengthy, but straightforward calculations
$\overset{*}{=}$	non-trivial equality that cannot be derived without additional input
$\overset{\circ}{\rightarrow}$	"it is easy to show"
$\overset{*}{\rightarrow}$	"it is *not* easy to show"
$\Rightarrow$	logical implication
$\wedge$	logical conjunction
$\vee$	logical disjunction
$\square$	repeated expression
$\blacksquare$	anonymous reference
w/o	"without"
w/	"with"
→	internal forward reference ("see below/later")
←	internal backward reference ("see above/before")
↑	external reference to advanced concepts ("have a look at an advanced textbook on…")
↓	external reference to basic concepts ("remember your basic course on…")
➔	reference to previous or upcoming exercises
★	optional choice/item
⁂	implicit or explicit definition of a new technical term ("so called …")
‡	Aside
$\equiv$	Synonymous terms
$:=$	Definition

The following scientific abbreviations are used in this document:

BRT	Belinfante-Rosenfeld tensor
CERN	European Organization for Nuclear Research
COE	Center of energy
COM	Center of mass | Center of momentum
CO	Continuity
DFST	Dual field-strength tensor
EM	Electromagnetic
EMT	Energy momentum tensor
EOM	Equation of motion
ES	Einstein synchronization
FLRW	Friedmann–Lemaître–Robertson–Walker (metric)
FST	Field-strength tensor
GR	General relativity
HME	Homogeneous Maxwell equations
HO	Homogeneity
IC	Invariance of coincidence
IME	Inhomogeneous Maxwell equations
IN	Inertial (test)
IRF	Instantaneous rest frame
IRS	Instantaneous rest system
IS	Inertial system | Isotropy
ISS	International space station
IT	Infinitesimal transformation
KG	Klein-Gordon
KGE	Klein-Gordon equation
LT	Lorentz transformation
ME	Maxwell equation(s)
OC	Orthonormal Cartesian (coordinates)
PDE	Partial differential equation
QED	Quantum electrodynamics
QFT	Quantum field theory
RI	Reparametrization invariance
SE	Schrödinger equation
SI	Système international d'unités
SL	Speed of light
SR	Special relativity
UV	Ultraviolett

↓ **Lecture 1** [17.10.23]

Setting the Stage

Terminology

The most important terms in this course and their German correspondence:

$$
\begin{aligned}
\textsc{Relativity} &= \textit{Relativitätstheorie} \\
\textsc{Special Relativity} &= \textit{Spezielle Relativitätstheorie (SRT)} \\
\textsc{General Relativity} &= \textit{Allgemeine Relativitätstheorie (ART)}
\end{aligned}
$$

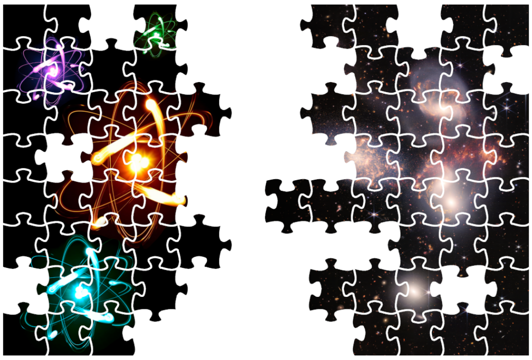
Relation of the theories:

$$
\textsc{Relativity} \left\{ \begin{array}{l} \textsc{Special Relativity} \\ \textsc{General Relativity} \end{array} \right.
$$

Motivation

RELATIVITY is arguably the most popular of scientific theories, for it speaks about an entity of every day experience: space and time. This popularity comes with a caveat:

The "Mona Lisa perspective"
The popular status of RELATIVITY in physics parallels that of the Mona Lisa in arts: Einstein's magnum opus inherits an aura of perfection and finality.



The "Puzzle Perspective"
RELATIVITY is interesting because it describes *some*, but *not all* facets of reality. Its incompatibility with quantum mechanics hints at a reality even stranger than its pieces.

¡! You should *not* view RELATIVITY as the "Mona Lisa of physics" but as the harbinger of quantum gravity[1] that, most likely, will come with a reformulation of reality so profound that the "strangeness" of quantum mechanics and RELATIVITY alike will pale in comparison (→ *Excursions*).

[1] I use the term "quantum gravity" here very loosely and essentially synonymous with "theory of everything".

Ontology

1 | The ⁂ *ontology of physics* is the collection of "things that exist" (⁂ *entities*):

$$\text{Ontology} = \Big\{ \underbrace{\underbrace{\text{Leptons, Hadrons, Higgs,}}_{\text{Matter: Atoms} \dots} \;\; \underbrace{\text{Gauge bosons}}_{\text{Interactions: Photons} \dots}}_{\text{Standard Model of Particle Physics}} \Big\}$$

2 | Physical theories are *models* that describe how these entities behave.

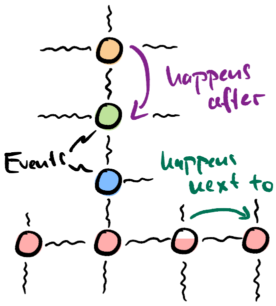
Examples:

Classical mechanics describes the dynamics *of* matter on macroscopic scales.
Quantum mechanics describes the dynamics *of* matter on microscopic scales.
Electrodynamics describes the dynamics *of* electromagnetic fields on macroscopic scales.

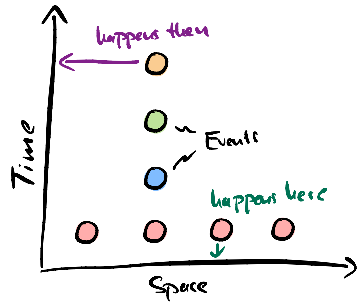
Note that these can be *effective* (approximate) descriptions that are restricted to finite scales of validity (length, energy, time).

3 | What is RELATIVITY a theory *of*?

i | ◁ Two notions of space and time:

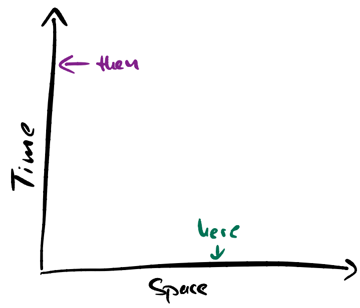


⁂ *Relational space & time*



⁂ *Newtonian space & time*

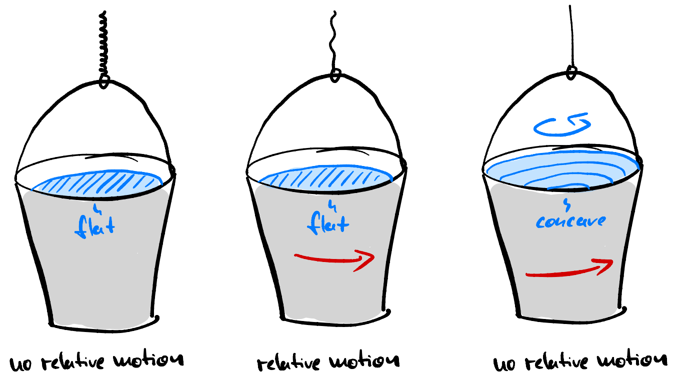
ii | ◁ Delete all entities from the world:



Nothing! Newtonian space & time left!

Question: Which notion describes reality?

iii | ◁ Newton's bucket:



Question: Rotation with respect to *what* determines the shape of the water surface?

Tentative answer: Rotation with respect to Newtonian space!

¡! Today, Newtonian space & time (sometimes called neo-Newtonian or Galilean spacetime) is *not* seen as a preferred ("absolute") coordinate system, with respect to which absolute *positions*, *times* and *velocities* can be measured; it is the entity that is responsible for the absolute notion of *acceleration* in Newtonian physics (which is also present in RELATIVITY). It is "the thing" that determines the reference frames that are *inertial* [5].

→ *Space & time (Spacetime)* is an independent "thing that exists."

The correct answer to the bucket experiment in RELATIVITY will be: The rotation with respect to the *local inertial frame*—which is determined by the local gravitational field—determines the shape of the water surface. This field is determined by the large-scale distribution of mass and energy in the universe, i.e., the fixed stars; the (rotating) mass of the earth has a non-zero but tiny effect as well (→ *Frame dragging*).

4 | Thus we should extend our ontology:

$$\text{Extended Ontology} = \{\, \underbrace{\underbrace{\text{Leptons, Hadrons, Gauge bosons, Higgs}}_{\text{Standard Model of Particle Physics}},\ \underbrace{\textbf{Spacetime}}_{\text{RELATIVITY}}}\,\}$$

Core Theory $\xleftarrow{\text{IR}\quad\text{Energy scale}\quad\text{UV}}\!\!\longrightarrow$ Theory of Everything (?)

The ⁂ *Core Theory* [16] (→ *below*) is an effective (quantum) field theory that encompasses the standard model and RELATIVITY. It describes all entities know to us *on our scales*—but is expected to fail on the Planck scale (in the "UV limit"). The theory that the Core Theory renormalized to in this UV limit is the famous "Theory of Everything". This is uncharted territory and we do not know what this theory looks like.

The extended ontology above is known as ⁂ *substantivalism* in the philosophy of science, see [17] for a review and [18] for a supportive account of this ontology. Opposing substantivalism is ⁂ *relationalism*, which defends the view that spacetime is not an independent entity but an emergent description of *relations* between entitites (↑ *The Hole Argument*). Relationalism is exemplified by ↑ *Mach's principle*, which has been historically influential in the development of GENERAL RELATIVITY (though Einstein later changed his views). In the light of non-trivial solutions (of the Einstein field equations) for "empty" universes in GENERAL RELATIVITY, and the (now experimentally confirmed) existence of gravitational waves, I take a substantivalist stance in this course.

5 | This extended ontology allows us to answers the question:

> RELATIVITY is the theory *of* spacetime (on macroscopic scales),
> just as electrodynamics is the theory *of* the electromagnetic field.

¡! Despite these conceptual similarities, there is a fundamental difference between RELATIVITY and electrodynamics (→ *below*): Whereas electrodynamics describes the dynamics of the electromagnetic field *on* spacetime, the gravitational field of RELATIVITY does *not* evolve *on* spacetime; it *is* spacetime!

‡ The Core Theory

The ⁂ *Core Theory* S_* is the ⁂ *effective field theory* that describes all entities on the energy scales relevant for our everyday life [16]. As typical for a field theory, it is best expressed as a ↑ *path integral*:

$$A_* = \int_{k<\Lambda} \mathcal{D}g\, \mathcal{D}G\, \mathcal{D}\psi\, \mathcal{D}\phi\, \exp\left(\frac{i}{\hbar} S_*[g, G, \psi, \phi]\right) .$$

Momentum cutoff | Gravitational field (Metric) | Gauge fields (Photons…) | Fermion fields (Electrons…) | Higgs field (Higgs boson) | Action

What makes this an *effective* theory is the momentum cutoff Λ: The theory describes the dynamics of the fields only up to some finite momentum/energy cutoff Λ. In [16] it is argued that $\Lambda \sim 10^{11}$ eV is a reasonable cutoff; since this is well below the Planck scale of 10^{28} eV, A_* does not describe the physics on these energy scales (e.g., what happens in black holes or near the Big Bang is *not* encoded in A_*). This reflects the lack of a consistent theory of quantum gravity.

The action S_* splits into two parts (plus one additional, technical term that we can savely ignore here):

$$S_*[g, G, \psi, \phi] = S_{\mathrm{EH}}[g] + S_{\mathrm{SM}}[g, G, \psi, \phi] .$$

The first part is the famous ⁂ *Einstein-Hilbert action* (G is the gravitational constant) and describes the gravitational field g:

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G} \int \mathrm{d}^4x\, \sqrt{-g}\, R(g) .$$

We will encounter this action in the second part of this course as it encodes the (source-free) ⁂ *Einstein field equations*; there you will learn what $R(g)$ is.

The second part is the action of the ⁂ *standard model of particle physics* (coupled to gravity via g) and describes all the stuff in our world (matter and interactions) *except gravity*:

$$S_{\mathrm{SM}}[g, G, \psi, \phi] = \int \mathrm{d}^4x\, \sqrt{-g} \left[i\bar{\psi}\, \not{D}\psi - \frac{1}{4}G^2 + |D\phi|^2 - V(\phi) + \left(\bar{\psi}_L^i Y_{ij} \phi \psi_R^j + \text{h.c.}\right)\right] .$$

Dirac (Fermion ke&i) | Yang-Mills (Gauge boson ke&i) | Klein-Gordon (Higgs boson ke&i) | Higgs potential (Symmetry breaking) | Yukawa coupling (Fermion masses)

Here "ke&i" stands for *kinetic energy* and *interactions* (with gauge bosons). The standard model action $S_{\text{SM}}[G,\psi,\phi] \equiv S_{\text{SM}}[\eta,G,\psi,\phi]$ on a static, flat spacetime $g=\eta$ is typically discussed in a course on quantum field theory with focus on high energy physics (↑ *Section 10.2 of my script on QFT [19]*). In this course on RELATIVITY, the existence of S_{SM} will leave its (classical) mark on the Einstein field equations in form of the ⁂ *energy-momentum tensor*.

Relation to other theories

1 | RELATIVITY is similar to other theories in that it is a theory *of* an entity that makes up reality. However, it is also different in that this very entity makes an appearance in most other theories:

Classical mechanics describes the macr. dynamics of **matter** *on* **spacetime**: $\vec{x}(t)$.
Quantum mechanics describes the micr. dynamics of **matter** *on* **spacetime**: $\Psi(\vec{x},t)$.
Electrodynamics describes the macr. dynamics of **EM fields** *on* **spacetime**: $\boldsymbol{E}(\vec{x},t)$, $\boldsymbol{B}(\vec{x},t)$.

In the light of the extended ontology (where spacetime is an idependent entity described by RELATIVITY), it can be useful to reframe the objective of various theories as follows:

Classical mechanics describes the macr. dynamics of **matter** *interacting with* a (static) **spacetime**.
Quantum mechanics describes the micr. dynamics of **matter** *interacting with* a (static) **spacetime**.
Electrodynamics describes the macr. dynamics of **EM fields** *interacting with* a (static) **spacetime**.

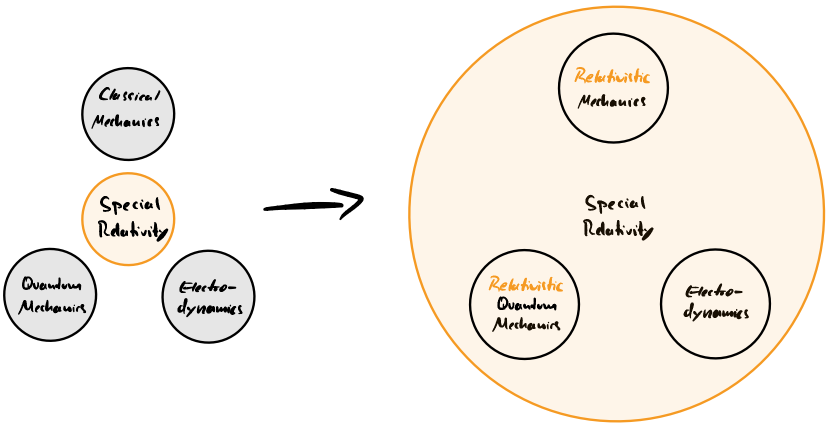
Note that this reading is manifest in the background-independent formulation of the Core Theory $S_\star[g,G,\psi,\phi]$ where the metric g and the other fields are treated on the same footing.

→ The properties of spacetime (as posited by RELATIVITY) must be reflected by these theories!

This means that we might have to *modify* known theories to be consistent with RELATIVITY. These modifications must adhere to the ⁂ *correspondence priciple*: The "old" (non-relativistic) versions of the theories must be included in the "new" (relativistic) versions as limiting cases.

2 | Incorporating the tenets of SPECIAL RELATIVITY leads to ...

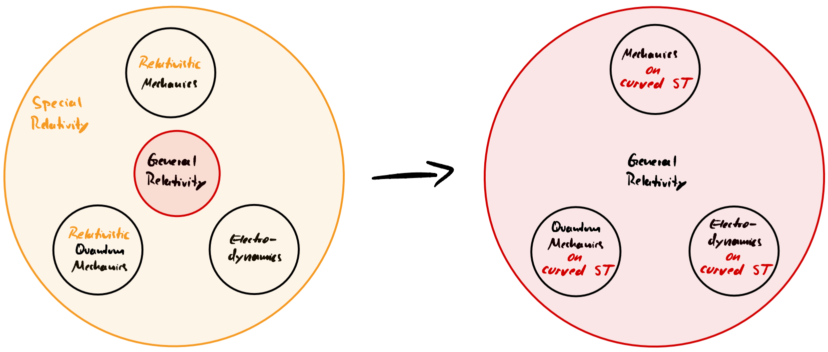
- *Relativistic* mechanics
- *Relativistic* quantum mechanics (Dirac equation, Klein-Gordon equation)
- *Relativistic* electrodynamics (= classical electrodynamics)



Luckily, classical electrodynamics is already consistent with SPECIAL RELATIVITY and needs no modification. By contrast, both classical mechanics and the quantum mechanics you learned in your previous courses must be *modified* to reflect the symmetries of spacetime posited by SPECIAL RELATIVITY.

3 | Incorporating the tenets of GENERAL RELATIVITY leads to ...

- (Relativistic) Mechanics *on curved spacetimes*
- (Relativistic) Quantum mechanics *on curved spacetimes*
- (Relativistic) Electrodynamics *on curved spacetimes*



In this course, we will discuss the modifications needed for *mechanics* and *electrodynamics* to fit the framework of GENERAL RELATIVITY. We won't discuss quantum mechanics on curved spacetimes.

¡! Quantum mechanics (describing matter and gauge bosons) on a curved spacetime is *not* "quantum gravity!" Quantum gravity is a theory where the metric field g *itself* is quantized (which we do not know how to do).

Spoiler

The gist of RELATIVITY can be summarized as follows:

$$\boxed{\begin{aligned} \text{Spacetime} &\leftrightarrow \text{Four dimensional Lorentzian manifold } (M, g) \\ \text{Gravitational field} &\leftrightarrow \text{Metric tensor field } g \end{aligned}}$$

This is what is meant by the popular statement that gravity "is not a force" but a geometrical deformation ("curvature") of spacetime.

and

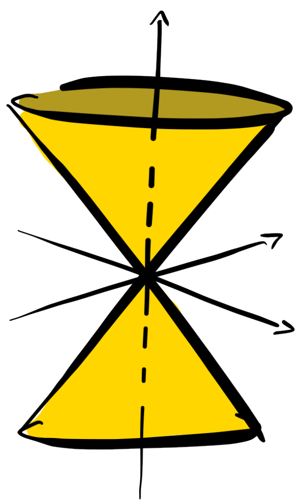
$$\boxed{\begin{aligned} \text{SPECIAL RELATIVITY}: &\ g \text{ has signature } (1,3)\ (\textit{Lorentz symmetry}) \\ \text{GENERAL RELATIVITY}: &\ g \text{ is a dynamical field } (\textit{Background independence}) \end{aligned}}$$

You most likely do not understand these statements at this point. That's fine! To provide you with the background knowledge to do so is the purpose of this course.

So let's start ...

Part I.

Special Relativity



1. Conceptual Foundations

◊ **Concepts**

- Events, Observations, Coincidences, Observers, Reference frames, Einstein synchronization, Cartesian coordinates, Inertial frames, Inertial coordinate systems, Coordinate transformations, Laws of nature, Physical models and theories
- Newtonian mechanics, Form-invariance and covariance, Invariance group, Active and passive transformations, Galilei transformations, Galilei group, Galilean principle of relativity
- Maxwell equations, Aether, Michelson Morley experiment, Principle of Special Relativity
- Isotropy, Homogeneity, Affine transformations
- Special Lorentz transformations, Lorentz Boosts, Lorentz group, Lorentz factor, Limiting velocity, Lorentz covariance, Addition of collinear velocities, Finite speed of causality
- Spacetime interval, Invariant interval, Time-like, Space-like, Light-like, Light cone, Invariant hyperbolae, Causality, Time-like trajectories, Partial order of events, Causal automorphism
- Relativity principles, Symmetries of spacetime, Simplicity of nature, Compressibility, Anthropic principle

1.1. Events, frames, laws, and models

1 | Events:

i | A. Einstein writes in his 1905 paper "Zur Elektrodynamik bewegter Körper" [9]:

> *Wir haben zu berücksichtigen, daß alle unserer Urteile, in welchen die Zeit eine Rolle spielt, immer Urteile über gleichzeitige Ereignisse sind. Wenn ich z. B. sage: "Jener Zug kommt hier um 7 Uhr an," so heißt dies etwa: "Das Zeigen des kleinen Zeigers meiner Uhr auf 7 und das Ankommen des Zuges sind gleichzeitige Ereignisse."*

And in his 1916 review "Die Grundlage der allgemeinen Relativitätstheorie" [20]:

> *Alle unsere zeiträumlichen Konstatierungen laufen stets auf die Bestimmung zeiträumlicher Koinzidenzen hinaus. Bestände beispielsweise das Geschehen nur in der Bewegung materieller Punkte, so wäre letzten Endes nichts beobachtbar als die Begegnungen zweier oder mehrerer dieser Punkte. Auch die Ergebnisse unserer Messungen sind nichts anderes als die Konstatierung derartiger Begegnungen materieller Punkte unserer Maßstäbe mit anderen materiellen Punkten bzw. Koinzidenzen zwischen Uhrzeigern, Zifferblattpunkten und ins Auge gefaßten, am gleichen Orte und zur gleichen Zeit stattfindenden Punktereignissen.*

We condense this into the following postulate:

> **§ Postulate: Invariance of coincidence** `IC`
>
> - Observations are *coincidences* of *events* local in space and time.
> - Coincidences of events are absolute and observer independent.

ii | Example:

$$
\begin{aligned}
\text{Event } e_1 &= \langle \text{Clock A shows time 11:30} \rangle \\
\text{Event } e_2 &= \langle \text{Detector B detects electron} \rangle \\
\text{Event } e_3 &= \langle \text{Clock C shows time 9:45} \rangle
\end{aligned}
$$

If detector B and clock A are at the same location (spatial coincidence), and clock A shows 11:30 when detector B detects and electron (temporal coincidence), we say that the events e_1 and e_2 coincide: $e_1 \sim e_2$.

→ Collect all events e_i that coincide into an equivalence class E:

$$e_1 \sim e_2 \sim e_3 \sim \dots \quad \rightarrow \quad E = \{e_1, e_2, e_3, \dots\}$$

In a slight abuse of nomenclature we call the coincidence class E also *event*.

Note that this abuse of nomenclature is also used in everyday life: What makes up an "event" (like a party) is the set of all "little events" (like you meeting your friend) that happen (roughly) at the same location and the same time.

iii | Assumption:

The set $\mathcal{E} = \{E_1, E_2, \dots\}$ of all coincidence classes is a complete, observer independent record of reality.

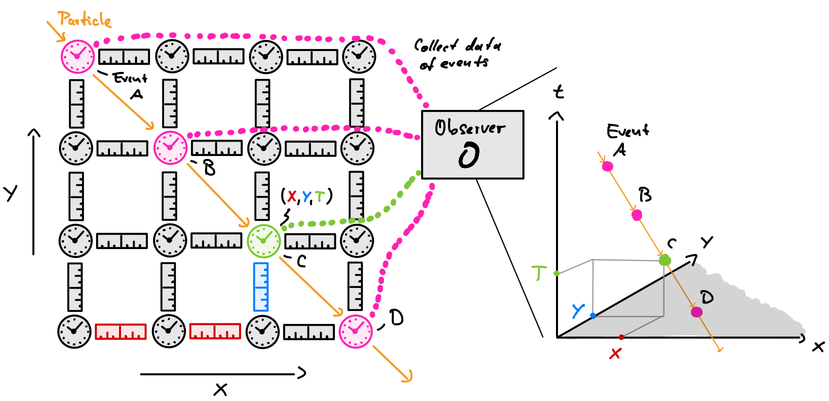
We call the information stored in $\mathcal{E}$ *absolute* because all observers agree on it.

2 | ⁂ *Observer* $\mathcal{O}$ ≡ ⁂ *(Reference) Frame* $\mathcal{O}$:

Goal: Systematic description of physical phenomena in terms of models.

Question: How to systematically observe reality and encode these observations?

:= Experimental setup to collect data about events in space & time:



↓ **Lecture 2** [24.10.23]

Assumptions:

- The rods and clocks are conceptual: they do not affect physical experiments.
- All rods and clocks are identical (when brought together, the rods have the same, time-independent length and the clocks tick with the same rate).
- The lattice is "infinitely dense": there is a clock at every point in space.
- Each clock is assigned a unique position label $\vec{x}$ and the reference frame label $\mathcal{O}$.

  For example, a unique position label $\vec{x}$ for a clock can be obtained by counting the rods in x-, y- and z-direction that one has to traverse to reach the clock from the origin. The origin O is, by definition, a "special" clock that is *assigned* the position label $\vec{x}_O = \vec{0}$.

¡! Observers are *not* sitting at the origin, looking at their wristwatch, and observing the events with binoculars! They are simply collecting and processing the data that is accumulated by the contraption we call a reference frame.

Since we assume that (ideally) there is one clock at every point in space:

→ For every observer $\mathcal{O}$ and every coincidence class E there is a unique event $e_{\mathcal{O}}$

$$E \ni e_{\mathcal{O}} = \langle \text{Clock with frame label } \mathcal{O} \text{ and position label } \vec{x} \text{ shows time } t \rangle \tag{1.1a}$$

$$=: (t, \vec{x})_{\mathcal{O}} \quad \Leftrightarrow \quad [E]_{\mathcal{O}} = (t, \vec{x}) \tag{1.1b}$$

for some position label $\vec{x}$ and clock reading t.

We refer to the event $(t, \vec{x})_{\mathcal{O}}$ as the *spacetime coordinates* of E with respect to frame $\mathcal{O}$. A different observer $\mathcal{O}'$ will use its own clocks and therefore other events ("coordinates") $(t', \vec{x}')_{\mathcal{O}'} \in E$ to refer to E.

In the real world, the ↑ *tracking detectors* of particle colliders are reminiscent of this ideal setup: They are comprised of 3D arrangements of semiconductor-based particle detectors that all report to a central computer that then reconstructs the trajectories of scattering products from the combination of all detection events.

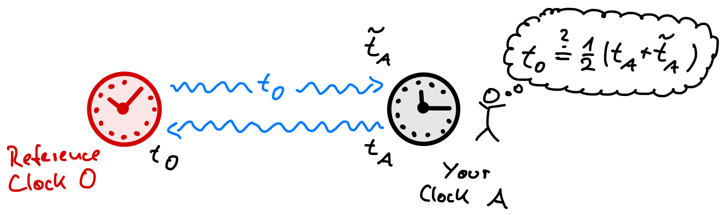
3 | ⁂ *Inertial (coordinate) systems:*

The setup of a reference frame $\mathcal{O}$ above is incomplete and actually very hard to work with: Without additional constraints on the geometry of the lattice and the correlations of clocks (their "calibration"), the record of events is essentially arbitrary. Let us therefore impose some deterministic "calibration procedure" (the same for all frames) that determines how to lay out the rod lattice and how to synchronize the clocks. This procedure endows our reference frame with a specific *coordinate system*, a labeling scheme to describe events.

i | Clock calibration: ⁂ *(Poincaré-)Einstein synchronization* **ES**

The conventional synchronization procedure (which is actually in practical use) is (Poincaré-)Einstein synchronization:

$$t_O \stackrel{?}{=} \tfrac{1}{2}\left(t_A + \tilde{t}_A\right) \tag{1.2}$$



You will study this particular procedure and its properties in ➔ Problemset 1.

In brief, the procedure goes as follows: Consider a reference clock O and some other clock A you wish to synchronize with O.

(1) To do so, you send a light signal from A to O and note the time t_A your clock A reads when the signal is emitted.

(2) When the signal arrives at O, it is immediately reflected back to A together with the reading t_O of clock O at this very moment.

(3) When the signal arrives back at your clock A (together with the timestamp t_O), you note again the reading of your clock as $\tilde{t}_A$.

(4) You are now in the possession of three timestamps: $(t_A, t_O, \tilde{t}_A)$. The idea of Einstein synchronization is to *postulate* the reciprocity of the speed of light: We *declare* that the speed of the signal from A to O is the same as on its way back from O to A (note that we cannot *measure* this reciprocity because we would need already synchronized clocks to do so!). Under this assumption, the readings of *synchronized clocks* must satisfy [9]

$$\Delta t_{A\to O} \equiv t_O - t_A \overset{!}{=} \tilde{t}_A - t_O \equiv \Delta t_{O\to A} \quad \Leftrightarrow \; t_O \overset{!}{=} \frac{1}{2}(t_A + \tilde{t}_A)\,, \tag{1.3}$$

which you can locally check with your data $(t_A, t_O, \tilde{t}_A)$. Note that you do not need to know the distance from O to A, nor the numerical value of the speed of light c for this procedure to work!

(5) Now if you just powered on your shiny new clock A for the first time, it is very unlikely that the condition Eq. (1.3) will be satisfied:

$$t_O = \frac{1}{2}(t_A + \tilde{t}_A) + \delta t = \frac{1}{2}[(t_A + \delta t) + (\tilde{t}_A + \delta t)]\,. \tag{1.4}$$

Here δt is an offset that you might encounter. But then you can just recalibrate your clock A by δt such that the new readings are $t_A + \delta t$ and $\tilde{t}_A + \delta t$.

Repeating this procedure for all clocks of the frame $\mathcal{O}$ allows you to establish a synchronization relation between arbitrary pairs of clocks. The fact that (under some reasonable and experimentally verified assumptions) the order in which you synchronize your clocks does not matter (the established relation is an *equivalence* relation, ➔ Problemset 1 and Ref. [21]) makes Einstein synchronization a very useful and peculiar *convention* [22–24]. However, one can show that it is the only convention that yields a non-trivial equivalence relation of simultaneity that is consistent with the causal structure on $\mathcal{E}$ (→ *later*) [25].

ii | Lattice calibration: ⁂ *Orthonormal Cartesian coordinates* **OC**:

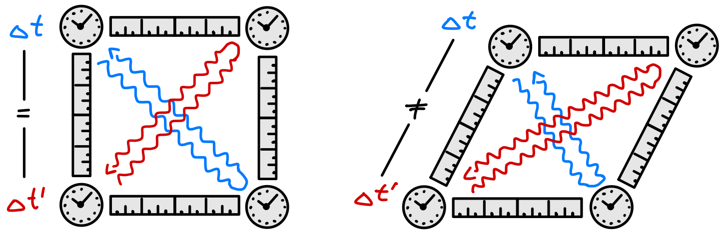
The layout of the lattice of rods assigns coordinates $\vec{x} = (x, y, z)$ to each clock. Depending on the actual shape of the lattice, we will denote events by different position labels. (Note that even with rigid rods connected in the topology of a cubic lattice the geometry is not fixed; for example, you can shear the lattice.) If we assume that space (not spacetime!) is a flat Euclidean space where all the facts of Euclidean geometry hold good (angles of triangles add up π, the Pythagorean theorem holds, the area of circles is πr^2, etc.), we can parametrize it without loss of generality by *orthonormal Cartesian coordinates*. In these coordinates, distances can be calculated by the Pythagorean formula:

Spatial distance between clocks at $\vec{x}$ and $\vec{y}$:

$$\underbrace{d(\vec{x},\vec{y})}_{\text{“Physics”}} = \underbrace{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}}_{\text{“Mathematics”}} \tag{1.5}$$

The fact that the coordinates of a point (x, y, z) are *distances* along paths parallel to the coordinate axes makes the coordinates *Cartesian*. The fact that Eq. (1.5) holds makes them *orthonormal* (i.e., the axes are orthogonal and have the same scale, as suggested by the sketch above). Coordinates are an intrinsically mathematical concept, they are “labels” to identify points on a manifold of physical points (or *events*, if you consider space*time* coordinates). By contrast, distances carry physical significance: You can measure them with light signals or rods. The prevalence of Cartesian coordinates makes it easy to conflate these two concepts (this will become particularly important in GENERAL RELATIVITY).

Here is a way to check whether your lattice satisfies the **OC** condition using the clocks of $\mathcal{O}$ (and the assumption of the isotropy of the two-way speed of light):



iii | “Inertial Test” (⁂ *law of inertia*):

Once you have arranged your rods and synchronized your clocks and thereby established a Cartesian coordinate system and a (allegedly) well-defined notion of simultaneity, you can perform the following test and check whether your particular reference frame $\mathcal{O}$ passes it or not:

IN Free particles move at constant velocity and in straight lines.
(⁂ *Homogeneity of Inertia*)

- It is implied that this statement is true *everywhere*, *anytime*, and *in all directions*.
- Velocities are computed as the time derivative of trajectories in the frame: $\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t}$.
- The property **IN** implies a certain form of *homogeneity* in space and time (since free particles must move in straight lines *anywhere* and *anytime*) and *isotropy* in space (they must move in straight lines in *any direction*). Without additional empirical input, this does not automatically imply that *every experiment* yields the same result anywhere, anytime and in any direction. This more general form of homogeneity and isotropy will be introduced later as **HO** and **IS**. Empirical evidence shows that spacetime indeed is homogeneous **HO** and space isotropic **IS** (in the absence of gravity). With this additional input, the “Inertial Test” to establish **IN** can be simplified to only one particle moving in a straight line at one place for some finite time (which is actually doable). If you presuppose homogeneity **HO** but *not* isotropy **IS**, you could observe multiple free particles starting at the same point but moving in different (linearly independent) directions.

> Frames equipped with a coordinate system defined by **ES** + **OC** which satisfy **IN** are called ⁂ *inertial coordinate systems*.

To distinguish arbitrary frames $\mathcal{O}$ (with arbitrary coordinates) from the special frames (equipped with Cartesian coordinates and synchronized clocks) that passed the inertial test, we label these coordinate systems by K, K', K'' etc. (if we refer to arbitrary inertial systems) and by A, B, C etc. (if we refer to specific inertial systems); the set of all inertial systems is denoted $\mathcal{I}$.

Alternative definitions:

There seem to be as many definitions of inertial systems as there are texts on SPECIAL RELATIVITY. Some are equivalent, some are not. Some more useful, others less so (none are "wrong", though, because *definitions* cannot be wrong). Some are operational in nature (like the one above), some purely mathematical. Here I only want to point out two ways one can modify the above definition without changing the concept of an inertial system:

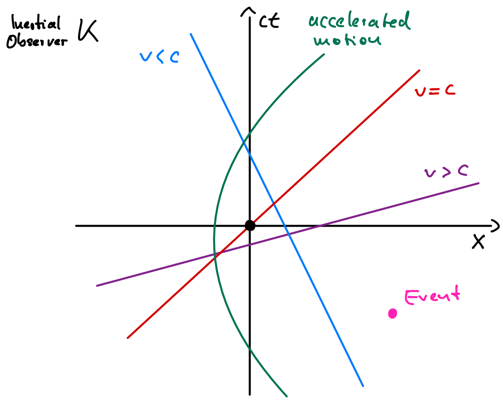
- The "inertial test" is crucial to the concept of an inertial frame. It rules out accelerated frames (both linear or rotating). An alternative to throwing test masses in different directions and recording their trajectories is to *repeat* the **ES** procedure periodically to *test* whether the clocks stay in sync. That is, to setup the coordinate system one synchronizes the clocks *once* (by recalibrating the clocks) and then repeats the procedure periodically to check whether the Einstein-synchronization condition remains valid ($\delta t = 0$ in our description above). As it will turn out in GENERAL RELATIVITY, your clocks will not stay in sync in frames that do not pass **IN** (and vice versa). This is essentially the definition given by Schutz [4].
- Instead of "hiding" the law of inertia in the synchronization of clocks, one can do a somewhat reverse modification and "hide" the synchronization of clocks in (an extension of) the law of inertia. To this end one extends the "inertial test" by a second class of tests/experiments, namely:

IN* Two identical particles that are initially adjacent and at rest, and then interact to repel each other, fly apart with the same velocity in opposite directions. (⁂ *Isotropy of Inertia)*

This statement about the isotropy of inertia implies an operational definition of simultaneity that is (empirically) equivalent to **ES**: You synchronize your clocks *such that* **IN*** is satisfied, for example by performing the experiment described by **IN*** equidistant between two clocks. When the particles reach the clocks, you reset both to $t = 0$. In this synchronization **IN*** is satisfied *by construction*; experiments show that clocks synchronized in this way are also synchronized according to **ES** (and vice versa).

4 | ⁂ *Spacetime diagram*

:= Data structure that encodes the collected data of an inertial coordinate system K:



- Often we draw only one dimension of space for the sake of simplicity.

- Because it will prove useful later, we measure time in units of length by multiplying t with the speed of light c. The choice of c is arbitrary at this point.

Notation: $\triangleleft$ Two inertial systems K and K':

We use the following shorthand notations to refer to the coordinates of events in the spacetime diagrams of K and K', respectively:

$$(t,\vec{x})_K \equiv (x)_K \equiv x \equiv (t,\vec{x}) \quad \text{and} \quad (t',\vec{x}')_{K'} \equiv \left(x'\right)_{K'} \equiv x' \equiv (t',\vec{x}') \tag{1.6}$$

When it is clear to which inertial system the coordinates belong we drop the subscripts K and K'.

↓ Interlude: Reconstructing spacetime diagrams from $\mathcal{E}$

If you are given the set $\mathcal{E}$ of events you can reconstruct the spacetime diagram of an inertial system K by looking in each coincidence class $E \in \mathcal{E}$ for the clock event $(t,\vec{x})_K \in E$. You then place E (or some sub-event you are interested in) graphically at the coordinate $(t,\vec{x})$ on a sheet of paper. The resulting picture is the spacetime diagram of K. In a another inertial system K' the events are arranged differently because different clock events $(t',\vec{x}')_{K'} \in E$ and hence coordinates $(t',\vec{x}')$ are used to draw the spacetime diagram. How $(t,\vec{x})$ and $(t',\vec{x}')$ are related is unclear at this point.

5 | Empirical facts:

The following facts cannot be bootstrapped from logical thinking alone. They are facts about our physical reality that we have strong experimental evidence for.

- Inertial systems exist (at least in some approximation).

  Examples would be an unaccelerated spaceship floating far away from the solar system or the interior of the international space station (if you do not measure too precisely). In SPECIAL RELATIVITY we assume that these systems can be extended to encompass all of spacetime.

- Constructing inertial systems (of arbitrary size) is not possible everywhere.

  → GENERAL RELATIVITY

  We will find in our discussion of GENERAL RELATIVITY that in a gravitational field the construction of inertial systems is only possible locally. For example: If you extend the ISS inertial system rigidly beyond the ISS itself, at some point you will find the trajectories of free particles to deviate from straight lines due to the inhomogeneity of the gravitational field. We will also see that the synchronization procedure used to calibrate the clocks fails in gravitational fields (you cannot keep your clocks in sync). For our discussion of SPECIAL RELATIVITY we ignore this and assume that our inertial systems cover all of spacetime.

6 | Relations between inertial systems:

i | There are three *straightforward* ways to construct a new inertial system K' from a given one K. They have in common that the two observers do *not* move with respect to one another so that pairs of clocks from K and K' spatially coincide for all times (this implies in particular that you can check that these pairs of clock run at the same rate):

(1) Translation in time by $s \in \mathbb{R}$ (→ 1 parameter)

Procedure:
Duplicate all clocks & rods in place. Label the new clocks with K' and the old position labels. Shift the reading of all clocks by a constant value $-s$:

$$(t',\vec{x}')_{K'} \sim (t,\vec{x})_K \quad \text{with} \quad t' = t - s \quad \text{and} \quad \vec{x}' = \vec{x}\,. \tag{1.7}$$

It is easy to see that this modification does not invalidate **ES**, **OC** or **IN**. In particular, the Einstein synchronization condition Eq. (1.2) remains valid:

$$t_O = \tfrac{1}{2}\left(t_A + \tilde{t}_A\right) \quad \Leftrightarrow \quad (t_O - s) = \tfrac{1}{2}\left[(t_A - s) + (\tilde{t}_A - s)\right]. \tag{1.8}$$

How to check from K:
At $(t)_K = 0$ the reading of the origin clock of K' is shifted by $-s \in \mathbb{R}$.

(2) Translation in space by $\vec{b} \in \mathbb{R}^3$ (→ 3 parameters)

Procedure:
Duplicate all clocks & rods and translate the whole lattice by $\vec{b}$ (since all clocks are type-identical, you can also simply modify the position labels without moving anything). Label the new clocks with K' and keep their synchronization:

$$(t', \vec{x}')_{K'} \sim (t, \vec{x})_K \quad \text{with} \quad t' = t \quad \text{and} \quad \vec{x}' = \vec{x} - \vec{b}. \tag{1.9}$$

¡! If you move the lattice K' in direction $\vec{b}$, the origin clock of K with position label $\vec{x} = \vec{0}$ will spatially coincide with a clock of K' with position label translated in the *opposite* direction, namely $-\vec{b}$. The same happens for rotations (→ *below*) and translations in time (← *above*).

It is easy to see that this modification does not invalidate **ES**, **OC** or **IN**. In particular, distances can still be computed with Eq. (1.5) since

$$d(\vec{x}, \vec{y}) = d(\vec{x} - \vec{b}, \vec{y} - \vec{b}) \quad \text{for } \vec{b} \in \mathbb{R}^3. \tag{1.10}$$

How to check from K:
At $(t)_K = 0$ the origin of K' is translated by $\vec{b} \in \mathbb{R}^3$ wrt. the origin of K.

(3) Rotation in space by $R \in \mathrm{SO}(3)$ (→ 3 parameters)

Procedure:
Duplicate all clocks & rods and rotate the whole lattice by the axis and angle defined by the rotation matrix R (since all clocks are type-identical, you can again simply modify the position labels without moving anything). Label the new clocks with K' and keep their synchronization:

$$(t', \vec{x}')_{K'} \sim (t, \vec{x})_K \quad \text{with} \quad t' = t \quad \text{and} \quad \vec{x}' = R^{-1}\vec{x}. \tag{1.11}$$

It is easy to see that this modification does not invalidate **ES**, **OC** or **IN**. In particular, distances can still be computed with Eq. (1.5) since

$$d(\vec{x}, \vec{y}) = d(R^{-1}\vec{x}, R^{-1}\vec{y}) \quad \text{for } R^{-1} \in \mathrm{SO}(3). \tag{1.12}$$

How to check from K:
The spatial axes of K' are rotated by $R \in \mathrm{SO}(3)$ wrt. the spatial axes of K.

¡! You can add spatial *reflections* to these transformations (↓ *improper rotations*), i.e., $R \in \mathrm{O}(3)$ instead of $R \in \mathrm{SO}(3)$. In our discussions we will omit these and only comment on them where necessary.

The combination of spatial rotations (proper and improper, i.e., including reflections) and spatial translations form the ↑ *Euclidean group* $\mathrm{E}(3) = \mathrm{ISO}(3)$.

However, experiments (and everyday experience) tell us that there is a fourth possibility how two inertial systems can be related:

Empirical fact:

(4) Uniform linear motion (⁂ *Boost*) by $\vec{v} \in \mathbb{R}^3$ (→ 3 parameters)

You experience this fact whenever you have a very smooth flight: If you don't look out the window (and cover your ears) everything behaves just as if the airplane were standing still on the ground; there is no evidence that you move with several hundred kilometers per hour relative to the ground.

How to check from K:
The origin of K' moves with constant velocity $(\vec{v})_K = \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)_K \in \mathbb{R}^3$.

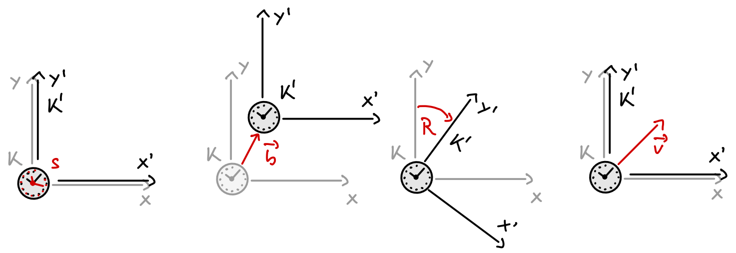
Note that just from this observation one cannot distinguish between a *pure boost* and a boost combined with a spatial rotation of the axes (because one probes only for the trajectory of a single point). We will → *later* be more precise about this distinction.

¡! We cannot write down the coordinate transformations for this relation (yet). The fundamental difference to (1)-(3) is that now the clocks of K' move wrt. the clocks of K. We cannot interpret this as a simple relabeling of fixed clocks. We cannot even be sure that the K- and K'-clocks "run at the same rate" (even if they are type-identical) because to check this we would have to compare the reading of a pair of clocks (one in K and one in K') at *two* consecutive points in time. To do this, however, the two clocks must be at the same place (remember that we can only observe coincidences!). But this is not possible: Since the two frames move uniformly, two clocks can never meet twice! As it will turn out, it is this relation (4) [and its concatenations with (1)-(3)] that harbors the essence of SPECIAL RELATIVITY.

ii | *Empirical fact:* The relations (1)-(4) are exhaustive.

With this we mean that whenever you encounter two inertial systems K and K' (i.e., both observers certify that they satisfy our definition of an inertial system, in particular, the "Inertial Test" **IN**), then you will find that the relation between the two is one of the four relations (1)-(4) or a combination of them.

→ The relation of two inertial systems K and K' is given by 10 parameters:

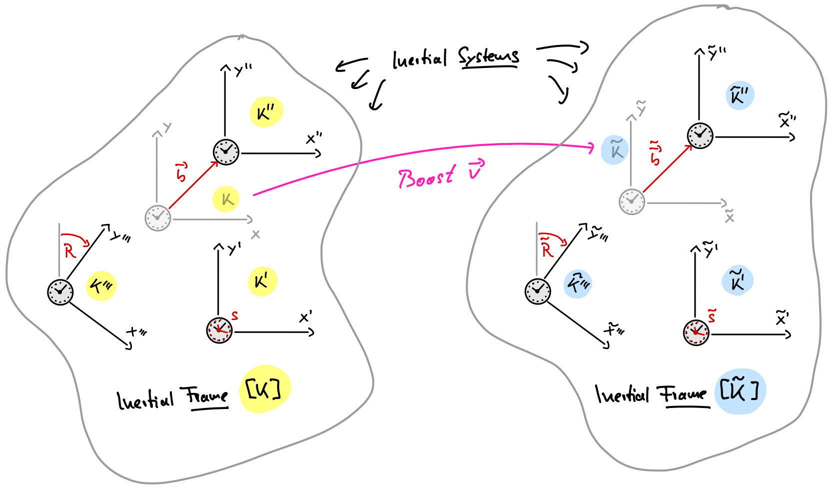


Note that all these relations can be operationally defined and measured within the frame K.

¡! The first three sketches can be taken at face value: For example, a translation in time really corresponds to the situation where all clocks are shifted by s an all spatial labels (in particular the axes) remain unaffected. However, for the boost (the last sketch on the right) we do *not* know (yet) how the coordinates transform (neither time nor space) except that the origin clock of K' follows a trajectory in K with uniform velocity $\vec{v}$. This implies that you should *not* take the sketch for a boost at face value: For example, we do not know whether the axes remain parallel as suggested by the sketch (spoiler: in general they will not).

iii | Since the transformations (1)-(3) do not change the state of motion of the observer (and can therefore be interpreted as a simple relabeling of the position labels and clock readings), it makes sense to collect all inertial frames K that can be connected in this way into an equivalence class $[K]$ which we call …

⁂ *Inertial frame* := Equivalence class $[K]$ of all inertial coordinate systems K related by spacetime translations and spatial rotations.



Inertial *frames* $[K]$ therefore correspond to the physical notion of a "state of motion." Physically, an inertial *frame* corresponds to the class of all freely moving particles in the universe that are mutually at rest. Given such a "state of motion" (e.g., by declaring one of the particles as reference point), you can then construct various Cartesian coordinate systems (e.g., using said reference particle as your origin) to describe events; these are the inertial *systems* that make up the equivalence class $[K]$.

iv | *Notation:*

We denote these relations between two inertial systems with the following shorthand notations:

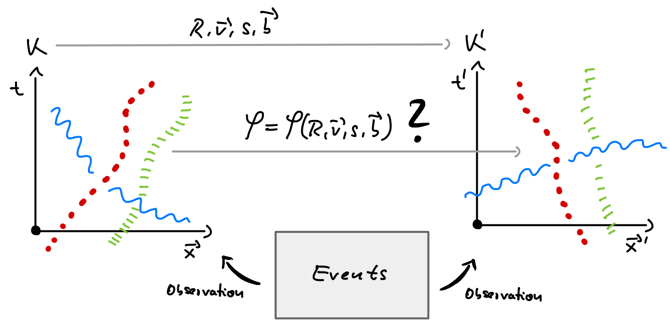
$$K \xrightarrow{R,\vec{v},s,\vec{b}} K',\ \ K \xrightarrow{R,\vec{v}} K',\ \ K \xrightarrow{\vec{v}} K',\ \ K \xrightarrow{v_x} K' \tag{1.13}$$

From left to right the relations become increasingly specialized.

¡! These relations are *not* symmetric (as indicated by the arrow). For example, $K \xrightarrow{v_x} K'$ specifies the situation where the (origin of) system K' moves with velocity v_x in x-direction *as measured in system K*.

v | Coordinate transformations:

$\triangleleft$ Two descriptions of the *same* events:



→ Transformation between these descriptions?

$$\boxed{\varphi(K \to K') :\ (t, \vec{x})_K \mapsto (t', \vec{x}')_{K'} \quad \text{⁂ } \textit{Coordinate transformation}}$$

Finding the functional form of φ (for the non-trivial case $\vec{v} \neq 0$) will be our main goal and central result of this chapter. However, before we can tackle this problem, we first have to introduce a few more concepts.

↓ Interlude: Relative information

We called the data in $\mathcal{E}$ *absolute* because all observers agree on the coincidence of events. However, this data cannot include arbitrary statements, e.g., the event "the particle has velocity $\vec{v}$" cannot be part of $\mathcal{E}$ because we know from experience that different observers in general do not agree on the velocity of an object. However, following Einstein, we postulated that coincidences are all we can ever observe; thus all there is to know must be encoded in $\mathcal{E}$! How is this consistent with the fact that velocities (for example) cannot show up in $\mathcal{E}$?

To understand this, it is instructive to think about quantities that can be *derived* from the absolute data in $\mathcal{E}$ by means of prescribed *algorithms*. An algorithm $\mathcal{A}$ is simply a program using data from $\mathcal{E}$ to compute other data (it can use potentially multiple events $E_1, E_2, \dots, E_N \in \mathcal{E}$ to do so). Furthermore, we allow the algorithm to take the label of an inertial system $K \in \mathcal{I}$ as input:

$$\mathcal{A} : \mathcal{E}^N \times \mathcal{I} \to \text{Output data} \tag{1.14}$$

As a constraint, we require that the algorithm *must not* use any (static) labels $A, B, \dots \in \mathcal{I}$ of inertial systems. The only reference to a frame it can use is the *variable* K. This somewhat arbitrary sounding restriction formalizes the notion that there are no inertial systems that are "special". Since all inertial systems must be treated equal, the algorithm cannot refer to any specific frame. (This → principle of relativity will take the center stage later and turns out to be crucial for the derivation of the transformation φ.)

Let us now contrive two algorithms to compute two quantities that are clearly physically relevant but are *not* contained in $\mathcal{E}$:

- Example 1: Velocity

  First think about how you would measure the velocity of a particle in the lab: You would detect the particle at two different (but nearby) locations, measure the time it requires to get from one to the other, and then compute the difference quotient of distance traveled by the time needed. Note that there is no way to measure the velocity at one point in space and time; you always need two points!

  To formalize this, consider two events E_1 and E_2 that both contain the sub-event "particle detected". The algorithm $\mathcal{V}(E_1, E_2; K)$ computes the (average) velocity between the two events as follows:

  1. `Select the event` $(t_1, \vec{x}_1)_K \in E_1$.
  2. `Select the event` $(t_2, \vec{x}_2)_K \in E_2$.
  3. `Compute and return the value` $\vec{v} = \frac{\vec{x}_2 - \vec{x}_1}{t_2 - t_1}$.

  It is important that this algorithm can be used *without modifications* by all observers $K \in \mathcal{I}$. To do so, each observer K plugs into $\mathcal{V}$ the two events (which are objective) an its *own* label K (since this is the only non-random choice possible).

  But then two *different* observers K and K' will pick *different* coordinates $(t_i, \vec{x}_i)$ (measured by different clocks) to compute their value of $\vec{v}$, which obviously can yield different outcomes (as expected for velocities). Note that for the velocities to be really different it must be $[K'] \neq [K]$, i.e., the two inertial *systems* must belong to different *frames*.

- Example 2: Duration & Simultaneity

  A very natural question is how much time passed between two events E_1 and E_2. The formal prescription how to answer this question is given by the algorithm $\mathcal{T}(E_1, E_2; K)$:

  1. `Select the event` $(t_1, \vec{x}_1)_K \in E_1$.
  2. `Select the event` $(t_2, \vec{x}_2)_K \in E_2$.

3. `Compute and return the value` $\Delta t = t_2 - t_1$.

For the very same reason as for the velocity algorithm above, the return value of course will depend on the chosen "clock events" $(t_i, \vec{x}_i)$. And so for the very same reason that velocities can be observer-dependent, time intervals can be as well. Since we define "simultaneity" as the property $\Delta t = 0$, this possibility for observer-dependent results directly transfers to our notion of simultaneity!

Note that we did not make *quantitative* statements about the outcomes for different observers. We neither showed *how* velocities depend on the frame nor whether simultaneity really *is* relative. (It could just be the case that in our world $t_2 - t_1$ always equals $t_2' - t_1'$ for a fixed event.) This depends on the actual numbers of the coordinates. Such statements therefore require quantitative statements about the relation of $(t, \vec{x})_K \in E$ and $(t', \vec{x}')_{K'} \in E$, which we do not know at this point (this is exactly the question for the functional form of the coordinate transformation φ).

However, what we did show is the *possibility* that simultaneity is relative, just as we already expect velocities to be! So when we later find the correct transformation φ and (surprise!) that indeed simultaneity is not an observer independent fact, you should not be surprised.

Question: Can the values of the electric and magnetic fields $\vec{E}$ and $\vec{B}$ be included in $\mathcal{E}$? If not, can you think of an algorithm that determines the electric and magnetic fields $\vec{E}$ and $\vec{B}$ using only coincidence data available in $\mathcal{E}$? Do you expect the electromagnetic field to be observer-dependent?

7 | Henceforth:

> Unless noted otherwise, all frames will be *inertial* (with Cartesian coordinates).
> → We will (almost exclusively) work with *inertial coordinate systems*.

We use the concept of inertial systems because to describe physics by equations, coordinates are a useful tool. As it turns out, Cartesian coordinates allow for particularly simple equations (at least if space is Euclidean). So our concept of inertial systems as defined above is the most useful one.

8 | Physical Models:

Let us fix a bit of terminology:

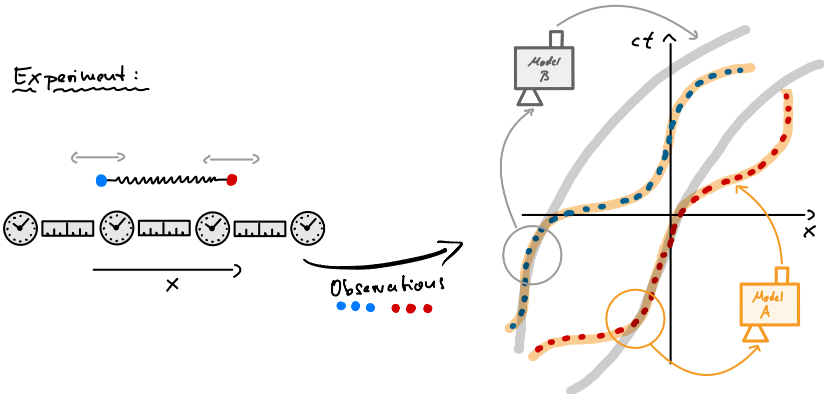
- ⁂ *(Physical) laws* are ontic features of reality (↑ *scientific realism*).

  Physical laws can only be *discovered*; they can neither be invented nor modified.

- ⁂ *(Physical) models* are algorithms used to describe reality.
  These algorithms are typically encoded in the language of mathematics.

  Physical models are *invented* and can be *modified*; I will use the terms *model* and *theory* interchangeably.

¡! These definitions are by no means conventional and you will find many variations in the literature. For the following discussion, it is only important that the terms we use have precise meaning.



¡! The validity of models cannot be *proven*; we can only gradually increase our trust in a model by repeated observations (experiments) – or reject it as invalid by demonstrating that its predictions contradict reality (↑ *Karl Popper*). Note that models might describe reality only approximately and in specific parameter regimes and still be useful.

You may dismiss this focus on terminology as "philosophical banter." Conceptual clarity, however, is absolutely crucial for science – in particular for RELATIVITY. Whenever there is confusion in physics, it is often rooted in the conceptual fuzziness of our thinking.

1.2. Galilei's principle of relativity

9 | Example: Newtonian mechanics

i | Definition of the model:

- $\sphericalangle$ Closed system of N massive particles with masses m_i and positions $\vec{x}_i$.
- $\sphericalangle$ Force exerted by k on i:

$$F_{k\to i}(\vec{x}_k - \vec{x}_i) = (\vec{x}_k - \vec{x}_i)\, f_{k\leftrightarrow i}(|\vec{x}_k - \vec{x}_i|) \tag{1.15}$$

It is $f_{k\leftrightarrow i} = f_{i\leftrightarrow k}$ and therefore $F_{k\to i}(\vec{x}_k - \vec{x}_i) = -F_{i\to k}(\vec{x}_i - \vec{x}_k)$.

→ *Newtonian equations of motion* (in some inertial system K):

$$m_i \frac{\mathrm{d}^2 \vec{X}_i}{\mathrm{d}t^2} = \sum_{k\neq i} \vec{F}_{k\to i}(\vec{X}_k - \vec{X}_i) \tag{1.16}$$

We denote with $\vec{X}_i \equiv \vec{X}_i(t)$ coordinate-valued *functions*; i.e., $\vec{x}_i = \vec{X}_i(t)$ determines a spatial point $\vec{x}_i$ for given t.

Remember: This model fully implements "Newton's laws of motion":

1. Lex prima:

> *A body remains at rest, or in motion at a constant speed in a straight line, unless acted upon by a force.*

This is the ↓ *principle of inertia*. It is part of the definition of the concept of a Newtonian force used in Eq. (1.16). Note that it is *not* a consequence of Eq. (1.16) for $F_{k\to i} \equiv 0$. It rather defines (together with the lex tertia below) the frames and coordinate systems (← *inertial systems*) in which Eq. (1.16) is valid (recall `IN`).

2. Lex secunda:

   When a body is acted upon by a net force, the body's acceleration multiplied by its mass is equal to the net force.

   This is just the functional form of Eq. (1.16) in words.

3. Lex tertia:

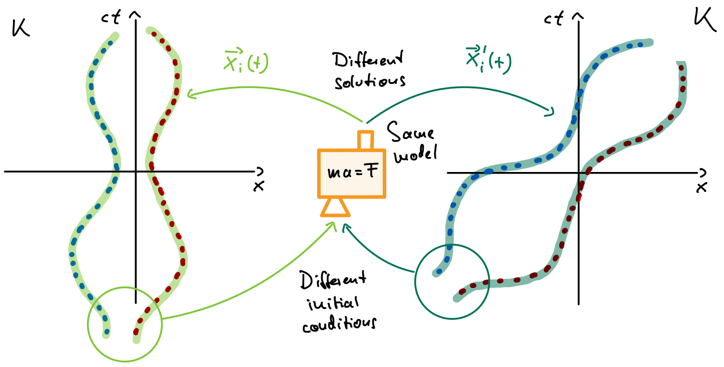
   If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

   This is guaranteed by the property $F_{k\to i} = -F_{i\to k}$ of the forces. Together with the lex secunda this is an expression of momentum conservation. For two particles:

$$m_1\frac{\mathrm{d}v_1}{\mathrm{d}t} + m_2\frac{\mathrm{d}v_2}{\mathrm{d}t} = \frac{\mathrm{d}p_1}{\mathrm{d}t} + \frac{\mathrm{d}p_2}{\mathrm{d}t} = F_{2\to 1} + F_{1\to 2} = 0 \tag{1.17}$$

   This implies in particular that two identical particles ($m_1 = m_2$) that are both at rest at $t = 0$ must obey $v_1(t) = -v_2(t)$ for all times (recall `IN*`).

ii | Application of the model:



As a working hypothesis, let us assume that the model Eq. (1.16) describes the dynamics of massive particles perfectly (from experience we know that there are at least regimes where it is good enough for all practical purposes).

iii | Symmetries of Newtonian mechanics:

To understand the solution space of Eq. (1.16) better, it is instructive to study transformations that map solutions to other solutions.

a | ⁂ *Galilei transformations*:

We define the following coordinate transformation:

$$\boxed{G\;:\mathbb{R}^4 \to \mathbb{R}^4\;:\quad \begin{cases} t' = t + s \\ \vec{x}' = R\vec{x} + \vec{v}t + \vec{b} \end{cases}} \tag{1.18}$$

A Galilei transformation G is characterized by 10 real parameters:

- $s \in \mathbb{R}$: Time translation (1 parameter)
- $\vec{b} \in \mathbb{R}^3$: Space translation (3 parameters)
- $\vec{v} \in \mathbb{R}^3$: Boost (3 parameters)
- $R \in \mathrm{SO}(3)$: Spatial rotation (3 parameters; rotation axis: 2, rotation angle: 1)

The set of all transformations forms (the matrix representation of) a group:

$$\mathscr{G}_+^\uparrow = \{G(R,\vec{v},s,\vec{b})\} \quad ⁂ \textit{ Proper orthochronous Galilei group} \tag{1.19}$$

with group multiplication

$$G_3 = G_1 \cdot G_2 = G(\underbrace{R_1R_2}_{R_3}, \underbrace{R_1\vec{v}_2 + \vec{v}_1}_{v_3}, \underbrace{s_1 + s_2}_{s_3}, \underbrace{R_1\vec{b}_2 + \vec{v}_1 s_2 + \vec{b}_1}_{\vec{b}_3}) \tag{1.20}$$

You derive this multiplication in ➔ Problemset 1 and show that the group axioms are indeed satisfied.

As a special case, the multiplication yields the rule for addition of velocities in Newtonian mechanics:

$$G(\mathbb{1},\vec{v}_1,0,\vec{0}) \cdot G(\mathbb{1},\vec{v}_2,0,\vec{0}) = G(\mathbb{1},\underbrace{\vec{v}_1+\vec{v}_2}_{\vec{v}_3},0,\vec{0}) \tag{1.21}$$

The full Galilei group is generated by the proper orthochronous transformations together with space and time inversion:

$$\mathscr{G} = \langle \mathscr{G}_+^\uparrow \cup \{P,T\}\rangle \quad ⁂ \textit{ Galilei group} \tag{1.22a}$$

$$P:\; (t,\vec{x}) \mapsto (t,-\vec{x}) \quad \text{Space inversion (parity)} \tag{1.22b}$$

$$T:\; (t,\vec{x}) \mapsto (-t,\vec{x}) \quad \text{Time inversion} \tag{1.22c}$$

b | Galilei covariance & Form-invariance:

Details: ➔ Problemset 1

⊲ Coordinate transformation Eq. (1.18)

We express the total differential and the trajectory in the new coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}t'}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t'} = \frac{\mathrm{d}}{\mathrm{d}t'} \tag{1.23}$$

and

$$\vec{X}_i'(t') = R\vec{X}_i(t) + \vec{v}t + \vec{b} = R\vec{X}_i(t'-s) + \vec{v}(t'-s) + \vec{b} \tag{1.24a}$$

$$\Leftrightarrow \quad \vec{X}_i(t) = R^{-1}\left[\vec{X}_i'(t') - \vec{v}(t'-s) - \vec{b}\right] \tag{1.24b}$$

Thus the left-hand side of the Newtonian equation of motion Eq. (1.16) reads in new coordinates:

$$m_i\frac{\mathrm{d}^2\vec{X}_i(t)}{\mathrm{d}t^2} = m_i\frac{\mathrm{d}^2}{\mathrm{d}t'^2}R^{-1}\left[\vec{X}_i'(t') - \vec{v}(t'-s) - \vec{b}\right] = R^{-1}\,m_i\frac{\mathrm{d}^2\vec{X}_i'(t')}{\mathrm{d}t'^2} \tag{1.25}$$

Note that the quantity $m_i\frac{\mathrm{d}^2}{\mathrm{d}t^2}\vec{X}_i(t)$ is not *invariant*; it transforms with an $R^{-1} \in \mathrm{SO}(3)$.

And the right-hand side:

$$\sum_{k\neq i}\vec{F}_{k\to i}(\vec{X}_k(t) - \vec{X}_i(t)) = R^{-1}\sum_{k\neq i}\vec{F}_{k\to i}(\vec{X}_k'(t') - \vec{X}_i'(t')) \tag{1.26a}$$

Here we used the form of the force Eq. (1.15), that $\vec{X}_k(t) - \vec{X}_i(t) = R^{-1}[\vec{X}'_k(t') - \vec{X}'_i(t')]$ and $|\vec{X}_k(t) - \vec{X}_i(t)| = |\vec{X}'_k(t') - \vec{X}'_i(t')|$ because of $R \in \mathrm{SO}(3)$.

Note that the force on the right-hand side is not *invariant* either; luckily, it transforms with *the same* $R^{-1} \in \mathrm{SO}(3)$; it "co-varies" with the left-hand side!

In conclusion, Newton's equation of motion Eq. (1.16) reads in the new coordinates:

$$\underbrace{R^{-1}\, m_i \frac{\mathrm{d}^2 X'_i(t')}{\mathrm{d}t'^2} = R^{-1} \sum_{k\neq i} \vec{F}_{k\to i}(\vec{X}'_k(t') - \vec{X}'_i(t'))}_{\to\ \textit{Covariance}} \tag{1.27a}$$

$$\overset{\times R}{\Leftrightarrow} \quad \underbrace{m_i \frac{\mathrm{d}^2 X'_i(t')}{\mathrm{d}t'^2} = \sum_{k\neq i} \vec{F}_{k\to i}(\vec{X}'_k(t') - \vec{X}'_i(t'))}_{\to\ \textit{Form-invariance}} \tag{1.27b}$$

(You can easily check that this holds for P and T as well.)

$\rightarrow$

> Newton's EOMs Eq. (1.16) are *form-invariant* under Galilei transformations.
>
> *Or:* Newton's EOMs Eq. (1.16) are Galilei-*covariant*.

↓ Interlude: Nomenclature

Let X be some group of coordinate transformations (here: $X = \mathcal{G}$ the Galilei group).

- A *quantity* is called X*-invariant* if it does not change under the coordinate transformation. Such quantities are called X*-scalars*.

  An example is the mass m in Eq. (1.16) (which is also constant).
- A *quantity* is called X*-covariant* if it transforms under some given representation of the X-group. If this representation is the trivial one (i.e., the quantity does not change at all) this particular X-covariant quantity is then also an X-scalar.

  An example of a Galilei-covariant (but not invariant) quantity is the force $\vec{F}_{k\to i}$ which transforms under a representation of $\mathcal{G}$.
- An *equation* is called X*-covariant* if the quantity on the left-hand side and on the right-hand side are X-covariant (under the same X-representation).

  An example is Newton's lex secunda Eq. (1.16) where $m_i \frac{\mathrm{d}^2}{\mathrm{d}t^2} x_i(t)$ transforms in the same (non-trivial) representation as $\vec{F}_{k\to i}$.
- X-covariant *equations* have the feature that a X-transformation leaves them *form-invariant*, i.e., they "look the same" after X-transformations because their left- and right-hand side vary in the same way (they "co-vary"). Note that the quantities in a form-invariant equation do not have to be *in*variant.

  An example is again Eq. (1.16) as we just showed. Note that $\vec{x}'_i(t')$ and $\vec{x}_i(t)$ are *different* vectors such that the two sides of the equation as not *in*variant (but *co*variant).

↓Lecture 3 [31.10.23]

c | Active symmetries:

There is something additional and particularly useful to be learned from the coordinate transformation above. We showed:

$$\text{If}\quad \vec{X}_i(t)\quad \text{satisfies}\quad m_i\frac{\mathrm{d}^2\vec{X}_i(t)}{\mathrm{d}t^2}=\sum_{k\neq i}\vec{F}_{k\to i}(\vec{X}_k(t)-\vec{X}_i(t)) \tag{1.28a}$$

$$\text{then}\quad \vec{X}_i'(t')\quad \text{satisfies}\quad m_i\frac{\mathrm{d}^2\vec{X}_i'(t')}{\mathrm{d}t'^2}=\sum_{k\neq i}\vec{F}_{k\to i}(\vec{X}_k'(t')-\vec{X}_i'(t')) \tag{1.28b}$$

But t' in the lower statement is just a dummy variable that can be renamed to whatever we want:

$$\text{If}\quad \vec{X}_i(t)\quad \text{satisfies}\quad m_i\frac{\mathrm{d}^2\vec{X}_i(t)}{\mathrm{d}t^2}=\sum_{k\neq i}\vec{F}_{k\to i}(\vec{X}_k(t)-\vec{X}_i(t)) \tag{1.29a}$$

$$\text{then}\quad \vec{X}_i'(t)\quad \text{satisfies}\quad m_i\frac{\mathrm{d}^2\vec{X}_i'(t)}{\mathrm{d}t^2}=\sum_{k\neq i}\vec{F}_{k\to i}(\vec{X}_k'(t)-\vec{X}_i'(t)) \tag{1.29b}$$

Use colors to highlight the changes.

→ $\vec{X}_i'(t)=R\vec{X}_i(t-s)+\vec{v}(t-s)+\vec{b}$ is a *new solution* of Eq. (1.16)!

Note that for $s=0$ it is $\vec{X}_i'(0)=R\vec{X}_i(0)+\vec{b}$ and $\dot{\vec{X}}_i(0)=R\dot{\vec{X}}_i(0)+\vec{v}$, i.e., the solution $\vec{X}_i'(t)$ satisfies different initial conditions.

→ We say:

The Galilei group $\mathcal{G}$ is an ⁂ *invariance group* or an *(active) symmetry* of Eq. (1.16).

↓ Interlude: Active and passive transformations

It is important to understand the conceptual difference between the two last points:

- In the previous step we took a specific trajectory (solution of Newton's equation) and expressed it in different coordinates. We then found that the differential equation obeyed by *the same physical trajectory* in these new coordinates "looks the same" as in the old coordinates. We called this peculiar feature of the differential equation "Galilei-covariance" or "form-invariance". This type of a transformation is called *passive* because we keep the physics the same and only change our description of it.
- In the last step, we have shown that there is a dual interpretation to this: If a differential equation is form-invariant under a coordinate transformation, then we can exploit this fact to construct *new solutions* from given solutions (in the same coordinate system!). This type of transformation is called *active* because we keep the coordinate frame fixed and actually *change the physics*. You can therefore think of active transformations/symmetries as "algorithms" to construct new solutions of a differential equation (a quite useful feature since solving differential equations is often tedious).

10 | Galilean relativity:

i | Remember:

The law of inertia holds (by definition) in all inertial systems.

→ The "inertial test" **IN** cannot be used to distinguish inertial systems.

This is a tautological statement because we *define* inertial systems in this way!

Empirical fact:

Every mechanical experiment (not just the "inertial test") yields the same result in all inertial systems.

This is not a tautology but an empirically tested feature of reality.

This motivates the following postulate (first given by Galileo Galilei):

> **§ Postulate: Galilei's principle of Relativity GR**
>
> No mechanical experiment can distinguish between inertial systems.

¡! In this formulation, **GR** encodes a (so far uncontested) empirical fact. In particular, it does neither refer nor rely on (the validity of) any *physical model*, e.g., Newtonian mechanics. As such we should expect that it survives our transition to SPECIAL RELATIVITY.

Here is a more operational formulation of **GR**: You describe a detailed experimental procedure using equipment governed by mechanics (springs, pendula, masses, …) that can be performed in a closed (but otherwise perfectly equipped) laboratory. Then you copy these instructions without modifications and hand them to scientists with labs in different inertial systems. They all perform your instructions and get some results (e.g. the final velocities of a complicated contraption of pendula). When they report back to you, their results will all be identical. This is the essence of **GR**.

ii | In the language of *models* that describe the mechanical laws faithfully, **GR** can be reformulated:

> **§ Postulate: Galilei's principle of Relativity GR'**
>
> The equations that describe mechanical phenomena *faithfully* have the same form in all inertial systems.

If this would not be the case you could distinguish between different inertial systems by checking which formula you have to use to describe your observations. Imagine a rotating (non-inertial) frame where you have to use a modified version of Newton's EOMs (that include additional terms for the Coriolis force) to describe your observations.

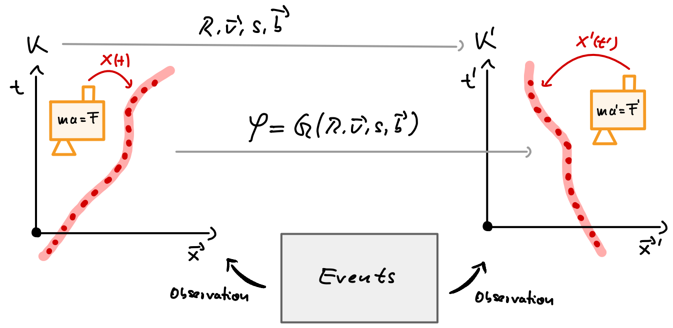
Note that "the same form" actually means that the models are *functionally equivalent* (have the same solution space). Functional equivalence is equivalent to the *possibility* to formulate the model (= equation of motion) in the same form.

iii | *Under the assumption* (!) that Newtonian physics (in particular Eq. (1.16)) describes mechanical phenomena *faithfully*, this implies:

Newton's equations of motion have the same form in all inertial systems.

¡! This statement is *not* equivalent to `GR` or `GR'` as it relies on an independent empirical claim (namely the validity of Newton's equation as a model of mechanical phenomena).

We can now combine this claim with our (purely mathematical!) finding concerning the invariance group of Newton's equations:



→ Preliminary/Historical conclusion:

$$\boxed{\varphi(K \xrightarrow{R,\vec{v},s,\vec{b}} K') \overset{?}{=} G(R^{-1}, -\vec{v}, -s, -\vec{b}) \;\in \mathcal{G}}$$

Recall that rotating the coordinate *axes* by R makes the coordinates of fixed events rotate in the *opposite* direction R^{-1}; the same is true for the other transformations.

Since this is a course on RELATIVITY, we should be skeptical (like Einstein) and ask:

Is this true?

1.3. Einstein's principle of special relativity

11 | Mathematical fact:

The Maxwell equations of electrodynamics are *not* Galilei-covariant.

Proof: ➔ Problemset 1

Here for your (and my) convenience the Maxwell equations in vacuum (in cgs units):

$$\text{Gauss's law (electric):} \quad \nabla \cdot \boldsymbol{E} = 0 \tag{1.30a}$$

$$\text{Gauss's law (magnetic):} \quad \nabla \cdot \boldsymbol{B} = 0 \tag{1.30b}$$

$$\text{Law of induction:} \quad \nabla \times \boldsymbol{E} = -\frac{1}{c}\partial_t \boldsymbol{B} \tag{1.30c}$$

$$\text{Ampère's circuital law:} \quad \nabla \times \boldsymbol{B} = \frac{1}{c}\partial_t \boldsymbol{E} \tag{1.30d}$$

"Handwavy explanation" for the absence of Galilei symmetry:

The Maxwell equations imply the wave equation for both fields:

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right)\boldsymbol{X} = \boldsymbol{0} \quad \text{for } \boldsymbol{X} \in \{\boldsymbol{E}, \boldsymbol{B}\}\,. \tag{1.31}$$

Here the speed of light c plays the role of the phase and group velocity of the waves; i.e., all light signals propagate with c. Form-invariance under some coordinate transformation φ implies that the same light signal propagates with *the same velocity* c in all coordinate systems related by φ. This is clearly incompatible with the Galilean law for adding velocities (according to which a signal with velocity u'_x in frame K' propagates with velocity $u_x = u'_x + v_x$ in frame K if $K \xrightarrow{v_x} K'$).

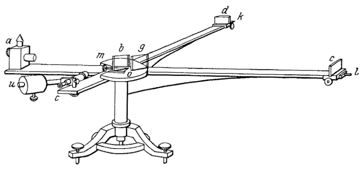
12 | The simplest escape from our predicament:

Maybe there is no relativity principle for electrodynamics?

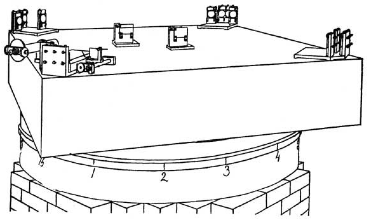
Reasoning: If we cling to the validity of Newtonian mechanics and Galilean relativity **GR**, we are forced to assume $\varphi = G$ as the transformation between inertial systems. Since the Maxwell equations are *not* form-invariant under these transformations, they look differently in different inertial systems. So there must be a (class of) designated inertial coordinate systems $[K_0]$ in which the Maxwell equations in the specific form Eq. (1.30) you've learned in your electrodynamics course are valid.

→ $[K_0]$ = Frame in which the *"luminiferous aether"* is at rest (?)

13 | Michelson Morley experiment (plots from [26, 27]):



Michelson's original setup (1881)



Michelson & Morley's improved setup (1887)

→ The speed of light is the same in all directions.

→ There is no "luminiferous aether" $[K_0]$.
(Or it is pulled along by earth – which contradicts the observed ↑ *aberration of light*.)

→ The speed of light c *cannot* be fixed wrt. some designated reference frame $[K_0]$.

→ *No experimental evidence* that the Maxwell equations *do not hold* in all inertial systems.

→ *Relativity principle* for electrodynamics?!

- Historical note:

  A. Einstein writes in a letter to F. G. Davenport (see Ref. [28]):

  > *[...] In my own development Michelson's result has not had a considerable influence. I even do not remember if I knew of it at all when I wrote my first paper on the subject (1905). The explanation is that I was, for general reasons, firmly convinced how this could be reconciled with our knowledge of electro-dynamics. One can therefore understand why in my personal struggle Michelson's experiment played no role or at least no decisive role.*

  → The Michelson Morley experiment did *not* kickstart SPECIAL RELATIVITY.

- Modern Michelson-Morley like tests of the isotropy of the speed of light achieve much higher precision than the original experiment. The authors of Refs. [29, 30], for example, report an upper bound of $\Delta c/c \sim 10^{-17}$ on potential anisotropies of the speed of light by rotating optical resonators.

14 | Two observations:

(1) *No* evidence that there is *no* relativity principle for electrodynamics.

(2) Why does Galilean relativity GR treat mechanics differently anyway?

Put differently: Why should mechanics, a branch of physics artificially created by human society, be different from any other branch of physics? This is not impossible, of course, but it certainly lacks simplicity! (To Galilei's defence: At his time "mechanics" was more or less identical to "physics".)

→ A. Einstein writes in §2 of Ref. [9] as his first postulate:

1. Die Gesetze, nach denen sich die Zustände der physikalischen Systeme ändern, sind unabhängig davon, auf welches von zwei relativ zueinander in gleichförmiger Translationsbewegung befindlichen Koordinatensystemen diese Zustandsänderungen bezogen werden.

We reformulate this into the following postulate:

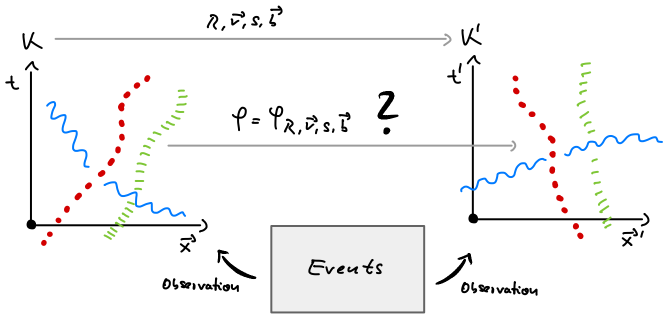
> **§ Postulate: (Einstein's principle of) Special Relativity SR**
>
> No ~~mechanical~~ experiment can distinguish between inertial systems.

Note the difference to Galilean relativity GR according to which no experiment *governed by classical mechanics* can distinguish between inertial systems. Einstein simply extended this idea to all of physics – no special treatment for mechanics!

¡! There are various names used in the literature to refer to SR. Here we call it the principle of *special* relativity, where the "special" refers to its restriction on *inertial systems* – as compared to the principle of *general* relativity in GENERAL RELATIVITY that refers to *all* frames (→ *later*). To emphasize its difference to Galilean relativity GR, some authors call SR the *universal* principle of relativity, where "universal" refers to its applicability on *all* laws of nature (not just the realm of classical mechanics).

15 | But now that there are more contenders (mechanics, electrodynamics, quantum mechanics) all of which must be invariant under the same transformation φ, we have to open the quest for φ again:

$$\boxed{\text{What is } \varphi?}$$



The differently colored/shaped trajectories symbolize phenomena of mechanics (red), electrodynamics (blue), and quantum mechanics (green). According to SR, *all* of them must be form-invariant under a common coordinate transformation φ.

¡! To reiterate: This is *not* a question about symmetry properties of equations or models! It is an experimentally testable fact about reality. There is only *one* correct φ and it is just as real as the three-dimensionality of space.

1.4. Transformations consistent with the relativity principle

Since this is a theory lecture, so we cannot do experiments. Let us therefore weaken the question slightly:

What is most general form of φ consistent with reasonable assumptions about reality?

§ Assumptions

- **SR** *Special Relativity*: There is no distinguished inertial system.
- **IS** *Isotropy:* There is no distinguished direction in space.
- **HO** *Homomgeneity:* There is no distinguished place in space or point in time.
- **CO** *Continuity:* φ is a continuous function (in the origin).

Something is "distinguished" if there exists an experiment that can be used to identify it unambiguously.

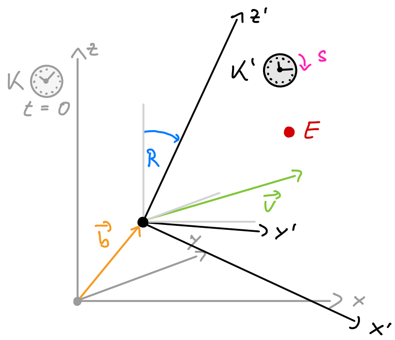
This derivation follows Straumann [8] with input from Schröder [1] and Pal [31].

Detailed calculations: ➔ Problemset 2

1 | Setup:

$\sphericalangle$ Two inertial systems $K \xrightarrow{R,\vec{v},s,\vec{b}} K'$.

$\sphericalangle$ Event $E \in \mathcal{E}$ with coordinates $x \equiv (t,\vec{x})_K \in E$ and $x' \equiv (t',\vec{x}')_{K'} \in E$:



We are interested in the transformation $\varphi \equiv \varphi_{R,\vec{v},s,\vec{b}}$ with

$$x' = \varphi(x)\,. \tag{1.32}$$

Note that **SR** forbids us to use the inertial system labels K or K' in the definition of φ! We can only use the relative parameters $(R,\vec{v},s,\vec{b})$ measured in K *wrt* K'.

2 | Affine structure:

Our first goal is to show that φ must be an affine map.

i | $\sphericalangle$ Event $\tilde{E} \in \mathcal{E}$ with coordinates $\tilde{x} = x + a$ in K for some shift $a \in \mathbb{R}^4$.

ii | Homogeneity **HO** $\rightarrow$

$$\varphi(x+a) - \varphi(x) \stackrel{!}{=} a'(\varphi, a) \tag{1.33}$$

$a'(\varphi, a)$: Shift in K' *independent* of x (this reflects homogeneity in space and time)

Imagine the right-hand side $a'(\varphi, a)$ where *not* independent of x. Then there would be an interval (say, a rod of spatial extend $\vec{a}$) that has the same length $\vec{a}$ in K no matter where it is located, but *variable* length $\vec{a}(\varphi, \vec{a}, \vec{x})$ in K' as a function of $\vec{x}$. The observer in K' can then use this "magic rod" to pinpoint absolute positions in space (the same argument works in time, then with a clock instead of a rod).

iii | For $x = 0$: $a'(\varphi, a) = \varphi(a) - \varphi(0) \rightarrow$

$$\varphi(x + a) = \varphi(x) + \varphi(a) - \varphi(0)\,. \tag{1.34}$$

iv | Let $\Psi(x) := \varphi(x) - \varphi(0) \rightarrow$

$$\Psi(x + a) = \Psi(x) + \Psi(a) \quad \text{and} \quad \Psi(0) = 0\,. \tag{1.35}$$

This would be satisfied if Ψ were *linear*! But we do not know this yet ...

v | Claim: $\Psi(x)$ continuous at $x = 0$ (follows from CO) $\Rightarrow$ Ψ is *linear*.

a | Eq. (1.35) $\rightarrow \Psi(nx) = n\Psi(x)$ for $n \in \mathbb{N}$ (show by induction!)

b | Eq. (1.35) $\rightarrow \Psi(-x) = -\Psi(x)$ (use $\Psi(0) = 0$) $\rightarrow \Psi(nx) = n\Psi(x)$ for $n \in \mathbb{Z}$

c | $\sphericalangle$ Rational number $r = \frac{m}{n}$, $m, n \in \mathbb{Z} \rightarrow$

$$r\Psi(x) = \frac{m}{n}\Psi(x) = \frac{1}{n}\Psi(mx) = \frac{1}{n}\Psi(nrx) = \frac{n}{n}\Psi(rx) = \Psi(rx)\,. \tag{1.36}$$

d | $\Psi(x)$ continuous at $x = 0 \xrightarrow{\text{Eq. (1.35)}} \Psi(x)$ continuous everywhere.
Show this using the definition of continuity, i.e., $\lim_{x\to 0} \Psi(x) = \Psi(0)$!

e | $r\Psi(x) = \Psi(rx)$ for $r \in \mathbb{Q} \xrightarrow{\Psi \text{ continuous}} r\Psi(x) = \Psi(rx)$ for $r \in \mathbb{R}$
Remember that real numbers are defined in terms of (equivalence classes of) limits of rational numbers, i.e., $\mathbb{Q}$ is dense in $\mathbb{R}$.

f | In conclusion:

$$\Psi(x + a) = \Psi(x) + \Psi(a) \quad \text{and} \quad \Psi(rx) = r\Psi(x) \tag{1.37}$$

$\rightarrow \Psi$ is *linear*.

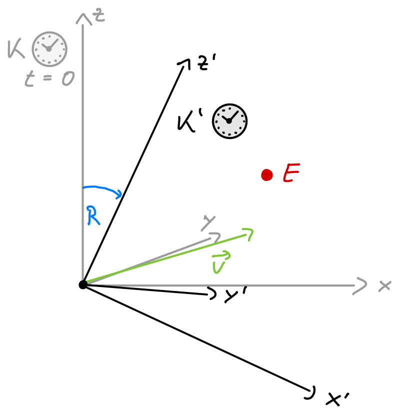
vi | If Ψ is linear, $\varphi(x) = \Psi(x) + \varphi(0)$ is *affine*:

$$\boxed{\varphi(x) = \Lambda x + a} \tag{1.38}$$

with $\Lambda = \Lambda(R, \vec{v}, s, \vec{b})$ a 4×4 matrix and $a = a(R, \vec{v}, s, \vec{b})$ a 4-dimensional vector.

3 | The spacetime translation a is simply $a = (-s, -\vec{b})$ [recall Eqs. (1.7) and (1.9)].
$\rightarrow \sphericalangle$ Homogeneous transformations ($a = 0$) in the following:

$$x' = \varphi(x) = \Lambda x\,. \tag{1.39}$$



4 | We already know from our discussion of inertial systems [recall Eq. (1.11)]:

Rotation group SO(3) must be part of the transformations φ with representation

$$x' = \Lambda_{R^{-1}} x \quad \text{with} \quad \Lambda_R := \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right) \quad \text{where} \quad R \in \mathrm{SO}(3)\,. \tag{1.40}$$

This is just a fancy way to rewrite Eq. (1.11).

5 | ⁂ *Pure boost* $K \xrightarrow{\mathbb{1},\vec{v},0,\vec{0}} K'$:

i | $\sphericalangle\,(t)_K = 0 \to \vec{x}\,' = \mathcal{M}\vec{x}$ for an invertible matrix $\mathcal{M} \in \mathbb{R}^{3\times 3}$:
This is the most general transformation for the position labels of the K and K'-clocks at $t = 0$. Note that we make no statements on the times t' displayed by the K'-clocks at $t = 0$.

$$\mathcal{M} = R_1 D R_2 = R_1 D R_1^T R = MR \tag{1.41}$$

with $R \in \mathrm{O}(3)$ and $M^T = M$.

This follows from the ↓ *singular value decomposition* of real matrices with $R_1, R_2 \in \mathrm{O}(3)$ and D a diagonal matrix.

ii | With spatial rotations Eq. (1.40) we can always transform the K-coordinates by $\vec{x} \mapsto R^{-1}\vec{x}$ such that $\vec{x}\,' = \mathcal{M}\vec{x} = M\vec{x}$ at $t = 0 \to$

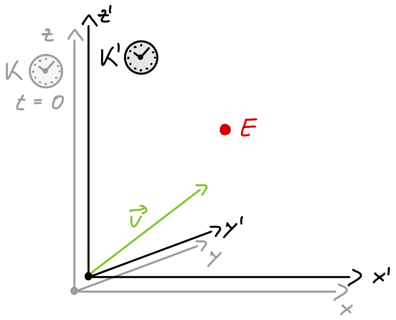
⁂ *Pure boost* $K \xrightarrow{\mathbb{1},\vec{v},0,\vec{0}} K'$:

$$x' = \Lambda_{\vec{v}} x \quad \Leftrightarrow \quad \begin{cases} t' = a(\vec{v})\,t + \vec{b}(\vec{v}) \cdot \vec{x} \\ \vec{x}\,' = M(\vec{v})\,\vec{x} + \vec{e}(\vec{v})\,t \end{cases} \tag{1.42}$$

- a: $\vec{v}$-dependent scalar
- $\vec{b}, \vec{e}$: $\vec{v}$-dependent vectors
- $M^T = M$: $\vec{v}$-dependent 3×3-matrix

Pure boosts are therefore characterized by a symmetric transformation of the spatial coordinates at $t = 0$ in K. Geometrically, this implies that there are *three* (orthogonal) lines through the origin of K which are mapped onto themselves under the boost (spanned by the eigenvectors of $M(\vec{v})$). The only other possibility is that there is a *single* invariant line, which then coincides with the rotation axis of a spatial rotation mixed into the boost. The *pure* boosts are therefore those boosts *without* any rotation mixed in.

→ We focus on pure boosts in the remainder of this derivation:



¡! Our characterization of a pure boost does *not* imply that at $t = 0$ the axes of the two systems K and K' align (as suggested by the sketch and naïvely expected). If this were the case, the eigenbasis of $M(\vec v)$ would be given by the basis vectors $\hat e_i$ in K. Since we do not know the form of $M(\vec v)$ (yet), we cannot make this assumption! So do not take this sketch literally, it only illustrates *symbolically* the situation of a pure boost in an arbitrary direction.

6 | Isotropy:

Here are two lines of arguments that use isotropy **IS** to restrict the form of Eq. (1.42) further:

- Argument A:

i | We claim that isotropy **IS** requires the following multiplicative structure for pure boosts and rotations:

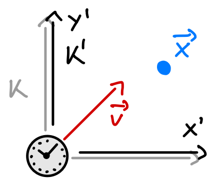
$$\Lambda_R \Lambda_{\vec v} \Lambda_{R^{-1}} \overset{!}{=} \Lambda_{R\vec v} \quad\Leftrightarrow\quad \forall_x : \Lambda_R \Lambda_{\vec v} x = \Lambda_R x' \overset{!}{=} \Lambda_{R\vec v} \Lambda_R x \,. \tag{1.43a}$$

$$\Leftrightarrow\quad \forall_x : \Lambda_{\vec v} x \overset{!}{=} \Lambda_{R^{-1}} \Lambda_{R\vec v} (\Lambda_R x) \,. \tag{1.43b}$$

The reasoning goes as follows:

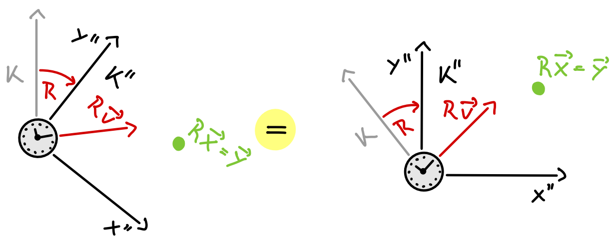
1. $\sphericalangle$ Left-hand side of Eq. (1.43b):

   $x = (t, \vec x)$ are the coordinates of some event in K and $\Lambda_{\vec v} x$ of the same event in K':



2. $\sphericalangle$ Right-hand side of Eq. (1.43b):

   We consider $y = (t, \vec y) := \Lambda_R x = (t, R\vec x)$ as an *active* transformation, i.e., y denotes a different event that is spatially rotated from x by R. To state our isotropy claim **IS**, we now rotate the coordinate system K'' in which we want to express this event *in the same way*. This implies a rotated boost $\Lambda_{R\vec v}$ and a subsequent rotation of the coordinate axes by R via $\Lambda_{R^{-1}}$. (Remember that when rotating the coordinate *axes* by R, the *coordinates* of an event transform by $\Lambda_{R^{-1}}$.):



3. Spatial isotropy **IS** is the property that the event x as seen from K' cannot be distinguished from the rotated event y as seen from the rotated system K''; this is Eq. (1.43b).

ii | Now we can use Eq. (1.42) to rewrite Eq. (1.43a) as

$$t' \stackrel{!}{=} a(R\vec{v})\,t + \vec{b}(R\vec{v}) \cdot R\vec{x} \tag{1.44a}$$

$$R\vec{x}' \stackrel{!}{=} M(R\vec{v})\,R\vec{x} + \vec{e}(R\vec{v})\,t \tag{1.44b}$$

iii | A comparison with Eq. (1.42) (for all t and $\vec{x}$ and arbitrary $\vec{v}$ and R) leads to constraints on the unknown functions:

- $a(\vec{v}) \stackrel{!}{=} a(R\vec{v}) \to a(\vec{v}) = a_v$ with $v = |\vec{v}|$

  Functions invariant under arbitrary rotations can only depend on the norm $|\vec{v}|$.

- $\vec{b}(\vec{v}) \stackrel{!}{=} R^T\vec{b}(R\vec{v}) \to \vec{b}(\vec{v}) = b_v\vec{v}$

  Note that $\vec{b}(R\vec{v})\cdot R\vec{x} = [R^T\vec{b}(R\vec{v})]\cdot\vec{x}$. Let $R_{\hat{v}}$ be some rotation with axis $\hat{v} = \vec{v}/v$ such that $R_{\hat{v}}\vec{v} = \vec{v}$; then $\vec{b}(\vec{v}) \stackrel{!}{=} R_{\hat{v}}^T\vec{b}(\vec{v})$ and therefore $\vec{b}(\vec{v}) \propto \vec{v}$ since rotation matrices have only a single eigenvector.

- $RM(\vec{v}) \stackrel{!}{=} M(R\vec{v})R \to M(\vec{v}) = c_v\,\mathbb{1} + d_v\,\hat{v}\hat{v}^T$

  First recall that $M^T(\vec{v}) = M(\vec{v})$ such that $M(\vec{v})$ can be written as sum of orthogonal projectors (projecting onto its eigenspaces). It is in particular $R_{\hat{v}}M(\vec{v})R_{\hat{v}}^T \stackrel{!}{=} M(\vec{v})$ such that one of the eigenvectors must be $\hat{v} \propto \vec{v}$. The remaining two eigenvectors are orthogonal to $\hat{v}$ and can therefore be mapped onto each other by $R_{\hat{v}}$. Since $R_{\hat{v}}$ commutes with $M(\vec{v})$, their eigenvalues must be degenerate such that the two-dimensional subspace orthogonal to $\hat{v}$ is a degenerate eigenspace. The most general spectral decomposition of $M(\vec{v})$ is then the one given above.

- $R\vec{e}(\vec{v}) \stackrel{!}{=} \vec{e}(R\vec{v}) \to \vec{e}(\vec{v}) = e_v\vec{v}$

  This is the same argument as for $\vec{b}(\vec{v})$.

- Argument B:

  A shorter (but less rigorous) line of arguments goes as follows:

  i | To define the unknown functions algebraically, we are only allowed to use the vector $\vec{v}$ and constant scalars. We *cannot* use $\vec{x}$ or t due to linearity, and any other constant vector (like $\hat{e}_x = (1,0,0)^T$) would pick out some direction and therefore violate isotropy **IS**.

  ii | Since the only *scalar* one can construct from a single vector is its norm, $|\vec{v}|^2 = \vec{v}\cdot\vec{v}$, it must be $a(\vec{v}) = a_v$.

  iii | Similarly, since the only *vector* one can construct from a single vector is a scalar multiplied by the vector itself, it must be $\vec{b}(\vec{v}) = b_v\vec{v}$ and $\vec{e}(\vec{v}) = e_v\vec{v}$.

  iv | Lastly, since $M^T(\vec{v}) = M(\vec{v})$, we can decompose the matrix into orthogonal projectors: $M(\vec{v}) = \sum_i \lambda_i(v)P_i(\vec{v})$. The only projectors that can be defined by a single vector are $P_0 = \hat{v}\hat{v}^T$ and $P_1 = \mathbb{1} - P_0 = \mathbb{1} - \hat{v}\hat{v}^T$ which leads to the most general form $M(\vec{v}) = c_v\,\mathbb{1} + d_v\,\hat{v}\hat{v}^T$.

Both arguments lead to the same form for pure boosts $\Lambda_{\vec{v}}$ consistent with isotropy **IS**:

$$t' = a_v\,t + b_v\,(\vec{v}\cdot\vec{x}) \tag{1.45a}$$

$$\vec{x}' = c_v\,\vec{x} + \frac{d_v}{v^2}\,\vec{v}(\vec{v}\cdot\vec{x}) + e_v\,\vec{v}\,t \tag{1.45b}$$

with $v = |\vec{v}| = |R\vec{v}|$ and $(R\vec{v}\cdot R\vec{x}) = (\vec{v}\cdot\vec{x})$.

7 | $\sphericalangle$ Trajectory of origin O' of K':

- In K': $\vec{x}'_{O'} = 0$ (This is the operational *definition* of the origin O'.)
- In K: $\vec{x}_{O'} = \vec{v}t$ (This is the operational *definition* of $\vec{v}$ in $K \xrightarrow{\mathbb{1},\vec{v},0,\vec{0}} K'$.)

In Eq. (1.45b):

$$\vec{0} = c_v\,\vec{v}t + \tfrac{d_v}{v^2}\,\vec{v}(\vec{v}\cdot\vec{v})t + e_v\,\vec{v}\,t \tag{1.46a}$$

$$\vec{v} \neq \vec{0}\ \&\ \forall t \quad \Rightarrow \quad 0 = c_v + d_v + e_v \tag{1.46b}$$

8 | <u>Reciprocity:</u>

i | $\sphericalangle$ Inverse transformation $K' \xrightarrow{\mathbb{1},\vec{v}',0,\vec{0}} K$ from K' to K:

$$\Lambda_{\vec{v}'}\Lambda_{\vec{v}} = \mathbb{1} \quad \Leftrightarrow \quad \Lambda_{\vec{v}'} = \Lambda_{\vec{v}}^{-1}\,. \tag{1.47}$$

Note that $\vec{v}'$ is the velocity of the origin O of K as measured in K'.

In general: $\vec{v}' = \vec{V}(\vec{v})$ with unknown function $\vec{V}$.

We assume *reciprocity*: $\vec{v}' = -\vec{v}$ such that

$$\Lambda_{\vec{v}}^{-1} = \Lambda_{-\vec{v}}\,. \tag{1.48}$$

While this is clearly the most reasonable/intuitive assumption, it is not trivial! Recall that $\vec{v}$ is the speed of the origin O' of K' measured with the clocks in K, whereas $\vec{v}'$ is the speed of the origin O of K measured with *different* clocks in K'. So without additional assumptions we cannot conclude that the results of these measurements yield reciprocal results.

However, the assumption of reciprocity can be rigorously derived from relativity **SR**, isotropy **IS** and homogeneity **HO**, see Ref. [32]. Reciprocity is therefore not an independent assumption.

ii | $\sphericalangle$ Inverse transformation in Eq. (1.45):

$$t = a_v\,t' - b_v\,(\vec{v}\cdot\vec{x}') \tag{1.49a}$$

$$\vec{x} = c_v\,\vec{x}' + \tfrac{d_v}{v^2}\,\vec{v}(\vec{v}\cdot\vec{x}') - e_v\,\vec{v}\,t' \tag{1.49b}$$

iii | Eq. (1.49) in Eq. (1.45) & Eq. (1.46b) $\overset{\circ}{\rightarrow}$ (we suppress the v dependence)

$$c^2 = 1\,, \tag{1.50a}$$

$$a^2 - ebv^2 = 1\,, \tag{1.50b}$$

$$e^2 - ebv^2 = 1\,, \tag{1.50c}$$

$$e(a+e) = 0\,, \tag{1.50d}$$

$$b(a+e) = 0\,. \tag{1.50e}$$

To show this, use $\vec{v} = (v_x, 0, 0)^T$ with $v_x \neq 0$ and remember that the equations you obtain from plugging Eq. (1.49) into Eq. (1.45) must be valid for all t' and $\vec{x}'$. Use Eq. (1.46b) to replace $c_v + d_v$ by $-e_v$.

We can conclude:

- $\xrightarrow{\text{Eq. (1.50a)}} c = 1$ ($c = -1$ contradicts $\lim_{v\to 0} \Lambda_{\vec{v}} \stackrel{!}{=} \mathbb{1}$)
- $\xrightarrow{\text{Eq. (1.50c)}} e \neq 0 \xrightarrow{\text{Eq. (1.50d)}} a + e = 0$
  $\rightarrow$ Eq. (1.50b) $\equiv$ Eq. (1.50c) & Eq. (1.50e) satisfied

9 | Collecting results from Eq. (1.50) & Eq. (1.46b):

$$c = 1\,, \quad e = -a\,, \quad d = a - 1\,, \quad b = \tfrac{1-a^2}{av^2}\,. \tag{1.51}$$

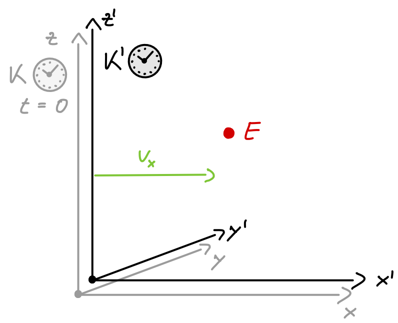
$d = a - 1$ follows from Eq. (1.46b) and the first two equations.

Eq. (1.45) $\xrightarrow{\text{Eq. (1.51)}}$

$$t' = a_v\, t + \tfrac{1-a_v^2}{va_v}\, (\hat{v} \cdot \vec{x}) \tag{1.52a}$$
$$\vec{x}' = \vec{x} + [a_v - 1]\, \hat{v}(\hat{v} \cdot \vec{x}) - va_v\, \hat{v}\, t \tag{1.52b}$$

with $\hat{v} := \vec{v}/|\vec{v}|$.

10 | $\sphericalangle$ Special boost $\vec{v} = (v_x, 0, 0)^T$ in x-direction:



$$t' = a_v\, t + \tfrac{1-a_v^2}{v_x a_v}\, x \tag{1.53a}$$
$$x' = a_v\, x - v_x a_v\, t \tag{1.53b}$$
$$y' = y \tag{1.53c}$$
$$z' = z \tag{1.53d}$$

Note that $v = |v_x|$ with $v_x \in \mathbb{R}$.

Matrix form:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{\left(\begin{array}{cc|cc} a_v & \tfrac{1-a_v^2}{v_x a_v} & & \\ -v_x a_v & a_v & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right)}_{=:\Lambda_{v_x}} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{1.54}$$

In the following, we refer to the upper 2×2-block as $A(v_x)$.

11 | Group structure:

i | Relativity principle **SR** $\rightarrow$

$$\varphi(K' \xrightarrow{R_2, \vec{v}_2, s_2, \vec{b}_2} K'') \circ \varphi(K \xrightarrow{R_1, \vec{v}_1, s_1, \vec{b}_1} K') \stackrel{!}{=} \varphi(K \xrightarrow{R_3, \vec{v}_3, s_3, \vec{b}_3} K'') \tag{1.55}$$

for some parameters $(R_3, \vec{v}_3, s_3, \vec{b}_3)$ that are a function of $(R_i, \vec{v}_i, s_i, \vec{b}_i)_{i=1,2}$.

In words:

The concatenation of a coordinate transformations from K to K' and from K' to K'' must be another coordinate transformation *that is parametrized by data that relates the reference systems K with K'' directly (without referring to K' in any way).*

You may ask why Eq. (1.55) is a constraint on φ in the first place. After all, we could just *define* that

$$\varphi(K \xrightarrow{R_3,\vec{v}_3,s_3,\vec{b}_3} K'') := \varphi(K' \xrightarrow{R_2,\vec{v}_2,s_2,\vec{b}_2} K'') \circ \varphi(K \xrightarrow{R_1,\vec{v}_1,s_1,\vec{b}_1} K') \,. \tag{1.56}$$

The problem is that the function defined such generically depends on 8 (!) parameters $R_1, \vec{v}_1, s_1, \vec{b}_1, R_2, \vec{v}_2, s_2, \vec{b}_2$ – it is a non-trivial functional constraint on φ that these can be compressed to four parameters $R_3, \vec{v}_3, s_3, \vec{b}_3$. This "compression" is mandated by the relativity principle SR according to which all inertial systems must be treated equally. In particular, the transformation between two systems K and K'' can only depend on parameters that can be experimentally determined from within these two systems. (The existence of) a third frame K' cannot be of relevance for this transformation as this would make K' special.

Combined with the existence of an inverse transformation (← *above*):

→ The set of all transformations forms a ↓ (multiplicative) group.

Note that *associativity* is implicit since we talk about the concatenation of linear/affine maps.

ii | In particular:

$$\Lambda_{v_x}\Lambda_{u_x} \overset{!}{=} \Lambda_{w_x} \quad \Leftrightarrow \quad A(v_x)A(u_x) \overset{!}{=} A(w_x) \tag{1.57}$$

where $w_x = W(v_x, u_x)$ has to be determined.

- ¡! Using the restricted form of the boost Eq. (1.54) that followed from previous arguments, it follows indeed that the concatenation of two pure boosts *in the same direction* has again the form of a pure boost (in the same direction). For the arguments that follow, this is sufficient.

  However, in general, the multiplicative group structure Eq. (1.55) allows for two boosts to concatenate to a *combination* of boosts and rotations. As we will see → *later*, this is indeed what happens: The concatenation of two pure boosts (in different directions) produces a boost with a rotation mixed in (↑ *Thomas-Wigner rotation*).
- Note that due to Eq. (1.43a) all that follows holds for any pair of *collinear* velocities $\vec{v}$ and $\vec{u}$ (there is nothing special about the x-direction). Indeed, let R be a rotation that maps $\vec{v}$ and $\vec{u}$ to vectors on the x-axis, $\vec{v}_x := R\vec{v}$ and $\vec{u}_x := R\vec{u}$. Then

  $$\Lambda_{\vec{v}}\Lambda_{\vec{u}} \overset{1.43a}{=} \Lambda_{R^{-1}}\Lambda_{\vec{v}_x}\Lambda_{\vec{u}_x}\Lambda_R \overset{!}{=} \Lambda_{R^{-1}}\Lambda_{\vec{w}_x}\Lambda_R \overset{1.43a}{=} \Lambda_{\vec{w}} \tag{1.58}$$

  where $\vec{w}$ is again collinear with $\vec{v}$ and $\vec{u}$.

$\overset{\circ}{\rightarrow}$ (use that the diagonal elements of $A(w_x)$ must be equal)

$$\forall_{v_x,u_x}: \quad \frac{1-a_v^2}{v_x^2 a_v^2} \overset{!}{=} \frac{1-a_u^2}{u_x^2 a_u^2} \tag{1.59}$$

→ Universal constant:

$$\kappa := \frac{a_v^2 - 1}{v_x^2 a_v^2} = \text{const} \tag{1.60}$$

Note: $[\kappa] = \text{Velocity}^{-2}$

→

$$a_v = \frac{1}{\sqrt{1-\kappa v_x^2}}\,. \tag{1.61}$$

We use the positive solution for a_v since $\lim_{v\to 0} A(v) \overset{!}{=} \mathbb{1}$, i.e., $\lim_{v\to 0} a_v \overset{!}{=} 1$.

iii | With this we check: $A(v_x)A(u_x) \stackrel{\circ}{=} A(w_x)$ with

$$w_x = W(v_x, u_x) \stackrel{\circ}{=} \frac{v_x + u_x}{1 + u_x v_x \kappa}\,. \tag{1.62}$$

Eq. (1.62) becomes important later: it tells us how to add velocities in SPECIAL RELATIVITY.

12 | Preliminary result:

Eq. (1.52) & Eq. (1.60) → Boost $\Lambda_{\vec{v}}$ in direction $\hat{v}$ with velocity $\vec{v} = v\hat{v}$:

$$\boxed{\begin{aligned} t' &= a_v\left[t - \kappa\,(\vec{v}\cdot\vec{x})\right] && (1.63a)\\ \vec{x}' &= \vec{x} + [a_v - 1]\,\hat{v}(\hat{v}\cdot\vec{x}) - a_v\,\vec{v}\,t && (1.63b)\end{aligned}}$$

with

$$\boxed{a_v = \frac{1}{\sqrt{1-\kappa v^2}}\,.} \tag{1.64}$$

This is the most general transformation between two inertial coordinate systems that move with relative velocity $\vec{v}$ (with coinciding axes at $t = 0$) that is consistent with our basic assumptions stated at the beginning of this section: `SR`, `HO`, and `IS`.

The only undetermined parameter left is κ.

1.5. The Lorentz transformation

The purpose of this section is to select the value for κ that describes our reality.

13 | Since $[\kappa] = \text{Velocity}^{-2}$ define formally: $\kappa \equiv 1/v_{\text{max}}^2$.

Why we subscribe the velocity v_{max} with "max" will become clear below.

14 | Three cases:

- $\kappa = 0 \;\Leftrightarrow\; v_{\text{max}} = \infty$:

$$\boxed{\text{Eq. (1.63)} \;\Rightarrow\; \left.\begin{aligned} t' &= t\\ \vec{x}' &= \vec{x} - \vec{v}\,t\end{aligned}\right\}\;\; ⁂\ \textit{Galilei boost}} \tag{1.65a}$$

→ Maxwell equations are *not* form-invariant under φ.

→ Maxwell equations cannot be correct and must be modified.

→ Experiment that shows the invalidity of Maxwell equations?

Note that we cannot conclude the *validity* of classical mechanics from this; Newton's equations may still require modifications (without spoiling the Galilean symmetry, of course).

- $\kappa > 0 \quad \Leftrightarrow \quad v_{\max} < \infty$:

$$\boxed{\text{Eq. (1.63)} \quad \Rightarrow \quad \left.\begin{aligned} t' &= \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{v_{\max}^2}\right) \\ \vec{x}' &= \vec{x} + (\gamma - 1)\,\hat{v}(\hat{v}\cdot\vec{x}) - \gamma\,\vec{v}\,t \end{aligned}\right\} \quad ⁂\ \textit{Lorentz boost}} \tag{1.66a}$$

with the ⁂ *Lorentz factor*

$$\boxed{\gamma_v \equiv \gamma := \frac{1}{\sqrt{1-\beta^2}} \quad \text{and} \quad \beta := v/v_{\max}\,.} \tag{1.67}$$

→ Newton's equations are *not* form-invariant under φ.

→ Classical mechanics cannot be correct and must be modified.

→ Experiment that shows the invalidity of Newton's equations?

Similarly, we cannot conclude the *validity* of electrodynamics from this; Maxwell equations may still require modifications (without spoiling the Lorentz symmetry).

- $\kappa < 0$: Physically not relevant. (➔ Problemset 2; we ignore this solution in the following.)

  This solution is not self-consistent (see e.g. Ref. [31]) and immediately leads to implications that are not observed in nature.

  For example, the rule Eq. (1.62) to compute the velocity w_x between K/K'' from the velocities v_x and u_x between K/K' and K'/K'' reads for $\kappa < 0$

$$w_x = \frac{v_x + u_x}{1 - u_x v_x |\kappa|}\,. \tag{1.68}$$

  Let $u_x, v_x > 0$ be positive, i.e., K' moves in *positive* x-direction *wrt* K and K'' moves also in *positive* x-direction *wrt* K'. But for large enough velocities $u_x v_x > 1/|\kappa|$ we find $w_x < 0$ such that K'' moves in *negative* x-direction *wrt* K.

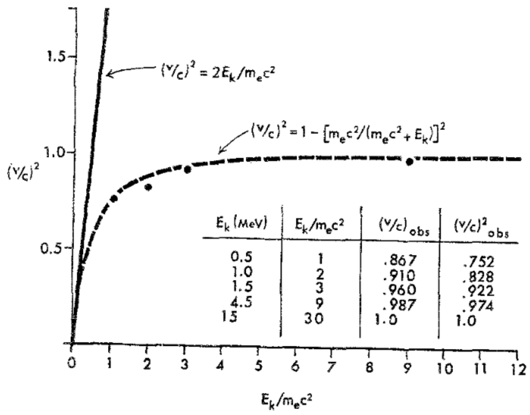
  No such effect has ever been observed; if you do, let us know!

Note that at no point we used or claimed that $v_{\max}$ is the speed of light!

$$\boxed{\text{Which transformation describes reality: } v_{\max} < \infty \text{ or } v_{\max} = \infty\text{?}}$$

15 | Evidence:

- Maximum velocity $v_{\max} \approx c < \infty$ for electrons (plot from Ref. [33]):



→ Newton's equations are clearly invalid for high velocities!

See Refs. [33, 34] for more technical details. Note that these results were obtained decades after Einstein published his seminal paper in 1905.

- *By contrast:*

  No evidence for the invalidity of Maxwell equations (on the macroscopic level).

  Electrodynamics, as encoded by the Maxwell equations, is of course not a truly fundamental theory as it is the classical limit of a quantum theory: Quantum electrodynamics (QED). For example, the linearity of the Maxwell equations (= EM waves cannot scatter off each other) is an *approximation*; in QED photons *can* (weakly) scatter off each other! This is why I emphasize that Maxwell theory is experimentally valid only on the *macroscopic level*. Note, however, that QED has the same spacetime symmetry group as electrodynamics, namely Lorentz transformations.

16 | Hence it is reasonable stipulate $v_{\text{max}} < \infty$ and postulate:

> The transformations φ between inertial systems are given by *Lorentz transformations*.
> These transformations must be (part of) the spacetime symmetries of *all* physical theories.

The last statement is often rephrased as follows:

> All (fundamental) theories must be *form-invariant* (covariant) under Lorentz transformations.

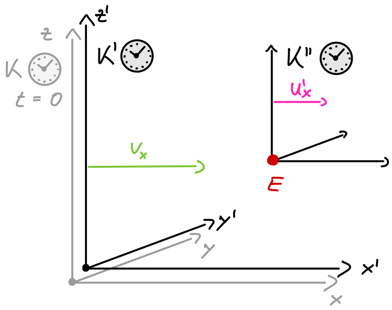
This is just **SR** all over again: The equations of models that describe reality must "look the same" (more precisely: be functionally equivalent) in all inertial systems. Since the transformations between inertial systems are given by Lorentz transformations (and not Galilean transformations, as historically anticipated), this requires their form-invariance under Lorentz transformations.

→ SPECIAL RELATIVITY restricts the structure of *all fundamental theories of physics*!

This is what is meant by the statement that SPECIAL RELATIVITY is a *theoretical framework* (German: *Rahmentheorie*) or "meta theory": It provides a "recipe" (*ordering principle*) of how to construct consistent theories of physics. The Standard Model of particle physics, for example, is form-invariant under Lorentz transformations, and if you propose an extension thereof (for example to give neutrinos a mass) you better make sure that the terms you write down are also form-invariant under Lorentz transformations (otherwise you will not be taken seriously!). Note, however, that this perspective prevents an important insight: What we really study is an entity called *spacetime*, and this entity has a property: Lorentz symmetry. Since all our (fundamental) physical theories are formulated *on* spacetime, it should not come as a surprise that the Lorentz symmetry of spacetime shows up all over the place.

17 | Interpreation of $v_{\max}$:

i | $\sphericalangle$ Systems $K \xrightarrow{v_x} K'$ and signal with velocity $\frac{\mathrm{d}x'}{\mathrm{d}t'} = u'_x$:



Question: What is the velocity $u_x = \frac{\mathrm{d}x}{\mathrm{d}t}$ of this signal in K?

ii | Remember (Group structure!):

$$\varphi(K' \xrightarrow{v_2} K'') \circ \varphi(K \xrightarrow{v_1} K') = \varphi(K \xrightarrow{v_3} K'') \quad \text{with} \quad v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{v_{\max}^2}} \,. \tag{1.69}$$

Let $v_1 = v_x$ and $v_2 = u'_x$ so that $v_3 = u_x$ (i.e., the signal is at rest in the origin of K'').

You can also derive this by computing the time derivative of the position of the signal in K using a Lorentz transformation; you will do this properly when you derive a more general addition of velocities (→ Problemset 2).

iii | Addition formula for collinear velocities:

$$\boxed{u_x = \frac{v_x + u'_x}{1 + \frac{v_x u'_x}{v_{\max}^2}}} \tag{1.70}$$

Because of isotropy **IS** this formula must be true in all directions (not just in x-direction) *as long as the two velocities to be added are parallel*. We still keep the index x to signify that these are not absolute values of velocities.

- Note that for $v_{\max} \to \infty$ we get back the "conventional" (= Galilean) additivity of velocities:

$$u_x = (v_x + u'_x)\left[1 - \frac{v_x u'_x}{v_{\max}^2} + \dots\right] \overset{v_{\max}\to\infty}{=} v_x + u'_x \tag{1.71}$$

  From this expansion and the validity of classical mechanics for small velocities (in particular its law for adding velocities), we can also conclude that $v_{\max}$ must be *large compared to everyday experience*.

- A historically influential experiment that (in hindsight) can be explained by the relativistic addition of velocities Eq. (1.70) is the ↑ *Fizeau experiment* [35, 36] (see also ↑ *Fresnel drag coefficient*). The Fizeau experiment was one of the crucial hints that led Einstein to SPECIAL RELATIVITY.

iv | $\sphericalangle\, 0 \leq v_x, u'_x \leq v_{\max}$: ($\tilde{v}_x := v_x/v_{\max}$ so that $0 \leq \tilde{v}_x, \tilde{u}_x \leq 1$)

$$u_x = v_{\max} \frac{\tilde{v}_x + \tilde{u}'_x}{1 + \tilde{v}_x \tilde{u}'_x} \leq v_{\max} \tag{1.72}$$

Here we used that $a + b \leq 1 + ab$ for numbers $0 \leq a, b \leq 1$.

$\rightarrow$ "Addition" of velocities Eq. (1.70) never exceeds $v_{\max}$.

$\rightarrow$ $v_{\max}$ plays the role of a *maximum velocity*.

v | $\sphericalangle$ Signal with maximum velocity in K': $u'_x = v_{\max}$:

$$u_x = \frac{v_{\max} + v_x}{1 + \frac{v_{\max} v_x}{v_{\max}^2}} = v_{\max} \frac{v_{\max} + v_x}{v_{\max} + v_x} = v_{\max} \tag{1.73}$$

Note that the result is completely independent of the velocity v_x of K'!

$\rightarrow$ Whatever moves with the maximum velocity $v_{\max}$ does so *in all inertial systems*!

Please appreciate how counterintuitive this effect is *from the perspective of everyday experience*! But also notice that we didn't have to postulate it: The relativity principle **SR** together with the *existence* of a (finite) maximum velocity is sufficient.

If you think about it: Assuming a maximum velocity (in the absence of a preferred reference frame) automatically invalidates the simple Galilean law of additive velocities. So it is actually not surprising at all that the maximum velocity must be independent of the reference system.

↓Lecture 4 [07.11.23]

18 | Experiments (in particular: the validity of Maxwell equations) show:

$$\boxed{v_{\max} = c = 299\,792\,458\,\mathrm{m\,s^{-1}}} \tag{1.74}$$

Note that since 1983 the value of c in the international system of units (SI) is *exact* by definition.

A. Einstein incorporated this insight in §2 of Ref. [9] as his second postulate:

> *2. Jeder Lichtstrahl bewegt sich im "ruhenden" Koordinatensystem mit der bestimmten Geschwindigkeit V, unabhängig davon, ob dieser Lichtstrahl von einem ruhenden oder bewegten Körper emittiert ist.*

Note that at the time it was conventional to denote the speed of light with a capital V. The convention switched to our now standard lower-case c just a few years later. For more historical background:

➔ https://math.ucr.edu/home/baez/physics/Relativity/SpeedOfLight/c.html

We can condense this into:

§ Postulate: Constancy of the speed of light SL

The speed of light is independent of the inertial system in which it is measured.

Comments:

- *If* you take the validity of the Maxwell equations for granted, then $v_{\max} = c < \infty$ (and thereby SL) follows immediately from the relativity principle SR because then the Maxwell equations must be valid in all inertial systems. But you've learned in your course on electrodynamics that the wavelike solutions of these equations always propagate with group velocity c in vacuum. This is only possible if the speed of light plays the role of the limiting velocity: $v_{\max} = c$.

  Einstein acknowledges as much at the beginning of Ref. [10]. However, SL is empirically weaker than claiming the validity of Maxwell's equations (after all, there could be alternative equations that also predict the velocity c of wavelike solutions). At the time when Einstein formulated SL in [9], he also worked on the photoelectric effect (another of his *annus mirabilis* papers [37]). The postulation of "quanta of light" is the foundation of quantum mechanics, but cannot be explained by Maxwell's equations. It is therefore reasonable to assume that Einstein didn't want to rely on the validity of this specific theory when formulating his SPECIAL RELATIVITY. He therefore opted for the empirically weaker (but still sufficient) assumption SL.

- If you derive the transformation φ using *both* postulates SR and SL the derivation is shorter (see e.g. [1] or [4]); one then of course doesn't find the Galilei transformations as an option. Note, however, that the relativity principle SR is a reasonable and intuitive starting point that doesn't need much convincing (after all, we witness the relativity of Newtonian mechanics in our everyday life). By contrast, the speed of light postulate SL clashes directly with our everyday experience (how velocities add up, that is). Through our elaborate derivation we learned how much is already implied by the simple, reasonable assumption of relativity. We only had to check whether there is any evidence of a finite maximum velocity $v_{\max}$. The counterintuitive feature that this velocity is the same viewed from all inertial systems was then a necessary conclusion from our derivation.

† Note: Finite speed of causality (Locality)

Another insight from our SR-based derivation of the Lorentz transformation is that the formulation of the speed-of-light postulate SL is conceptually misleading:

- The constant $v_{\max}$ and its role as a maximum velocity followed *without* referring to light (or electrodynamics) in any way!

  Put bluntly: SPECIAL RELATIVITY is *not* about the "strange behavior" of light!

- The relevant speed for SPECIAL RELATIVITY is the *speed of causality*: How fast can information travel, i.e., one event affect another. $v_{\max}$ is the maximum speed of *causal interactions*, irrespective of the mediator of these interactions.

  In our world, the fastest and most salient information carrier just happens to be the electromagnetic field ("light"). For example, to synchronize our clocks with light signals, it wasn't the light *per se* we were interested in; we just used it as carrier of information to correlate the clocks.

- Given the relativity principle SR and our derivation in Section 1.4, we showed that there are only two possibilities: (1) There is *no* upper bound on velocities (Galilean symmetry) or (2) there *is* such an upper bound $v_{\max}$ (Lorentz symmetry). In the latter case, every signal that propagates with $v_{\max}$ in some frame automatically does so in all inertial systems. (Which immediately leads to the counterintuitive conclusion, akin to SL, that there are signals the velocity of which does not depend on the velocity of the observer.)

- We could replace SL therefore by the (empirically weaker) postulate that there are *no instantaneous actions* at a distance (this is essentially a statement about *locality*). This modified postulate implies the existence of a maximal velocity $v_{\max} < \infty$ which, in turn, selects the Lorentz transformation as the correct symmetry. That $v_{\max} = c$ is then a fact to be discovered by experiments.

- It turns out that *everything with vanishing rest mass* travels at this maximum speed $v_{\max} = c$. Since photons are the only elementary particles that are massless and can be easily detected, we just happen to refer to this maximum velocity as "speed of light."

  For example: Without Higgs symmetry breaking, the $W^{\pm}$ and Z bosons of the weak interaction are massless and would propagated with light velocity, just as the photon (the weak interactions would then be no longer "weak"). For a long time it was believed that neutrinos are massless as well, and therefore would also propagate with the speed of light (today we know that they have a very tiny mass).

19 | Special Lorentz transformations = Lorentz boosts:

Now that everything is settled, let us write down our final result in their conventional form.

¡! These are not the most general (homogeneous) Lorentz transformations since we omit rotations, parity and time inversion. We will discuss the structure of the full homogeneous Lorentz group and its inhomogeneous generalization (→ *Poincaré group*) later. To discuss the "fancy" phenomena of SPECIAL RELATIVITY, the transformations below are sufficient.

i | Boost in arbitrary directions ($\vec{v} = v\hat{v}$ with $\hat{v} \equiv \vec{v}/|\vec{v}|$):

$$\boxed{\Lambda(K \xrightarrow{\vec{v}} K') : \begin{cases} ct' = \gamma\left(ct - \beta \vec{x} \cdot \hat{v}\right) \\ \vec{x}' = \vec{x} + (\gamma - 1)(\vec{x} \cdot \hat{v}) \cdot \hat{v} - \gamma \vec{v} t \end{cases}} \tag{1.75}$$

(Since we now settled on Lorentz transformation for φ, we write $\varphi = \Lambda$ henceforth.)

with $\beta \equiv v/c$ and the *Lorentz factor*

$$\boxed{\gamma_v \equiv \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}\,.} \tag{1.76}$$

ii | Special case: Boost in x-direction ($\vec{v} = v_x \hat{x}$):

$$\boxed{\Lambda(K \xrightarrow{v_x} K') : \begin{cases} ct' = \gamma\left(ct - \frac{v_x}{c} x\right) \\ x' = \gamma(x - v_x t) \\ y' = y \\ z' = z \end{cases}} \tag{1.77}$$

20 | <u>State of affairs:</u>

Now that we know the spacetime symmetry φ of reality, we have quite a to-do list:

- We will have to *modify* Newton's equations to replace their Galilean by a Lorentz symmetry, without changing their predictions for small velocities $v \ll c$ (↓ *correspondence principle*).

  → Relativistic mechanics

- We can keep the Maxwell equations in their current form ☺.

  Note that we still have to check that they are really Lorentz covariant (➔ Problemset ?)!

  In the end we will come up with a neat notation that allows us to rewrite (not modify!) the Maxwell equations in a compact form to make their Lorentz symmetry apparent.

- Similar to classical mechanics, we will have to replace the *Schrödinger equation* in quantum mechanics by a modified version with Lorentz symmetry.

  → Relativistic quantum mechanics (Klein-Gordon and Dirac equation)

But before we do all the heavy work:

Simple implications of this transformation? (→ *below* and next lectures)

With "simple" we refer to implications that follow without imposing a model-specific dynamics (= equation of motion). We will refer to these implications as *kinematic* because they follow from fundamental constraints on the degrees of freedom of all relativistic theories.

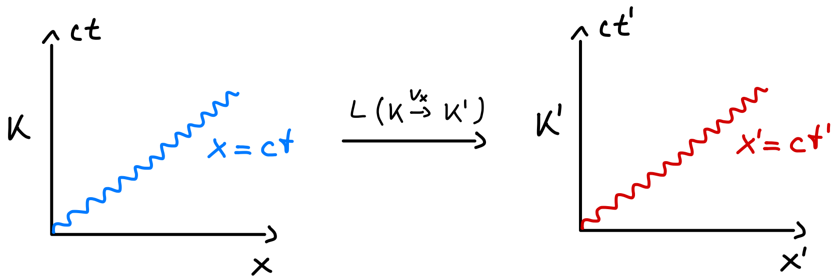
1.6. Invariant intervals and the causal partial order of events

1 | $\triangleleft$ Trajectory of a light signal in x-direction in K:

$$x(t) = ct,\ y = 0,\ z = 0 \tag{1.78}$$

Trajectory of the same signal in K' with $K \xrightarrow{v_x} K'$:

$$x'(t') = ct',\ y' = 0,\ z' = 0 \tag{1.79}$$



This follows from our previous discussion: signals propagating with $c = v_{\max}$ do so in all inertial systems!

You can also simply calculate this using the Lorentz boost Eq. (1.77):

$$ct' = \gamma\left(ct - \tfrac{v_x}{c} ct\right) \tag{1.80a}$$

$$\text{and} \quad x' = \gamma(ct - v_x t) = ct'\,. \tag{1.80b}$$

→

$$(ct)^2 - x^2 = 0 = (ct')^2 - (x')^2 \quad \text{is a } \textit{frame-independent} \text{ quantity.} \tag{1.81}$$

Note that the separate summands [$(ct)^2$ etc.] are *not* frame-independent!

This finding motivates the definition of the …

2 | ⁂ *Spacetime interval*:

Details: ➔ Problemset 2

$\triangleleft$ Two events $E_1 \ni (t_1, \vec{x}_1)_K$ and $E_2 \ni (t_2, \vec{x}_2)_K$ with temporal and spatial separation

$$(\Delta t)_K := t_1 - t_2 \quad \text{and} \quad (\Delta \vec{x})_K := \vec{x}_1 - \vec{x}_2\,. \tag{1.82}$$

Then the *spacetime interval* between E_1 and E_2 is denoted $(\Delta s)^2 \equiv \Delta s^2$ and defined as

$$\boxed{(\Delta s)^2 := (c\Delta t)_K^2 - (\Delta \vec{x})_K^2\,.} \tag{1.83}$$

We omit the subscript K from Δs because it is frame-independent (→ *next*).

In our example above it was $\Delta t = t - 0$ and $\Delta \vec{x} = (x - 0, 0 - 0, 0 - 0)$, i.e., we considered the interval between the event in the origin $x_O = (0, \vec{0})$ and the events along the trajectory $(ct, x(t), 0, 0)$ of the light signal.

3 | The importance of Δs^2 stems from the following fact:

> The spacetime interval Δs^2 is independent of the frame in which it is calculated.

This means that given two events, all observers agree on the numerical value of the interval Δs^2 between these two events.

Proof: Use Eq. (1.75) to calculate (Details: ➔ Problemset 2)

$$(ct')^2 = \left[\gamma\left(ct - \beta\vec{x}\cdot\hat{v}\right)\right]^2 \tag{1.84a}$$

$$(\vec{x}')^2 = \left[\vec{x} + (\gamma-1)(\vec{x}\cdot\hat{v})\cdot\hat{v} - \gamma\vec{v}t\right]^2 \tag{1.84b}$$

$$\Rightarrow \quad (ct')^2 - (\vec{x}')^2 \stackrel{\circ}{=} (ct)^2 - (\vec{x})^2 + \underbrace{\dots}_{=0} \tag{1.84c}$$

Note that we do not have to do the computation for two events and an interval Δt and $\Delta\vec{x}$ since the special Lorentz transformations are *linear*.

This proves the invariance under *special* Lorentz transformations (= Lorentz boosts). It is easy to see that the invariance is also valid for inhomogneous shifts in time and space (these drop out in the intervals Δt etc.) and spatial rotations Λ_R [since $(\Delta\vec{x})^2$ is clearly invariant under rotations]. We will come back to this when we discuss the structure of the Lorentz group in more detail (→ *later*).

4 | Two events E_1 and E_2 are in one of three possible (frame-independent) relations:

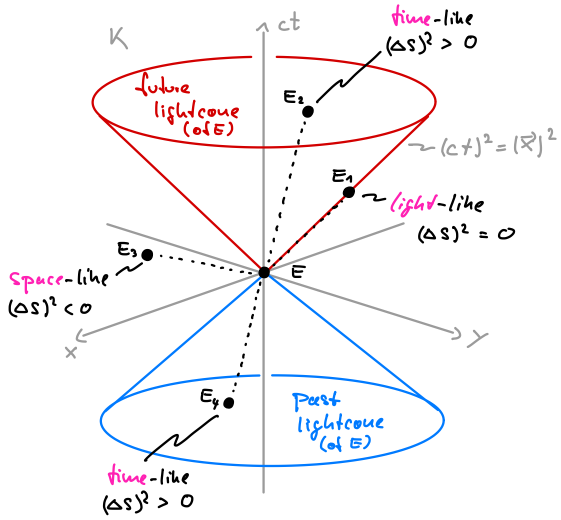
$$\Delta s^2 \begin{cases} > 0 & E_1 \text{ and } E_2 \text{ are ⁂ \textit{time-like} separated} \\ = 0 & E_1 \text{ and } E_2 \text{ are ⁂ \textit{light-like} separated} \\ < 0 & E_1 \text{ and } E_2 \text{ are ⁂ \textit{space-like} separated} \end{cases} \tag{1.85}$$

Note that Δs^2 can be *negative* so that Δs^2 should be read as a symbol rather than defining an imaginary number Δs. For the special case of *time-like* intervals, however, Δs^2 indeed defines a real number $\Delta s = \sqrt{\Delta s^2}$ which we will later relate to the time measured by moving clocks (the so called *proper time*).

All events that are light-like separated from an event E (*wlog* in the origin) satisfy

$$\Delta s^2 = 0 \quad \Leftrightarrow \quad (ct)^2 = (\vec{x})^2 \quad \Leftrightarrow \quad |ct| = |\vec{x}| \tag{1.86}$$

which determines the ⁂ *light cone* of E:



Here we show the light cone of an event E in a space time with *two* spatial dimensions x and y. The light cone in our $3+1$ dimensional space time is a higher-dimensional generalization which obeys the same equations.

- Time-like events satisfy $\Delta s^2 > 0 \quad \Leftrightarrow \quad |ct| > |\vec{x}|$ which characterizes the (disconnected) interior of the light cone. The manifold with $ct > |\vec{x}| \geq 0$ is called ⁂ *future light cone* (of E) whereas the events with $-ct > |\vec{x}| \geq 0$ make up the ⁂ *past light cone* (of E).
- Space-like events satisfy $\Delta s^2 < 0 \quad \Leftrightarrow \quad |ct| < |\vec{x}|$ which characterizes the (connected) spacetime volume outside the light cone.

5 | Causality:

The importance of the threefold classification of spacetime intervals stems from the following observations.

i | ⊲ Actions of (homogeneous) Lorentz transformations:

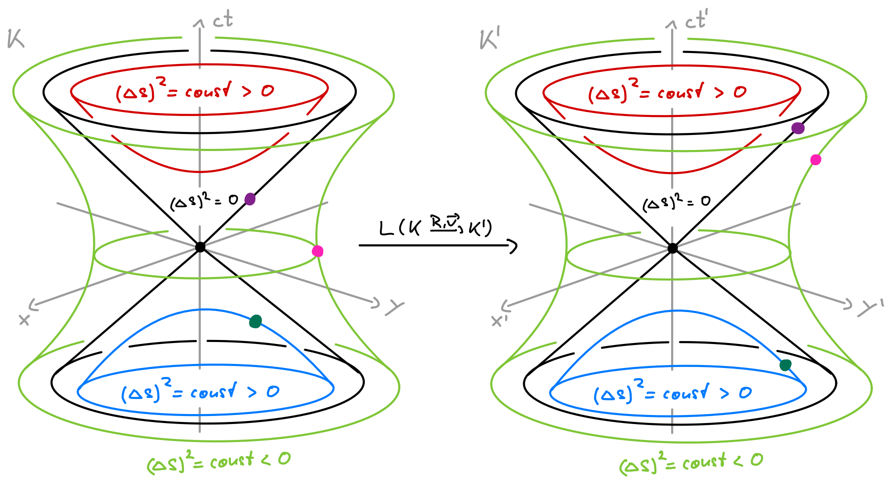
Since Δs^2 is invariant under Lorentz transformations, the manifold of events characterized by a specific value $\Delta s^2 = \pm C$ ($C \geq 0$) must be mapped onto itself under these transformations: Events on these hyperbolic manifolds cannot leave their manifolds under Lorentz transformations.

Invariant hyperbolae:

$$\text{time-like:} \quad \Delta s^2 = C > 0 \quad \Rightarrow \quad ct = \pm\sqrt{C + |\vec{x}|^2} \tag{1.87a}$$

$$\text{light-like:} \quad \Delta s^2 = C = 0 \quad \Rightarrow \quad ct = \pm|\vec{x}| \tag{1.87b}$$

$$\text{space-like:} \quad \Delta s^2 = -C < 0 \quad \Rightarrow \quad ct = \pm\sqrt{|\vec{x}|^2 - C} \tag{1.87c}$$



This picture leads immediately to two conclusions:

ii | ⊲ Two *distinct* events $E_1 \ni (t_1, \vec{x}_1)_K$ and $E_2 \ni (t_2, \vec{x}_2)_K$ with coordinates in K:

- If $\Delta s^2 \geq 0$ (= time-like or light-like), then

$$\text{either} \quad \forall_K : \ (t_1)_K > (t_2)_K \quad \text{or} \quad \forall_K : \ (t_1)_K < (t_2)_K \, . \tag{1.88}$$

  This means that for time-like or light-like separated events all observers agree on their temporal ordering! Note that they do not necessarily agree on the time $(t_1)_K - (t_2)_K$ elapsed between the two events.

  Proof: Assume $(t_1)_A < (t_2)_A$ and $(t_1)_B > (t_2)_B$ for two inertial systems A and B. Because of the continuity of Lorentz transformations there must exist a frame C with $(t_1)_C = (t_2)_C$. But in this frame $(\Delta s)^2_C = -(\Delta\vec{x})^2_C \geq 0$ such that $(\vec{x}_1)_C = (\vec{x}_2)_C$ and therefore $E_1 = E_2$ (which contradicts our assumption that the two events are distinct).

  Proof by picture!

- If $\Delta s^2 < 0$ (= space-like), then

$$\exists_{A,B} : \ (t_1)_A > (t_2)_A \quad \text{and} \quad (t_1)_B < (t_2)_B \, . \tag{1.89}$$

  This means that for space-like separated events there are always observers who see E_1 happening before E_2 while other observers see E_1 happening after E_2. The temporal order of space-like separated events is therefore observer-dependent!

  Proof: ➔ Problemset ?

  Proof by picture!

iii | Conventional relation of time order and causality:

$$E_1 \text{ can causally affect } E_2 \ \Rightarrow \ E_1 \text{ happens before } E_2 \tag{1.90}$$

Since causality should be an objective, observer-independent fact, and we just showed that only time- and light-like separated events have an observer-independent temporal order, it is reasonable to define the following …

… ↓ *(strict) partial order* $\prec$ on the set $\mathcal{E}$ of events:

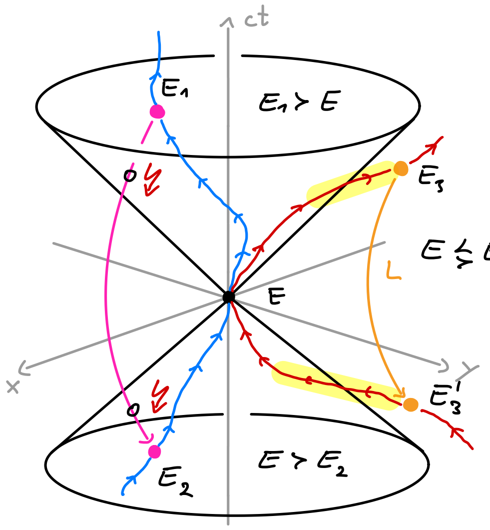
$$E_1 \prec E_2 \quad :\Leftrightarrow \quad \Delta s^2 \geq 0 \ \text{ and } \ t_1 < t_2 \ : \ \text{"}E_1 \text{ can affect } E_2\text{"} \tag{1.91}$$

$$E_1 \succ E_2 \quad :\Leftrightarrow \quad \Delta s^2 \geq 0 \ \text{ and } \ t_1 > t_2 \ : \ \text{"}E_2 \text{ can affect } E_1\text{"} \tag{1.92}$$

This is a *partial* order because for $\Delta s^2 < 0$ there is *no* relation between E_1 and E_2 (we denote this by $E_1 \between E_2$).

To be a partial order, one has to show *irreflexivity* (which is trivial since $t < t$ is not true) and *transitivity*. To show transitivity, show that $\Delta s^2_{1,2} \geq 0$ and $\Delta s^2_{2,3} \geq 0$ together with $t_2 > t_1$ and $t_3 > t_2$ implies $\Delta s^2_{1,3} \geq 0$ and $t_3 > t_1$ (use the triangle inequality).

iv | This definition of causality is consistent with our previous findings that no signal can travel faster than the speed of light c:



- $E \prec E_1$: There exists a signal trajectory $\vec{x}(t)$ with $\left|\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t}\right| \leq c$ connecting the two events (blue in the sketch).
- $E \between E_3$: Any trajectory $\vec{x}(t)$ connecting the two events (red in the sketch) has some segment with $\left|\frac{\mathrm{d}\vec{x}(t)}{\mathrm{d}t}\right| > c$ (yellow in the sketch). Since this is physically impossible, there is no signal of any kind that can mediate causal influence from E to E_3 (and vice versa).

  This follows from an application of (a generalization of) the ↓ *mean value theorem*.

6 | Since the causal structure $(\mathcal{E}, \prec)$ is observer independent:

$$\boxed{\text{There is } \textit{no} \text{ relativity of causality in SPECIAL RELATIVITY!}}$$

If one observer states that E_1 can causally affect E_2, then *all* observers will agree on this statement.

7 | *Fun fact:*

If one starts from the causal structure $(\mathcal{E}, \prec)$ and derives the group of ↑ *causality-preserving automorphisms* Φ,

$$E_1 \prec E_2 \quad \Leftrightarrow \quad \Phi(E_1) \prec \Phi(E_2)\,, \tag{1.93}$$

one again finds the homogeneous Lorentz transformations (boosts & rotations) that we constructed above (plus space-inversion, spacetime dilations and translations), see Ref. [38] for more details. Most interestingly, for the proof neither a continuity assumption on Φ nor a topology on $\mathcal{E}$ is required; all this follows (at least in $2+1$ spacetime dimensions and more) from the partial order $\prec$.

1.7. ‡ Relativity, compressibility, and the anthropic principle

The statements in this section are not specific to Einstein's relativity principle **SR**.

1 | Relativity principles …

- … are statements about (the existence of) *symmetries* of spacetime.
- … imply the versatility of models to predict events *from many viewpoints*.
- … are statements about an *a priori* unnecessary *simplicity* of nature.

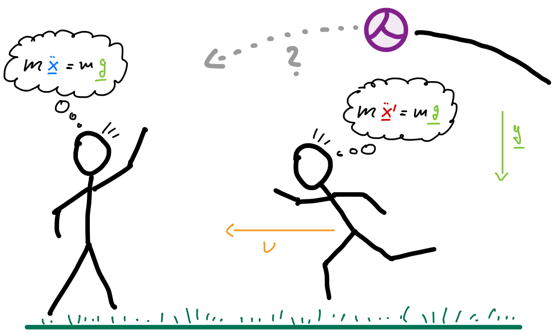
2 | Imagine a world *without any* relativity principle:

The equations (models) that capture physical laws faithfully are different from frame to frame.

→ Your brain must learn arbitrary many *different* models adapted to all possible reference frames to anticipate the future in all situations.

→ Biologically impossible (your brain capacity is finite, building models is expensive)

3 | Example: Catching balls:



Notice that most reference frames that we naturally encounter are (approximately) inertial only in x and y direction (the axes that are locally parallel to earth's surface) and constantly accelerated in z direction (the axis perpendicular to earth's surface; the acceleration is $g \approx 9.81\,\mathrm{m/s^2}$). The non-relativistic symmetries that relate these frames are a *subgroup* of the full Gallilei group (excluding rotations around the x and y axes as well as "large" translations). Our brain contains only models for these frames (equipped with Cartesian coordinates). Have you ever tried throwing or catching a ball in frames with acceleration in x or y directions (like a centrifuge)?

➔ YouTube Video: The artificial gravity lab (Tom Scott)

Note that it is not *impossible* to train specific models for other frames to which the relativity principle of our everyday experience does not apply (after some practice you *can* throw and catch balls in a centrifuge of constant angular velocity). But this is just *one* additional model and even this is not implemented in our brains by default!

4 | Relativity principle

→ Descriptions of natural phenomena are highly *compressible*.

→ Only *few* models (equations) are necessary to anticipate the future.

5 | Anthropic principle:

Question: Why are there spacetime symmetries / relativity principles in the first place?

Answer: Because if there were none, evolution would most likely be impossible, hence we would be unable to ask the question.

Note that evolution relies on the somewhat reliable proliferation of information over time. This seems only possible if the individuals carrying this information survive. Surviving in environments with life-threatening phenomena (thunderstorms, predators, ...) relies on its (approximate) *predictability* by (approximate) *models* that are learned evolutionary and/or by experience.

For this argument to work some form of "ensemble interpretation" of reality is required (e.g. ↑ *multiverses*) [39].

2. Kinematic Consequences

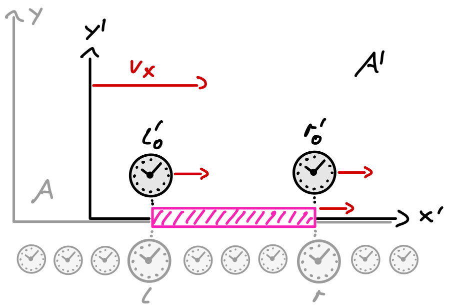
In this chapter we study implications of the special Lorentz transformations Eq. (1.75) and Eq. (1.77) that follow without imposing a model-specific dynamics (= equations of motion). We refer to these implications as *kinematic* because they follow from fundamental constraints on the degrees of freedom of all relativistic theories. The phenomena we will encounter are therefore *features of spacetime itself* – and not of some entities that live on/in (or couple to) spacetime.

¡! The phenomena we will encounter are *not* "illusions" (in the sense that we "see" things differently than they "really are"). Remember that we precisely defined what we mean by observers/reference frames; in particular, we emphasized that we do not "look" at anything, we *measure* events in a systematic way, using a well-defined structure called ← *inertial system*. All phenomena we will encounter are derived from and to be understood in this operational, physically meaningful context.

2.1. Length contraction and the Relativity of Simultaneity

1 | ⊲ Inertial systems $A \xrightarrow{v_x} A'$ with rod on x'-axis and at rest in A':

Remember that $A \xrightarrow{v_x} A'$ denotes a boost in x-direction with v_x (as measured in A) where the spatial axes of both A and A' coincide at $t = 0$:



In such situations, we refer to A' as the ⁂ *rest frame* of the rod and A as the ⁂ *lab frame* (some call A the ⁂ *stationary frame*). In the following, coordinates of events in the inertial system A' are marked by primes.

2 | First, we have to define what we mean by the "lenght" of an object:

"Length" is an intrinsically non-local concept. It is not something you can measure or define at a single point in space. Consequently, there are no "length-events" in $\mathcal{E}$. Thus we need an algorithm (= operational definition) of what we mean by "length".

⊲ Two event *types*:

$$\{e_L\} = \{\langle\text{Left end of rod detected}\rangle\} \tag{2.1a}$$
$$\{e_R\} = \{\langle\text{Right end of rod detected}\rangle\} \tag{2.1b}$$

Think of an event *type* as a set (equivalence class) of all elementary events that you deem ↑ *type-identical* (but not ↑ *token-identical*). In the example given here, there will be many events e_L in spacetime that signify "Left end of rod detected" (if there is one rod, there will be one such event for each time t); these are *different* events of the *same type* $\{e_L\}$.

One could even declare that the event *type* $\{e_L\}$ *is* what we refer to as "the left end of the rod."

→ Algorithm `LENGTH` to compute *"Length of Rod"* in system K at time t:

LENGTH:
→ **Input:** Coincidences $\mathscr{E}$, Inertial system label K, Time t
← **Output:** Length l_K of rod at time t as measured in K

1. Find (unique) event $L \in \mathscr{E}$ with $\{e_L\} \in L$ and $(t, \vec{l})_K \in L$.
2. Find (unique) event $R \in \mathscr{E}$ with $\{e_R\} \in R$ and $(t, \vec{r})_K \in R$.
3. Return $l_K := |\vec{l} - \vec{r}|$.

Here, $\{e_L\} \in L$ is shorthand for $\{e_L\} \cap L \neq \emptyset$. In words: the coincidence class L contains an event of the *type* "Left end of rod detected".

Note that we define "length" as the spatial distance between the two ends of the rod *at the same time* t (as measured by the clocks in K). I hope you agree that this is what one typically means by "length."

↓Lecture 5 [14.11.23]

3 | We now apply this algorithm twice, in the lab frame A and the rest frame A':

i | Rest frame A':

⁂ *Proper length* ≡ ⁂ *Rest length* := Length of rod in A':

$$l_0 := \mathsf{LENGTH}(\mathcal{E}, t'_0; A') = |\vec{l}'_0 - \vec{r}'_0| = |l'_0 - r'_0| \tag{2.2}$$

with simultaneous clock events $(t'_0, \vec{l}'_0)_{A'} \in L_0$ and $(t'_0, \vec{r}'_0)_{A'} \in R_0$.

The time t'_0 that we choose is irrelevant since the rod is (by definition) at rest in A'. Since the rod lies on the x'-axis, it is $\vec{l}'_0 = (l'_0, 0, 0)$ and $\vec{r}'_0 = (r'_0, 0, 0)$.

The subscript "0" in L_0 indicates that this is a specific event (coincidence class) we selected in A' to compute the length of the rod. It does *not* mean "as seen from the rest frame A'" or anything like that. Remember that coincidence classes in $\mathcal{E}$ are objective information!

ii | Lab frame A:

Length of moving rod in A:

$$l := \mathsf{LENGTH}(\mathcal{E}, t; A) = |\vec{l} - \vec{r}| \tag{2.3}$$

with simultaneous clock events $(t_l, \vec{l})_A \in L$ and $(t_r, \vec{r})_A \in R$ with $t_l = t_r = t$.

The time t that we choose might be irrelevant as well, but we do not know this yet.

¡! There is no reason to assume that the events L_0/R_0 chosen in A' to measure the length of the rod are identical to the events L/R used in A: $L_0 \neq L$ and $R_0 \neq R$ in general.

4 | How does l_0 relate to l?

i | In Section 1.5 we did a lot of hard work to compute the transformation φ which transforms the coordinates of an event in one inertial system into the coordinates of *the same event* in another inertial system. We identified the transformation as the Lorentz transformation:

$$\Lambda(A \xrightarrow{v_x} A') : \; [E]_A = (t, \vec{x}) = x \mapsto \Lambda_{v_x} x = x' = (t', \vec{x}') = [E]_{A'} \tag{2.4}$$

ii | So let us use this tool [namely Eq. (1.77)] to obtain the coordinates of the events L and R (used for the length measurement in A) in the rest frame A' of the rod:

$$[L]_{A'} = \begin{cases} ct'_l = \gamma\left(ct_l - \frac{v_x}{c} l_x\right) \\ l'_x = \gamma(l_x - v_x t_l) \\ l'_y = l_y \\ l'_z = l_z \end{cases} \quad \text{and} \quad [R]_{A'} = \begin{cases} ct'_r = \gamma\left(ct_r - \frac{v_x}{c} r_x\right) \\ r'_x = \gamma(r_x - v_x t_r) \\ r'_y = r_y \\ r'_z = r_z \end{cases} \tag{2.5}$$

Here we use $\vec{l} = (l_x, l_y, l_z)$ and $\vec{r} = (r_x, r_y, r_z)$. Since we declared that the rod is fixed on the x'-axis of A', and $\{e_L\} \in L$ and $\{e_R\} \in R$, it must be $l'_y = l'_z = r'_y = r'_z = 0$, and therefore $\vec{l} = (l_x, 0, 0)$ and $\vec{r} = (r_x, 0, 0)$. That is, the rod is not rotated by the boost and always lies on the x-axis of A as well. In particular: $l = |\vec{l} - \vec{r}| = |l_x - r_x|$.

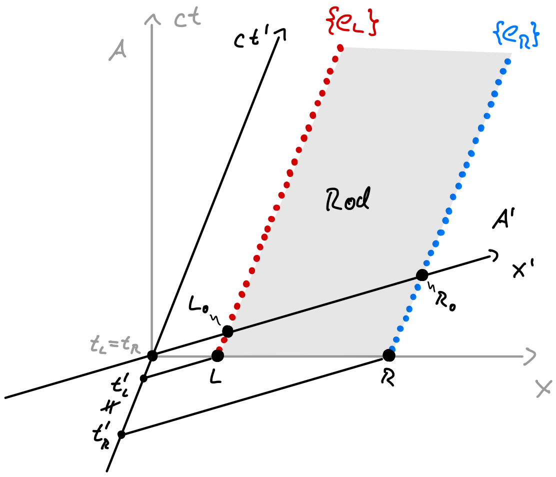
→ Two immediate conclusions:

a | In A' the two events L and R are *no longer simultaneous*:

$$t_l = t_r \text{ in } A \quad \text{but} \quad t'_l \neq t'_r \text{ in } A' \text{ (since } l_x \neq r_x\text{)}. \tag{2.6}$$

$\rightarrow$ The simultaneity of events is *observer-dependent*.

This ambiguity of simultaneity can be graphically illustrated in a spacetime diagram (for details on how to draw the (t', x')-axes in A: ➔ Problemset 2):



- As a side note, this calculation implies that not only is it generally *not* true that $L_0 = L$ and $R_0 = R$, it is actually *impossible* (at least for both pairs).
- In the sketch above, the "interior of rod"-events are painted gray. One is tempted to ask: Which "line" of these events *is* the rod? The counterintuitive answer is that this depends on the observer: For A-observers, *horizontal* lines of gray events make up "the rod", whereas for the A'-observer *tilted* lines are "the rod". It is actually more reasonable to think of the complete area of gray events as "the rod", just as the event type $\{e_L\}$ is "the left edge" of the rod. This suggests that our intuitive concept of the *instantaneous existence of extended objects* – which feels so natural to us – is, to some extend, misleading.

b | In A' the coordinate distance is different:

$$|l'_x - r'_x| \overset{t_l = t_r}{=} \gamma |l_x - r_x| \overset{v_x \neq 0}{\neq} |l_x - r_x| = l \tag{2.7}$$

¡! The time-dependence cancels so that the expressions are time-independent.

At this point, it is a bit premature to identify the left-hand side as the *rest length* l_0 of the rod because these are spatial coordinates of events that are *not simultaneous*! (Remember that the length of any object in any frame is defined as the coordinate distance of *simultaneous* events.)

However, since A' is (by definition) the *rest frame* of the rod, the position labels of the A'-clocks adjacent to the ends of the rod are the same for all events:

$$\left.\begin{aligned} l'_x &\overset{\{e_L\} \ni L}{=} l'_0 \\ r'_x &\overset{\{e_R\} \ni R}{=} r'_0 \end{aligned}\right\} \Rightarrow |l'_x - r'_x| = |l'_0 - r'_0| = l_0 \tag{2.8}$$

→ ⁂ *Length contraction* ≡ ⁂ *Lorentz contraction:*

A rod of rest length l_0 is *shorter* if measured from an inertial system in relative motion:

$$\boxed{l = l_0\sqrt{1-\frac{v^2}{c^2}} \overset{v\neq 0}{<} l_0} \tag{2.9}$$

- ¡! Due to isotropy, this result is true for any length of extended objects *in the direction of the boost*. A rod along the y'-axis, for example, is contracted according to Eq. (2.9) for a boost in y-direction, but not for a boost in x-direction.
- The rod is just a proxy for *any* physical object; the Lorentz contraction therefore affects all physical objects in the same way. The contraction is not a dynamical feature of the object itself (like a force that compresses the atomic lattice) but an intrinsic property of space(time).
- Note that we say above "if measured from ..." and not "as viewed from ...." This distinction is important: If you ask how you would *visually perceive* extended objects flying by (or how they look on a picture taken by a camera) you have to factor in that the photons bouncing of the object at different points take different times to reach your eye (our the camera sensor). If you do the math (➔ Problemset 3), this additional optical effect leads to the surprising result that 3D objects actually do *not look* "squeezed" but *rotated*. This implies in particular that a moving sphere still *looks* like a sphere and not like an ellipse (↑ *Penrose-Terrell effect* [40, 41], see also Ref. [42]).

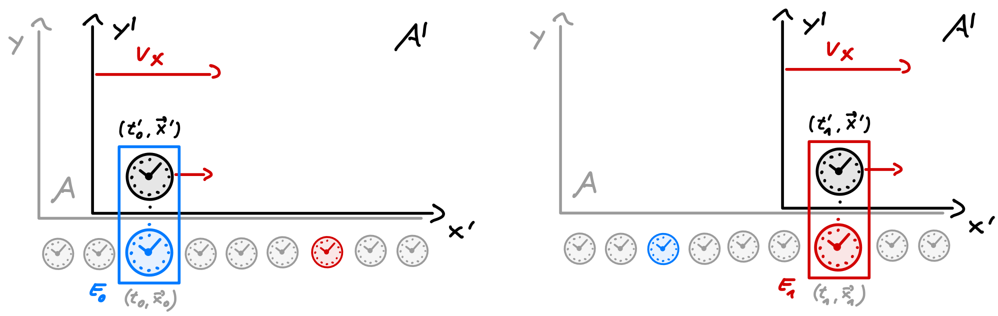
  You can experience this effect (among others) in the educational game "A Slower Speed of Light," which has been developed by the MIT Game Lab for educational purposes, and can be downloaded here for Windows, Mac, and Linux (➔ Problemset 3):

  ➔ Download "A Slower Speed of Light"

  You should always keep in mind, however, that this "looking" is *not* what we refer to as *observing* in RELATIVITY; the latter has been defined operationally as a measurement procedure at the beginning of this course.

2.2. Time dilation

1 | ⊲ Inertial systems $A \xrightarrow{v_x} A'$ and a clock $\vec{x}'$ at rest in A':



2 | ⊲ Two events:

$$A'\text{-Clock } \vec{x}' \text{ meets } A\text{-clock } \vec{x}_0:\quad (t'_0, \vec{x}')_{A'} \sim (t_0, \vec{x}_0)_A \in E_0 \tag{2.10a}$$

$$A'\text{-Clock } \vec{x}' \text{ meets } A\text{-clock } \vec{x}_1:\quad (t'_1, \vec{x}')_{A'} \sim (t_1, \vec{x}_1)_A \in E_1 \tag{2.10b}$$

¡! The two events E_0 and E_1 relate *three* different clocks: The single A'-clock $\vec{x}'$ and two *different* A-clocks $\vec{x}_0$ and $\vec{x}_1$.

3 | As for length, the concept of "duration" cannot be defined locally in spacetime. We therefore need an operational definition (algorithm) of "duration":

```
DURATION:
→ Input: Two events E_0 and E_1, Inertial system label K
← Output: Time interval Δt_K between events as measured in K
   1. Find (unique) clock event (t_0, x⃗_0)_K ∈ E_0.
   2. Find (unique) clock event (t_1, x⃗_1)_K ∈ E_1.
   3. Return Δt_K := t_1 − t_0.
```

Hopefully you agree that this is a reasonable definition of the duration (or time interval) between two events.

4 | We can now apply this algorithm to determine the time elapsed between E_0 and E_1:

$$\text{In } A': \quad \Delta t' = \mathtt{DURATION}(E_0, E_1; A') = t_1' - t_0' \quad \text{Measured by a single clock!} \tag{2.11a}$$

$$\text{In } A: \quad \Delta t = \mathtt{DURATION}(E_0, E_1; A) = t_1 - t_0 \quad \text{Measured by two clocks!} \tag{2.11b}$$

5 | <u>How does Δt relate to $\Delta t'$?</u>

i | Since $(t_0', \vec{x}')_{A'} \sim (t_0, \vec{x}_0)_A$ and $(t_1', \vec{x}')_{A'} \sim (t_1, \vec{x}_1)_A$, we can use the Lorentz transformation to translate between the coordinates:

$\xrightarrow{\text{Inverse of Eq. (1.77)}}$

Remember that $\Lambda_{\vec{v}}^{-1} = \Lambda_{-\vec{v}}$ because of reciprocity; the inverse Lorentz transformation can then be obtained by substituting $v_x \mapsto -v_x$:

$$[E_0]_A = \begin{cases} ct_0 = \gamma\left(ct_0' + \frac{v_x}{c}x'\right) \\ x_0 = \gamma(x' + v_x t_0') \end{cases} \quad \text{and} \quad [E_1]_A = \begin{cases} ct_1 = \gamma\left(ct_1' + \frac{v_x}{c}x'\right) \\ x_1 = \gamma(x' + v_x t_1') \end{cases} \tag{2.12}$$

We omit the other two coordinates since they are invariant anyway; the transformation of the spatial coordinate is also not necessary for the following derivation.

ii | Subtracting the equations for the time coordinate of both events yields:

$$c(t_1 - t_0) = \gamma c(t_1' - t_0') \tag{2.13}$$

Note that in the inverse Lorentz transformation Eq. (2.12) the position coordinate in A' is x' for *both* events because the *same* A'-clock takes part in both coincidences.

iii | ⁂ <u>*Time dilation:*</u>

→ The moving clocks in A' run slower than the stationary clocks in A:

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \overset{v \neq 0}{>} \Delta t_0} \tag{2.14}$$

We renamed $\Delta t' \equiv \Delta t_0$ to emphasize the analogy to the *proper length* l_0:

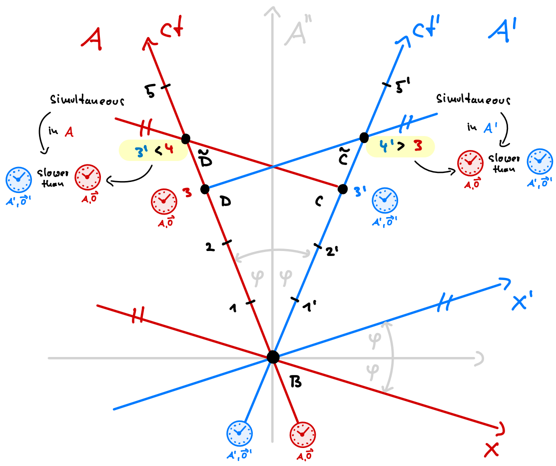
Δt_0: ⁂ *Proper time* elapsed in A' between E_0 and E_1
Δt: Time elapsed in A between E_0 and E_1

- The characteristic feature of the *proper time* Δt_0 between two (time-like separated) events E_0 and E_1 is that it can be measured by a *single* inertial clock that takes part in both events. All other time intervals must be measured by subtracting the reading of *two different* clocks. Eq. (2.14) tells you that these time intervals are always longer than the proper time Δt_0.
- ¡! Due to isotropy, our result above is true for boosts in any direction.

  Note that in the derivation above, we did *not* impose any special constraints on the positions of the clocks (except that they coincide pairwise at E_0 and E_1). In particular, we did not assume (despite the sketch suggesting this) that the clocks are located on the x/x'-axis. *All* clocks in A' are slowed down in the same way, irrespective of their location!
- This result does *not* contradict our assumption that all clocks are type-identical (= run with the same rate if put next to each other at rest) because the two events needed to compare the tick rate of moving clocks necessarily describe coincidences between *different* pairs of clocks.

6 | Relativity principle:

Because of the relativity principle **SR** time dilation must be completely *symmetrical*: The A'-clocks run slower compared to the A-clocks, and the A-clocks run slower compared to the A' clocks. That this is indeed that case (without being a clock "paradox") is best illustrated in a symmetric spacetime diagram:



The existence of the "median frame" A'' between $A \xrightarrow{v_x} A'$ can be easily shown with the addition for collinear velocities Eq. (1.70). This symmetric form of a spacetime diagram is sometimes called ↑ *Loedel diagram* [43] and makes the symmetry between inertial frames manifest; in particular, the units on the axes of A and A' are identical (they are not identical to the units of A'', tough). In this symmetric form, the t'-axis is orthogonal to the x-axis and the t-axis to the x'-axis. Note that because of the relativistic addition of velocities, it is $A'' \xrightarrow{\tilde{v}_x} A'$ and $A'' \xrightarrow{-\tilde{v}_x} A$ with $\tilde{v}_x = v_x \frac{\gamma}{1+\gamma}$ and $\tan(\varphi) = \frac{\tilde{v}_x}{c}$ (➲ Problemset 3). Only in the non-relativistic limit $v_x/c \to 0$ one finds $\tilde{v}_x = \frac{v_x}{2}$ as naïvely expected.

Note that due to the relativity of simultaneity, the two observers use *different* pairs of clock-events to decide which of the two origin clocks runs slower:

- For A the two clock events $\tilde{D}$ and C are simultaneous such that one has to conclude that the (blue) A'-clock runs slower than the (red) A-clock.

- By contrast, for the observer A' the two events D and $\tilde{C}$ are simultaneous such that one has to conclude that the (red) A-clock runs slower than the (blue) A'-clock.

It is evident from the diagram that there is no disagreement about coincidences of events (or readings of clocks). It is just the observer-dependent concept of simultaneity that leads to the seemingly "paradoxical" reciprocity of time dilation.

7 | Experiments:

- *Muon decay [44]:*

  Muons quickly decay into electrons (and neutrinos):

  $$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e \,. \tag{2.15}$$

  This decay can be readily observed in storage rings of particle colliders like CERN. The lifetime of muons *at rest* (measured by clocks in an inertial laboratory frame) is $\tau_\mu^0 \approx 2.1948(10)\,\mu\mathrm{s}$. However, the lifetime of muons in flight (close to the speed of light) is measured to be $\tau_\mu \approx 64.368(29)\,\mu\mathrm{s}$, i.e., much longer! If one carefully takes into account the speed of the muons and additional experimental imperfections, this result fits Eq. (2.14) with deviations of only $\sim 0.1\,\%$ [44].

  Notes:

  - In the rest frame of the flying *muons* one would measure the usual lifetime $\tau_\mu^0 \approx 2.1948(10)\,\mu\mathrm{s}$. However, in this frame, the *laboratory* is *Lorentz contracted* such that the muon reaches exactly the same point in space where it decays in this "shorter" lifetime. Note how time-dilation and Lorentz contraction provide different explanations for the same experimental obervation.
  - One can also use different particle species to study time dilation, for example *pions* (a sort of meson, i.e., a hadron with one quark and one antiquark) [45].

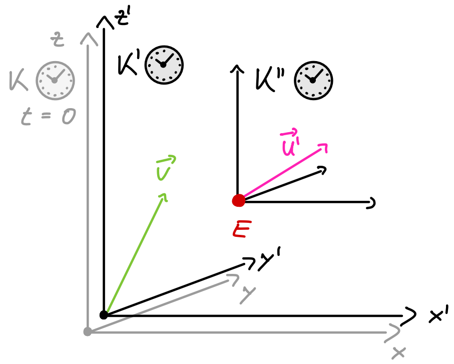
- *Hafele-Keating experiment [46, 47]:*

  In 1971, J.C. Hafele and R. E. Keating took four Cesium atomic clocks along commerical jet flights around the globe twice: once eastward and once westward. Compared to a reference clock on the ground, the clocks on the eastward flight lost on average $\sim 59\,\mathrm{ns}$ (= they ran slower) and the clocks on the westward flight gained $\sim 273\,\mathrm{ns}$ (= they ran faster). To understand this qualitatively, note that the reference clock on the ground is *rotating* (together with earth) and therefore is *not* an inertial clock. Therefore imagine an (approximately) inertial reference system flying along earth around the sun, and from this system look down on the north pole; earth is now slowly rotating beneath you. From this inertial system, the eastward flight has higher velocity than the reference clock, which, in turn, has higher velocity than the westward flight. Thus you find that the eastward clock runs slower than the reference clock which runs slower than the westward clock (this is also true if the clocks are accelerated, → *below*). These theoretical considerations are explained in [46].

2.3. Addition of velocities

Details: ➔ Problemset 2

1 | ∢ Particle moving with $\vec{u}' = \frac{d\vec{x}'}{dt'}$ in system K' and inertial system K with $K \xrightarrow{\vec{v}} K'$:



2 | Velocity $\vec{u}$ in K:

$$\boxed{\vec{u} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \equiv \vec{v} \oplus \vec{u}' \stackrel{\circ}{=} \frac{1}{1+\frac{\vec{v}\cdot\vec{u}'}{c^2}} \left[\vec{v} + \frac{\vec{u}'}{\gamma_v} + \frac{\gamma_v}{c^2(1+\gamma_v)}(\vec{u}'\cdot\vec{v})\,\vec{v}\right]} \tag{2.16}$$

Proof: Use Eq. (1.75) (➔ Problemset 2).

¡! The relativistic addition of velocities $\oplus$ is in general not commutative ($\vec{v} \oplus \vec{u} \neq \vec{u} \oplus \vec{v}$) nor associative [$\vec{v} \oplus (\vec{u} \oplus \vec{w}) \neq (\vec{v} \oplus \vec{u}) \oplus \vec{w}$]. As you can easily see from Eq. (2.16), it is also not linear: $(\lambda\vec{v}) \oplus (\lambda\vec{u}) \neq \lambda(\vec{v} \oplus \vec{u})$. Be careful: There are different notations (in particular: orderings) used in the literature.

3 | $\sphericalangle$ Non-relativistic limit ($c \to \infty \Rightarrow \gamma_v \to 1$):

$$\lim_{c\to\infty} \vec{v} \oplus \vec{u}' = \lim_{c\to\infty} \vec{u}' \oplus \vec{v} = \vec{v} + \vec{u}' \tag{2.17}$$

→ Galilean addition of velocities

4 | Special case: $\vec{v} = (v_x, 0, 0)$:

$$u_x \stackrel{\circ}{=} \frac{v_x + u'_x}{1+\frac{v_x u'_x}{c^2}}\,, \quad u_y \stackrel{\circ}{=} \frac{u'_y/\gamma_v}{1+\frac{v_x u'_x}{c^2}}\,, \quad u_z \stackrel{\circ}{=} \frac{u'_z/\gamma_v}{1+\frac{v_x u'_x}{c^2}}\,. \tag{2.18}$$

¡! Note that also the transverse components of $\vec{u}'$ are modified, but in a different way than the collinear component u'_x. For $\vec{u}' = (u'_x, 0, 0)$ we get our previous result for collinear velocities Eq. (1.70) back.

5 | Thomas-Wigner rotation [48, 49]:

Remember that for *collinear* addition of velocities the concatenation of two boosts yields another boost: $\Lambda_{v_x}\Lambda_{u_x} = \Lambda_{w_x}$ [recall Eq. (1.57)].

As a straightforward (but tedious) calculation using two general boosts Eq. (1.75) shows, this is *not* true in general: $\Lambda_{\vec{v}}\Lambda_{\vec{u}} \neq \Lambda_{\vec{w}}$ with $\vec{w} = \vec{u} \oplus \vec{v}$. Rather one finds

$$\Lambda_{\vec{v}}\Lambda_{\vec{u}} = \Lambda_{\vec{u}\oplus\vec{v}}\Lambda_{R(\vec{u},\vec{v})} \tag{2.19}$$

with the ⁂ *Thomas-Wigner rotation* $R(\vec{u},\vec{v}) \in \mathrm{SO}(3)$ (we omit the expliclit form of $R(\vec{u},\vec{v})$ here).

This is not in contradiction with our general addition for velocities above because there we were only interested in the velocity of a moving particle (which you can identify with the origin of its rest frame K''); we completely ignored the *axes* of K''. The Thomas-Wigner rotation tells you that the concatenation of two *pure* boosts is *not* a pure boost in general.

2.4. Proper time and the twin "paradox"

1 | $\sphericalangle$ Time-like trajectory $\mathcal{P} \subseteq \mathcal{E}$ of a spaceship with departure $D \in \mathcal{P}$ and arrival $A \in \mathcal{P}$.

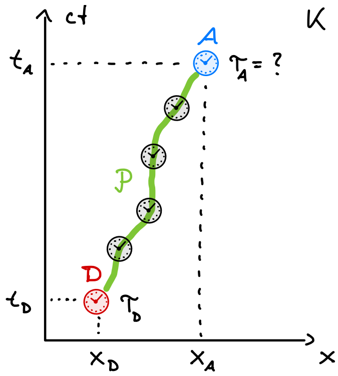
$\sphericalangle$ Coordinate parametrization $\vec{x}(t)$ of $\mathcal{P}$ in system K with

$$\text{departure } [D]_K = (t_D, \vec{x}_D) \quad \text{and} \quad \text{arrival } [A]_K = (t_A, \vec{x}_A) : \tag{2.20}$$

Formally, $\mathcal{P}$ is a set of coincidence classes parametrized in K by the clock events $(t, \vec{x}(t))_K$:

$$\mathcal{P} = \{ [(t, \vec{x}(t))_K] \,|\, t \in [t_D, t_A] \} \subseteq \mathcal{E} \,. \tag{2.21}$$

This suggests the formal notation $[\mathcal{P}]_K = (t, \vec{x}(t))$.



2 | Thought experiment:

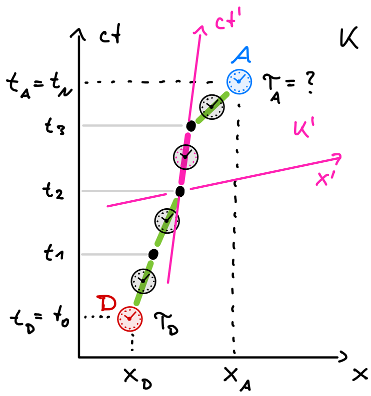
The spaceship takes a clock along and resets it to $\tau_D = \tau(t_D)$ at departure D.

What is the reading $\tau_A = \tau(t_A)$ of the clock at arrival A?

We assume that the clock in the spaceship is type-identical to the clocks used for inertial observers.

3 | *Idea:*

Approximate the trajectory by a *polygon* of N segments $i = 1, \ldots, N$ separated by time steps t_i (with $t_0 := t_D$ and $t_N := t_A$):



i | Let $\Delta t_i := t_{i-1} - t_i$ and $\Delta \vec{x}_i := \vec{x}(t_{i-1}) - \vec{x}(t_i)$

For each segment, there is an inertial frame K' with a t'-axis that follows the spacetime segment (because all segments are time-like!). This is the instantaneous rest frame of the spaceship where the clock in the spaceship and the origin clock of K' are at the same place and at rest relative to each other. Since the clocks are type-identical, the time $\Delta\tau_i$ accumulated by the spaceship clock on this segment is identical to the time $\Delta t_i'$ elapsed for the origin clock of K' on this segment: $\Delta\tau_i = \Delta t'_i$. This time is equal to the spacetime interval $(\Delta s'_i)^2 = (c\Delta t'_i)^2 - 0$ because the origin clock is at rest in K' (so that $\Delta\vec{x}'_i = \vec{0}$). But remember that the spacetime interval $(\Delta s'_i)^2$ is Lorentz invariant so that we can calculate *the same number* in any inertial system: $(\Delta s'_i)^2 = (\Delta s_i)^2 = (c\Delta t_i)^2 - (\Delta\vec{x}_i)^2$.

In summary, on the ith interval, the spaceship clock accumulates the time

$$\Delta\tau_i = \frac{\Delta s_i}{c} := \frac{\sqrt{\Delta s_i^2}}{c} = \frac{\sqrt{(c\Delta t_i)^2 - (\Delta\vec{x}_i)^2}}{c} = \Delta t_i \sqrt{1 - \frac{(\Delta\vec{x}_i/\Delta t_i)^2}{c^2}} \tag{2.22}$$

The above chain of arguments provided us with a *physical interpretation* for the Lorentz invariant spacetime interval $(\Delta s)^2 > 0$ of time-like separated events: It measures (up to a factor of c) the time accumulated by an inertial (= unaccelerated) clock that takes part in both events.

ii | Continuum limit $N \to \infty$ ($v(t) := |\vec{v}(t)| = |\dot{\vec{x}}(t)|$):

$$\boxed{\mathrm{d}\tau = \frac{\mathrm{d}s}{c} = \mathrm{d}t\sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} \quad \Leftrightarrow \quad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{v(t)}} \tag{2.23}$$

Note that this is just an infinitesimal version of the time-dilation formula Eq. (2.14) with $\Delta t \to \mathrm{d}t$ and $\Delta t_0 \to \mathrm{d}\tau$.

Since $(\Delta s)^2 = (\Delta s')^2$ is *Lorentz invariant*:

$$K \xrightarrow{\Lambda} K': \quad \mathrm{d}t\sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} = \frac{\mathrm{d}s}{c} = \frac{\mathrm{d}s'}{c} = \mathrm{d}t'\sqrt{1 - \frac{\dot{\vec{x}}'(t')^2}{c^2}} \tag{2.24}$$

You can check this also explicitly using the Lorentz transformation Eq. (1.75).

↓**Lecture 6** [21.11.23]

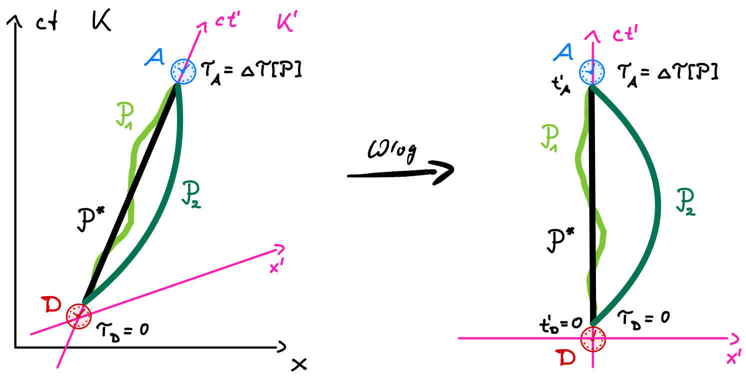
iii | → ⁂ *Proper time* accumulated by the spaceship clock along the trajectory $\mathcal{P}$:

$$\boxed{\Delta\tau[\mathcal{P}] = \lim_{N\to\infty}\sum_{\substack{\text{Segment}\\ i=1}}^{N}\Delta\tau_i = \int_{\mathcal{P}} \mathrm{d}\tau = \int_{\mathcal{P}}\frac{\mathrm{d}s}{c} := \int_{t_D}^{t_A} \mathrm{d}t\,\sqrt{1-\frac{\dot{\vec{x}}(t)^2}{c^2}}} \tag{2.25}$$

- As constructed, the proper time $\Delta\tau[\mathcal{P}]$ of a time-like trajectory $\mathcal{P}$, parametrized by $\vec{x}(t)$ for $t \in [t_0, t_1]$, is the time elapsed by a clock that follows this trajectory in spacetime.
- ¡! This result is valid for *accelerated* clocks.

  In general, SPECIAL RELATIVITY *can* described the physics of accelerated objects as long as the descpription of the process is given in an inertial coordinate system (as is the case here).
- ¡! The right-most expression in Eq. (2.25) yields the same result *in all inertial systems* K [recall Eq. (2.24)]. This is why $\tau[\mathcal{P}]$ is a function of the *event trajectory* $\mathcal{P}$ and not its coordinate parametrization $\vec{x}(t)$. This is important: It tells us that all inertial observers will agree on the reading of the spaceship clock τ_A at arrival A (although their *parametrization* $\vec{x}(t)$ may look different).
- Note that since $\vec{x}(t)$ is assumed to be time-like, it is $\forall_t\,:\,|\dot{\vec{x}}(t)| < c$ such that the radicand is always non-negative.
- $\tau[\bullet]$ is a *functional* of the trajectory $\mathcal{P}$; this is why we use square-brackets.

4 | Which trajectory $\mathcal{P}^*$ between the two events D and A *maximizes* the proper time $\Delta\tau$?



i | D and A are *time-like* separated → ∃ Inertial system $K' = K(D, A)$ with

$$[D]_{K'} = (t'_D = 0, \vec{x}'_D = \vec{0}) \quad \text{and} \quad [A]_{K'} = (t'_A, \vec{x}'_A = \vec{0}) \tag{2.26}$$

That is, without loss of generality, we can Lorentz transform into an inertial system where the two events happen at the *same* location (and by translations we can assume that this location is the origin $\vec{0}$ and that the coordinate time is $t'_D = 0$ at D). We label the time and space coordinate in K' by t' and $\vec{x}'$. Because of the relativity principle SR, K' is as good as any system to describe events.

ii | Time of an arbitrary path $\mathcal{P} \ni D, A$ with $[\mathcal{P}]_{K'} = (t', \vec{x}'(t'))$:

$$\Delta\tau[\mathcal{P}] = \int_{t'_D}^{t'_A} \mathrm{d}t'\sqrt{1-\frac{\dot{\vec{x}}'(t')^2}{c^2}} \overset{\vec{x}'(t')\equiv\vec{0}}{\leq} \int_{t'_D}^{t'_A}\mathrm{d}t' = t'_A - t'_D = \Delta\tau[\mathcal{P}^*] \tag{2.27}$$

Here $\mathcal{P}^*$ is the trajectory between D and A that is parametrized by the constant function $\vec{x}'(t') \equiv \vec{0}$ in K'. In other inertial systems, this trajectory will not be constant; however, it is inertial, i.e., $\mathcal{P}^*$ is described by a trajectory between D and A with uniform velocity.

Check this by applying a Lorentz transformation to the coordinates $(t', \vec{0})_{K'}$!

→ Clocks that travel along the *inertial trajectory* $\mathcal{P}^*$ between D and A collect the largest proper time $\tau^* = \Delta\tau[\mathcal{P}^*]$.

Collecting the "largest time" means that the these clocks run the *fastest*.

5 | It is important to let this result sink in:

Let K' be the rest frame of earth (which is located in the origin $\vec{0}$) and consider two twins of age τ_D:

- **Twin S** departs with a **S**paceship at D, flies away from earth, turns around and returns to earth at A. **Twin S** therefore follows a trajectory similar to $\mathcal{P}_2$ in the sketches above.
- **Twin E** stays on **E**arth. He follows the inertial trajectory $\mathcal{P}^*$ in the sketches above.

We just proved above:

$$\boxed{\langle\text{Age of } \textbf{Twin S} \text{ at } A\rangle = \Delta\tau[\mathcal{P}_2] + \tau_D < \Delta\tau[\mathcal{P}^*] + \tau_D = \langle\text{Age of } \textbf{Twin E} \text{ at } A\rangle}$$

This is the famous ⁂ *Twin "paradox"*: **Twin S** aged less than **Twin E**.

6 | Why there is *no* paradox:

- If you don't see why the above result should be paradoxical:

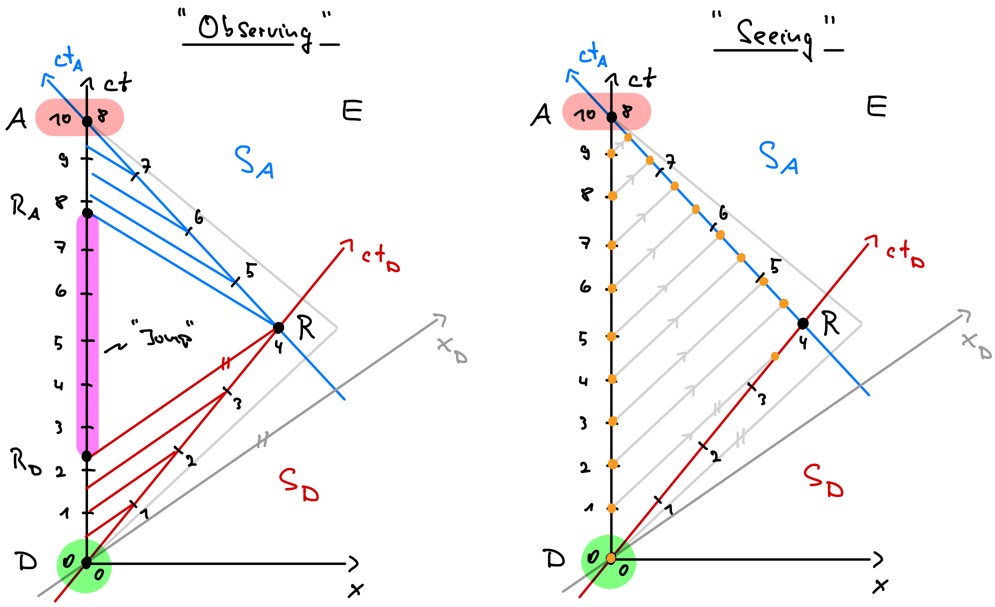
  Good! Move along. Nothing to see here! ☺
- Why one *could* conclude that the above result is paradoxical (= logically inconsistent):
  - From the view of **Twin E**, **Twin S** speeds around quickly, thus time-dilation tells him that **Twin S** should age slower. And indeed, when **Twin S** returns, he actually didn't age as much.
  - Now, you conclude, due to the relativity principle **SR**, we could also take the perspective of **Twin S** (i.e., our system of reference is now attached to the spaceship). Then **Twin S** would conclude that time-dilation makes **Twin E** (who now, together with earth, speeds around quickly) age more slowly. But this does not match up with the above result that, when both twins meet again at A, **Twin S** is the younger one! *Paradox!*

  The resultion is quite straightforwad:

  The invocation of the relativity principle **SR** in the last point is not admissible! Remember that **SR** only makes claims about the equivalence of *inertial systems*. Now have a look at the trajectory $\mathcal{P}_2$ of the spaceship again: it is clearly accelerated and cannot be inertial. And that there *is* at least a period where the spaceship (and **Twin S**) is accelerating is a *neccessity* for **Twin S** to *return* to **Twin E** (at least in flat spacetimes, but not so in curved ones [50])! This implies that the reunion of both twins at A requires at least one of them to *not* stay in an inertial system. This breaks the symmetry between the two twins and explains why the result can be (and is) asymmetric.
- ¡! For historical (and anthropocentric) reasons, the "twin paradox" is called a "paradox." We stick to this term because we have to – and not because it is appropriate name. The term "paradox" suggests an intrinsic inconsistency of RELATIVITY. As we explained above: *This is not the case.* All "paradoxes" in RELATIVITY are a consequence of unjustified, seemingly "intuitive" reasoning. The root cause is almost always an inappropriate, vague notion of "absolute simultaneity" that cannot be operationalized.

- An overview on different geometric approaches to rationalize the phenomenon can be found in Ref. [51].

  Below are two widely used spacetime diagrams of an idealized version where **Twin S** changes inertial systems only once from S_D to S_A halfway through the journey at R. You can think of this as an instantenous acceleration at the kink. Note, however, that the acceleration itself is dynamically irrelevant for the arguments; it is only important that the inertial frames in which **Twin S** departs and returns are not the same:



  - In the left diagram the *slices of simultaneity* in the two systems S_D and S_A are drawn. As predicted by time-dilation (and mandated by SR), **Twin S** observes the clocks of **Twin E** to run *slower* during his "inertial periods", i.e., while he stays in a single inertial system. However, the moment **Twin S** "jumps" from S_D to S_A at R, his notion of simultaneity changes instantaneously: In S_D, R and R_D are simultaneous; in S_A, however, R and R_A are simultaneous. Due to this jump, the record of **Twin S** contains now a temporal gap for events on earth (highlighted interval). It is this "missing" time interval that overcompensates the slower running clocks on earth (as observed from S_D and S_A) and makes **Twin S** conclude that **Twin E** ages faster (in agreement with the actual outcome of the experiment).

    If you wonder what happened to the (missing) observations of events in the triangle $R_A R R_D$: there is a nice explanation in Schutz [4]. (The bottom line is that **Twin S** constructs a bad coordinate system by stopping the recording of events in system S_D when he reaches R.)

  - In the right diagram, we draw *light signals* ("pings") of an earth-bound clock next to **Twin E** sent to **Twin S**. **Twin S** receives these signals and measures their period. This idealizes how **Twin S** *sees* (not *observes*!) the clocks ticking on earth (and, by proxy, how fast **Twin E** ages). It is important to understand the difference between this "seeing" and our operational definition of *observing* (using the contraption called an ← *inertial system*, as used in the left diagram). As demonstrated by the diagram, **Twin S** first sees the clock on earth ticking *slower*; but when he turns around at R, the clocks on earth (apparently) speed up significantly. In the end, this speedup overcompensates for the slowdown during the first part of the journey so that **Twin S** again arrives at the (correct) conclusion that **Twin E** ages faster. Note that the speedup of the earth-bound clock *seen* by **Twin S** during the second half of his journey does not contradict time-dilation because *seeing* is not *observing*. This is similar to the ↑ *Penrose-Terrell effect* in that a genuine relativistic effect (here: time-dilation) is distorted by an additional "imaging effect" due to the finite speed of light.

- In our careful derivation above, we not only showed that **Twin S** ages less than **Twin E**; we also showed that this conclusion is *independent* of the inertial observer! Thus we know that there will be *no dispute* about the different ages between different inertial observers.
- The Hafele-Keating experiment [46, 47] and the muon decay experiments [44], mentioned previously in the context of time-dilation, are experimental confirmations of the twin "paradox." So our theoretical prediction above (that **Twin S** ages less than **Twin E**) is experimentally confirmed. End of discussion.
- Our derivation of the accumulated proper time along trajectories in spacetime is both mathematically sound and experimentally confirmed. This qualifies SPECIAL RELATIVITY as a successfull theory of physics. *Operationally* there is nothing to complain about: the theory does its job to produce quantiative predictions of real phenomena. So why do so many people (physicists included) – despite the various efforts to visualize the phenomenon – have this nagging feeling of dissatisfaction that they cannot get rid of? The reason, so I would argue, is the human brain and its proclivity to inject concepts of absolute simulateneity into its model building. This qualifies the historical overemphasis of the twin "paradox" as a *meta problem*: The question to study is not how to "solve" the twin "paradox" (as we showed above, there is nothing to solve); the question to study is why so many peoply thought (and still think) that there is a problem in the first place. This *meta problem* is an actual problem to study; but it falls into the domain of cognitive science, and not physics!

7 | Two lessons to be learned from this:

> You can live longer than your inertial-system-dwelling peers by changing inertial systems (= accelerating) at least once.

The mere fact that our universe *really* allows for this (at least in theory) makes it much more interesting than its boring alternative: a Galilean universe.

and

> Phenomena like length contraction and the twin "paradox" are physically *real*. Their "paradoxical" flavor is a phenomenon of human cognition, not physics.

This is why we put "paradox" always in quotes in the context of RELATIVITY.

3. Mathematical Tools I: Tensor Calculus

In this chapter we introduce tensor calculus (↑ *Ricci calculus*) for general coordinate transformations φ (which will be useful both in SPECIAL RELATIVITY and GENERAL RELATIVITY). The coordinate transformations φ relevant for SPECIAL RELATIVITY are Lorentz transformations (and therefore linear) which simplifies expressions often significantly (→ *Chapter 4*). However, this special feature of coordinate transformations in SPECIAL RELATIVITY is not crucial for the discussions in this chapter.

Goal: Construct Lorentz covariant (form invariant) equations
(for mechanics, electrodynamics, quantum mechanics)

Question: How to do this *systematically*?

Note that (we suspect that) Maxwell equations *are* Lorentz covariant. Clearly this is not obvious and requires some work to prove; we say that the Lorentz covariance is *not manifest*: it is there, but it is hard to see. Conversely, without additional tools that make Lorentz covariance more obvious, it is borderline impossible to *construct* Lorentz covariant equations from scratch (which we must do for mechanics and quantum mechanics!).

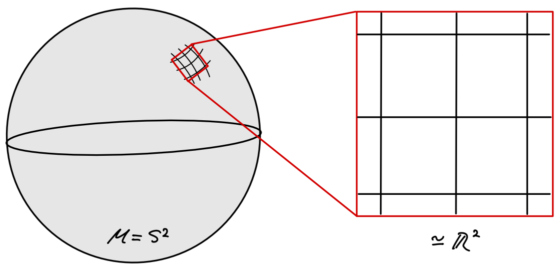
We are therefore looking for a "toolkit" that provides us with elementary "building blocks" and a set of rules that can be used to construct Lorentz covariant equations. This toolbox is known as *tensor calculus* or ↑ *Ricci calculus*; the "building blocks" are tensor fields and the rules for their combination are given by index contractions, covariant derivatives, etc. The rules are such that the expressions (equations) you can build with tensor fields are *guaranteed* to be Lorentz covariant. This implies in particular that if you can rewrite any given set of equations (like the Maxwell equations) in terms of these rules, you automatically show that the equations were Lorentz covariant all along. We then say that the Lorentz covariance is *manifest*: one glance at the equation is enough to check it.

Later, in GENERAL RELATIVITY, our goal will be to construct equations that are invariant under *arbitrary* (differentiable) coordinate transformations (not just global Lorentz transformations). Luckily, the formalism we introduce in this chapter is powerful enough to allow for the construction of such → *general covariant* equations as well. This is why we keep the formalism in this chapter as general as possible, and specialize it to SPECIAL RELATIVITY in the next Chapter 4. The discussion below is therefore already a preparation for GENERAL RELATIVITY; it is based on Schröder [1] and complemented by Carroll [52].

3.1. Manifolds, charts and coordinate transformations

1 | D-dimensional Manifold

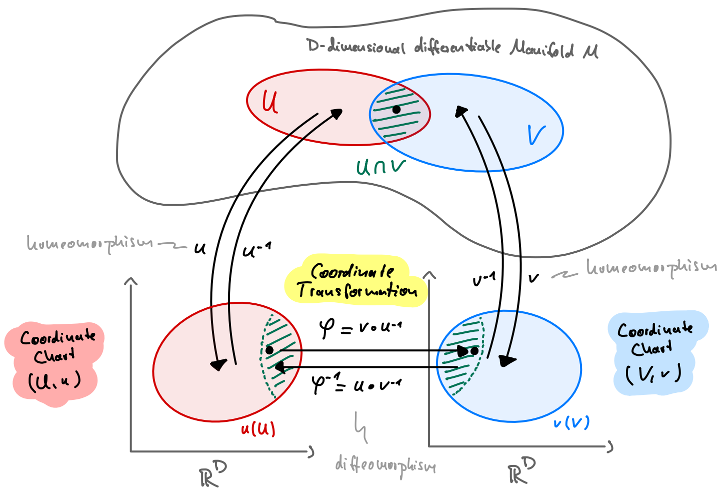
= Topological space that *locally* "looks like" D-dimensional Euclidean space $\mathbb{R}^D$:



- ¡! In RELATIVITY, the manifold of interest is the set of coincidence classes $\mathcal{E}$; it makes up the $D = 4$-dimensional manifold we call *spacetime*.
- A space that "locally looks like $\mathbb{R}^D$" is formalized as a ↑ *topological space* that is locally ↑ *homeomorphic* to Euclidean space $\mathbb{R}^D$. The structure defined in this way is then called a ↑ *topological manifold*.

2 | Differentiable Manifolds:

We want to formalize this idea and introduce additional structure to the manifold so that we can differentiate functions on it:



i | ⁂ *Coordinate system / Chart* (U, u):

$$u: \quad U \subseteq M \to u(U) \subseteq \mathbb{R}^D \tag{3.1a}$$

$$u^{-1}: u(U) \subseteq \mathbb{R}^D \to \quad U \subseteq M \tag{3.1b}$$

$U \subseteq M$: open subset of M; u and u^{-1} are continuous and $u \circ u^{-1} = \mathbb{1}$.

$U = M$ is allowed. This is the situation we assumed so far in SPECIAL RELATIVITY: Our inertial coordinate systems cover all of spacetime $M = \mathcal{E}$.

ii | ⊲ Two charts (U, v) and (V, v) and let $U \cap V \neq 0$:

$$\varphi := v \circ u^{-1} : u(U \cap V) \to v(U \cap V) \tag{3.2a}$$

$$\varphi^{-1} := u \circ v^{-1} : v(U \cap V) \to u(U \cap V) \tag{3.2b}$$

φ: ⁂ *Coordinate transformation / Transition map*

$U = M = V$ and $U \cap V = M$ is allowed. This is the situation we assume so far in SPECIAL RELATIVITY where $(U = \mathcal{E}, u)$ and $(V = \mathcal{E}, v)$ correspond to the coordinate systems of two different inertial systems. The coordinate transformation φ would then be a Lorentz transformation (defined on $U \cap V = \mathcal{E}$).

iii | ⁂ *Atlas* := Family of charts $(U_i, u_i)_{i \in I}$ such that $M = \bigcup_{i \in I} U_i$

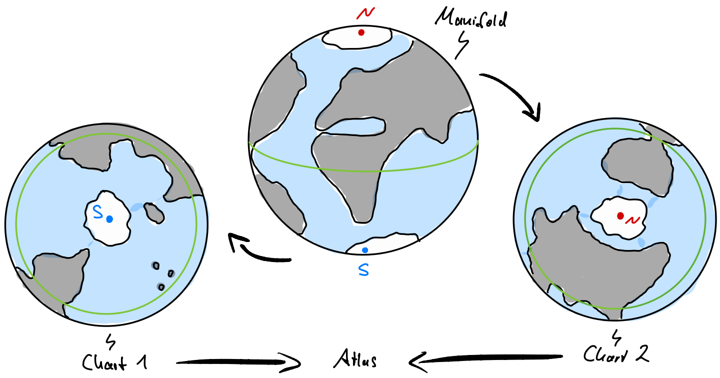
This definition of an atlas formalizes the notion of an atlas in real life (of the book variety): It contains many charts that, taken together, cover the complete manifold (typically earth). The different charts (on different pages of the book) all overlap on their edges such that you can draw any route on earth without gaps.

All φ, φ^{-1} differentiable → M: ⁂ *Differentiable Manifold*

- φ and φ^{-1} are maps from $\mathbb{R}^D$ to itself. It is therefore clear what "differentiable" means.
- In mathematics one is of course more precise about the degree of differentiability of the transition functions, and subsequently assigns this degree to the manifold. For example, if all coordinate transformations are infinitely often differentiable (= smooth), the manifold is called a ↑ *smooth manifold*. We are sloppy in this regard: For us all functions are differentiable as often as we need them to be.

In RELATIVITY we will only be concerned with *differentiable* manifolds.

3 | Example:



→ In general, a manifold *cannot* be covered by a single chart (Earth, mathematically S^2, needs at least two charts). In SPECIAL RELATIVITY this is not a problem: There we assume that spacetime is a flat (pseudo-)Euclidean space $\mathcal{E} \simeq \mathbb{R}^4$ and the coordinates given by our inertial systems cover all of spacetime. Later, in GENERAL RELATIVITY, this will not necessarily be the case.

3.2. Scalars

4 | ⁂ *Scalar (field)* := Function $\phi : M \to \mathbb{R}/\mathbb{C}$

- If ϕ maps to $\mathbb{R}$ ($\mathbb{C}$), we call ϕ a real (complex) scalar field.
- ¡! ϕ is a *geometric object* because it only depends on the manifold itself. It does not rely on charts/coordinates and does not depend on a particular set of charts you might choose to parametrize the manifold. The notion of a mathematical object to be "geometric in nature" or "independent of the choice of coordinates" is *absolutely crucial* for the understanding of GENERAL RELATIVITY. The reason why these "geometric objects" are so important for physics is the following insight that took physicists (including Einstein) a long time to fully comprehend and implement mathematically:

> Coordinates (charts) do *not* represent physical entities.
> They are (useful) "mathematical auxiliary structures."

- One reason why it is so hard for us to grasp and implement the "physical irrelevance" of coordinates is, so I believe, that the first (and often only) coordinates we encounter in school are *Cartesian coordinates*. They are particularly intuitive because they are simply the *distances* of a point to some coordinate axes. Distances are a geometric property and physically relevant (you can measure them with rods); they are not the invention of mathematicians. This makes students draw the (wrong) conclusion that coordinates in general have intrinsic physical meaning. The problem is that coordinates *are* inventions of mathematicians; they do not share the ontological status of physical quantities like lengths etc. To undo this misconception is key to understand GENERAL RELATIVITY (→ *much later*).

- Since both M and $\mathbb{R}/\mathbb{C}$ are ↑ *topological spaces*, it makes sense to ask whether (or require that) ϕ is *continuous*. It does *not* make sense to ask whether ϕ is *differentiable* (and what is derivative is) because, in general, M does neither come with a notion of "distance" between two points in M nor can you add or subtract points (M does not have to be a ↓ *metric space* and/or a ↓ *linear space*). So an expression like $\partial_p \phi(p)$ does not make sense (→ *below*)!

5 | We just declared that coordinates are "not physical." The problem is that *without* coordinates it is really hard (at least for physicists) to do actual calculations with the geometric objects we are interested in (for example: compute derivatives). In addition, comparing theoretical predictions with experimental observations typically requires some sort of coordinate representation. Our ← *inertial systems*, for example, are elaborate measurement devices that produce a specific coordinate representation of the observed events.

This is why we always assume in the following that we have one (or more) charts that allow us to parametrize a (part of the) manifold, and then express the geometric quantities as functions of these coordinates. This means for the scalar field:

$\triangleleft$ Two overlapping charts u and v:

$$\Phi(x) := \phi(u^{-1}(x)) \quad x \in u(U \cap V) \tag{3.3a}$$

$$\bar{\Phi}(\bar{x}) := \phi(v^{-1}(\bar{x})) \quad \bar{x} \in v(U \cap V) \tag{3.3b}$$

Φ and $\bar{\Phi}$ are functions on (subsets of) $\mathbb{R}^D$; in contrast to ϕ which is a function on the manifold M. In an abuse of notation, some authors do not make this distinction and write ϕ and $\bar{\phi}$ instead.

$\overset{\circ}{\rightarrow}$

$$\boxed{\bar{\Phi}(\bar{x}) = \Phi(x) \quad \text{for} \quad \bar{x} = \varphi(x) \quad \text{with} \quad \varphi = v \circ u^{-1}\,.} \tag{3.4}$$

Note that $\bar{\Phi}(\bar{x}) \overset{\text{def}}{=} \phi(p) \overset{\text{def}}{=} \Phi(x)$ with $u^{-1}(x) = p = v^{-1}(\bar{x})$.

- In RELATIVITY we typically work in a particular chart (coordinate system). Thus we write our fields as functions of coordinates (and not points on the manifold); e.g., when working with scalars, we typically work with Φ (and not ϕ).
- ¡! The special transformation of a field Eq. (3.4) (given as function of coordinates) tells us that it actually encodes a geometric, chart-independent function ϕ (given as function of points on the manifold). This idea will be prevalent throughout this chapter and is the basis of our modern formulation of RELATIVITY: We work with functions that depend on specific coordinates (and therefore change when we transition to another chart); however, these functions satisfy certain transformation laws [like Eq. (3.4)] that guarantee that they actually encode geometric, chart-independent objects (which is what physics is about).
- As a function of coordinates, scalar fields are those fields the values of which do not change under coordinate transformations. A typical example would be the temperature as a function of position: When you move your coordinate system, the temperature of a particular point in space still is the same (only your coordinates of this particular point have changed!). This is exactly what Eq. (3.4) demands.

Note that being a scalar (field) does not simply mean "being a number." The z-component of the electric field strength $E_z(x)$, for example, assigns a number to every point x; however, it does *not* transform like Eq. (3.4) under coordinate transformations. (Do you see why? What happens to E_z if you rotate your coordinate system?)

- In the literature, you will find the notation $\bar{\Phi} = \Phi$ to characterize scalars. This does *not* mean $\bar{\Phi}(x) = \Phi(x)$ for all $x \in \mathbb{R}^D$ (which characterizes *form-invariance* or *functional equivalence*), but rather $\bar{\Phi}(\bar{x}) = \Phi(x)$ (which characterizes scalar fields). Note that with $x = \varphi^{-1}(\bar{x})$ it follows $\bar{\Phi}(\bar{x}) = \Phi(\varphi^{-1}(\bar{x}))$ such that the function $\bar{\Phi}$ is typically *not* functionally equivalent to Φ. This ambiguity is the price we have to pay if we want to express geometric objects in terms of coordinates.
- Since $\Phi : \mathbb{R}^D \to \mathbb{R}$, it is well-defined what "differentiability" of Φ means. So expressions like $\frac{\partial \Phi(x)}{\partial x^k}$ make sense now (if Φ is differentiable). One then defines that ϕ is differentiable on M iff Φ is differentiable for all charts of an atlas of M.

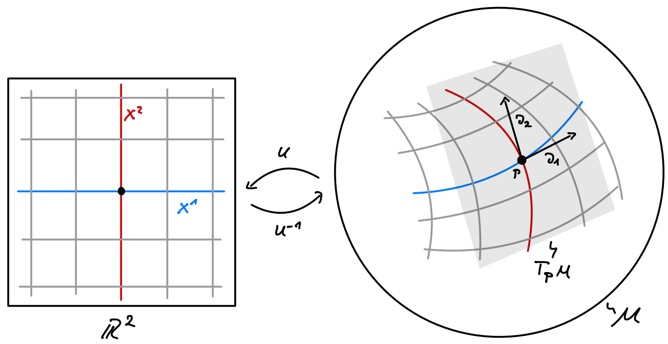
3.3. Covariant and contravariant vector fields

Are scalar fields the only geometric objects that can be defined on a manifold? The answer is *no*, there are many more! And these objects are not just toys for mathematicians: they are necessary to represent physical quantities like the electromagnetic field. Unfortunately, the definition of these quantities is not so straightforward as for scalars. We will not be mathematically precise in our discussion; however, it is important to understand the conceptual ideas:

6 | ⁂ *Tangent space T_pM at $p \in M$*

= *Vector space* of directional derivative operators with evaluation at $p \in M$ (=derivations)

These operators can be applied to differentiable functions on the manifold (i.e., scalar fields).



- The tangent space T_pM is the mathematical formalization of the intuitive concept of the plane $\mathbb{R}^2$ that you can attach tangentially at any point p of a two-dimensional manifold. The problem with this picture is that it only works if you *embed* the manifold M into a higher-dimensional Euclidean space. Mathematically, such an approach is not satisfying because it presupposes additional structure to characterize the manifold (which, as it turns out, is not needed). Physically, the approach is also problematic: The manifold we are interested in is all of spacetime $\mathcal{E}$. But $\mathcal{E}$ is all there is, it is (to the best of our knowledge) not embedded into anything. It is therefore crucial that we can work with manifolds "stand alone", without assuming any embedding into a higher-dimensional space. The price we have to pay is that tangent vectors must be defined, rather abstractly, as directional derivative operators.

- There is a different tangent space $T_p M$ at every point $p \in M$; these vector spaces all have the same dimension D (like the manifold) and are therefore all isomorphic. However, without additional structure, there is no natural connection (isomorphism) between these different vector spaces at different points. The disjoint union of all tangent spaces is called ↑ *tangent bundle* TM.
- Mathematically, the vectors in the tangent space can be defined as equivalence classes of smooth curves through p with the same derivative (with respect to their parametrization) at p. This equivalence class corresponds to a particular directional derivative that one can apply to smooth functions on the manifold at p. We do not need this abstract "bootstrapping procedure" for $T_p M$ in the following.

$\vartriangleleft$ Chart (U, u) with coordinates $x = (x^0, x^1, \dots, x^D)$

→ ⁂ *Coordinate basis* $\{\partial_i \equiv \frac{\partial}{\partial x^i}\}$ for $T_p M$

Recall that partial derivatives are special kinds of directional derivatives (namely in the direction where you keep all but one coordinate fixed). You can therefore think of ∂_i as the tangent vector at $p \in M$ that points into the x^i-direction mapped by u^{-1} onto the manifold.

↓**Lecture 7** [28.11.23]

7 | Since T_pM is a vector space for each point p of the manifold M, we can define *fields* on M that assign to each point p a tangent vector:

⁂ *Vector field*: $A(p) = \sum_{i=1}^{D} A^i(x)\partial_i$ with $x = u(p)$

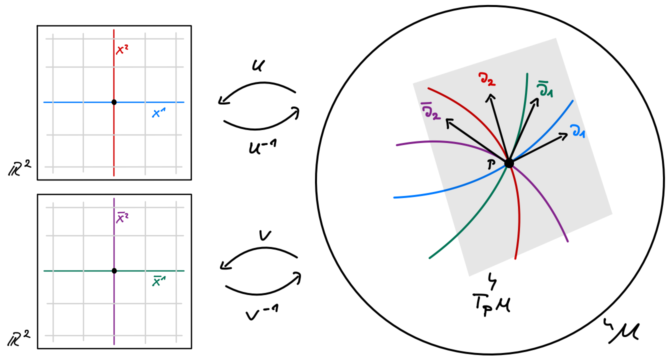
At every point $p \in M$ the vector field yields a tangent vector $A(p) = \sum_i A^i(u(p))\partial_i \in T_pM$.

8 | $\sphericalangle$ Coordinate transformation $\bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x})$

→ Chain rule:

$$\underbrace{\frac{\partial}{\partial \bar{x}^i}}_{\bar{\partial}_i} = \sum_{k=1}^{D} \frac{\partial x^k}{\partial \bar{x}^i} \underbrace{\frac{\partial}{\partial x^k}}_{\partial_k} \tag{3.5}$$

→ For $x = u(p)$ and $\bar{x} = v(p)$ this is a *basis change* on the tangent space T_pM from one coordinate basis $\{\partial_i\}$ to another coordinate basis $\{\bar{\partial}_i\}$ via the (invertible) matrix $\frac{\partial x^k}{\partial \bar{x}^i}$:



9 | $\sphericalangle$ Vector field A and expand it in different coordinate bases:

$$\sum_i A^i(x)\partial_i = A(p) = \sum_i \bar{A}^i(\bar{x})\bar{\partial}_i \tag{3.6}$$

with $x = u(p)$ and $\bar{x} = v(p)$.

- ¡! The vector field A is a geometric object, just as the scalar field ϕ was. That it does *not* depend on the chosen chart is the statement of this equation.
- You learned this (with different notation and without the x/p-dependency) in your first course on linear algebra: Given a vector space V, a vector $\vec{v} \in V$, and a basis $\{\vec{e}_i\}$ with $V = \operatorname{span}\{e_i\}$, you can encode the vector in a basis-dependent set of numbers v_i called *components* via linear combination: $\vec{v} = \sum_i v_i\vec{e}_i$. The *same* vector can be encoded by *different* components v_i' in a *different* basis $\{\vec{e}_i'\}$: $\vec{v} = \sum_i \vec{v}_i'\vec{e}_i'$. In our terminology, the vector $\vec{v}$ is a "geometric object" that does not depend on your choice of basis; only its components do. In this context, the gist of the story is that $\vec{v}$ represents something physical (like the velocity of a particle). The *components* v_i do so only indirectly because they depend on your choice of the basis $\{\vec{e}_i\}$ – and this choice does not bear any physical meaning.

Eq. (3.6) →

$$A = \sum_i A^i(x)\partial_i \stackrel{!}{=} \sum_i \bar{A}^i(\bar{x})\bar{\partial}_i \stackrel{\text{Eq. (3.5)}}{=} \sum_k \underbrace{\left[\sum_i \frac{\partial x^k}{\partial \bar{x}^i}\bar{A}^i(\bar{x})\right]}_{\stackrel{!}{=} A^k(x)} \partial_k \tag{3.7}$$

This motivates the following definition (we replace $x \leftrightarrow \bar{x}$ and the indices $i \leftrightarrow k$):

10 | $\sphericalangle$ D-tuple $\{A^i(x)\}$ of fields (in some chart with coordinates x):

$$\boxed{⁂\ \textit{Contravariant vector field}\ \{A^i(x)\} \quad :\Leftrightarrow \quad \bar{A}^i(\bar{x}) = \sum_{k=1}^{D} \frac{\partial \bar{x}^i}{\partial x^k} A^k(x)} \tag{3.8}$$

*Contra*variant vector (field) $\rightarrow$ *Super*script indices!

This is a *convention* which relates syntax and semantics and is at the heart of *tensor calculus*. The idea is that whenever you are given a collection of fields $A^i(x)$, you immediately know that they transform like Eq. (3.8) under coordinate transformations. (Unfortunately, there are exceptions to this rule, e.g., the → *Christoffel symbols*.)

- ¡! Not every D-tuple of fields transforms as Eq. (3.8). To deserve the name "contravariant vector (field)," (and superscript indices) one has to check this transformation law explicitly!
- The rationale of Eq. (3.8) is the same as that of Eq. (3.4): Whenever we find a family of fields that transform under coordinate transformations as Eq. (3.8), we immediately know that together they encode a geometric, chart-independent object on the manifold that can be used to describe a physical quantity.

11 | <u>(Counter)Examples:</u>

- $\sphericalangle$ Only *linear* coordinate transformations: $\bar{x} = \varphi(x) = \Lambda x$

  $\sphericalangle$ Coordinate functions $X^i(x) := x^i$ as fields:

$$\underbrace{\bar{X}^i(\bar{x})}_{\bar{x}^i} = \sum_{k=1}^{D} \Lambda^i_k \underbrace{X^k(x)}_{x^k} = \sum_{k=1}^{D} \underbrace{\frac{\partial \bar{x}^i}{\partial x^k}}_{\Lambda^i_k} X^k(x) \tag{3.9}$$

  $\rightarrow$ Coordinate functions are contravariant vectors for linear transition maps.

  This is useful in SPECIAL RELATIVITY because there we only consider global Lorentz transformations (which are linear).

- $\sphericalangle$ D *scalar* fields $\Phi^i(x)$ ($i = 1, \dots, D$):

$$\text{For general } \bar{x} = \varphi(x): \quad \bar{\Phi}^i(\bar{x}) = \Phi^i(x) \neq \sum_{k=1}^{D} \underbrace{\frac{\partial \bar{x}^i}{\partial x^k}}_{\neq \delta^i_k} \Phi^k(x) \tag{3.10}$$

  $\rightarrow$ $\{\Phi^i(x)\}$ are *not* components of a contravariant vector field.

  - You see: not every collection of D fields is a vector!
  - ¡! δ^i_k is the Kronecker symbol: $\delta^i_k = 1$ for $i = k$ and $\delta^i_k = 0$ for $i \neq k$. The notation δ_{ik} is *not* used in tensor calculus (→ *later*).

12 | <u>Reminder: ↓ *Dual spaces*</u>

i | Remember: Linear algebra

Consider the vector space $V = \mathbb{R}^D$ and a column vector $\vec{v} = (v_1, \dots, v_D)^T \in V$ (a $1 \times D$-matrix). Let $\vec{w}^T = (w_1, \dots, w_D)$ be a row vector (a $D \times 1$-matrix). We can then perform a matrix multiplication between the vectors and interpret it as a linear map $\vec{w}^T$ acting on the vector $\vec{v}$ and producing a number:

$$\vec{w}^T : \vec{v} \in V \mapsto \vec{w}^T \cdot \vec{v} = \begin{pmatrix} w_1 & \dots & w_D \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_D \end{pmatrix} = \sum_i w_i v_i \in \mathbb{R}\,. \tag{3.11}$$

In mathematical parlance $\vec{w}^T$ is a *linear functional* on the vector space V. All linear functionals of this form make up another vector space V^* called the ↓ *dual space* of V. You can think of V^* as the vector space of all D-dimensional *row* vectors and V as the vector space of all D-dimensional *column* vectors. The elements of the dual space are referred to as a ↓ *covectors*.

ii | Remember: Quantum mechanics

In quantum mechanics, the state of a physical system is described by ↓ *state vectors* in some Hilbert space $\mathcal{H}$ (which is a special kind of vector space). Vectors in this space are written as ↓ *kets*: $|\Psi\rangle \in \mathcal{H}$. You can produce a ↓ *bra* $\langle\Psi| = |\Psi\rangle^\dagger$ by applying the complex transpose operator. As in the example above, the bra $\langle\Psi|$ is a covector from the dual space $\mathcal{H}^*$; indeed, it acts as a linear functional on state vectors via the inner product of the Hilbert space:

$$\langle\Psi||\Phi\rangle := \langle\Psi|\Phi\rangle \in \mathbb{C}\,. \tag{3.12}$$

This is the gist of the famous ↓ *Dirac bra-ket notation*.

iii | Hopefully these examples convinced you that the dual space is just as important and useful as the vector space itself.

→ Dual space of the tangent space $T_p M$?

Given a coordinate basis $\{\partial_i\} \in T_p M$ of a vector space, there is a standard way to define a basis of of the dual space $T_p^* M$:

↓ *Dual basis* $\{\mathrm{d}x^i\}$ with

$$\mathrm{d}x^i(\partial_j) := \delta^i_j = \frac{\partial x^i}{\partial x^j} \tag{3.13}$$

→ $\{\mathrm{d}x^i\}$ is a basis of the ⁂ *Cotangent space* $T_p^* M$

$T_p^* M$ is the dual space of $T_p M$; it is common to write $T_p^* M$ and not $(T_p M)^*$.

13 | Since $T_p^* M$ is just another vector space for each point p of the manifold M, we can again define *fields* on M that map into this space:

⁂ *Covector field*: $B(p) = \sum_{i=1}^D B_i(x)\,\mathrm{d}x^i$ with $x = u(p)$

14 | Just like the coordinate basis, the dual coordinate basis depends on the chart and changes under coordinate transformations:

⊲ Coordinate transformation $\bar{x} = \varphi(x)$:

$$\mathrm{d}\bar{x}^i = \sum_{k=1}^D \frac{\partial \bar{x}^i}{\partial x^k}\,\mathrm{d}x^k \tag{3.14}$$

- Check that this is the correct transformation for the dual coordinate basis:

$$\mathrm{d}\bar{x}^i(\bar{\partial}_j) = \left[\sum_k \frac{\partial \bar{x}^i}{\partial x^k}\mathrm{d}x^k\right]\left(\sum_l \frac{\partial x^l}{\partial \bar{x}^j}\partial_l\right) = \sum_{k,l}\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}\underbrace{\mathrm{d}x^k(\partial_l)}_{\delta^k_l} = \underbrace{\sum_k \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^k}{\partial \bar{x}^j}}_{\frac{\partial \bar{x}^i}{\partial \bar{x}^j}} = \delta^i_j \quad ☺ \tag{3.15}$$

- You might recognize Eq. (3.14): This is simply the rule to compute the ↓ *total differential* of the function $\bar{x} = \varphi(x)$. This is no coincidence and explains why we use the differential notation $\mathrm{d}x^i$ for the dual vectors: The objects $\mathrm{d}x^i$ that we physicists like to illustrate as "infinitesimal shifts" in x^i are actually *linear functionals* (↑ *1-forms*).

15 | Now we can play the same game on T_p^*M as before on T_pM:

⊲ Covector field B and expand it in different dual coordinate bases:

$$\sum_i B_i(x)\mathrm{d}x^i = B(p) = \sum_i \bar{B}_i(\bar{x})\mathrm{d}\bar{x}^i \tag{3.16}$$

with $x = u(p)$ and $\bar{x} = v(p)$.

¡! The covector field B is another geometric object, just as the vector field A was. That it does *not* depend on the chosen chart is the statement of this equation.

Eq. (3.16) →

$$B = \sum_i B_i(x)\,\mathrm{d}x^i \stackrel{!}{=} \sum_i \bar{B}_i(\bar{x})\,\mathrm{d}\bar{x}^i \stackrel{\text{Eq. (3.14)}}{=} \sum_k \underbrace{\left[\sum_i \frac{\partial \bar{x}^i}{\partial x^k}\bar{B}_i(\bar{x})\right]}_{\stackrel{!}{=}B_k(x)}\mathrm{d}x^k \tag{3.17}$$

This motivates the following definition (we replace $x \leftrightarrow \bar{x}$ and the indices $i \leftrightarrow k$):

16 | ⊲ D-tuple $\{B_i(x)\}$ of fields (in some chart with coordinates x):

$$\boxed{⁂\ \textit{Covariant vector field}\ \{B_i(x)\} \quad :\Leftrightarrow \quad \bar{B}_i(\bar{x}) = \sum_{k=1}^{D}\frac{\partial x^k}{\partial \bar{x}^i}B_k(x)} \tag{3.18}$$

*Co*variant vector (field) → *Sub*script indices!

The rationale of Eq. (3.18) is the same as that of Eq. (3.8): Whenever we find a family of fields that transform under coordinate transformations as Eq. (3.18), we immediately know that together they encode a geometric, chart-independent object on the manifold that can be used to describe a physical quantity. To indicate that this object is a *covariant* vector field, we use *sub*script indices.

17 | Example:

First, let us introduce an even shorter notation for partial derivatives: $\Phi_{,i} \equiv \partial_i\Phi$

Following our index convention, the lower index in these expressions is only warranted *if* the field transforms as a covariant vector field according to Eq. (3.18). Let us check this:

$$\bar{\Phi}_{,i}(\bar{x}) = \bar{\partial}_i\bar{\Phi}(\bar{x}) \stackrel{\substack{\text{Eq. (3.4)}\\ \text{Eq. (3.5)}}}{=} \sum_{k=1}^{D}\frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \Phi(x)}{\partial x^k} = \sum_{k=1}^{D}\frac{\partial x^k}{\partial \bar{x}^i}\Phi_{,k}(x) \tag{3.19}$$

→ The gradient of a scalar is a covariant vector field.

18 | What happens if we apply a covector field on a vector field at each point $p \in M$?

$$\phi(p) := B(p)A(p) = \sum_{i,j} B_i(x)A^j(x)\,\underbrace{\mathrm{d}x^i(\partial_j)}_{\delta^i_j} = \sum_i A^i(x)B_i(x) =: \Phi(x) \tag{3.20}$$

→ $\Phi(x)$ must be a scalar!

This is a good point to introduce a new (and very convenient) notation:

⁂ *Einstein sum convention:*

$$\sum_{i=1}^{D} A^i(x)B_i(x) \equiv \underbrace{A^i(x)B_i(x)}_{\text{⁂ Einstein summation} \atop \text{⁂ Contraction}} = A^l(x)B_l(x) \tag{3.21}$$

The *Einstein sum convention* or *Einstein summation* is a syntactic convention according to which a sum is automatically implied (but not written) whenever two indices show up twice in an expression and one is up (contravariant) and one down (covariant). Note that such indices are "dummy indices" in the sense that you can rename them to whatever you want (as long as you do not use the same letter for other indices already!). The sum over one co- and one contravariant index is called a *contraction*.

With this new notation it is straightforward to check that Φ transforms according to Eq. (3.4) by using the transformations Eq. (3.8) and Eq. (3.18):

$$\bar{\Phi}(\bar{x}) = \bar{A}^i(\bar{x})\bar{B}_i(\bar{x}) = \left[\frac{\partial \bar{x}^i}{\partial x^k}A^k(x)\right]\left[\frac{\partial x^l}{\partial \bar{x}^i}B_l(x)\right] \tag{3.22a}$$

$$= \underbrace{\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^i}}_{\text{Chain rule} \to \delta^l_k} A^k(x)B_l(x) = A^l(x)B_l(x) = \Phi(x) \tag{3.22b}$$

The intermediate expression contains *three* sums over the colored indices (which we don't write)!

→ The contraction of a contra- and a covariant vector field yields a scalar field.

19 | Note on nomenclature:

- If you compare Eq. (3.18) with Eq. (3.5) you find that the *components* B_i of a covector field transform like the *basis vectors* ∂_i of the tangent space. We say the components *co*vary ("vary together") with the basis. This is why they are called *covariant*.
- A comparison of Eq. (3.8) and Eq. (3.14) shows that the components A^i of a vector field transform like the basis $\mathrm{d}x^i$ of the cotangent space – which is the *inverse* ("opposite") transformation as for the basis of the tangent space ∂_i. Thus we say the components A^i *contra*vary ("vary opposite to") the basis ∂_i. This is why they are called *contravariant*.

3.4. Higher-rank tensors

You learned in your linear algebra course that two vector spaces V and W can be used to construct a new vector space $V \otimes W$ called the ↓ *tensor product*. This allows us to generalize the notion of contra- and covariant *vector* fields to *tensor* fields, all of which are geometric, chart-independent objects defined on the manifold that are needed to describe physical quantities:

20 | An ⁂ *(absolute)* (p,q)*-tensor (field)* T of rank $r = p + q$

$$T^{i_1 i_2 \dots i_p}{}_{j_1 j_2 \dots j_q} \equiv T^{i_1 i_2 \dots i_p}{}_{j_1 j_2 \dots j_q}(x) \quad \text{or} \quad T^I{}_J \equiv T^I{}_J(x)\,, \tag{3.23}$$

with ↓ *multi-indices* $I = (i_1 \dots i_p)$ and $J = (j_1 \dots j_q)$,
transforms like the tensor product of p contravariant and q covariant vector fields:

$$\boxed{\overbrace{\bar{T}^{i_1 \dots i_p}{}_{j_1 \dots j_q}(\bar{x})}^{=\bar{T}^I{}_J(\bar{x})} = \underbrace{\left[\frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}}\right]}_{=:\frac{\partial \bar{x}^I}{\partial x^M}} \underbrace{\left[\frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}}\right]}_{=:\frac{\partial x^N}{\partial \bar{x}^J}} \underbrace{T^{m_1 \dots m_p}{}_{n_1 \dots n_q}(x)}_{=T^M{}_N(x)}} \tag{3.24}$$

There are $r = p + q$ sums in this transformation rule (Einstein summation!).

- ¡! It is important that we do *not* write contra- and covariant indices above each other like so: T^i_j (at least not with additional knowledge about the tensor). This will become important below.
- Henceforth we always encode tensor fields by their chart-dependent *components*. The actual tensor field is of course chart-independent and maps each point $p \in M$ to an element of the tensor product

$$\underbrace{T_pM \otimes \cdots \otimes T_pM}_{p \text{ factors}} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{q \text{ factors}}\,. \tag{3.25}$$

  like so

$$T(p) = \sum_{I,J} T^{i_1 \dots i_p}{}_{j_1 \dots j_q}(x)\, \partial_{i_1} \otimes \cdots \otimes \partial_{i_p} \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_q}\,. \tag{3.26}$$

- Note that while tensors (more precisely: tensor components) are indicated by upper and lower indices (corresponding to their rank), not every object that is conventionally written with upper and lower indices does encode a tensor. For example, the transformation matrices $\frac{\partial \bar{x}^i}{\partial x^m}$, which describe a basis change on T_p^*M, do not encode a tensor field.

21 | Examples:

$$\begin{aligned} \text{Scalar } \Phi(x) \quad &\to \quad (0,0)\text{-tensor} \\ \text{Contravariant vector } A^i(x) \quad &\to \quad (1,0)\text{-tensor} \\ \text{Covariant vector } B_i(x) \quad &\to \quad (0,1)\text{-tensor} \\ \text{Tensor product } T^i{}_j(x) := A^i(x) B_j(x) \quad &\to \quad (1,1)\text{-tensor (Check this!)} \end{aligned}$$

22 | Properties:

- *Equality:*

$$A = B \quad :\Leftrightarrow \quad \forall_{i_1 \dots i_p} \forall_{j_1 \dots j_q} :\; A^{i_1 \dots i_p}{}_{j_1 \dots j_q} = B^{i_1 \dots i_p}{}_{j_1 \dots j_q} \tag{3.27}$$

- *Symmetry:*

$$T \text{ (anti-)symmetric in } k \text{ and } l \quad :\Leftrightarrow \quad T^{\dots k \dots l \dots}{}_{\dots} = (-)\, T^{\dots l \dots k \dots}{}_{\dots} \tag{3.28}$$

Every contra- or covariant rank-2 tensor can be decomposed into a sum of symmetric and antisymmetric tensors:

$$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{=:T_{(ij)}} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{=:T_{[ij]}} = T_{(ij)} + T_{[ij]}\,. \tag{3.29}$$

23 | Constructing tensors:

New tensors can be constructed from known tensors as follows (Proofs: ➔ Problemset 4):

- *Sum of (p,q)-tensors A and B yields (p,q)-tensor C:*

$$C^{i_1\dots i_p}{}_{j_1\dots j_q} := A^{i_1\dots i_p}{}_{j_1\dots j_q} + B^{i_1\dots i_p}{}_{j_1\dots j_q} \tag{3.30a}$$

$$\text{or} \quad C^I{}_J := A^I{}_J + B^I{}_J \tag{3.30b}$$

- *Product of (p,q)-tensor A and scalar Φ yields (p,q)-tensor C:*

$$C^I{}_J := \Phi\, A^I{}_J \tag{3.31}$$

- *Tensor product of (p,q)-tensor A and (r,s)-tensor B yields $(p+r, q+s)$-tensor C:*

$$C^{IK}{}_{JL} := A^I{}_J \cdot B^K{}_L \tag{3.32}$$

- *Contractions:*

  Summing over a pair of contra- and covariant indices yields a tensor of rank $(p-1, q-1)$:

$$\tilde{A}^{i_1\dots\bullet\dots i_p}{}_{j_1\dots\bullet\dots j_q} := A^{i_1\dots k\dots i_p}{}_{j_1\dots k\dots j_q} \tag{3.33}$$

  The • indicates that the index summed over on the right side is missing in the list.

  Proof: ➔ Problemset 4

  A special case of a contraction (in combination with a tensor product) is the scalar obtained from a contra- and a covariant vector field above:

$$\Phi = C^i{}_i = A^i B_i\,. \tag{3.34}$$

- *Quotient theorem:*

$$\overset{\sqcap}{AB} = C \text{ tensor for all tensors } B \quad \Rightarrow \quad A \text{ is tensor} \tag{3.35}$$

  Here, $\overset{\sqcap}{AB}$ denotes (potentially multiple) contractions between indices of A and B (but not within A and B).

  – As an example, rewrite an arbitrary contravariant vector A^i as $A^i = \delta^i_j A^j$ with Kronecker symbol δ^i_j. The above theorem then implies that δ^i_j transforms as a $(1,1)$-tensor (verify this using the definition!). Hence we actually should write $\delta^i{}_j$ instead of δ^i_j. However, because the Kronecker symbol is symmetric in its indices, this simplified notation is allowed (→ *later*).

– Special case:

$$A_{ik}B^k = C_i \text{ covector for all vectors } B^k \quad \Rightarrow \quad A_{ik} \text{ is } (0,2)\text{-tensor} \tag{3.36}$$

Proof: ➔ Problemset 4

24 | Relative tensors:

i | Relative tensor are a generalization of the (absolute) tensors defined above. This generalization is useful because most of the rules for computing with tensors discussed so far carry over to relative tensors.

A ⁂ *relative tensor of weight* $w \in \mathbb{Z}$ picks up an additional power w of the ↓ *Jacobian determinant* under coordinate transformations:

$$\boxed{\bar{R}^I{}_J(\bar{x}) = \det\left(\frac{\partial x}{\partial \bar{x}}\right)^w \frac{\partial \bar{x}^I}{\partial x^M}\frac{\partial x^N}{\partial \bar{x}^J} R^M{}_N(x) \quad \text{with weight } w \in \mathbb{Z}} \tag{3.37}$$

and Jacobian determinant

$$\det\left(\frac{\partial x}{\partial \bar{x}}\right) := \sum_{\sigma \in S_D} (-1)^\sigma \prod_{i=1}^{D} \frac{\partial x^i}{\partial \bar{x}^{\sigma_j}} \,. \tag{3.38}$$

Here S_D is the group of permutations σ on D elements.

Since $\bar{x} = \varphi(x)$ is invertible, $x = \varphi^{-1}(\bar{x})$, it is $\frac{\partial \bar{x}}{\partial x} = \left(\frac{\partial x}{\partial \bar{x}}\right)^{-1}$ and therefore $\det\left(\frac{\partial \bar{x}}{\partial x}\right) = \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{-1}$.

ii | Examples:

- *(Absolute) tensors* $\equiv$ Relative tensors of weight $w = 0$
- *Volume form*: Relative tensor of weight $w = -1$:

$$\mathrm{d}^D\bar{x} = \mathrm{d}^D x \det\left(\frac{\partial \bar{x}}{\partial x}\right) = \mathrm{d}^D x \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{-1} \tag{3.39}$$

Remember the rule for integration by substitution with multiple variables!

- ⁂ *Tensor density* $\mathcal{L}(x) :=$ Relative tensor of weight $w = +1 \rightarrow$

$$S = \int \underbrace{\mathrm{d}^D x \mathcal{L}(x)}_{\text{Absolute tensor}} = \int \mathrm{d}^D\bar{x}\bar{\mathcal{L}}(\bar{x}) \tag{3.40}$$

In this example, we assume that $\mathcal{L}(x)$ is a *scalar* tensor density such that its integral is a (absolute) *scalar* quantity.

In ↑ *relativistic field theories* (like electrodynamics), the *Lagrangian density* $\mathcal{L}(x)$ is a scalar tensor density such that the ↓ *action* S becomes a scalar.

- Let $i_1, i_2, \dots, i_D \in \{1, 2, \dots, D\}$ and define the ⁂ *Levi-Civita symbol* as

$$\boxed{\varepsilon^I \equiv \varepsilon^{i_1 i_2 \dots i_D} := \begin{cases} +1 & I \textit{ even} \text{ permutation of } 1,2,\dots,D \\ -1 & I \textit{ odd} \text{ permutation of } 1,2,\dots,D \\ 0 & \text{(at least) two indices equal} \end{cases}} \tag{3.41}$$

An even (odd) permutation of $1, 2, \dots, D$ is constructed by an even (odd) number of *transpositions* (= exchanges of only two indices).

$\overset{\circ}{\rightarrow}$

$$\bar{\varepsilon}^I = \varepsilon^I \overset{\circ}{=} \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{+1} \frac{\partial \bar{x}^I}{\partial x^J} \varepsilon^J \tag{3.42}$$

$\rightarrow \varepsilon^I = \varepsilon^{i_1 i_2 \dots i_D}$ is a $(D, 0)$-tensor density

- ¡! $\bar{\varepsilon}^I = \varepsilon^I$ is true by definition: ε is a *symbol* defined by Eq. (3.41); this definition is independent of the coordinate system. In Eq. (3.42) we compare this trivial transformation with that of a (relative) tensor and conclude that it is equivalent to the statement that ε^I transforms as a $(D, 0)$-tensor density with weight $w = +1$. This knowledge is helpful in tensor calculus to construct covariant expressions that contain Levi-Civita symbols (→ *below*).
- To show this, note that the Levi-Civita symbol can be used to compute determinants:

$$\det\left(\frac{\partial \bar{x}}{\partial x}\right) = \sum_{\sigma \in S_D} (-1)^\sigma \prod_{i=1}^{D} \frac{\partial \bar{x}^i}{\partial x^{\sigma_j}} = \frac{\partial \bar{x}^1}{\partial x^{j_1}} \cdots \frac{\partial \bar{x}^D}{\partial x^{j_D}} \varepsilon^{j_1 \dots j_D} \,. \tag{3.43}$$

Details: ➔ Problemset 4

↓Lecture 8 [05.12.23]

3.5. The metric tensor

A differentiable manifold M does not automatically allow us to measure the length of curves, the angles of intersecting lines, or the area/volume of subsets of the manifold; to do so, we need a *metric* on M (which is an additional piece of information). While the continuity structure (an atlas) that comes with M determines its *topology*, the metric determines its *geometry* (= shape). The same manifold M can be equipped with *different* metrics; this corresponds to different geometries of the same topology (a potato and an egg both have the topology of a sphere, nonetheless they are geometrically distinct).

A differentiable manifold together with a (pseudo-)metric is called ↑ *(pseudo-)Riemannian manifold*. In SPECIAL RELATIVITY and GENERAL RELATIVITY, spacetime is modeled by such (pseudo-)Riemannian manifolds where the metric is used to represent spatial and temporal distances between events.

25 | Motivation:

On linear spaces V, it is convenient to define an ↓ *inner product* (like in quantum mechanics where you consider Hilbert spaces and use their inner product to compute probabilities and transition amplitudes).

Recall the definition of a (real) inner product:

$$\langle\bullet|\bullet\rangle : V\times V \to \mathbb{R} \quad \text{with} \dots \tag{3.44a}$$

$$\text{Symmetry:} \quad \langle x|y\rangle = \langle y|x\rangle \tag{3.44b}$$

$$\text{(Bi)linearity:} \quad \langle ax+by|z\rangle = a\langle x|z\rangle + b\langle y|z\rangle \tag{3.44c}$$

$$\text{Positive-definiteness:} \quad x\neq 0 \;\Rightarrow\; \langle x|x\rangle > 0 \tag{3.44d}$$

Once you have an inner product, you get a norm, and subsequently a metric for free:

$$\underbrace{\langle x|y\rangle}_{\text{Inner product}} \;\Rightarrow\; \underbrace{\|x\| := \sqrt{\langle x|x\rangle}}_{\text{Norm}} \;\Rightarrow\; \underbrace{d(x,y) := \|x-y\|}_{\text{Metric}} \tag{3.45}$$

Thus an inner product is a rather versatile structure and nice to have!

Problem: We cannot define a inner product on the manifold directly because M is not a linear space.

However: We can introduce an inner product on each of its tangent spaces T_pM! →

26 | ⁂ *Riemannian (Pseudo-)Metric* $\mathrm{d}s^2$:= Symmetric, non-degenerate $(0,2)$-tensor field:

$$\mathrm{d}s^2 : M \ni p \mapsto \underbrace{\left(\mathrm{d}s_p^2 : T_pM\times T_pM \to \mathbb{R}\right)}_{\text{Bilinear \& symmetric \& non-degenerate}} \tag{3.46a}$$

$$\mathrm{d}s_p^2 \text{ bilinear} \;\Rightarrow\; \mathrm{d}s_p^2 \in T_p^*M\otimes T_p^*M$$

$$\Rightarrow\; \mathrm{d}s_p^2 = \sum_{i,j=1}^{D} g_{ij}(x)\,\mathrm{d}x^i\otimes \mathrm{d}x^j \equiv g_{ij}(x)\,\mathrm{d}x^i\mathrm{d}x^j \tag{3.46b}$$

with $g_{ij} = g_{ji}$ (symmetry) and $g = \det(g_{ij}) \neq 0$ (non-degeneracy).

- The tensor product is *non-commutative*: $\mathrm{d}x^i \otimes \mathrm{d}x^j \neq \mathrm{d}x^j \otimes \mathrm{d}x^i$. However, you can always decompose a tensor product as

$$\mathrm{d}x^i \otimes \mathrm{d}x^j = \underbrace{\frac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i)}_{=:\mathrm{d}x^i \vee \mathrm{d}x^j} + \underbrace{\frac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}x^j - \mathrm{d}x^j \otimes \mathrm{d}x^i)}_{=:\mathrm{d}x^i \wedge \mathrm{d}x^j} \tag{3.47}$$

  with the symmetrized tensor product $\mathrm{d}x^i \vee \mathrm{d}x^j$ and the anti-symmetrized tensor product $\mathrm{d}x^i \wedge \mathrm{d}x^j$ (↑ *wedge product*).

  Since g_{ij} is assumed to be symmetric, only the symmetric component survives:

$$g_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij}(x)\mathrm{d}x^i \vee \mathrm{d}x^j \equiv g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j \tag{3.48}$$

  This means that when writing $\mathrm{d}x^i\mathrm{d}x^j$ in the above formula, you can be sloppy and either mean $\mathrm{d}x^i \otimes \mathrm{d}x^j$ or, equivalently, $\mathrm{d}x^i \vee \mathrm{d}x^j$. You will find both conventions in the literature. I will use $\mathrm{d}x^i\mathrm{d}x^j \equiv \mathrm{d}x^i \vee \mathrm{d}x^j$ so that $\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^j\mathrm{d}x^i$.

- It would be more appropriate to write $g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ for the metric $(0,2)$-tensor; it is conventional, however, to reserve g for the determinant $\det(g_{ij})$ so that we are stuck with $\mathrm{d}s^2$ for the metric. Note that the d in $\mathrm{d}s^2$ does *not* refer to an ↑ *exterior derivative*, it is purely symbolical.

- To define a proper ↓ *inner product* on T_pM, we should demand ↓ *positive-definiteness* instead of *non-degeneracy*. This, however, is often (for example in RELATIVITY) too restrictive; as it turns out, non-degeneracy is all we need for an isomorphism between T_pM and T_p^*M ("pulling indices up and down", → *below*). This is why *negative* eigenvalues of g_{ij} are fine for many purposes, and motivates the concept of a → *signature*:

27 | Signature:

Since $g_{ij}(x) = g_{ji}(x)$ and $\det(g_{ij}(x)) \neq 0$

$\rightarrow g_{ij}(x)$ has r *positive* and s *negative* real eigenvalues for all $p \in M$

Since $\det(g_{ij}(x)) \neq 0$, these numbers must be the *same* for all $p \in M$.

$\rightarrow (r,s)$: ⁂ *Signature* of the metric $\mathrm{d}s^2$

This classification does not depend on the coordinate basis (↑ *Sylvester's law of inertia*).

- $(r > 0, s = 0)$

  $\rightarrow \mathrm{d}s^2$: *Riemannian metric* $\rightarrow (M, \mathrm{d}s^2)$: ⁂ *Riemannian manifold*

  I.e., g_{ij} has only positive eigenvalues for all $p \in M$ and is therefore ↓ *positive-definite*. This produces a true, positive-definite inner product on T_pM.

- $(r > 0, s > 0)$

  $\rightarrow \mathrm{d}s^2$: *pseudo-Riemannian metric* $\rightarrow (M, \mathrm{d}s^2)$: ⁂ *pseudo-Riemannian manifold*

  I.e., g_{ij} has both positive and negative eigenvalues and is therefore ↓ *indefinite*.

  - $(r > 0, s = 1)$ or $(r = 1, s > 0)$:

    $\rightarrow \mathrm{d}s^2$: *Lorentzian metric* $\rightarrow (M, \mathrm{d}s^2)$: ⁂ *Lorentzian manifold*

    In RELATIVITY we are only interested in metric tensors with one positive and three negative eigenvalues (equivalently: three positive and one negative eigenvalue). Mathematically speaking, spacetime is then a four-dimensional Lorentzian manifold and a special case of a pseudo-Riemannian manifold.

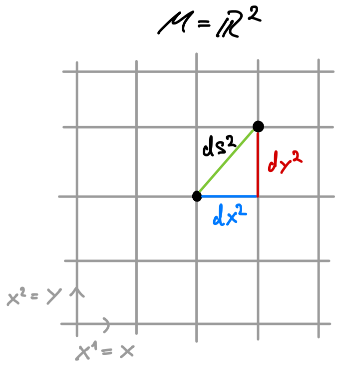
28 | Example: (Details: ➔ Problemset 4)

i | $\sphericalangle$ $D = 2$ Euclidean space $E_2 \equiv (\mathbb{R}^2, \mathrm{d}s_E^2)$

The Euclidean metric in Cartesian coordinates $x^1 = x$ and $x^2 = y$ reads:

$$\mathrm{d}s_E^2 := \mathrm{d}x^2 + \mathrm{d}y^2 = g_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j \quad \text{with} \quad (g_{ij}) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\text{Signature} \\ (2,0)}} . \tag{3.49}$$

This is consistent with the notion of $\mathrm{d}x$ and $\mathrm{d}y$ as infinitesimal shifts in coordinates and $\mathrm{d}s^2$ as the infinitesimal distance (squared) that corresponds to this shift:



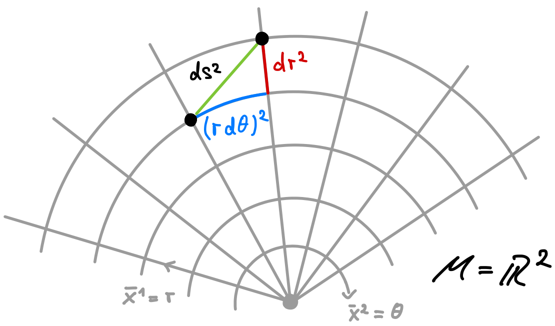
ii | We can now transition to a new chart, namely polar coordinates $\bar{x}^1 = r$ and $\bar{x}^2 = \theta$. The induced basis change on the cotangent space is given by the total differential of the coordinate functions Eq. (3.14):

$$\varphi^{-1} : \begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \overset{\text{Eq. (3.14)}}{\Rightarrow} \begin{aligned} \mathrm{d}x &= \cos(\theta)\,\mathrm{d}r - r\sin(\theta)\,\mathrm{d}\theta \\ \mathrm{d}y &= \sin(\theta)\,\mathrm{d}r + r\cos(\theta)\,\mathrm{d}\theta \end{aligned} \tag{3.50}$$

iii | We find the components of the metric tensor field in the new basis $\{\mathrm{d}\bar{x}^1 = \mathrm{d}r, \mathrm{d}\bar{x}^2 = \mathrm{d}\theta\}$:

$$\mathrm{d}s^2 \stackrel{\circ}{=} \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 = \bar{g}_{ij}(\bar{x})\,\mathrm{d}\bar{x}^i\,\mathrm{d}\bar{x}^j \quad \text{with} \quad (\bar{g}_{ij}) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}_{\substack{\text{Signature} \\ (2,0)}} . \tag{3.51}$$

This expression is again compatible with infinitesimal shifts in the (new) coordinates r and θ:



- The Euclidean plane E_2 is therefore an example for a Riemannian manifold with metric signature $(2,0)$; its distinctive feature is that it is *flat*.

- Note that here we compute *the same* infinitesimal length in different coordinates (with the same result)! We did not change the *metric*, only the *coordinates* and thereby the coordinate basis in which we express the metric tensor. This is *flat* Euclidean space in ↑ *curvilinear coordinates*. By contrast, later in GENERAL RELATIVITY we will study curved (non-flat, non-Euclidean) metric tensors, i.e., we will modify the geometry of space(time) itself.

29 | Since the metric $\mathrm{d}s^2$ is a $(0,2)$-tensor field:

$$\bar{g}_{ij}(\bar{x})\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j = \mathrm{d}s^2 = g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j \tag{3.52}$$

Eq. (3.14) $\overset{\circ}{\to}$

$$\bar{g}_{ij}(\bar{x}) = \frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j}g_{lm}(x) \tag{3.53}$$

The metric (components) transforms as any other $(0,2)$ tensor. Nothing special!

Side note:

Let $g := \det(g_{ij})$ and $\bar{g} := \det(\bar{g}_{ij}) \xrightarrow{\text{Eq. (3.53)}}$

$$\sqrt{|\bar{g}|} = \left|\det\left(\frac{\partial x}{\partial \bar{x}}\right)\right|\sqrt{|g|} \tag{3.54}$$

$\to \sqrt{|g|}$ is a *pseudo* scalar tensor density of weight $w = +1$. The "pseudo" indicates that the absolute value of the Jacobian determinant shows up, cf. Eq. (3.37).

$\sphericalangle\; g < 0 \xrightarrow{\text{Eq. (3.39)}} \mathrm{d}^D x\sqrt{-g}$ is a scalar (→ *later*)!

30 | Length of curves on M:

One immediate benefit of having a Riemannian manifold is that we can now compute the length of curves $\gamma(t)$ on M (parametrized by $t \in [a,b]$ and given in some chart):

$$\boxed{\begin{aligned} L[\gamma] \equiv \int_\gamma \mathrm{d}s &:= \int_a^b \sqrt{g_{ij}(\gamma(t))\frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j(t)}{\mathrm{d}t}}\,\mathrm{d}t && (3.55)\\ &\equiv \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)}\,\mathrm{d}t && (3.56)\end{aligned}}$$

¡! If $\mathrm{d}s^2$ is a true *pseudo* metric (i.e., g_{ij} has at least one negative eigenvalue), one must make sure that the chosen curve γ does not produce negative values under the square root. In RELATIVITY these will be ↑ *time-like* curves.

Example:

Let γ be the circle with radius R in the Euclidean plane E_2. A possible parametrization in Cartesian coordinates (with origin in the center of the circle) is $\vec{\gamma}_{xy}(t) = (x_t, y_t) = (R\cos(t), R\sin(t))$ with $0 \leq t < 2\pi$ so that one finds for the circumference:

$$L = \int_\gamma \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_0^{2\pi}\sqrt{\dot{x}_t^2 + \dot{y}_t^2}\,\mathrm{d}t \overset{\circ}{=} 2\pi R \tag{3.57}$$

The same length can of course be calculated with the parametrization $\vec{\gamma}_{r\theta}(t) = (r_t, \theta_t) = (R, t)$ and $0 \leq t < 2\pi$ in polar coordinates:

$$L = \int_\gamma \sqrt{\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2} = \int_0^{2\pi} \sqrt{\dot{r}_t^2 + r_t^2 \dot{\theta}_t^2}\,\mathrm{d}t \stackrel{\circ}{=} 2\pi R \tag{3.58}$$

Details: ➔ Problemset 4

31 | Besides computing lengths of curves (and other geometric quantities, → *later*), there is another benefit of having a metric tensor:

Pulling indices *down*:

$$\boxed{\tilde{T}^{i_1 \dots \square \dots i_p \square \dots \square}_{\square \dots\, i\, \dots \square\, j_1 \dots j_q} := g_{ik} T^{i_1 \dots\, k\, \dots i_p \square \dots \square}_{\square \dots \square \dots \square\, j_1 \dots j_q}} \tag{3.59}$$

→ $\tilde{T}$ is a tensor of type $(p-1, q+1)$

- In Eq. (3.59) we indicate "empty" slots for indices by □ to emphasize that in each index "column" an index can either be *up* (contravariant) or *down* (covariant). It is conventional to omit the □-markers. Note that this explains why you never should write two indices directly above each other (except for special cases, → *below*).

  Furthermore, since g is fixed, it makes sense to label $\tilde{T}$ again by T (note that the difference between the original tensor and the new one is manifest in the different index patterns!):

$$\tilde{T}^{i_1 \dots \square \dots i_p \square \dots \square}_{\square \dots\, i\, \dots \square\, j_1 \dots j_q} \quad \mapsto \quad T^{i_1 \dots \quad \dots i_p}{}_{\quad\quad i \quad\quad j_1 \dots j_q} \tag{3.60}$$

  Example:

$$A^{i}{}_{j}{}^{k}{}_{l} := g_{jm} A^{imk}{}_{l} \tag{3.61}$$

- This convention matches perfectly with the computation of an inner product (which is determined by the metric tensor g) of two contravariant vectors:

$$\langle A, B \rangle \stackrel{\text{def}}{=} g_{ij} A^i B^j \stackrel{\text{def}}{=} \underbrace{A^i B_i}_{\text{Scalar}} \tag{3.62}$$

32 | Pulling indices *up*:

We would like to have a $(2,0)$-tensor g^{ij} with the property

$$\delta^k_j T^j = T^k \stackrel{!}{=} g^{ki} T_i \stackrel{\text{def}}{=} g^{ki} g_{ij} T^j\,. \tag{3.63}$$

g^{ij} allows us to revert the pulling-down of indices defined by the metric g_{ij}. Note that g^{ij} is a *different* tensor than g_{ij}, we could call it $\tilde{g}^{ij}$; however, it is conventional to denote it with the same label due to the following close relationship with g:

$$\boxed{g^{ki} g_{ij} \stackrel{!}{=} \delta^k_j} \tag{3.64}$$

This is an implicit equation for g^{ki}!

→ g^{ij} is the *inverse matrix* of g_{ij}
(Which always exists because $\mathrm{d}s^2$ is non-degenerate: $\det(g_{ij}) \neq 0$.)

→ In general:

$$\boxed{\tilde{T}^{i_1\dots i_p\,\square\dots\, j\,\dots\square}{}_{\square\dots\square\, j_1\dots\square\dots j_q} := g^{jk}\, T^{i_1\dots i_p\,\square\dots\square\dots\square}{}_{\square\dots\square\, j_1\dots\, k\,\dots j_q}} \tag{3.65}$$

→ $\tilde{T}$ is a tensor of type $(p+1, q-1)$

- Again we relabel $\tilde{T}$ to T and omit the $\square$-markers:

$$\tilde{T}^{i_1\dots i_p\,\square\dots\, j\,\dots\square}{}_{\square\dots\square\, j_1\dots\square\dots j_q} \quad\mapsto\quad T^{i_1\dots i_p}{}_{j_1\dots}{}^{j}{}_{\dots j_q} \tag{3.66}$$

- Example:

$$A^{ijkl} := g^{lm} A^{ijk}{}_m \tag{3.67}$$

- With these new definitions, we can now raise and lower contractions:

$$A^i B_i = A^i \delta_i^j B_j = A^i g_{ik} g^{kj} B_j = A^i g_{ik} B^k = A_k B^k = A_i B^i \tag{3.68}$$

- What happens if you pull the indices of the Kronecker symbol up or down?

$$\delta^{ij} := g^{jk}\delta^i{}_k = g^{ij} \quad\text{and}\quad \delta_{ij} := g_{ik}\delta^k{}_j = g_{ij} \tag{3.69}$$

¡! $\delta^{ij} \equiv g^{ij}$ and $\delta_{ij} \equiv g_{ij}$ denote the metric and its inverse!

→ We never use the notation δ^{ij} and δ_{ij} to prevent confusion!

- Note that in general

$$g^{jk} T^i{}_k = T^{ij} \neq T^{ji} = g^{jk} T_k{}^i\,. \tag{3.70}$$

This means that the "column" in which the index is located is *important*, and notations like T^i_k are ill defined (if you pull k up by g^{jk}, do you get T^{ij} or T^{ji}?). However, if the tensor is *symmetric*, $T^{ij} = T^{ji}$, this does not matter and you can get away with the sloppy notation T^i_k. This explains why writing δ^i_k for the Kronecker symbol is fine: $g^{ji} = g^{jk}\delta^i_k$ is symmetric.

33 | Mathematical side note:

"Pulling indices up and down" is mathematically the application of an ↓ *isomorphism* between T_pM and T_p^*M:

$$g(\bullet,\bullet):\ T_pM \ni A \mapsto g(A,\bullet) \in T_p^*M \tag{3.71}$$

This has nothing to do with differential geometry or manifolds in particular; it is a general feature of non-degenerate bilinear forms on vector spaces. In differential geometry, this canonical isomorphism between the tangent bundle TM and the cotangent bundle T^*M is know as ↑ *musical isomorphism*.

For example, you are using the same kind of isomorphism all the time in quantum mechanics, namely whenever you "dagger" a *ket* $|\Psi\rangle$ to obtain a *bra* $\langle\Psi|$:

$$(\bullet)^\dagger:\ \mathcal{H} \ni |\Psi\rangle \mapsto \langle\Psi| \equiv |\Psi\rangle^\dagger \in \mathcal{H}^* \quad\text{with}\quad \langle\Psi||\Phi\rangle \stackrel{!}{=} \langle\Psi|\Phi\rangle_{\mathcal{H}} \quad\text{for all } |\Phi\rangle \in \mathcal{H}\,. \tag{3.72}$$

Note how the bra *bra* $\langle\Psi|$ associated to the *ket* $|\Psi\rangle$ is *defined* via the inner product $\langle\bullet|\bullet\rangle_{\mathcal{H}}$ (and therefore metric) of the Hilbert space (↑ *Riesz representation theorem*)!

This leads to a nice dictionary between concepts in tensor calculus (and therefore RELATIVITY) and the bra-ket formalism of quantum mechanics:

	Relativity (fixed $p \in M$)	**Quantum mechanics**
Inner product space	$T_p M$	$\mathcal{H}$
Basis	$\{\partial_i\}$	$\{\|i\rangle\}$
Vector	$A = A^i \partial_i$	$\|\Psi\rangle = \Psi_i \|i\rangle$
Dual space	$T_p^* M$	$\mathcal{H}^*$
Dual basis	$\{\mathrm{d}x^i\}$	$\{\langle i\|\}$
...	$\mathrm{d}x^i(\partial_j) = \delta^i_j$	$\langle i\|j\rangle = \delta_{ij}$
Covector	$B = B_i \mathrm{d}x^i$	$\langle\Psi\| = \Psi_i^* \langle i\|$
Inner product	$g(A_1, A_2) = g_{ij} A_1^i A_2^j$	$\langle\Psi\|\Phi\rangle$
Tensor	$A = A^{ij} \partial_i \otimes \partial_j$	$\|\Psi\rangle \otimes \|\Phi\rangle \equiv \|\Psi\rangle\|\Phi\rangle$
...	$B = B_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$	$\langle\Psi\| \otimes \langle\Phi\| \equiv \langle\Psi\|\langle\Phi\|$
Operator	$T = T^i{}_j \partial_i \otimes \mathrm{d}x^j$	$\|\Phi\rangle \otimes \langle\Psi\| \equiv \|\Phi\rangle\langle\Psi\|$
Trace	$T^i{}_i$	$\mathrm{Tr}[\|\Phi\rangle\langle\Psi\|]$
Scalar	$BA = B_i A^i = g_{ij} B^i A^j$	$\langle\Psi\|\|\Phi\rangle = \langle\Psi\|\Phi\rangle$
Pulling indices down	$A_i = g_{ij} A^j$	$\langle\Psi\| = \|\Psi\rangle^\dagger$
Pulling indices up	$A^i = g^{ij} A_j$	$\|\Psi\rangle = \langle\Psi\|^\dagger$

3.6. Differentiation of tensor fields

34 | Remember: $\partial_i \Phi$ is covariant vector if Φ is scalar. However:

$\triangleleft$ Contravariant vector A^i:

$$\bar{A}^i{}_{,k} \equiv \frac{\partial \bar{A}^i}{\partial \bar{x}^k} = \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial}{\partial x^m} \left[\frac{\partial \bar{x}^i}{\partial x^l} A^l \right] = \underbrace{\frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^l} \frac{\partial x^m}{\partial \bar{x}^k} A^l}_{\neq 0 \text{ (in general) ☹}} + \underbrace{\frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial A^l}{\partial x^m}}_{(1,1)\text{-tensor ☺}} \tag{3.73}$$

Here we used the transformation of $\bar{A}^i$ [Eq. (3.8)] and $\bar{\partial}_k$ [Eq. (3.5)] and the product rule.

$\rightarrow$ In general: $\frac{\partial \bar{A}^i}{\partial \bar{x}^k}$ is not a tensor!

35 | How to define a derivative of tensor fields that again transforms as a tensor?

To solve this problem, we first need a new field:

$\rightarrow$ ⁂ *Christoffel symbols (of the second kind)*:

$$\boxed{\Gamma^i{}_{kl} := \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right)} \tag{3.74}$$

- The Christoffel symbols are symmetric in the lower two indices: $\Gamma^i{}_{kl} = \Gamma^i{}_{lk}$
- ¡! Despite the index notation, the Christoffel symbols are *not* tensors:

$$\bar{\Gamma}^i{}_{kl} \stackrel{\circ}{=} \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial x^p}{\partial \bar{x}^l} \Gamma^m{}_{np} - \underbrace{\frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial x^p}{\partial \bar{x}^l} \frac{\partial^2 \bar{x}^i}{\partial x^n \partial x^p}}_{\text{No tensor!}} \tag{3.75}$$

This is why they are called "symbols" and not "tensors"!

- There are also Christoffel symbols of the *first* kind:

$$\Gamma_{ikl} := g_{ij}\Gamma^{j}{}_{kl} = \frac{1}{2}\left(g_{ik,l} + g_{il,k} - g_{kl,i}\right) \tag{3.76}$$

- Mathematically, the Christoffel symbols are the coefficients (in some basis) of the ↑ *Levi-Civita connection* which is determined by the metric tensor g^{ij} (→ *later*).

36 | ◁ Contravariant vector $\bar{A}^i$ and contract it with $\bar{\Gamma}^i{}_{kl}$:

$$\bar{\Gamma}^i{}_{kl}\bar{A}^l = \underbrace{\frac{\partial\bar{x}^i}{\partial x^m}\frac{\partial x^n}{\partial\bar{x}^k}\Gamma^m{}_{np}\underbrace{\left[\frac{\partial x^p}{\partial\bar{x}^l}\bar{A}^l\right]}_{A^p}}_{(1,1)\text{-tensor} ☺} - \underbrace{\frac{\partial x^n}{\partial\bar{x}^k}\frac{\partial^2\bar{x}^i}{\partial x^n\partial x^p}\underbrace{\left[\frac{\partial x^p}{\partial\bar{x}^l}\bar{A}^l\right]}_{A^p}}_{\text{Problematic term in Eq. (3.73)}} \tag{3.77}$$

Idea: Add Eq. (3.73) and Eq. (3.77) to cancel the problematic term:

$$\bar{A}^i{}_{,k} + \bar{\Gamma}^i{}_{kp}\bar{A}^p = \underbrace{\frac{\partial x^m}{\partial\bar{x}^k}\frac{\partial\bar{x}^i}{\partial x^l}\left[A^l{}_{,m} + \Gamma^l{}_{mp}A^p\right]}_{(1,1)\text{-tensor} ☺☺} \tag{3.78}$$

37 | This motivates the definition of the ⁂ *Covariant derivative:*

$$\text{Scalar:} \quad \Phi_{;k} := \Phi_{,k} \tag{3.79a}$$

$$\text{Contravariant vector:} \quad A^i{}_{;k} := A^i{}_{,k} + \Gamma^i{}_{kl}A^l \tag{3.79b}$$

$$\text{Covariant vector:} \quad B_{i;k} := B_{i,k} - \Gamma^l{}_{ik}B_l \tag{3.79c}$$

- With this definition, $A^i{}_{;k}$ is a $(1,1)$-tensor and $B_{i;k}$ is a $(0,2)$-tensor!
- With this definition, the product rule is valid for the covariant derivative:

$$(A^iB_i)_{;k} = (A^iB_i)_{,k} \stackrel{\circ}{=} A^i{}_{;k}B_i + A^iB_{i;k} \tag{3.80}$$

- The construction of higher-rank tensors by tensoring contra- and covariant vectors Eq. (3.32) and the definitions of the covariant derivative above Eq. (3.79) can be used to construct covariant derivatives of arbitrary tensor fields. For example:

$$T^i{}_{k;l} := T^i{}_{k,l} + \Gamma^i{}_{ml}T^m{}_k - \Gamma^m{}_{kl}T^i{}_m \tag{3.81}$$

- With this generalization, we can apply the covariant derivative multiple times. For example:

$$A^i{}_{;k;l} \equiv \left(A^i{}_{;k}\right)_{;l} \tag{3.82}$$

- The covariant derivative is *not commutative* in general:

$$A^i{}_{;k;l} - A^i{}_{;l;k} \neq 0 \tag{3.83}$$

→ Riemann curvature tensor → GENERAL RELATIVITY (→ *later*)

(This is not the case for the "normal" derivative: $A^i{}_{,k,l} = A^i{}_{,l,k}$.)

38 | Conclusion:

If you can formulate an equation that describes a physical theory in terms of tensors, it can always be brought into the form

$$T^I{}_J(x) = 0\,. \tag{3.84}$$

(This equation is meant to hold for all values of indices I and J and all coordinate values x.)

Here is an example:

The (inhomogeneous) Maxwell equations on an arbitrary (potentially curved) spacetime read:

$$\underbrace{F^{\mu\nu}{}_{;\nu} + \frac{4\pi}{c}J^\mu}_{=:T^\mu(x)} = 0 \tag{3.85}$$

with current density J^μ and field strength tensor $F^{\mu\nu} = g^{\mu\rho}g^{\nu\pi}(A_{\pi;\rho} - A_{\rho;\pi})$.

How does Eq. (3.84) look like in any other coordinate system $\bar{x} = \varphi(x)$?

Easy:

$$\bar{T}^I{}_J(\bar{x}) = \frac{\partial \bar{x}^I}{\partial x^M}\frac{\partial x^N}{\partial \bar{x}^J}\underbrace{T^M{}_N(x)}_{=0} = 0 \quad \Leftrightarrow \quad \bar{T}^I{}_J(\bar{x}) = 0\,. \tag{3.86}$$

This means:

Tensor equations are automatically form-invariant under *arbitrary* coordinate transformations; we say they exhibit ⁂ *(manifest) general covariance.*

The "manifest" means that checking general covariance is just a matter of checking whether the equation "looks right", i.e., whether it is built from tensors following the rules discussed in this chapter. If a property of an equation is manifest, you don't have to do calculations to verify it!

In the next chapter, we take a step back and specialize the allowed coordinate transformations to the Lorentz transformations of SPECIAL RELATIVITY. We can then use the form-invariance of equations built from "Lorentz tensors" to construct Lorentz covariant equations from scratch – which was our original goal!

↓Lecture 9 [12.12.23]

4. Formulation on Minkowski Space

In this section we briefly reformulate what we already learned about SPECIAL RELATIVITY in terms of tensor calculus. We use this notation in subsequent chapters to make classical and quantum mechanics relativistic, and reformulate electrodynamics in a form where its Lorentz covariance is manifest. It also allows a smooth transition into GENERAL RELATIVITY.

The formulation of SPECIAL RELATIVITY on a unified, four-dimensional spacetime manifold goes back to Hermann Minkowski, Albert Einstein's former professors of mathematics at ETH. Minkowski writes in the notes of his lecture "Raum und Zeit" delivered 1908 in Cologne [53]:

> *Die Anschauungen über Raum und Zeit, die ich Ihnen entwickeln möchte, sind auf experimentell-physikalischem Boden erwachsen. Darin liegt ihre Stärke. Ihre Tendenz ist eine radikale. Von Stund' an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und nur noch eine Art Union der beiden soll Selbständigkeit bewahren.*

Einstein, a physicist all through, didn't appreciate this mathematical reformulation of his theory at first. According to Sommerfeld, he (Einstein) commented:

> *Seit die Mathematiker über die Relativitätstheorie hergefallen sind, verstehe ich sie selbst nicht mehr.*

Einstein later changed his views and acknowledged that without Minkowski's introduction of spacetime as a four-dimensional manifold, the development of GENERAL RELATIVITY would have been impossible.

For a historical account on the role of Minkowski, and his relationship (or absence thereof) to Einstein, see Ref. [54].

4.1. Minkowski space

1 | Manifold:

$$\boxed{M = \langle\text{Spacetime of events / coincidence classes } \mathcal{E}\rangle \simeq \mathbb{R}^4} \tag{4.1}$$

It is a well founded, but nonetheless empirical assumption that the spacetime manifold of events has the *topology* of $\mathbb{R}^4$. Note that at this point we do not impose restrictions on the *geometry* of spacetime, e.g., whether it is flat or curved; this follows below when we settle on a metric.

2 | Charts:

In SPECIAL RELATIVITY, we restrict the coordinate systems to the ones that correspond to inertial observers / inertial coordinate systems:

$$\boxed{(\mathcal{E}, K) \quad \leftrightarrow \quad \text{Inertial (coordinate) systems } K \in \mathcal{I}} \tag{4.2}$$

The coordinates are the ones obtained by an ↑ *inertial observer*:

$$K : \mathcal{E} \ni E \mapsto K(E) := [E]_K = x \tag{4.3}$$

$$\text{with} \quad x^\mu = (x^0, x^1, x^2, x^3)^T = (ct, x, y, z)^T = (ct, \vec{x})^T \tag{4.4}$$

- ¡! Henceforth, *Greek* indices $\mu, \nu, \ldots$ run over $0, 1, 2, 3$ where $\mu = 0$ denotes the time component and $\mu = 1, 2, 3$ denote the spatial components. *Roman* indices $i, j, \ldots$ run only over the spatial components $1, 2, 3$.
- ¡! We multiply the time t with the speed of light to measure times and distances in the same units.
- Since we assumed that our inertial systems cover all of spacetime, the domains on which the coordinate functions are defined are the complete manifold.
- The notation above is very suggestive: You can think of our inertial systems, namely the calibrated latticework of clocks and rods, as physical manifestations of the coordinate map of the corresponding chart. That is, an inertial system is a measurement device, or function, which assigns to every event $E \in \mathcal{E}$ the coordinate tuple $x = K(E) = (ct, \vec{x})_K \in E$.

3 | Transition maps:

i | We worked hard in Section 1.4 to derive and select the correct coordinate transformations between different inertial systems. The most general ones have the form of ...

$$\left.\begin{array}{r} \textit{Inhomogenous}\text{ Lorentz transformations} \\ \text{Poincaré transformations} \end{array}\right\} : \quad \bar{x} = \varphi(x) = \Lambda x + a \tag{4.5}$$

with $a \in \mathbb{R}^4$ arbitrary and $\Lambda \in \mathbb{R}^{4\times 4}$ a ↑ *Lorentz transformation*.

For the special case $a = 0 \in \mathbb{R}^4$ we found:

$$\textit{Homogeneous}\text{ Lorentz transformations:} \quad \bar{x} = \varphi(x) = \Lambda x \tag{4.6}$$

ii | Since these transformations are affine, we find immediately:

$$\frac{\partial \bar{x}^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu \quad \text{and} \quad \frac{\partial x^\mu}{\partial \bar{x}^\nu} = (\Lambda^{-1})^\mu{}_\nu \equiv \Lambda_\nu{}^\mu \tag{4.7}$$

Recall that the derivative of a linear (affine) map is simply the matrix which defines the map.

¡! We use the tensor-inspired notation $\Lambda^\mu{}_\nu$ for the matrix elements of Λ to allow for well-defined contractions with the metric (→ *later*). In $\Lambda^\mu{}_\nu$, the upper index μ denotes the *rows*, the lower index ν the *columns* of the matrix. The notation $\Lambda_\nu{}^\mu$ for the components of the inverse transformation matrix Λ^{-1} is purely conventional at this point; it will turn out to be consistent with pulling indices up and down with the Minkowski metric (→ *below*).

This allows us to rewrite the coordinate transformation Eq. (4.5) in tensor notation:

$$\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{4.8}$$

¡! The matrix-vector product Λx is now given by the Einstein summation (index contraction) highlighted blue. We will stick to this notation whenever possible. Since we are now in the world of tensor calculus, it is strongly discouraged to think of and write rank-2 tensors as "matrices" and contractions as matrix-vector products Λx (even though Λ does not represent the components of a tensor). It is less error-prone (and simpler) to perform computations using the index notation introduced in Chapter 3.

iii | Writing down the most general homogeneous Lorentz transformation is very complicated (and unnecessary). Here we provide the two special Lorentz transformations (boosts) discussed earlier in the new matrix notation, and an example for a spatial rotation about the z-axis:

- Lorentz boost in x-direction $K \xrightarrow{v_x} \bar{K}$ ($\beta_x = v_x/c$):

$$\boxed{\text{Eq. (1.77)} \rightarrow \quad \Lambda^\mu{}_\nu = [\Lambda_{v_x}]^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta_x\gamma & 0 & 0 \\ -\beta_x\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}} \tag{4.9}$$

- Lorentz boost in $\hat{v}$-direction $K \xrightarrow{\vec{v}} \bar{K}$ ($v = |\vec{v}|$ and $\tilde{\gamma} := \gamma - 1$):

$$\boxed{\begin{gathered}\text{Eq. (1.75)} \rightarrow \quad \Lambda^\mu{}_\nu = [\Lambda_{\vec{v}}]^\mu{}_\nu = \\ \begin{pmatrix} \gamma & -\beta_x\gamma & -\beta_y\gamma & -\beta_z\gamma \\ -\beta_x\gamma & 1+\tilde{\gamma}{v_x}^2/v^2 & \tilde{\gamma}v_xv_y/v^2 & \tilde{\gamma}v_xv_z/v^2 \\ -\beta_y\gamma & \tilde{\gamma}v_xv_y/v^2 & 1+\tilde{\gamma}{v_y}^2/v^2 & \tilde{\gamma}v_yv_z/v^2 \\ -\beta_z\gamma & \tilde{\gamma}v_xv_z/v^2 & \tilde{\gamma}v_yv_z/v^2 & 1+\tilde{\gamma}{v_z}^2/v^2 \end{pmatrix}_{\mu\nu}\end{gathered}} \tag{4.10}$$

- Spatial rotation $K \xrightarrow{R_z(\theta),\vec{0}} \bar{K}$ by θ in xy-plane:

$$\boxed{\Lambda^\mu{}_\nu = [R_z(\theta)]^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}} \tag{4.11}$$

4 | <u>Metric tensor:</u>

i | We elevate the spacetime manifold M to a pseudo-Riemannian (and Lorentzian) manifold by introducing the following pseudo-Riemannian metric tensor (given in inertial coordinates):

$$⁂ \textit{Minkowski metric } \mathrm{d}s^2 \begin{cases} := (c\mathrm{d}t)^2 - (\mathrm{d}\vec{x})^2 \\ = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2 \\ = \eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu \end{cases} \tag{4.12a}$$

$$\text{with metric components} \quad \eta_{\mu\nu} = \eta^{\mu\nu} = \underbrace{\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}}_{\text{Signature } (1,3) \equiv (+,-,-,-)} . \tag{4.12b}$$

- The components $\eta_{\mu\nu}$ of this metric tensor in Eq. (4.12b) are the same for all inertial coordinate systems [→ *Eq.* (4.21) *below*].
- Recall that η^{muv} is the matrix inverse of $\eta_{\mu\nu}$.

→ We call the spacetime manifold equipped with this metric …

$$⁂ \textit{Minkowski space}: \quad \mathbb{R}^{1,3} \equiv (\mathcal{E} \simeq \mathbb{R}^4, \mathrm{d}s^2) \tag{4.13}$$

- We will always use $\eta_{\mu\nu}$ to denote the components of the Minkowski metric (in an inertial coordinate chart) to distinguish it from a generic metric g_{ij}.
- Note that, informally speaking, $\mathrm{d}s^2$ this is the infinitesimal form of the ← *invariant spacetime interval* Eq. (1.83) we introduced earlier (→ *below*).
- Minkowski space is therefore an example of a ← *Lorentzian manifold*. By fixing a metric, we fixed the *geometry* of spacetime. As we will see in our discussion of GENERAL RELATIVITY, the distinctive feature of Minkowski space is that it is *flat* (it has no curvature). It will turn out that, in reality, this assumption is only valid to some degree: The tenet of GENERAL RELATIVITY is that the *deviations* of spacetime from flat Minkowski space are what we expercience as gravity!

ii | With the metric we can measure "lengths" of trajectories on spacetime:

∢ Time-like trajectory $\gamma : s \mapsto x^\mu(s)$ for $s \in [s_a, s_b]$ in $\mathbb{R}^{1,3}$ →

$$L[\gamma] \stackrel{3.55}{=} \int_{s_a}^{s_b} \sqrt{\eta_{\mu\nu}\frac{\mathrm{d}x^\mu(s)}{\mathrm{d}s}\frac{\mathrm{d}x^\nu(s)}{\mathrm{d}s}}\,\mathrm{d}s \tag{4.14a}$$

$$\stackrel{4.12b}{=} \int_{s_a}^{s_b} \sqrt{[\dot{x}^0(s)]^2 - [\dot{x}^1(s)]^2 - [\dot{x}^2(s)]^2 - [\dot{x}^3(s)]^2}\,\mathrm{d}s \tag{4.14b}$$

$$\text{Choose parametrization } s := x^0/c \equiv t \tag{4.14c}$$

$$= \int_{t_a}^{t_b} \underbrace{\sqrt{c^2 - \vec{v}^2(t)}}_{>0 \text{ (time-like)}}\,\mathrm{d}t \tag{4.14d}$$

$$\stackrel{2.25}{=} c\Delta\tau[\gamma] \tag{4.14e}$$

Thus the "length" $L[\gamma]$ of time-like curves in $\mathbb{R}^{1,3}$ is the ← *proper time* $\Delta\tau[\gamma]$ along the curve defined in Eq. (2.25) (multiplied by c); this explains why the Minkowski metric $\mathrm{d}s^2$ is the right choice for SPECIAL RELATIVITY.

4.2. Four vectors and tensors

5 | Tensors are defined as in Chapter 3, with the restriction to $D = 4$ and that only homogeneous Lorentz transformations Eq. (4.7) are considered as transition maps. To emphasize this, we introduce a new nomenclature:

Tensor calculus	SPECIAL RELATIVITY
Contravariant vector A^i	Contraviarant ⁂ *Lorentz vector / 4-vector* A^μ
Covariant vector B_i	Covariant ⁂ *Lorentz vector / 4-vector* B_μ
(Mixed) tensor $T^i{}_j$	(Mixed) ⁂ *Lorentz tensor / 4-tensor* $T^\mu{}_\nu$
Scalar Φ	⁂ *Lorentz scalar* Φ

Then a generic (p, q) tensor transforms under the coordinate transformation Eq. (4.7) as:

$$\boxed{\bar{T}^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q}(\bar{x}) = [\Lambda^{\mu_1}{}_{\rho_1}\dots\Lambda^{\mu_p}{}_{\rho_p}][\Lambda_{\nu_1}{}^{\pi_1}\dots\Lambda_{\nu_q}{}^{\pi_q}]\,T^{\rho_1\dots\rho_p}{}_{\pi_1\dots\pi_q}(x)} \tag{4.15}$$

6 | With the Minkowski metric, we can reformulate our classification for 4-vectors [recall Eq. (1.85)]:

$$\boxed{\left.\begin{array}{l} X^\mu \text{ time-like} \\ X^\mu \text{ light-like} \\ X^\mu \text{ space-like}\end{array}\right\} \quad :\Leftrightarrow \quad X^2 = X^\mu X_\mu = (X^0)^2 - (\vec{X})^2 \left\{\begin{array}{l} > 0 \\ = 0 \\ < 0\end{array}\right.} \tag{4.16}$$

A *light-like* 4-vector is also called ⁂ *null*.

¡! We use this classification scheme also for generic Lorentz vectors that are not coordinate differences between a pair of events (→ *below*). Since the pseudo-norm $X^\mu X_\mu = X^2$ is a Lorentz scalar, this classification is independent of the inertial system.

7 | <u>Coordinate functions:</u>

It is a particular feature of *linear* coordinate transformations (here: homogeneous Lorentz transformations) that the coordinate functions themselves transform as contravariant vector fields:

$\lessdot$ Coordinate field $X^\mu(x) := x^\mu \rightarrow$

$$\underbrace{\bar{X}^\mu(\bar{x})}_{\bar{x}^\mu} = \Lambda^\mu{}_\nu \underbrace{X^\nu(x)}_{x^\nu} = \underbrace{\frac{\partial \bar{x}^\mu}{\partial x^\nu}}_{\Lambda^\mu{}_\nu} X^\nu(x) \tag{4.17}$$

We make the identification $X^\mu(x) \equiv x^\mu$ and don't write $X^\mu(x)$ henceforth.

Consequently, we can construct ⁂ *covariant coordinates* (a covariant vector field) via the metric by pulling the index down:

$$\boxed{x_\mu := \eta_{\mu\nu}x^\nu = (x^0, -x^1, -x^2, -x^3) = (ct, -\vec{x})} \tag{4.18}$$

¡! To pull the index of a contravariant vector down, you multiply the spatial components by -1.

8 | ⊲ Coordinates of two events x_A^μ and $x_B^\mu \to \Delta x^\mu := x_B^\mu - x_A^\mu$ Lorentz vector

$$\Delta x^2 \equiv \Delta x^\mu \Delta x_\mu \tag{4.19a}$$
$$\stackrel{\text{def}}{=} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \tag{4.19b}$$
$$= (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2 \tag{4.19c}$$
$$\stackrel{\text{def}}{=} \Delta s^2 \tag{4.19d}$$

Remember [Eq. (1.84)]: $\Delta s^2 = \Delta \bar{s}^2$ for arbitrary Lorentz transformations

$\rightarrow$

$$\underbrace{\eta_{\mu\nu}\, \Delta x^\mu \Delta x^\nu}_{\Delta s^2} = \underbrace{\eta_{\rho\pi}\, \Delta \bar{x}^\rho \Delta \bar{x}^\pi}_{\Delta \bar{s}^2} = \left[\eta_{\rho\pi} \Lambda^\rho{}_\mu \Lambda^\pi{}_\nu\right] \Delta x^\mu \Delta x^\nu \tag{4.20}$$

Since this is true for all events $\Delta x^\mu \stackrel{\circ}{\to}$

$$\boxed{\Lambda^\rho{}_\mu \Lambda^\pi{}_\nu \eta_{\rho\pi} = \eta_{\mu\nu}} \tag{4.21}$$

Concluding Eq. (4.21) from Eq. (4.20) is non-trivial because we consider "norms" $\eta_{\mu\nu}\, \Delta x^\mu \Delta x^\nu$ and not "inner products" $\eta_{\mu\nu}\, \Delta x^\mu \Delta y^\nu$. However, for *symmetric*, real matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, it is true that if $\vec{x}^T \boldsymbol{A} \vec{x} = \vec{x}^T \boldsymbol{B} \vec{x}$ for all real vectors $\vec{x}$, then $\boldsymbol{A} = \boldsymbol{B}$. This is so because $\boldsymbol{A} - \boldsymbol{B}$ is a symmetric matrix that can be diagonalized by an orthogonal matrix and $\vec{x}^T(\boldsymbol{A} - \boldsymbol{B})\vec{x} = 0$. The last condition implies that all eigenvalues of $\boldsymbol{A} - \boldsymbol{B}$ are zero and therefore $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{0}$. Alternatively you can use the ↓ *polarization identity* to show that the invariance of the Minkowski (pseudo) norm implies the invariance of the Minkowski (pseudo) inner product.

We say:

$$\boxed{\text{Lorentz transformations are ↓ \textit{isometries} of Minkowski space.}} \tag{4.22}$$

With $\det(\eta_{\mu\nu}) \neq 0$, a corollary of Eq. (4.21) is:

$$\boxed{\det\left(\Lambda^\mu{}_\nu\right) = \pm 1} \tag{4.23}$$

If you want to write Eq. (4.21) in the old matrix notation, make the identifcations $\Lambda^\mu{}_\nu = \boldsymbol{\Lambda}_{\mu\nu}$ and $\eta_{\mu\nu} = \boldsymbol{\eta}_{\mu\nu}$. Here, subscripts of bold symbols denote the entries of *matrices* as usual (first index: row; second index: column). Equations that contain matrices (bold symbols) do *not* comply with the syntax of tensor calculus (which is why you should avoid them!).

Eq. (4.21) then reads in matrix notation:

$$\boldsymbol{\Lambda}^T_{\mu\rho} \boldsymbol{\eta}_{\rho\pi} \boldsymbol{\Lambda}_{\pi\nu} = \boldsymbol{\eta}_{\mu\nu} \quad \Leftrightarrow \quad \boldsymbol{\Lambda}^T \boldsymbol{\eta} \boldsymbol{\Lambda} = \boldsymbol{\eta} \tag{4.24}$$

Here we defined the *transposed matrix* as $\boldsymbol{\Lambda}^T_{\mu\rho} := \boldsymbol{\Lambda}_{\rho\mu}$, i.e., the matrix where rows and columns are swapped. Eq. (4.24) immediately implies $\det(\boldsymbol{\Lambda}^T)\det(\boldsymbol{\eta})\det(\boldsymbol{\Lambda}) = \det(\boldsymbol{\eta})$; using $\det(\boldsymbol{\eta}) \neq 0$ and $\det(\boldsymbol{\Lambda}^T) = \det(\boldsymbol{\Lambda})$, we find $\det(\boldsymbol{\Lambda}) = \pm 1$.

9 | Eq. (4.21) →

$$\Lambda^{\rho}{}_{\mu}\left[\Lambda^{\pi}{}_{\nu}\eta_{\rho\pi}\eta^{\nu\sigma}\right] = \delta^{\sigma}_{\mu} \tag{4.25}$$

We can therefore conclude that:

$$\boxed{\Lambda_{\rho}{}^{\sigma} := \eta_{\rho\pi}\eta^{\nu\sigma}\Lambda^{\pi}{}_{\nu} = (\Lambda^{-1})^{\sigma}{}_{\rho}} \tag{4.26}$$

Note that this is consistent with our definition in Eq. (4.7).

In the literature (e.g. Schröder [1]) the concept of a "transposed" transformation is introduced. We refer to it as *"pseudo-adjoint"* transformation instead and label it by $*$. It is defined analogous to proper adjoints on proper inner product spaces:

$$\underbrace{\eta_{\mu\nu}\Lambda^{\nu}{}_{\rho}}_{=:\Lambda_{\mu\rho}} x^{\rho} y^{\mu} \overset{\text{def}}{=} \langle y, \Lambda x\rangle \overset{!}{=} \langle \Lambda^* y, x\rangle \overset{\text{def}}{=} \underbrace{\eta_{\mu\nu}(\Lambda^*)^{\mu}{}_{\rho}}_{=:(\Lambda^*)_{\nu\rho}} x^{\nu} y^{\rho}\,. \tag{4.27}$$

This yields as reasonable definition for the pseudo-adjoint:

$$(\Lambda^*)_{\mu\nu} := \Lambda_{\nu\mu} \quad \Rightarrow \quad (\Lambda^*)^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu} \overset{\text{Eq. (4.26)}}{=} (\Lambda^{-1})^{\mu}{}_{\nu}\,. \tag{4.28}$$

One can then define a corresponding matrix $\boldsymbol{\Lambda}^*$ such that $(\Lambda^*)^{\mu}{}_{\nu} = \boldsymbol{\Lambda}^*_{\mu\nu}$ and use $(\Lambda^{-1})^{\mu}{}_{\nu} = \boldsymbol{\Lambda}^{-1}_{\mu\nu}$ to rewrite the above equation as

$$\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}^{-1}\,. \tag{4.29}$$

Recall that the pseudo-adjoint is implicitly defined via the inner product. At no point did we claim that the pseudo-adjoint matrix is given by the *transposed* matrix $\boldsymbol{\Lambda}^T$ (which is defined by swapping rows and columns)! To find a relation to the latter, we can rewrite Eq. (4.26) in matrix language:

$$\boldsymbol{\Lambda}^{-1}_{\sigma\rho} = \boldsymbol{\eta}_{\rho\pi}\boldsymbol{\Lambda}_{\pi\nu}\boldsymbol{\eta}^{-1}_{\nu\sigma} = (\boldsymbol{\eta}\boldsymbol{\Lambda}\boldsymbol{\eta})_{\rho\sigma} = (\boldsymbol{\eta}\boldsymbol{\Lambda}^T\boldsymbol{\eta})_{\sigma\rho}\,. \tag{4.30}$$

Here we used that $\boldsymbol{\eta}^{-1} = \boldsymbol{\eta} = \boldsymbol{\eta}^T$ and that $\boldsymbol{M}^T_{ab} := \boldsymbol{M}_{ba}$ for any matrix $\boldsymbol{M}$. So finally:

$$\boldsymbol{\Lambda}^* = \boldsymbol{\Lambda}^{-1} = \boldsymbol{\eta}\boldsymbol{\Lambda}^T\boldsymbol{\eta}\,. \tag{4.31}$$

The take home message is that the *transpose* of a Lorentz transformation (given by swapping columns and rows) is *not* its inverse (there are additional minuses sprinkled in by the metric)! By contrast, the *pseudo-adjoint* (defined via the pseudo-inner product) *is* identical to the inverse.

Warning: In the literature you will find the notation ^{T} instead of * (e.g. Schröder [1]). Then one finds the (correct) relation $(\Lambda^T)^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}$. The problem is that this notation *suggests* that $(\Lambda^T)^{\mu}{}_{\nu} \overset{\text{\lightning}}{=} \boldsymbol{\Lambda}^T_{\mu\nu}$ and therefore $\boldsymbol{\Lambda}^{-1} \overset{\text{\lightning}}{=} \boldsymbol{\Lambda}^T$. As shown above, *both equations are wrong!*

10 | <u>Covariant derivative:</u>

i | Since in inertial coordinate systems the Minkowski metric is given by $\eta_{\mu\nu}$, it follows immediately for the Christoffel symbols Eq. (3.74):

$$\Gamma^{i}{}_{kl} = \frac{1}{2}\eta^{im}(\underbrace{\eta_{mk,l}}_{0} + \underbrace{\eta_{ml,k}}_{0} - \underbrace{\eta_{kl,m}}_{0}) = 0 \tag{4.32}$$

¡! If you would transform into *curvilinear (non-inertial) coordinates*, the Christoffel symbols would *not* vanish – even on flat Minkowski space (➔ Problemset 5). That simple partial derivatives produce Lorentz tensors is therefore a special feature of Minkowski space in inertial coordinates.

$\xrightarrow{\text{Eq. (3.79)}}$

$$\boxed{\begin{aligned} \text{Lorentz Scalar:} \quad & \Phi_{;\mu} := \Phi_{,\mu} = \partial_\mu \Phi && (4.33a) \\ \text{Contravariant Lorentz vector:} \quad & A^\mu{}_{;\nu} := A^\mu{}_{,\nu} = \partial_\nu A^\mu && (4.33b) \\ \text{Covariant Lorentz vector:} \quad & B_{\mu;\nu} := B_{\mu,\nu} = \partial_\nu B_\mu && (4.33c) \end{aligned}}$$

ii | ⁂ *4-Gradient*:

This allows us to think of the differential operator ∂_μ itself as a *covariant* Lorentz vector and motivates the introduction of its *contravariant* components:

$$\boxed{\begin{aligned} \partial_\mu \quad &= \frac{\partial}{\partial x^\mu} = \left(\tfrac{1}{c}\partial_t, +\vec{\nabla}\right)^T && (4.34a) \\ \partial^\mu := \eta^{\mu\nu}\partial_\nu &= \frac{\partial}{\partial x_\mu} = \left(\tfrac{1}{c}\partial_t, -\vec{\nabla}\right) && (4.34b) \end{aligned}}$$

Using Eq. (3.5), the transformation laws match that of co- and contravariant Lorentz vectors, respectively:

$$\boxed{\begin{aligned} \bar{\partial}_\mu &= \frac{\partial}{\partial \bar{x}^\mu} = \Lambda_\mu{}^\nu \frac{\partial}{\partial x^\nu} = \Lambda_\mu{}^\nu \partial_\nu && (4.35a) \\ \bar{\partial}^\mu &= \frac{\partial}{\partial \bar{x}_\mu} = \Lambda^\mu{}_\nu \frac{\partial}{\partial x_\nu} = \Lambda^\mu{}_\nu \partial^\nu \,. && (4.35b) \end{aligned}}$$

¡! The *covariant* 4-gradient (index *down*) is the partial derivative wrt. the *contravariant* coordinates (index *up*) and vice versa.

iii | These transformation properties immediately suggest two Lorentz scalars that can be constructed from 4-gradients ($A^\mu = (A^0, \vec{A})$):

$$\boxed{\begin{aligned} &⁂\ \textit{4-divergence}: \quad \partial A := \partial_\mu A^\mu = \partial^\mu A_\mu = \tfrac{1}{c}\partial_t A^0 + \vec{\nabla}\cdot\vec{A} && (4.36a) \\ &⁂\ \textit{4-Laplacian}: \quad \Box \equiv \partial^2 := \partial_\mu \partial^\mu = \left(\tfrac{1}{c}\partial_t\right)^2 - \vec{\nabla}^2 && (4.36b) \end{aligned}}$$

The 4-Laplacian $\Box$ is also known as ↓ *d'Alembert operator*.

Examples:

- In electrodynamics (→ *later*) the gauge potential transforms as a contravariant Lorentz vector $A^\mu = (\frac{1}{c}\varphi, \vec{A})$.

  The ↓ *Lorenz gauge* is defined as $\partial_\mu A^\mu = 0$; it is Lorentz invariant since the 4-divergence is a Lorentz scalar: $\bar{\partial}_\mu \bar{A}^\mu(\bar{x}) = \partial_\mu A^\mu(x)$.

  Note: The Lorenz gauge is named after ↑ *Ludvig Lorenz*; by contrast, the Lorentz transformation is named after ↑ *Hendrik Lorentz*. Thus: *The Lorenz gauge* (no "t") *is Lorentz invariant.*

- In vacuum (and in Lorenz gauge), the gauge field of electrodynamics satisfies the wave equation

$$\partial^2 A^\mu = \left[\left(\tfrac{1}{c}\partial_t\right)^2 - \vec{\nabla}^2\right] A^\mu = 0\,. \tag{4.37}$$

Since ∂^2 is a Lorentz scalar and A^μ a Lorentz vector, $\partial^2 A^\mu$ transforms as a contravariant Lorentz vector and the equation is *manifestly Lorentz covariant*:

$$\partial^2 A^\mu(x) = 0 \quad \Leftrightarrow \quad \bar{\partial}^2 \bar{A}^\mu(\bar{x}) = 0\,. \tag{4.38}$$

- If we have a scalar field Φ, we can construct a manifestly Lorentz covariant wave equation:

$$(\partial^2 + m^2)\Phi(x) = 0 \quad \Leftrightarrow \quad (\bar{\partial}^2 + m^2)\bar{\Phi}(\bar{x}) = 0\,. \tag{4.39}$$

The parameter m is arbitrary and plays the role of a mass (spectral gap) of the excitations. This equation is known as ↑ *Klein-Gordon equation* and describes, for example, the classical equation of motion of the Higgs field (without interactions).

11 | Relative tensors → ⁂ *Lorentz pseudo tensor*:

Since $\det(\Lambda) = \pm 1$, the classification of tensors simplifies:

$$\boxed{\begin{aligned} \text{Tensor:}\quad & \bar{T}^{M}{}_{N}(\bar{x}) = \ \Lambda^{M}{}_{R}\Lambda_{N}{}^{P}\,T^{R}{}_{P}(x) & (4.40\text{a}) \\ \textit{Pseudo}\ \text{tensor:}\quad & \bar{T}^{M}{}_{N}(\bar{x}) = \det(\Lambda)\,\Lambda^{M}{}_{R}\Lambda_{N}{}^{P}\,T^{R}{}_{P}(x) & (4.40\text{b}) \end{aligned}}$$

Here we use again a multi-index notation: $M = \mu_1, \dots, \mu_p$ etc. Recall that $\det(\Lambda) = \pm 1$; pseudo tensors therefore pick up an additional minus sign under parity or time inversion (→ *later*).

→ Relative tensors of *odd* weight w are pseudo tensors under Lorentz transformations.

Example:

The Levi-Civita symbol is a Lorentz pseudo tensor [recall Eq. (3.42)]:

$$\bar{\varepsilon}^{\mu\nu\rho\pi} = \varepsilon^{\mu\nu\rho\pi} = \det(\Lambda)\,\Lambda^{\mu}{}_{\mu'}\Lambda^{\nu}{}_{\nu'}\Lambda^{\rho}{}_{\rho'}\Lambda^{\pi}{}_{\pi'}\,\varepsilon^{\mu'\nu'\rho'\pi'}\,. \tag{4.41}$$

This means that if you contract a Levi-Civita symbol with an actual $(0,4)$ Lorentz tensor like $F_{\mu\nu}F_{\rho\pi}$ (the tensor product of two electromagnetic field trength tensors), you obtain a *pseudo (Lorentz) scalar*:

$$\bar{\Phi}(\bar{x}) := \bar{\varepsilon}^{\mu\nu\rho\pi}\,\bar{F}_{\mu\nu}\bar{F}_{\rho\pi} \stackrel{\circ}{=} \det(\Lambda)\,\varepsilon^{\mu\nu\rho\pi}\,F_{\mu\nu}F_{\rho\pi} = \det(\Lambda)\,\Phi(x)\,. \tag{4.42}$$

Since this is a quadratic (pseudo) scalar quantity, you might try to add it to the Lagrangian of Maxwell theory ($\theta \in \mathbb{R}$):

$$\tilde{\mathcal{L}} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \theta\varepsilon^{\mu\nu\rho\pi}F_{\mu\nu}F_{\rho\pi}\,. \tag{4.43}$$

(This Lagrangian is now only invariant under Lorentz transformations with $\det(\Lambda) = +1$.)

The new term is called ↑ θ-*term*. One can show that it is a total derivative and therefore does not affect the classical equations of motion (Maxwell's equations). However, for non-abelian generalizations of electrodynamics like ↑ *quantum chromodynamics* (↑ *Yang-Mills theories*), it *does* affect the theory (↑ *Strong CP-problem* [55]).

Note that we did not use the metric tensor $\eta_{\mu\nu}$ to construct the term $\varepsilon^{\mu\nu\rho\pi}F_{\mu\nu}F_{\rho\pi}$ (as compared to $F^{\mu\nu}F_{\mu\nu}$ where we need it to pull two indices up); this makes the θ-term an example of a so called ↑ *topological* term (↑ *topological field theory*): the term doesn't "see" the *geometry* of spacetime! In condensed matter physics, the term plays a role in the description of ↑ *topological insulators* [56].

12 | In the next chapter we want to construct a relativistic version of classical mechanics (using the framework of tensors calculus to make the equations Lorentz covariant). As a preparation, we can already define two 4-vectors with physical interpreation:

i | 4-velocity:

Question: What is a reasonable definition for a relativistic (= Lorentz covariant) velocity?

$\sphericalangle$ Particle trajectory $x^\mu(\lambda)$ parametrized by λ:

$$x^\mu(\lambda) = \begin{pmatrix} ct(\lambda) \\ \vec{x}(\lambda) \end{pmatrix} \quad \Rightarrow \quad \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = \begin{pmatrix} \frac{\mathrm{d}ct}{\mathrm{d}\lambda} \\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda} \end{pmatrix} \tag{4.44}$$

First try: $\lambda = t$ (coordinate time) $\rightarrow$

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \begin{pmatrix} c \\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} c \\ \vec{v}(t) \end{pmatrix} \tag{4.45}$$

with coordinate velocity $\vec{v}(t)$.

Problem:

$\frac{\mathrm{d}x^\mu}{\mathrm{d}t}$ is *not* a contravariant Lorentz vector because $\mathrm{d}t \neq \mathrm{d}\bar{t}$ is not a Lorentz scalar. That is:

$$\frac{\mathrm{d}\bar{x}^\mu}{\mathrm{d}\bar{t}} \neq \Lambda^\mu{}_\nu \frac{\mathrm{d}x^\nu}{\mathrm{d}t} \tag{4.46}$$

$\rightarrow$ Eq. (4.45) is useless to construct Lorentz covariant equations!

Idea: The ← *Proper time* τ is a Lorentz scalar [Eq. (2.24)]: $\mathrm{d}\tau = \mathrm{d}\bar{\tau}$

$\rightarrow$ Set $\lambda = \tau$:

$$\boxed{⁂\ \textit{4-velocity:} \quad u^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \begin{pmatrix} c\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} \end{pmatrix} = \gamma_v \begin{pmatrix} c \\ \vec{v} \end{pmatrix}} \tag{4.47}$$

Here we used $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{v(t)}$ [recall Eq. (2.23)].

By construction, the 4-velocity is a contravariant Lorentz vector: $\bar{u}^\mu = \Lambda^\mu{}_\nu u^\nu$.

$\sphericalangle$ Pseudo-norm:

$$\boxed{u^2 = \eta_{\mu\nu} u^\mu u^\nu = (u^0)^2 - (\vec{u})^2 \mathrel{\hat{=}} c^2 > 0} \tag{4.48}$$

$\rightarrow$ *Time-like* 4-vector

In Minkowski space, u^μ is the tangent at x^μ of the world line $x^\mu(\tau)$.

ii | 4-acceleration:

Following the same line of arguments above, the 4-acceleration is then defined as the derivative of the 4-velocity wrt. the proper time:

⁂ *4-acceleration:*

$$\boxed{b^\mu := \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \begin{pmatrix} c\frac{\mathrm{d}\gamma_{v(t)}}{\mathrm{d}\tau} \\ \frac{\mathrm{d}[\gamma_{v(t)}\vec{v}(t)]}{\mathrm{d}\tau} \end{pmatrix} \mathrel{\hat{=}} \begin{pmatrix} \gamma_v^4 \frac{\vec{v}\cdot\vec{a}}{c} \\ \gamma_v^2 \vec{a} + \gamma_v^4 \frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v} \end{pmatrix}} \tag{4.49}$$

Here $\vec{a} := \frac{d\vec{v}(t)}{dt}$ is the coordinate acceleration or 3-acceleration.

It is now easy to show that $b^2 = b_\mu b^\mu < 0$ is a *space-like* Lorentz vector and that

$$\boxed{\frac{d(u^\mu u_\mu)}{d\tau} = \frac{d(c^2)}{d\tau} = 0 \quad \Rightarrow \quad u^\mu b_\mu = 0\,,} \tag{4.50}$$

i.e., the 4-acceleration is always "orthogonal" (in terms of the Minkowski metric) to the 4-velocity.

4.3. The complete Lorentz group

Details: ➲ Problemset 5

1 | The Lorentz group is a matrix group defined as the homogenous isometry group of the Minkowski metric η:

$$\boxed{⁂\ \textit{Lorentz group}: \quad \mathrm{O}(1,3) := \left\{ \boldsymbol{\Lambda} \in \mathbb{R}^{4\times 4} \,\middle|\, \boldsymbol{\Lambda}^T \boldsymbol{\eta} \boldsymbol{\Lambda} = \boldsymbol{\eta} \right\}} \tag{4.51}$$

with identification $\Lambda^\mu{}_\nu = \boldsymbol{\Lambda}_{\mu\nu}$ and $\eta_{\mu\nu} = \boldsymbol{\eta}_{\mu\nu}$.

- As shown previously [Eq. (4.21) and Eq. (4.24)], the matrix constraint in Eq. (4.51) is equivalent to the property

$$\eta_{\mu\nu}\, x^\mu y^\nu \overset{\text{def}}{=} \eta(x,y) \overset{!}{=} \eta(\Lambda x, \Lambda y) \overset{\text{def}}{=} \left[\eta_{\rho\pi} \Lambda^\rho{}_\mu \Lambda^\pi{}_\nu\right] x^\mu y^\nu \tag{4.52}$$

  for all 4-vectors x, y. Namely, the transformations Λ do not change the inner product (and thereby length) of arbitrary vectors; maps with this feature are called ↑ *isometries*.

- If you replace the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}\,(+1,-1,-1,-1)$ by the Euclidean metric $\delta_{\mu\nu} = \mathrm{diag}\,(+1,+1,+1,+1)$, the homogeneous isometry constraint becomes $\boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \mathbb{1}$ since $\delta = \mathbb{1}$ is the identity matrix; this constraint characterizes *orthogonal matrices*. The homogenous isometry group of a $D = 4$ Euclidean space is therefore O(4): the group of four-dimensional *rotations* and *reflections*.

2 | Continuous Lorentz transformations:

i | Mathematical fact: $\mathrm{O}(1,3)$ is a ↑ *Lie group* (= a group that is also a differentiable manifold)

To be precise: $\mathrm{O}(1,3)$ is a 6-dimensional (→ *below*) ↑ *non-compact* ↓ *non-abelian* disconnected (→ *below*) real matrix Lie group with components that are not ↑ *simply connected*.

→ In a neighborhood of $\mathbb{1}$, elements of Lie groups can be written as exponentials:

$$\boldsymbol{\Lambda} = \exp(\boldsymbol{X}) \quad \text{with} \quad \boldsymbol{X} \in \mathfrak{o}(1,3) \tag{4.53}$$

where $\mathfrak{o}(1,3)$ denotes the ↑ *Lie algebra* (= vector space with a Lie bracket):

$$\mathfrak{o}(1,3) = \left\{ \boldsymbol{X} \in \mathbb{R}^{4\times 4} \,\middle|\, \exp(t\boldsymbol{X}) \in \mathrm{O}(1,3) \ \text{ for all } \ t \in \mathbb{R} \right\}. \tag{4.54}$$

ii | The isometry constraint on the group elements can be translated into the Lie algebra:

$$\boldsymbol{\Lambda}^T \eta \boldsymbol{\Lambda} = \eta \quad \overset{\text{Eq. (4.53)}}{\Longleftrightarrow} \quad \boldsymbol{X}^T = -\eta \boldsymbol{X} \eta \tag{4.55}$$

→ Most general form of $\boldsymbol{X}$:

$$\boldsymbol{X} = \begin{pmatrix} 0 & a & b & c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{pmatrix} \quad \text{with} \quad a, \dots, f \in \mathbb{R} \tag{4.56}$$

Proof: ➔ Problemset 5

→

- $\dim(\mathfrak{o}(1,3)) = 6$

  This is why $\mathrm{O}(1,3)$ is a 6-dimensional Lie group.
- $\mathrm{Tr}[\boldsymbol{X}] = 0 \;\Rightarrow\; \det \boldsymbol{\Lambda} = \det[\exp(\boldsymbol{X})] = \exp(\mathrm{Tr}[\boldsymbol{X}]) = 1$

  → All Lorentz transformations connected to the identity have positive determinant. Recall that we found previously $\det \Lambda = \pm 1$, so we should not expect to find *all* elements of $\mathrm{O}(1,3)$ in this way.

iii | Generators = Basis of $\mathfrak{o}(1,3)$ [57]:

We use the shorthand $+$ $(-)$ for $+1$ (-1).

$$\boxed{\begin{aligned} \boldsymbol{L}_x &= \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&-\\0&0&+&0 \end{pmatrix}, \quad \boldsymbol{L}_y = \begin{pmatrix} 0&0&0&0\\0&0&0&+\\0&0&0&0\\0&-&0&0 \end{pmatrix}, \quad \boldsymbol{L}_z = \begin{pmatrix} 0&0&0&0\\0&0&-&0\\0&+&0&0\\0&0&0&0 \end{pmatrix} \quad (4.57a)\\ \boldsymbol{K}_x &= \begin{pmatrix} 0&+&0&0\\+&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \boldsymbol{K}_y = \begin{pmatrix} 0&0&+&0\\0&0&0&0\\+&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \boldsymbol{K}_z = \begin{pmatrix} 0&0&0&+\\0&0&0&0\\0&0&0&0\\+&0&0&0 \end{pmatrix} \quad (4.57b) \end{aligned}}$$

Interpretation:

$$\exp(\varphi \boldsymbol{L}_x) \triangleq \begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&\cos\varphi&-\sin\varphi\\0&0&\sin\varphi&\cos\varphi \end{pmatrix} = \boldsymbol{\Lambda}_{R_x(\varphi)} \quad \to \textit{Rotation} \text{ around } x\text{-axis} \tag{4.58a}$$

$$\exp(-\theta \boldsymbol{K}_x) \triangleq \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0\\ -\sinh\theta & \cosh\theta & 0 & 0\\ 0&0&1&0\\0&0&0&1 \end{pmatrix} = \boldsymbol{\Lambda}_{v_x} \quad \to \textit{Boost} \text{ in } x\text{-direction} \tag{4.58b}$$

with ← *rapidity* $\tanh\theta = \frac{v_x}{c} \in (-1,1)$ (➔ Problemset 3) and rotation angle $\varphi \in [0, 2\pi)$.

→

$$\boxed{\begin{aligned} \boldsymbol{L}_x, \boldsymbol{L}_y, \boldsymbol{L}_z &: \quad \text{Generators of } \textit{rotations} \quad (4.59a)\\ \boldsymbol{K}_x, \boldsymbol{K}_y, \boldsymbol{K}_z &: \quad \text{Generators of } \textit{boosts} \quad (4.59b) \end{aligned}}$$

An arbitrary element of O(1, 3) that is connected to the identity can then be written as

$$\boldsymbol{\Lambda} = \exp\left(\sum_i \varphi_i \boldsymbol{L}_i - \sum_i \theta_i \boldsymbol{K}_i\right) \quad \text{with} \quad i \in \{x, y, z\}\,. \tag{4.60}$$

In particular [57]:

$$\text{Pure boost:} \quad \boldsymbol{\Lambda}_{\vec{v}} \equiv \boldsymbol{\Lambda}_{\vec{\theta}} = \exp\left(-\vec{\theta}\cdot\vec{\boldsymbol{K}}\right) \tag{4.61a}$$

$$\text{Pure rotation:} \quad \boldsymbol{\Lambda}_{R_{\vec{\varphi}}} = \exp\left(\vec{\varphi}\cdot\vec{\boldsymbol{L}}\right) \tag{4.61b}$$

with rotation angle $\varphi = |\vec{\varphi}|$, rotation axis $\hat{\varphi} = \vec{\varphi}/\varphi$, and rapidity vector

$$\vec{\theta} \equiv \vec{\theta}(\vec{v}) := \hat{v}\tanh^{-1}\left(\frac{v}{c}\right)\,. \tag{4.62}$$

¡! The rapidity vector $\vec{\theta}$ is *not* given by the rapidities $\tanh^{-1}\frac{v_i}{c}$ of the components v_i of $\vec{v}$.

iv | Lie algebra:

The Lie bracket (= commutator) on the Lie *algebra* determines the multiplicative structure of the Lie *group* via the ↓ *Baker-Campbell-Hausdorff formula*:

$$\exp(X)\cdot\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X, Y] + \ldots\right)\,. \tag{4.63}$$

→ The Lie algebra $\mathfrak{o}(1,3)$ determines the (local) group structure of O(1, 3):

Eq. (4.57) $\overset{\circ}{\rightarrow}$

$$\boxed{\begin{aligned} [L_i, L_j] &= \varepsilon^{ijk} L_k \\ [L_i, K_j] &= \varepsilon^{ijk} K_k \\ [K_i, K_j] &= -\varepsilon^{ijk} L_k \end{aligned}} \qquad \begin{aligned}(4.64\text{a})\\(4.64\text{b})\\(4.64\text{c})\end{aligned}$$

Some comments and implications:

- ¡! Because of Eq. (4.64) [and Eq. (4.63)], you *cannot* simply combine exponentials:

$$\exp\left(-\vec{\theta}\cdot\vec{\boldsymbol{K}}\right)\cdot\exp\left(\vec{\varphi}\cdot\vec{\boldsymbol{L}}\right) \neq \exp\left(\vec{\varphi}\cdot\vec{\boldsymbol{L}} - \vec{\theta}\cdot\vec{\boldsymbol{K}}\right)\,, \tag{4.65a}$$

$$\exp\left(-\vec{\theta}\cdot\vec{\boldsymbol{K}}\right)\cdot\exp\left(-\vec{\theta}'\cdot\vec{\boldsymbol{K}}\right) \neq \exp\left(-(\vec{\theta}+\vec{\theta}')\cdot\vec{\boldsymbol{K}}\right)\,, \tag{4.65b}$$

$$\exp\left(\vec{\varphi}\cdot\vec{\boldsymbol{L}}\right)\cdot\exp\left(\vec{\varphi}'\cdot\vec{\boldsymbol{L}}\right) \neq \exp\left((\vec{\varphi}+\vec{\varphi}')\cdot\vec{\boldsymbol{L}}\right)\,. \tag{4.65c}$$

  This is why the concatenation of Lorentz transformations is quite complicated in general.

- Eq. (4.64a) is written in physics often as $[L_i, L_j] = i\hbar\varepsilon^{ijk}L_k$ with angular momentum operators L_k. In this notation, they generate rotations $U_{\vec{\omega}} = \exp(\frac{i}{\hbar}\vec{\omega}\vec{L})$. The additional phase i in the commutation relation matches a corresponding factor in an alternative definition of the generators $\vec{L}$. (Recall that the L_i in Eq. (4.57) are *anti-Hermitian* whereas in physics we often prefer *Hermitian* operators.)
- Eq. (4.64a) shows that $\mathfrak{o}(3) := \operatorname{span}\{L_x, L_y, L_z\}$ forms a *subalgebra* of $\mathfrak{o}(1,3)$. On the group level, this means that the group of spatial rotations SO(3) is a subgroup of the full Lorentz group O(1, 3).

By contrast, Eq. (4.64c) shows that the boost generators $\{K_x, K_y, K_z\}$ do *not* form a subalgebra, but mix with rotations. This implies that there is no "subgroup of pure boosts" in $\mathrm{O}(1,3)$. In particular:

$$\Lambda_{\vec{v}}\Lambda_{\vec{u}} = \Lambda_{\vec{u}\oplus\vec{v}}\Lambda_{R(\vec{u},\vec{v})} \tag{4.66}$$

with the ← *Thomas-Wigner rotation* $R(\vec{u},\vec{v}) \in \mathrm{SO}(3)$ [recall Section 2.3].

- There is a more compact, 4-vector-inspired notation for the 6 generators in Eq. (4.57), namely [58]:

$$\left(J^{\alpha\beta}\right)^{\mu}{}_{\nu} \equiv \left(J^{\alpha\beta}\right)_{\mu\nu} := \eta^{\alpha\mu}\delta^{\beta}_{\nu} - \eta^{\beta\mu}\delta^{\alpha}_{\nu}\,. \tag{4.67}$$

Inspection shows that (➔ Problemset 5)

$$L_x = J^{23} = -J^{32}\,, \qquad K_x = J^{01} = -J^{10}\,, \tag{4.68a}$$
$$L_y = J^{31} = -J^{13}\,, \qquad K_y = J^{02} = -J^{20}\,, \tag{4.68b}$$
$$L_z = J^{12} = -J^{21}\,, \qquad K_z = J^{03} = -J^{30}\,. \tag{4.68c}$$

The three equations of the Lie algebra Eq. (4.64) can then be condensed into a single equation [58]:

$$[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}\,. \tag{4.69}$$

This form is useful to construct other representations of the Lorentz group, especially in relativistic quantum mechanics (→ *Dirac equation*).

v | It is a useful mathematical fact that every continuous Lorentz transformation of the form Eq. (4.60) can be decomposed *uniquely* as follows:

$$\boxed{\begin{gathered}\mathbf{\Lambda} = \mathbf{\Lambda}_{\vec{v}}\mathbf{\Lambda}_R = \mathbf{\Lambda}_R\mathbf{\Lambda}_{\vec{w}} \qquad (4.70a)\\ \text{with parameters:}\\ \frac{v_i}{c} = -\frac{\mathbf{\Lambda}_{i0}}{\mathbf{\Lambda}_{00}}\,,\quad \frac{w_i}{c} = -\frac{\mathbf{\Lambda}_{0i}}{\mathbf{\Lambda}_{00}}\quad\text{and}\quad R_{ij} = \mathbf{\Lambda}_{ij} - \frac{\mathbf{\Lambda}_{i0}\mathbf{\Lambda}_{0j}}{1+\mathbf{\Lambda}_{00}} \qquad (4.70b)\end{gathered}}$$

$\mathbf{\Lambda}_{\vec{v}}$ and $\mathbf{\Lambda}_R$ are defined in Eq. (4.61a) [or Eq. (1.75)] and Eq. (4.61b) [or Eq. (1.40)].

The proof can be found in Ref. [59]. This decomposition, sometimes referred to as ⁂ *rotation-boost decomposition*, relates to the mathematical concept of ↑ *Cartan decompositions* [60].

If we use the multiplicative law $\mathbf{\Lambda}_R\mathbf{\Lambda}_{\vec{v}}\mathbf{\Lambda}_{R^{-1}} = \mathbf{\Lambda}_{R\vec{v}}$ [recall Eq. (1.43a)] and choose R such that $R\vec{v} = (v_x, 0, 0)^T$, we can also find a decomposition of the form

$$\boxed{\mathbf{\Lambda} = \mathbf{\Lambda}_{R_1}\mathbf{\Lambda}_{v_x}\mathbf{\Lambda}_{R_2}} \tag{4.71}$$

with appropriately chosen rotations $R_1, R_2 \in \mathrm{SO}(3)$ and a boost in x-direction by v_x.

3 | Discrete generators:

It is easy to verify that the following two matrices also belong to $\mathrm{O}(1,3)$:

⁂ *Parity*:
$P\,:\,(t,\vec{x}) \mapsto (t,-\vec{x})$

$$\Rightarrow \quad P^{\mu}{}_{\nu} \equiv \boldsymbol{P}_{\mu\nu} := \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} \tag{4.72}$$

⁂ *Time reversal*:
$T\,:\,(t,\vec{x}) \mapsto (-t,\vec{x})$

$$\Rightarrow \quad T^{\mu}{}_{\nu} \equiv \boldsymbol{T}_{\mu\nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}_{\mu\nu} \tag{4.73}$$

In contrast to the continuous group elements above: $\det(P) = \det(T) = -1$

$\rightarrow$ P and T are not generated by boosts or rotations!

4 | Structure of the Lorentz group:

Combining the discrete transformation $\boldsymbol{P}$ and $\boldsymbol{T}$ with the continuous transformations $\boldsymbol{\Lambda} = \exp(\boldsymbol{X})$ yields the complete group $\mathrm{O}(1,3)$. Let us study its structure:

i | $\det(\Lambda) = \pm 1 \rightarrow$

$$O(1,3) = \underbrace{L_+}_{\det(\Lambda)=+1} \cup \underbrace{L_-}_{\det(\Lambda)=-1} \tag{4.74}$$

All Lorentz transformations that are continuously connected to $\mathbb{1}$ are in L_+. One can transition between L_+ and L_- by applying either T or P.

ii | In addition, we find:

$$1 = \eta_{00} \overset{4.21}{=} \left(\Lambda^0{}_0\right)^2 - \sum_{k=1}^{3} \left(\Lambda^k{}_0\right)^2 \leq \left(\Lambda^0{}_0\right)^2 \tag{4.75}$$

Thus $\Lambda^0{}_0 \neq 0$ and $\operatorname{sign}(\Lambda^0{}_0) = \pm 1$ can be used to characterize Lorentz transformations. Note that $\operatorname{sign}(P^0{}_0) = +1$ but $\operatorname{sign}(T^0{}_0) = -1$ and $\operatorname{sign}((PT)^0{}_0) = -1$.

iii | Neither $\det(\Lambda) = \pm 1$ nor $\operatorname{sign}(\Lambda^0{}_0) = \pm 1$ can be changed by continuously deforming a Lorentz transformation.

$\rightarrow$ Four disconnected components of $\mathrm{O}(1,3)$:

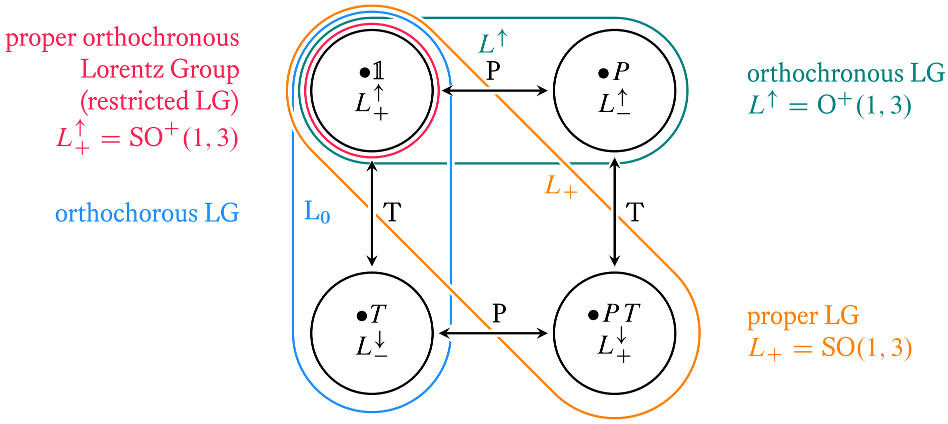
$$L_+^{\uparrow}: \quad \det(\Lambda) = +1 \quad \text{and} \quad \operatorname{sign}(\Lambda^0{}_0) = +1 \quad (\mathbb{1} \in L_+^{\uparrow}) \tag{4.76a}$$

$$L_-^{\uparrow}: \quad \det(\Lambda) = -1 \quad \text{and} \quad \operatorname{sign}(\Lambda^0{}_0) = +1 \quad (P \in L_-^{\uparrow}) \tag{4.76b}$$

$$L_+^{\downarrow}: \quad \det(\Lambda) = +1 \quad \text{and} \quad \operatorname{sign}(\Lambda^0{}_0) = -1 \quad (PT \in L_+^{\downarrow}) \tag{4.76c}$$

$$L_-^{\downarrow}: \quad \det(\Lambda) = -1 \quad \text{and} \quad \operatorname{sign}(\Lambda^0{}_0) = -1 \quad (T \in L_-^{\downarrow}) \tag{4.76d}$$

Graphically:



$$L_\pm^{\uparrow/\downarrow}\begin{cases}\uparrow & \text{no time inversion } (\operatorname{sign}{\Lambda^0}_0 = +1)\\ \downarrow & \text{time inversion } (\operatorname{sign}{\Lambda^0}_0 = -1)\\ + & \det\Lambda = +1 \text{ (proper)}\\ - & \det\Lambda = -1 \text{ (improper)}\end{cases}$$

iv | Subgroups: We can define the following four subgroups of $\mathrm{O}(1,3)$:

$$\text{⁂ } \textit{Proper LG}:\quad \mathrm{SO}(1,3) \equiv L_+ := L_+^\uparrow \cup L_+^\downarrow \tag{4.77a}$$

$$\text{⁂ } \textit{Orthochronous LG}:\quad \mathrm{O}^+(1,3) \equiv L^\uparrow := L_+^\uparrow \cup L_-^\uparrow \tag{4.77b}$$

$$\text{⁂ } \textit{Proper orthochronous LG}:\quad \mathrm{SO}^+(1,3) := L_+^\uparrow \tag{4.77c}$$

$$\text{⁂ } \textit{Orthchorous LG}:\quad L_0 := L_+^\uparrow \cup L_-^\downarrow \tag{4.77d}$$

Note that subgroups must contain the identity $\mathbb{1}$!

In Greek, "chrónos" (*χρόνος*) means "time" and "chóros" (*χώρος*) means "space".

According to modern physics, Einstein's principle of relativity **SR** reads formally:

> All fundamental theories of nature must be invariant under the *proper orthochronous Lorentz group* $\mathrm{SO}^+(1,3)$.

- This does not prevent specific theories to have *additional* symmetries. ↑ *Quantum electrodynamics* (QED), for example, is invariant under the *full* Lorentz group $O(1,3)$. This means that phenomena of electromagnetism – and its interaction with charged particles – are also symmetric under time inversion T and parity P.

  So far, observations suggest that, besides the electromagnetic force, also gravity and the strong force are symmetric under P and T. (Interestingly, there is no formal reason *why* the strong force should *not* break P and T; the fact that it does *not* violate these symmetries is called the ↑ *strong CP problem*).

- However, today we know that there *are* terms in the standard model of particle physics that *violate* both P and T. For example, the weak interaction (responsible for radioactive β-decay) violates parity P strongly (↑ *Wu experiment*). This means that you can use experiments that depend on the weak interaction to tell the difference between our world and its mirror image (or a right-handed and a left-handed coordinate system). There are also weak terms (concerning quarks) that violate time reversal T (↑ CP *violation*). As a consequence, the standard model as a whole is only invariant under the proper orthochronous Lorentz group $\mathrm{SO}^+(1,3)$.

This explains why we can only require symmetry under $\mathrm{SO}^+(1,3)$, and not the full Lorentz group $\mathrm{O}(1,3)$: We already *know* by experiments that the latter is *not* a fundamental symmetry of nature!

- The fact that there are processes that violate parity symmetry P contradicts our everyday experience: If you run an experiment using equipment found in a school physics lab and put a mirror next to it, there is no way to decide whether you are watching the experiment directly or via the mirror (i.e., parity inverted). The reason is that the physics we experience in everyday life is goverened by electrodynamics and gravity, both of which are invariant under P. To unveil that nature secretly violates P, you must perform an experiment that involves the weak interaction (that is: a particle physics experiment). This is what Chien-Shiung Wu did in her now famous ↑ *Wu experiment*. At the time, the result (that P is not a symmetry of nature) was unexpected and groundbreaking.

  So if you are surprised that P is not a symmetry of nature, you are not alone. Here is how Wolfang Pauli reacted to the result of the Wu experiment [61]:

  > *At one point, Temmer found himself in the presence of eminence grise Wolfgang Pauli, who asked for the latest news from the United States. Temmer told him that parity was no longer to be assumed conserved. "That's total nonsense" averred the great man. Temmer: "I assure you the experiment says it is not." Pauli (curtly): "Then it must be repeated!"*

4.4. ‡ Why is spacetime 3+1 dimensional?

Given the discussions in Chapter 3 and Chapter 4 it is clear that the mathematical formalism allows for straightforward generalizations to higher- (or lower-) dimensional spacetime manifolds with arbitrary signatures; these suggest spacetimes with various numbers of spatial and temporal dimensions.

It is therefore natural to ask:

Is there anything special about our $3+1$-dimensional world?

What follows is not a *proof* that spacetime must be $3+1$ dimensional. Our goal is to argue that all spacetimes, except ours with *three* space and *one* time dimension, face severe problems that, most likely, would not allow for complex life.

The following discussion is based on Tegmark [39, 62].

1 | ⊲ Pseudo-Riemannian manifold of signature (t,s) with metric

$$g_{ij} = \mathrm{diag}(\underbrace{+1,\dots,+1}_{t},\underbrace{-1,\dots,-1}_{s}) \tag{4.78}$$

- This is the generalization of Minkowski space to a (flat) spacetime manifold with, naïvely, t time and s space-dimensions.
- Most of our discussions in this chapter can be transferred to this more general setting.

2 | ⊲ ↑ *Klein-Gordon equation* for signature (t,s):

$$\boxed{(\partial^2+m^2)\,\Phi = \underbrace{\sum_{i=1}^{t}\frac{\partial^2\Phi}{\partial {x^i}^2}}_{t\,\times\,\text{Time (?)}} - \underbrace{\sum_{i=t+1}^{s+t}\frac{\partial^2\Phi}{\partial {x^i}^2}}_{s\,\times\,\text{Space (?)}} + m^2\Phi = 0} \tag{4.79}$$

- Recall that $\partial^2 = g^{ij}\partial_i\partial_j$ where g^{ij} is given by (the inverse of) Eq. (4.78).
- The Klein-Gordon equation (KGE) is the simplest covariant field equation. It describes the time evolution of a scalar field of mass m. It is ubiquitous in relativistic physics (especially in ↑ *quantum field theory*).
- For example, the components of the electromagnetic field in vacuum are described by the KGE for $m=0$ and $(t,s)=(1,3)$ (which is then referred to as ↓ *wave equation*):

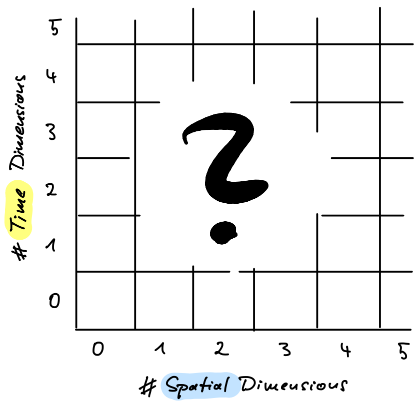
$$\partial^2 E_i = \tfrac{1}{c^2}\partial_t^2 E_i - \nabla^2 E_i = 0\,, \tag{4.80a}$$
$$\partial^2 B_i = \tfrac{1}{c^2}\partial_t^2 B_i - \nabla^2 B_i = 0\,. \tag{4.80b}$$

  This motivates in Eq. (4.79) the (tentative) identification of the coordinates with positive sign as "time coordinates", and the ones with a negative sign as "space coordinates":

  The difference between time and space is just a sign!

  In the following, we use the KGE as a proxy for more general relativistic field equations.

→ Possible combinations of t time and s space dimensions:



3 | Partial differential equations (PDE):

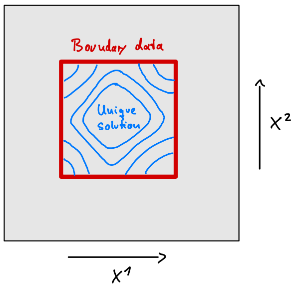
The general KGE in Eq. (4.79) is an example of a partial differential equation (PDE). The theory of PDEs has been thoroughly developed by mathematicians and a lot is known about their solvability. The problem of solving a PDE, given some boundary/initial conditions, is known as ↑ *Cauchy problem*:

- ⁂ *Well-posed (Cauchy) problem*: Given some boundary/initial data, there exists a *unique* solution to the PDE that satisfies these conditions, and this solution is *robust*. Here "robust" means that if you slightly modify the boundary/initial conditions, the solution also changes only slightly. Put differently: The solutions are not *chaotic* and you can use them to extrapolate reliably from boundary/initial states with finite errorbars. This is a crucial feature to use PDEs for *predictions* in the real world.
- ⁂ *Ill-posed (Cauchy) problem*: Given some boundary/initial data, there either exist *multiple* solutions to the PDE that satisfy these conditions, or the unique solution is not robust. In both cases, the PDE cannot be used for predictions in the real world.

i | $\sphericalangle$ $(t = 0, s)$ or $(t, s = 0)$ → Eq. (4.79) = ↑ *Elliptic PDE*

This corresponds to spacetimes that are ← *Riemannian manifolds*.

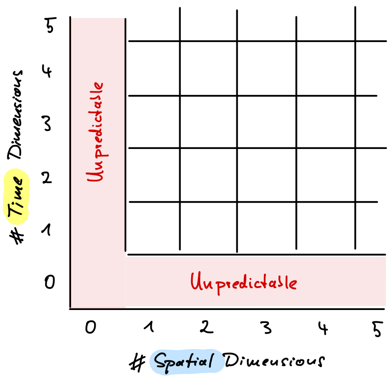
Elliptic PDEs have well-posed *boundary problems*:



→ One *cannot* use Eq. (4.79) to make predictions ☹

→ No coordinate in Eq. (4.79) qualifies as a "time coordinate".

→



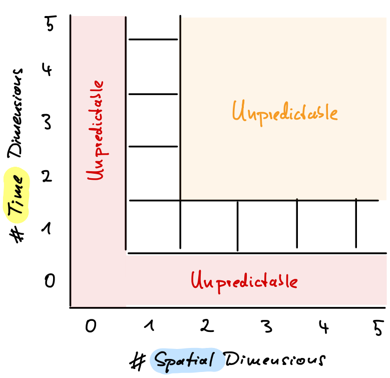
ii | $\sphericalangle$ $(t \geq 2, s \geq 2)$ or $(t \geq 2, s \geq 2)$ → Eq. (4.79) = ↑ *Ultrahyperbolic PDE*

This corresponds to spacetimes that are generic ← *pseudo-Riemannian manifolds*.

A similar but more involved chain of arguments holds also for ultrahyperbolic PDEs [39, 62].

→ One *cannot* use Eq. (4.79) to make predictions ☹

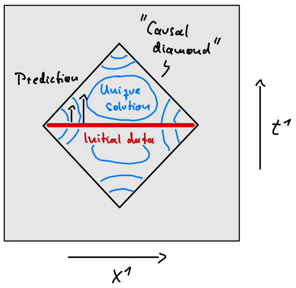
→



iii | $(t = 1, s \geq 1)$ or $(t \geq 1, s = 1)$ → Eq. (4.79) = ↑ *Hyperbolic PDE*

This corresponds to spacetimes that are ← *Lorentzian manifolds*.

Hyperbolic PDEs have well-posed *initial value problems*:



→ We *can* use Eq. (4.79) to make predictions ☺

4 | Stability:

- ∢ Newtonian Gravity in $s \geq 4$ spatial dimensions:

  → ↓ *Two-body problem* has no stable orbits (only scattering and attraction solutions).

  → No stable planetary systems possible ☹

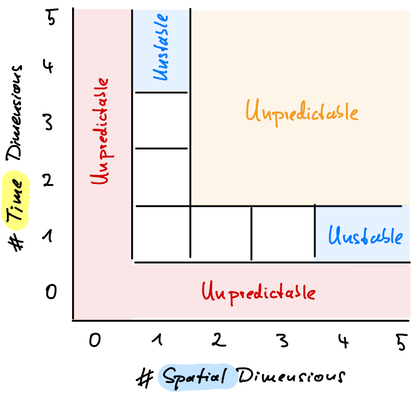
- ∢ Hydrogen atom in $s \geq 4$ spatial dimensions:

  → Schrödinger equation has no bound states.

  → No stable atoms possible ☹

The opposite cases with $t \geq 4$ and $s = 1$ are equivalent if one interprets space as time and vice versa (which is necessary to use hyperbolic PDEs to predict "the future", → *below*).

→



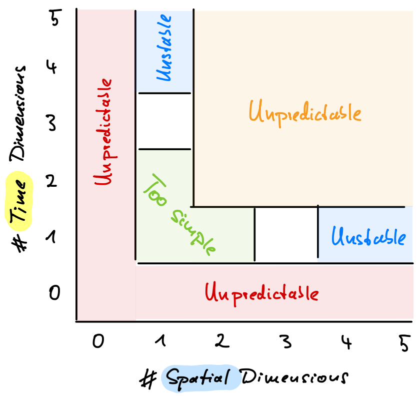
5 | Simplicity:

GENERAL RELATIVITY in $s \leq 2$ spatial dimensions → No gravity (→ *later*)!

→ No stars, no planets, no orbits ☹

The opposite cases with $t \leq 2$ and $s = 1$ are equivalent if one interprets space as time and vice versa (which is necessary to use hyperbolic PDEs to predict "the future", → *below*).

→



6 | Tachyon world:

¡! In the literature both Lorentzian signatures $(1,3)$ and $(3,1)$ are used to formulate SPECIAL RELATIVITY. Formulations in signature $(3,1)$ have nothing to do with the Tachyon sector discussed here since they compensate for the global minus in their equations. For example, the KGE in signature $(-,+,+,+,)$ reads $(-\partial^2+m^2)\Phi=0$ which is equivalent to the KGE $(\partial^2+m^2)\Phi=0$ in signature $(+,-,-,-)$. The point here is that we do *not* add this additional minus:

$$\text{Eq. (4.79)} \xrightarrow[\text{Time}\leftrightarrow\text{Space}]{(1,3)\mapsto(3,1)} (-\partial^2+m^2)\Phi=0 \;\Leftrightarrow\; (\partial^2-m^2)\Phi=0 \tag{4.81}$$

In more detail:

For $t=3$ and $s=1$ the KGE reads

$$\underbrace{\frac{\partial^2\Phi}{\partial(x^1)^2}+\frac{\partial^2\Phi}{\partial(x^2)^2}+\frac{\partial^2\Phi}{\partial(x^3)^2}}_{3\times\text{Time (?)}}-\underbrace{\frac{\partial^2\Phi}{\partial(x^4)^2}}_{1\times\text{Space (?)}}+m^2\Phi=0\,. \tag{4.82}$$

But because this an hyperbolic PDE, the Cauchy problem is only well-posed with initial conditions on a hypersurface spanned by $\{x^1,x^2,x^3\}$. Put differently: The PDE allows predictions only in x^4-direction! Thus we should interpret x^4 as *time* and $\{x^1,x^2,x^3\}$ as *space*:

$$\underbrace{\frac{\partial^2\Phi}{\partial(x^1)^2}+\frac{\partial^2\Phi}{\partial(x^2)^2}+\frac{\partial^2\Phi}{\partial(x^3)^2}}_{3\times\text{Space (!)}}-\frac{1}{c^2}\underbrace{\frac{\partial^2\Phi}{\partial t^2}}_{1\times\text{Time (!)}}+m^2\Phi=0 \tag{4.83}$$

with $ct\equiv x^4$. But this KGE is equivalent to

$$\left(\frac{1}{c^2}\partial_t^2-\nabla^2-m^2\right)\Phi=(\partial^2-m^2)\Phi=0\,. \tag{4.84}$$

Thus the "transposed" situation $(t\geq 1, s=1)$ is equivalent to the situation $(t=1, s\geq 1)$ with *negative* square-masses in the equations. Fields with negative square-mass (equivalently: imaginary mass) are called ↑ *tachyonic fields* or ↑ *tachyons* for short.

→ All massive particles are ↑ *tachyons* [63]

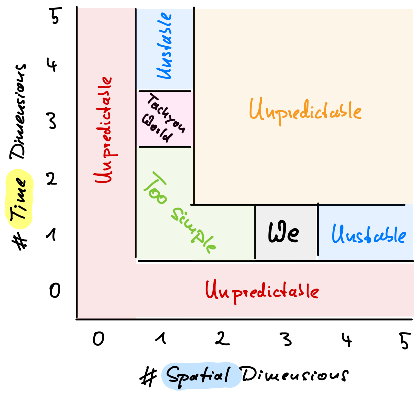
¡! Tachyonic fields are not science fiction; they do exist (→ *below*) but, contrary to the features assigned to them in science fiction, do *not* allow for faster-than-light propagation of information.

→ Tachyon fields herald vacuum instabilities [64] ☹

The spontaneous symmetry breaking of the ↑ *Higgs mechanism* is an example of this phenomenon: The Higgs field has a negative square-mass which is responsible for the "Mexican hat potential."

The consequence is spontaneous symmetry breaking, which, in this context, can be reframed as "tachyon condensation." On the new, symmetry broken vacuum, excitations are *not* tachyons with negative square-mass but Higgs bosons with positive square-mass.

→



7 | These arguments support the following *hypothesis*:

> Only a spacetime with **1 time** and **3 space** dimensions supports observers like us.

What does this line of arguments explain? Well, if you would randomly construct universes by dicing the number of space and time dimensions, only the ones with *one time* and *three space* dimensions have the chance to develop complex observers like us (who then wonder why their universe is $3+1$-dimensional). Thus the arguments above are important for "ensemble interpretations" of reality, like certain ↑ *multiverse hypotheses* or superstring theories (which can predict a plethora of different spacetime dimensions) [39, 65].

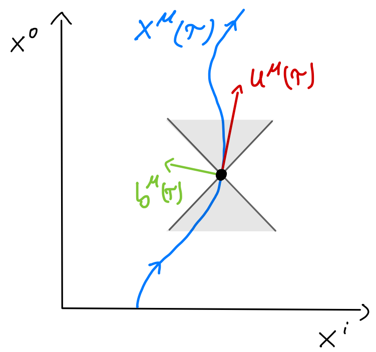
↓Lecture 10 [19.12.23]

5. Relativistic Mechanics

Equipped with the machinery of Chapter 4, we can finally construct a relativistic (Lorentz covariant) version of classical mechanics.

5.1. The relativistic point particle

1 | ⊲ Point particle in $\mathbb{R}^{1,3}$ with trajectory $x^\mu(\tau)$:



2 | It is reasonable to define the relativistic momentum of a massive particle as follows:

$$\boxed{\begin{aligned} &⁂\ \textit{4-momentum}: \quad p^\mu := m u^\mu = m\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \begin{pmatrix} m\gamma_v c \\ m\gamma_v \vec{v} \end{pmatrix} \equiv \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix} && (5.1) \\ &\text{with (rest) mass } m \text{ and } ⁂\ \textit{3-momentum } \vec{p}. && (5.2) \end{aligned}}$$

- ¡! The mass m is the good old (inertial) mass we would assign to the particle in classical mechanics; it is a measure of the particles resistance to changes in its state of motion. You can determine it by applying a (weak) force to the particle *at rest* and observing its initial acceleration: $m = F/a$. This mass is an intrinsic property of the particle and does not depend on velocity. It is sometimes called *rest mass*, but we will simply call it *mass*.
- Since the 4-velocity u^μ is a Lorentz vector, the 4-momentum is also a Lorentz vector; i.e., under a Lorentz transformation Λ the 4-momentum transforms as $\bar{p}^\mu = \Lambda^\mu{}_\nu p^\nu$.
- We will later rederive the expression for the 4-momentum as the conserved ↓ *Noether charge* for translations in spacetime.

3 | The spatial part of the momentum (the 3-momentum $\vec{p}$) is related to the velocity as follows:

$$\boxed{\vec{p} = m\gamma_v\vec{v} = \underbrace{\frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}}_{\text{RELATIVITY}} \xrightarrow{\beta \ll 1} \underbrace{m\vec{v}}_{\substack{\text{Newtonian}\\ \text{mechanics}}}} \tag{5.3}$$

- ¡! In SPECIAL RELATIVITY the kinetic momentum is no longer proportional to the velocity. In particular for $v \to c$ the momentum of a massive particle *diverges*.
- The non-relativistic limit ($v \ll c \Rightarrow \beta \ll 1 \Rightarrow \gamma_v \approx 1$) is consistent with the Newtonian (non-relativistic!) relation $\vec{p} = m\vec{v}$ for the kinetic momentum; the 3-momentum $\vec{p}$ is therefore the proper relativistic version of the momentum in Newtonian mechanics.
- This explains why the above definition for the 4-momentum is reasonable – and why the mass m must be identified with the mass used in Newtonian mechanics.
- At this point it is unclear how to interpret the time-component $p^0 = m\gamma_v c$ of p^μ (→ *below*).

4 | Eq. (5.1) $\xrightarrow{\text{Eq. (4.48)}}$

$$\boxed{p^2 = p^\mu p_\mu = (p^0)^2 - \vec{p}^2 \overset{\text{def}}{=} m^2 u^2 \overset{4.48}{=} m^2 c^2 > 0} \tag{5.4}$$

→ The mass m is a *Lorentz scalar*: $m^2 = p^2/c^2$

- The 4-momentum is a *time-like* 4-vector for massive particles.
- This means that the mass m can be measured/computed in every inertial system by measuring/computing the 4-momentum p^μ and its pseudo-norm p^2. The numerical result will always be the same, namely m^2c^2.

5 | Equation of motion (EOM):

i | We want an EOM that …

- …is manifestly Lorentz covariant → Lorentz tensor equation
- …reduces to Newton's equation of motion

$$m\vec{a} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F} \quad \text{with} \quad \vec{p} = m\vec{v} \tag{5.5}$$

in the non-relativistic limit (*correspondence principle*).

ii | Suggestion:

$$\boxed{mb^\mu = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = K^\mu \equiv \begin{pmatrix} K^0 \\ \vec{K} \end{pmatrix} \quad \text{with} \quad ⁂\ \textit{4-force}\ K^\mu\,.} \tag{5.6}$$

Because this is a equation built from Lorentz vectors, it is form-invariant (Lorentz covariant) by construction:

$$mb^\mu = K^\mu \;\Leftrightarrow\; m\,\Lambda^\nu{}_\mu b^\mu = \Lambda^\nu{}_\mu K^\mu \;\Leftrightarrow\; m\bar{b}^\nu = \bar{K}^\nu \tag{5.7}$$

This is of course only so if the 4-force transforms like a Lorentz vector.

iii | ∢ Instantaneous rest frame (IRF) K_0:

a | At any time there is an inertial coordinate system K_0 in which the (potentially accelerated) particle is at rest *at this very moment* (if the particle is accelerating, it is also accelerating in this frame).

$$mb_0^\mu \overset{4.49}{=} \begin{pmatrix} 0 \\ m\vec{a}_0 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 \\ \vec{F}_0 \end{pmatrix} = K_0^\mu \tag{5.8}$$

This follows from the correspondence principle: In the IRF the particle is in the non-relativistic, Newtonian limit. Thus its coordiante acceleration $\vec{a}_0$ must be given by Newton's equation of motion: $m\vec{a}_0 = \vec{F}_0$.

with

- ⁂ *Proper acceleration* $\vec{a}_0$

  The proper acceleration is the coordinate accelaration (3-acceleration) that you can measure (e.g., with an accelerometer) in the IRF K_0 of the particle.

  It follows immediately that the norm of the proper acceleration is a Lorentz scalar:

$$\boxed{b^2 = b^\mu b_\mu = -|\vec{a}_0|^2 < 0} \tag{5.9}$$

- ⁂ *Proper force* $\vec{F}_0$

  The proper force is the Newtonian force (3-force) you can measure (e.g., with a spring balance) in the IRF K_0 of the particle.

b | We *demand* that this equation is Lorentz covariant, i.e., that b_0^μ and K_0^μ transform as contravariant Lorentz 4-vectors. We can then use a Lorentz boost to transform back into the lab frame in which the particle has coordinate velocity $\vec{v}$:

$\xrightarrow{\text{Eq. (1.75)}}$

$$\text{4-acceleration:}\quad b^\mu = (\Lambda_{-\vec{v}})^\mu{}_\nu\, b_0^\nu \overset{1.75}{=} \begin{pmatrix} \gamma_v \frac{\vec{a}_0\cdot\vec{v}}{c} \\ \vec{a}_0 + \frac{\gamma_v-1}{v^2}(\vec{a}_0\cdot\vec{v})\vec{v} \end{pmatrix} \tag{5.10a}$$

$$\text{4-force:}\quad K^\mu = (\Lambda_{-\vec{v}})^\mu{}_\nu\, K_0^\nu \overset{1.75}{=} \begin{pmatrix} \gamma_v \frac{\vec{F}_0\cdot\vec{v}}{c} \\ \vec{F}_0 + \frac{\gamma_v-1}{v^2}(\vec{F}_0\cdot\vec{v})\vec{v} \end{pmatrix} \tag{5.10b}$$

We will use these expressions later!

iv | On the other hand, we can return to Eq. (5.6) and study the 4-force K^μ in more detail:

a | $\triangleleft$ Spatial components of Eq. (5.6):

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}\tau} = \gamma_{v(t)}\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{K} \quad\Leftrightarrow\quad \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\vec{K}}{\gamma_v} =: \vec{F} \quad\Leftrightarrow\quad \vec{K} = \gamma_v\vec{F} \tag{5.11}$$

with ⁂ *3-force* $\vec{F}$.

Here $\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$ denotes the change in momentum measured in coordinate time; it makes sense identify this quantity with the relativistic analog of the Newtonian force.

b | What is the time component K^0 of the 4-force? $\triangleleft$

$$0 \overset{4.50}{=} mb^\mu u_\mu \overset{5.6}{=} K^\mu u_\mu = K^0u^0 - \vec{K}\cdot\vec{u} \overset{4.47}{=} \gamma_v(K^0c - \vec{K}\cdot\vec{v}) \tag{5.12}$$

$\rightarrow$

$$K^0 = \frac{\vec{K}\cdot v}{c} \overset{5.11}{=} \frac{\gamma_v}{c}\,\vec{F}\cdot\vec{v} \tag{5.13}$$

c | In summary, the 4-force in terms of the 3-force and the 3-velocity reads

$$\boxed{\text{4-force:}\quad K^\mu = \begin{pmatrix} \gamma_v \frac{\vec{F}\cdot\vec{v}}{c} \\ \gamma_v \vec{F} \end{pmatrix}} \tag{5.14}$$

Example:

In our discussion of electrodynamics (→ *Chapter 6*) we will find the following expression for the 3-force acting on a charged particle in an electromagnetic field:

$$\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B} \tag{5.15}$$

This is the conventional ↓ *Lorentz force*.

This example demonstrates that the 3-force $\vec{F}$ is indeed the proper relativistic analog of Newtonian forces. Note, however, that it is only the *component* of the 4-force and thus does not transform nicely under Lorentz transformations.

d | Spatial part of Eq. (5.6) $\xrightarrow{\text{Eq. (5.14)}}$

$$\boxed{\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\gamma_v\vec{v}) = \langle\text{Change in 3-momentum}\rangle} \tag{5.16}$$

The Newtonian equation $\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t}$ therefore remains valid in SPECIAL RELATIVITY for the 3-force $\vec{F}$ and the 3-momentum $\vec{p}$. By contrast, $\vec{p} = m\gamma_v\vec{v}$ is different from the Newtonian relation $\vec{p} = m\vec{v}$ between momentum and velocity.

e | Temporal part of Eq. (5.6) $\xrightarrow{\text{Eq. (5.14)}}$

$$\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = \gamma_v\frac{\mathrm{d}p^0}{\mathrm{d}t} = \gamma_v\frac{\vec{F}\cdot\vec{v}}{c} \quad\Rightarrow\quad \frac{\mathrm{d}(cp^0)}{\mathrm{d}t} = \vec{F}\cdot\vec{v} \tag{5.17}$$

→ $\vec{F}\cdot\vec{v}$: Work performed by $\vec{F}$ on particle

→ $E = cp^0$: Total energy of particle

Note that we can actually only conclude $E = cp^0 + \text{const}$ from the differential equation above. We will later see that the constant must be set to zero because p^0 is the conserved Noether charge that derives from time translations.

The time component of the EOM Eq. (5.6) can therefore be written as:

$$\boxed{\vec{F}\cdot\vec{v} = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\gamma_v c^2) = \langle\text{Change in energy}\rangle} \tag{5.18}$$

We will discuss the expression for the energy in Section 5.2 below.

6 | Above we expressed the 4-force in terms of the proper force $\vec{F}_0$ and in terms of the 3-force $\vec{F}$. Equating the two expressions yields a relation between the 3-force $\vec{F}_0$ measured in the IRF and the 3-force $\vec{F}$ measured in the lab frame:

Eq. (5.10b) & Eq. (5.14) →

3-force $\vec{F}$ as function of proper force $\vec{F}_0$ and velocity $\vec{v}$:

$$\boxed{\vec{F} = \frac{\vec{F}_0}{\gamma_v} + \left(1 - \frac{1}{\gamma_v}\right) \frac{\vec{F}_0 \cdot \vec{v}}{v^2} \vec{v}} \tag{5.19}$$

Recall that the proper force is the Newtonian force you would measure with a spring scale in the IRS of the particle. In contrast to Newtonian mechanics, the force $\vec{F}$ measured from a frame in relative motion is *different* from $\vec{F}_0$. In the non-relativistic limit $\gamma_v \approx 1$ we find $\vec{F} \approx \vec{F}_0$ and this distinction becomes irrelevant (as assumed by Newtonian mechanics).

7 | A similar comparison yields a relation between the 3-acceleration in the IRF (the proper acceleration) and the 3-acceleration in the rest frame:

Eq. (5.10a) & Eq. (4.49) →

3-acceleration $\vec{a}$ as function of proper acceleration $\vec{a}_0$ and velocity $\vec{v}$:

$$\boxed{\vec{a} \stackrel{\circ}{=} \frac{1}{\gamma_v^2} \left[\vec{a}_0 - \left(1 - \frac{1}{\gamma_v}\right) \frac{\vec{v} \cdot \vec{a}_0}{v^2} \vec{v}\right]} \tag{5.20}$$

This is again in sharp contrast to Newtonian mechanics where, as a consequence of absolute time, acceleration does not depend on the velocity of the reference frame. In the non-relativistic limit for $\gamma_v \approx 1$ we find $\vec{a} \approx \vec{a}_0$, consistent with Newtonian mechanics.

8 | Sanity check:

If we integrate the equation of motion Eq. (5.16), we find:

$$\int_0^T \vec{F}\,\mathrm{d}t = \frac{m\vec{v}_T}{\sqrt{1 - \frac{v_T^2}{c^2}}} - \text{const}\,. \tag{5.21}$$

For a finite 3-force $|\vec{F}| < \infty$ and finite time $T < \infty$, and non-zero mass $m \neq 0$, it follows for the final velocity $\vec{v}_T$:

$$\frac{m|\vec{v}_T|}{\sqrt{1 - \frac{v_T^2}{c^2}}} < \infty \quad \Rightarrow \quad |\vec{v}_T| < c\,. \tag{5.22}$$

Thus the dynamics does not allow massive particles to reach the speed of light, no matter how strong the force or how long the acceleration! This is in direct contradiction to Newtonian mechanics and by now experimentally well-confirmed (→ *below*).

5.2. Momentum, Energy, and Mass

9 | To summarize, the 4-momentum of a *massive particle* can be written as:

$$\boxed{p^\mu = mu^\mu = \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \gamma_v mc \\ \gamma_v m\vec{v} \end{pmatrix}} \tag{5.23}$$

10 | The relativistic energy of a *massive particle* is then (as a function of 3-velocity):

$$\boxed{⁂\ \textit{Relativistic energy}: \quad E = cp^0 = \gamma_v mc^2 = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}} \tag{5.24}$$

With

$$m^2c^2 \overset{5.4}{=} p^2 = (p^0)^2 - (\vec{p})^2 = E^2/c^2 - \vec{p}^2 \tag{5.25}$$

we find the alternative expression as a function of 3-momentum:

$$\boxed{⁂\ \textit{Energy-momentum relation:} \quad E = \sqrt{\vec{p}^2c^2 + m^2c^4}} \tag{5.26}$$

- This expression is also valid in the massless case $m = 0$ (→ *below*).
- Eq. (5.25) has actually two solutions: $E = \pm\sqrt{\vec{p}^2c^2 + m^2c^4}$. In relativistic mechanics (and relativistic single-particle quantum mechanics), we can ignore the negative energy solution and consider only time-like 4-momenta p^μ that point into the future light-cone. In quantum field theory, where interacting particles can be destroyed and produced, these negative energy solutions necessitate the introduction of ↑ *antiparticles* (like the positron).
- For fixed mass m, Eq. (5.25) determines a 3-dimensional hypersurface in the 4-dimensional "energy-momentum space" spanned by 4-momenta $p^\mu = (p^0, \vec{p}) \in \mathbb{R}^4$. For $m \neq 0$ this hypersurface is a hyperboloid of two sheets $E = \pm\sqrt{\vec{p}^2c^2 + m^2c^4}$ (for $m = 0$ it is a cone: $E = \pm c|\vec{p}|$). This hypersurface is called ⁂ *mass shell*. If a 4-momentum satisfies the energy-momentum relation (with either sign) we say that it is "on-shell"; if not, it is "off-shell". In quantum field theory, real particles that can be measured are always on-shell; intermediate "virtual particles" in scattering processes can be off-shell.

11 | Rest energy:

i | ⊲ Rest frame K_0 of the particle where $\vec{p} = 0$:

$$p_0^\mu = \begin{pmatrix} p_0^0 \\ \vec{0} \end{pmatrix} = \begin{pmatrix} E_0/c \\ \vec{0} \end{pmatrix} \tag{5.27}$$

For these considerations, it does not matter whether the particle is accelerating and this is an IRF, or whether the particle is in inertial motion and has a fixed rest frame. Formally, since $p^2 = m^2c^2 > 0$ is a time-like Lorentz vector, there is always an inertial frame in which $p^0 \neq 0$ and $\vec{p} = 0$.

→

$$\boxed{⁂\ \textit{Rest energy}: \quad E_0 = mc^2} \tag{5.28}$$

This is Einstein's famous *principle of equivalence of (inertial) mass and (rest) energy*.

- ¡! The *total energy* E is the *time-component of a 4-vector*: $p^\mu = (E/c, \vec{p})^T$; thus it makes sense to refer to the *rest energy* E_0 – which is the component of this 4-vector in the rest frame K_0, i.e., the particular frame where $\vec{p} = \vec{0}$.

- ¡! By contrast, the *mass* is a *Lorentz scalar*, namely $p^2 = m^2c^2$; hence it is the same in all inertial systems and it does not make sense to refer to the *rest mass* m_0 as this term suggests that there is a "non-rest mass" (which there isn't).
- Einstein first derived the mass-energy equivalence in his Annus Mirabilis paper *Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?* [10]. In the paper, the equation is not given verbatim but encoded in the following statement:

  > *Gibt ein Körper die Energie L in Form von Strahlung ab, so verkleinert sich seine Masse um L/V^2.*

  Einstein concludes:

  > *Die Masse eines Körpers ist ein Maß für dessen Energieinhalt; [...]. Es ist nicht ausgeschlossen, daß bei Körpern, deren Energie in hohem Maße veränderlich ist (z.B. bei den Radiumsalzen), eine Prüfung der Theorie gelingen wird.*

  Einstein further elaborates on the relativistic energy relation and its implications in [66]. He provides self-contained step-by-step derivation in Ref. [67]. Additional insight was provided over the years with alterantive derivations by various authors [68–70].

  The derivation by Feigenbaum and Mermin in [70] is particularly insightful as it follows Einsteins original derivation in [10] closely without invoking electrodynamics. They also demonstrate that the heart of relativistic mechanics is actually Eq. (5.24) (where mc^2 appears as a *coefficient*), and not Eq. (5.28) (which is conventional).

Note 5.1: Some comments on $E_0 = mc^2$

Eq. (5.28) is arguably the most famous equation in physics. The popularization of scientific concepts is often accompanied by simplifications and distortions. This is also the case for $E_0 = mc^2$:

- $E_0 = mc^2$ is often written as $E = mc^2$. This is either wrong or misleading (depending on the interpretation of the symbols); in any case, it is not consistent with modern conventions in RELATIVITY (→ *below*).
- $E_0 = mc^2$ is by no means Einstein's most important equation. This is why it is not refered to as "Einstein equation;" this honor goes to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{5.29}$$

  which are also known as the → *Einstein field equations*; these form the basis of GENERAL RELATIVITY and are empirically of much greater value than Eq. (5.28). Luckily, the Einstein field equations look daunting and are not nearly as accessible as $E_0 = mc^2$; hence they weren't seized (and multilated) by pop culture like $E_0 = mc^2$ was.
- How statements are phrased determines our conceptualization of the world. The often heard phrase

  "$E_0 = mc^2$ says that mass can be converted into energy"

  makes me think of "mass" as a sort of coal that can be lighted and then produces energy (maybe in form of light and heat or an atomic explosion). I am quite convinced that there are many who got "conceptually derailed" by statements like this, and hence think of Einstein's revelation as modern-day equivalent of an early human realizing, perhaps by witnessing a lightning strike, that wood can be kindled to produce heat. This is completely off the mark.

$E_0 = mc^2$ says that rest energy and inertial mass are *equivalent*; not that they can be "converted" into each other. It means that the Lorentz symmetry of spacetime necessitates that our concepts of "energy" (as a quantity that can make things change in time) and "inertial mass" (as a quantity that measures how hard it is to make the state of motion of an object change in time) are like two sides of the same coin. Note that we did not arrive at the equation by studying the microscopic dynamics and interactions of matter (like we do in quantum mechanics, and especially quantum field theory); the equivalence of rest energy and mass is a consequence of the symmetries of spacetime alone. One can take $E_0 = mc^2$ thus as a hint at the unanswered questions "What is time?" and "What is inertia?" because energy is the generator of time translations (think of the time-evolution operator in quantum mechanics) and mass quantifies the phenomenon of inertia.

To drive the point home, here a few examples:

- An atom in an excited electronic state is heavier than the same atom in the ground state.
- A battery gets lighter when being discharged.
- A chunk of metal is heavier when it is hot.
- If you put an atomic bomb into an opaque, completely sealed "super box" that survives the explosion, the weight of the box does not change when the bomb goes off. This makes it clear that mass is not "converted" into energy.
- If the box is made out of "super glass" that lets only photons escape, the box gets lighter by E_{phot}/c^2 if the photons carry away the energy E_{phot}.

• For these reasons, $E_0 = mc^2$ is *not* a magical blueprint to build atomic bombs. The equation is only relevant in this context because it provides a nice "shortcut" to compute the energies that the fission (splitting) of isotopes can yield (or cost, depending on the isotopes). Because one could measure the rest masses of isotopes rather easily (using mass spectrometry [71]) – but had almost no clue how to describe the inner workings (and therefore binding energies) of said nulei – the equation allowed for a straightforward survey of the periodic table to identify suitable isotopes that would yield energy under fission. $E_0 = mc^2$ is not the *reason* why atomic weapons work, and these weapons are not so powerful "because they convert mass into energy." This is pure nonsense. If you discharge the battery of your phone, it *also* looses mass – because rest energy and mass are equivalent: $E_0 = mc^2$! And yes, this mass difference is much smaller than the mass difference accompanied by a nuclear explosion. But this is not the *reason*; the reason is that the strength of electromagnetic interactions – which govern chemical processes (like discharging your battery) – is dwarfed by the strength of the strong interaction (and its residual, the nuclear force) – which governs nuclear reactions.

In a nutshell:

When studying reaction processes (of any sort), the change of restmass predicted by $E_0 = mc^2$ is an ↑ *epiphenomenon*. The mass change is not causal; it cannot be, because it is a consequence of the symmetries of spacetime, and not of the inner workings of matter.

ii | Unfortunately, the notation and interpretation of SPECIAL RELATIVITY has changed since its inception. In former times it was conventional to introduce the concept of a

$$⁂\ \textit{Relativistic mass:}\quad m_r := \gamma_v m = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}} \tag{5.30}$$

which depends on velocity. With this definition, the relativistic relation between 3-velocity and 3-momentum reads $\vec{p} = m_r \vec{v}$ and parallels the Newtonian relation $\vec{p} = m\vec{v}$. The relativistic energy relation then reads $E = m_r c^2$.

The concept of a velocity-dependent, relativistic mass is avoided in most modern treatments of RELATIVITY (and in this script). While this is mostly a matter of concepts and semantics, there are good reasons why the concept of a velocity dependent mass is less useful than it might seem (→ *below*).

Here a few comments on various notations that you might encounter:

$E_0 = mc^2$	Correct ☺
$\cancel{E = mc^2}$	Only makes sense if $m = m_r$ (which we don't use).
$\cancel{E_0 = m_0 c^2}$	Why m_0? There is only m!
$\cancel{E = m_0 c^2}$	Energy is frame-dependent. Do you mean E_0? Otherwise: Wrong!

For more details and explanations see Refs. [72–74].

iii | → Take home message:

$$\boxed{\text{There is only one mass: the rest mass } m \text{ (which we call } \textit{mass}\text{). Thus mass does } \textit{not} \text{ depend on velocity.}}$$

This convention is used by almost all modern textbooks on RELATIVITY.

Unfortunately the old conventions (using relativistic, velocity-dependent masses) are still used by school books and popular science books.

iv | Aside: *Why introducing velocity depended masses leads nowhere.*

If you are still inclined to think in terms of a velocity-dependent, relativistic mass m_r, here is a compelling argument why this is a useless and artificial concept that needs to die:

The 3-component of the relativistic equation of motion Eq. (5.16) reads

$$\vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma_v\vec{v}\right) = m\gamma_v\vec{a} + m\gamma_v^3\frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v} \tag{5.31}$$

with two extreme cases:

$$\vec{v} \parallel \vec{a} \quad \Rightarrow \quad \vec{F} \stackrel{\circ}{=} m\gamma_v^3\vec{a} \tag{5.32a}$$

$$\vec{v} \perp \vec{a} \quad \Rightarrow \quad \vec{F} = m\gamma_v\vec{a} \tag{5.32b}$$

If you insist on introducing a "mass" as the proportionality factor between 3-force and 3-acceleration to quantify the inertial response of an object at finite velocity, you are not only forced (☺) to make this mass velocity dependent, you also need *two* masses:

$$\text{"Longitudinal mass"}: \quad m_\parallel := m\gamma_v^3 \tag{5.33}$$

$$\text{"Transverse mass"}: \quad m_\perp := m\gamma_v \tag{5.34}$$

The above result demonstrates that the concept of a mass as a measure for inertia is not very useful in SPECIAL RELATIVITY. More precisely, the result shows that the quantitites $m_\parallel$ and $m_\perp$ are *relational properties* between an object and an observer (they depend on the state of motion of the observer); they are not *intrinisic* properties of the object itself. Only the restmass m qualifies as such an intrinisc property. The velocity dependence of $m_\parallel$ and $m_\perp$ is not an intrinisc feature of matter, it is a feature of spacetime.

This is why in modern textbooks there is only one mass m (the rest mass) which does *not* depend on v, and one has to accept that the Newtonian relation $\vec{p} = m\vec{v}$ is no longer valid. The concepts of "longitudinal mass" and "transverse mass" (and velocity dependent mass, for that matter) are therefore no longer used in modern literature.

↓**Lecture 11** [09.01.24]

12 | $\sphericalangle$ Non-relativistic limit:

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \approx \underbrace{mc^2}_{\text{Rest energy}} + \underbrace{\frac{1}{2}mv^2}_{\text{Newtonian kinetic energy}} + \mathcal{O}\left(\beta^4\right) \tag{5.35}$$

This shows again that the correspondence principle is satisfied: For small velocities compared to c, the kinetic energy of Newtonian mechanics is (up to a constant shift given by the rest energy) a good proxy for the true energy of the particle.

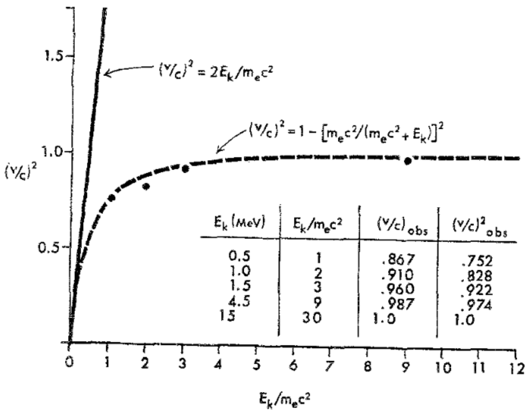
13 | The *kinetic energy* is: $E_{\text{kin}} = E - E_0 = E - mc^2$

→ The velocity of a relativistic particle as a function of its *kinetic energy* is:

$$\beta^2 = \left(\frac{v}{c}\right)^2 = 1 - \left[\frac{mc^2}{E_{\text{kin}} + mc^2}\right]^2 \xrightarrow{\beta \ll 1} \frac{2E_{\text{kin}}}{mc^2} \tag{5.36}$$

Note that in the non-relativistic limit it is $E_{\text{kin}} \ll mc^2$.

This velocity dependence has been confirmed experimentally to high precision; for example with accelerated electrons [33] (see Refs. [33, 34] for more technical details):



→ The relativistic energy relation Eq. (5.24) is correct ☺☺

14 | Massless particles:

So far we considered only particles with non-vanishing mass $m \neq 0$. The definition of the momentum Eq. (5.1) and the relativistic energy Eq. (5.24) cannot be directly applied to particles without mass. However:

i | $\sphericalangle$ Eq. (5.26) with $m \to 0$:

$$\boxed{E = |\vec{p}|c \quad \text{(linear dispersion)}} \tag{5.37}$$

$\sphericalangle$ Eq. (5.4) with $m \to 0$:

$$\boxed{p^2 = 0 \ \text{(light-like)} \quad \Rightarrow \quad p^\mu = \begin{pmatrix} |\vec{p}| \\ \vec{p} \end{pmatrix}} \tag{5.38}$$

¡! We take this as the *definition* of the 4-momentum for massless particles (it is the only definition that is consistent with $p^\mu = mu^\mu$ in the limit of vanishing mass). Note that there is no finite 4-velocity u^μ associated to massless particles.

ii | The fact that p^μ becomes light-like for massless particles already suggests that they move with the speed of light. We can verify this:

$$\left.\begin{aligned} E &= \gamma_v mc^2 \\ \vec{p} &= \gamma_v m\vec{v} \end{aligned}\right\} \quad \Rightarrow \quad E = |\vec{p}|\,\frac{c^2}{v} \quad \xrightarrow[\text{Eq. (5.37)}]{m\to 0} \quad |\vec{p}|\,c \tag{5.39}$$

This limit is only consistent if $v \to c$ for $m \to 0$:

$$\boxed{\text{All particles with vanishing mass move with the speed of light.}} \tag{5.40}$$

- Examples: Photons, Gravitons (if they exist)
- Massless particles do not have a rest frame.

  You would need a boost with $v = c$ to reach such a frame; but such boosts are not defined (because the Lorentz factor diverges in this limit).
- ¡! The relativistic energy $E = \gamma_v mc^2$ holds only for massive particles. For massless particles it does *not* follow $E = 0$ but rather $E = |\vec{p}|c \neq 0$. So photons do have energy and momentum, but no mass (neither rest- nor any other type of mass). You are also not allowed to use the "forbidden" equation $E = m_r c^2$ and declare $m_r = E/c^2 = |\vec{p}|/c$ as the "dynamic mass" of the photon because (1) we argued above that this concept is not as useful as it sounds, and (2) you only renamed momentum, so what's the point. And if you are afraid that later – in GENERAL RELATIVITY – our photons will not be deflected by stars or sucked into black holes because they "have no mass": I assure you, they will; they have energy and momentum, that's enough.
- This demonstrates why the "speed of light" is sort of a misnomer in this context, and we should have stuck to our v_{max} (but then all our equations would look different from the literature). Then it would be conceptually clear that *every* particle with vanishing rest mass "runs into" the universal speed limit v_{max}.

5.3. Action principle and conserved quantities

In this section we study a more formal (and more versatile) approach to describe the dynamics of relativistic systems, namely in terms of the Lagrangian and the action. We do this for the free particle (no force!) and consider electromagnetic forces in the next Chapter 6.

1 | Action of free massive particle:

i | $\sphericalangle$ Trajectory γ parametrized by
$x^\mu = x^\mu(\lambda)$ with $\lambda \in [\lambda_a, \lambda_b]$ and $x^\mu(\lambda_a) = a^\mu$, $x^\mu(\lambda_b) = b^\mu$

Remember the characteristic property of the trajectory of a free particle (Section 2.4):
The proper time (= Minkowski distance) is *maximized* along the trajectory!

$$\boxed{\rightarrow \text{Action:} \quad S[\gamma] := \alpha \int_\gamma \mathrm{d}s = \alpha \int_{\lambda_a}^{\lambda_b} \sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,\mathrm{d}\lambda} \tag{5.41}$$

with $\dot{x}^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$.

The prefactor α is undetermined so far (→ *next step*).

¡! The parameter λ has no physical interpretation in this formulation as this action is reparametrization invariant (→ *Section 5.4*).

ii | Correspondence principle $\rightarrow \alpha = -mc$

To determine the parameter α, consider the non-relativistic limit of the Lagrangian in coordinate time parametrization $\lambda = t$:

$$\tilde{L} = \alpha\sqrt{c^2 - \dot{\vec{x}}^2} = \alpha c\sqrt{1 - \frac{v^2}{c^2}} \xrightarrow{\beta \ll 1} L \approx \alpha c \underbrace{- \frac{\alpha v^2}{2c}}_{\overset{!}{=} \frac{1}{2}mv^2} \tag{5.42}$$

The non-relativistic limit yields – up to a constant that doesn't change the equations of motion – the Lagrangian with Newtonian kinetic energy if we set $\alpha = -mc$.

iii | Lagrangian:

$$\boxed{L(x^\mu, \dot{x}^\mu) = -mc\sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = -mc\sqrt{\dot{x}_\mu \dot{x}^\mu}} \tag{5.43}$$

- ¡! This Lagrangian is only valid for *massive* particles.
- The Lagrangian Eq. (5.43) is fully specified as is; there is no need to fix a specific parametrization. In this form, the Lagrangian [more precisely: the action Eq. (5.41)] has a gauge symmetry: the parametrization λ is arbitrary (→ *Section 5.4*).
- On the contrary, if you fix a parametrization (= fix a gauge), e.g., by identifying λ with the coordinate time $\lambda = t \equiv x^0/c$ ("static gauge") or the proper time $\lambda = \tau$ ("proper time gauge"), you obtain different (but physically equivalent) Lagrangians which have no longer a gauge symmetry:

$$\lambda \overset{!}{=} t \quad \Leftrightarrow \quad c\lambda \overset{!}{=} x^0 \quad \Rightarrow \quad \tilde{L}_t(\vec{x}, \dot{\vec{x}}) = -mc^2\sqrt{1 - \dot{\vec{x}}^2/c^2}\,, \tag{5.44a}$$

$$\lambda \overset{!}{=} \tau \quad \Leftrightarrow \dot{x}^\mu \dot{x}_\mu \overset{!}{=} c^2 \quad \Rightarrow \quad \tilde{L}_\tau(x^\mu, \dot{x}^\mu) = -mc^2\,. \tag{5.44b}$$

We denote gauge-fixed Lagrangians by $\tilde{L}$ and the gauge-invariant Lagrangian Eq. (5.43) by L. In the following we often work with the latter and choose specific parametrizations at the end of our calculations to express results in known quantities.

2 | Euler-Lagrange equations:

$$\delta S \overset{!}{=} 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\sigma} - \underbrace{\frac{\partial L}{\partial x^\sigma}}_{=0} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{-mc\dot{x}_\sigma}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} = 0 \tag{5.45}$$

These are 4 differential equations ($\sigma = 0, 1, 2, 3$)!

→ Equations of motion in the "proper time gauge" $\lambda = \tau$ [where $\dot{x}_\mu \dot{x}^\mu = u^2 = c^2$]:

$$\boxed{m\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = 0} \tag{5.46}$$

This is Eq. (5.6) for vanishing 4-force ☺

3 | ⊲ Action in "static gauge" $\lambda = t = x^0/c$:

$$\boxed{S[\gamma] \overset{\lambda=\frac{x^0}{c}}{\equiv} \tilde{S}_t[\vec{x}(t)] = \int_{t_a}^{t_b} \tilde{L}_t(\vec{x},\dot{\vec{x}})\,\mathrm{d}t = -mc^2 \int_{t_a}^{t_b} \sqrt{1-\frac{\dot{\vec{x}}^2}{c^2}}\,\mathrm{d}t} \tag{5.47}$$

i | Canonical momenta ($\vec{v} = \dot{\vec{x}}$):

$$\vec{p} = \frac{\partial \tilde{L}_t}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \tag{5.48}$$

This is the expression for the relativistic 3-momentum Eq. (5.3) we found before, now derived as the canonical momentum of a Lagrangian.

ii | Hamiltonian:

$$\tilde{H}_t = \vec{p}\cdot\vec{v} - \tilde{L}_t \overset{\circ}{=} \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} = cp^0 \overset{5.26}{=} c\sqrt{\vec{p}^2 + m^2c^2} \tag{5.49}$$

This is just the relativistic energy Eq. (5.24) we found before, now derived from a Lagrangian.

- ⊲ Non-relativistic limit:

$$\tilde{H}_t = mc^2\sqrt{1+\frac{\vec{p}^2}{m^2c^2}} \overset{\frac{\vec{p}^2}{2m}\ll mc^2}{\approx} \underbrace{mc^2}_{\text{Rest energy}} + \underbrace{\frac{\vec{p}^2}{2m}}_{\text{Newtonian kinetic energy}} \tag{5.50}$$

- ¡! Contrary to the action Eq. (5.47), this Hamiltonian also makes sense for *massless* particles:

$$\tilde{H}_t \overset{m=0}{=} |\vec{p}|c \tag{5.51}$$

4 | Noether's (first) theorem:

Details: ➔ Problemset 6

x^μ cyclic → Spacetime translations $x^\mu + \delta\varepsilon^\mu$ are *continuous symmetries* of S

These transformations correspond to the inhomogeneous part of Poincaré transformations: $\bar{x}^\mu = x^\mu + a^\mu$. Every relativistic theory must have this symmetry; for field theories one obtains then four conserved currents: → *Energy momentum tensor*.

↓ *Noether's theorem* → ↓ *Conserved Noether charges* Q_μ: (set $\lambda = t$ as the coordinate time)

$$Q_\mu \equiv \begin{Bmatrix} \text{Time translation} \Rightarrow \text{Energy } E/c \\ \text{Space translations} \Rightarrow \text{Momentum } \vec{p} \end{Bmatrix} \tag{5.52}$$

$$= -\frac{\partial L}{\partial \dot{x}^\mu} = \frac{mc\dot{x}_\mu}{\sqrt{c^2-\vec{v}^2}} = \begin{pmatrix} \frac{1}{c}\frac{mc^2}{\sqrt{1-\beta^2}} \\ -\frac{m\vec{v}}{\sqrt{1-\beta^2}} \end{pmatrix} = p_\mu \tag{5.53}$$

- Because x^μ are cyclic coordinates, we can obtain the Noether charges directly from the Lagrangian as $\frac{\partial L}{\partial \dot{x}^\mu}$; the additional minus is conventional to connect to our definition of the 4-momentum.
- ¡! This shows that our definition of the 4-momentum is consistent, and the identification of its time-component p^0 as the total energy was correct: By definition, *energy* is the Noether charge that corresponds to translation invariance in *time*. Similarly, *momentum* is the charge for translation invariance in *space*.

5 | Noether charges for *homogeneous* Lorentz transformations?

Any relativistic theory is also invariant under (proper orthochronous) Lorentz transformations, $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\mu$; for these there must exist additional conserved Noether charges:

Infinitesimal Lorentz transformations $x^\mu + \delta\varepsilon^\mu{}_\nu x^\nu$ are *continuous symmetries* of S

The infinitesimal transformation is antisymmetric: $\delta\varepsilon^\mu{}_\nu = -\delta\varepsilon_\nu{}^\mu$, ➔ Problemset 5.

$\overset{\circ}{\rightarrow}$ Conserved Noether charges:

$$\boxed{\text{⁂ } \textit{Angular momentum (tensor):} \quad L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu} \tag{5.54}$$

This is an example of an antisymmetric $(2,0)$ Lorentz tensor.

Proof: ➔ Problemset 6

i | $\triangleleft$ Spatial components:

$$\left.\begin{aligned} L^{23} &= x^2 p^3 - x^3 p^2 = l_1 \\ L^{31} &= x^3 p^1 - x^1 p^3 = l_2 \\ L^{12} &= x^1 p^2 - x^2 p^1 = l_3 \end{aligned}\right\} \quad \text{with 3-angular momentum } \vec{l} = \vec{x} \times \vec{p}. \tag{5.55}$$

→ 3-angular momentum $\vec{l}$ is *not* (part of a) Lorentz *vector* but of a $(2,0)$ *tensor*!

It is not surprising that invariance under spatial rotations $\mathrm{SO}(3) \subset \mathrm{O}(1,3)$ implies angular momentum conservation.

ii | $\triangleleft$ Mixed components:

$$\begin{aligned} L^{10} &= x^1 \gamma_v mc - ctp^1 = cn_1 \\ L^{20} &= x^2 \gamma_v mc - ctp^2 = cn_2 \\ L^{30} &= x^3 \gamma_v mc - ctp^3 = cn_3 \end{aligned} \tag{5.56}$$

with ⁂ *dynamic mass moment*

$$\vec{n} := m\gamma_v \left(\vec{x} - t\vec{v}\right) = \frac{E}{c^2}\vec{x} - t\,\vec{p} = \text{const}\,. \tag{5.57}$$

This is the relativistic version of the ↓ *center-of-mass theorem*.

The *center of mass* (COM) is now the *center of energy* (COE). Since $\vec{n}$ (and E) is conserved, we can set $t = 0$ to find $\vec{n} = E/c^2 \vec{x}_0$, which is the initial center of energy of the system (times E/c^2).

For many particles this is slightly less trivial: One finds analogously the conserved quantity

$$\vec{N} = \sum_i \vec{n}_i = \sum_i \left(\frac{E_i}{c^2}\vec{x}_i - t\,\vec{p}_i\right) = \text{const}\,. \tag{5.58}$$

Division by the total (also conserved) energy $E = \sum_i E_i$ yields

$$\vec{X}_{\text{COE}}(t) := \frac{\sum_i E_i \vec{x}_i}{\sum_i E_i} = t\frac{c^2 \vec{P}}{E} + \text{const} \equiv t\vec{V}_{\text{COE}} + \text{const} \tag{5.59}$$

with the total 3-momentum $\vec{P} = \sum_i \vec{p}_i$. Thus the ⁂ *center of energy* $\vec{X}_{\text{COE}}$ moves in a straight line with constant velocity $\vec{V}_{\text{COE}}$. Note that the center of energy becomes the Newtonian center of mass in the non-relativistic limit where $E_i \approx E_{i,0} = m_i c^2$.

6 | ⊲ Multiple particles (covariantly coupled by fields):

The above arguments can be directly generalized to many (non-interacting) particles. This immediately yields the sum of the 4-momenta of these particles as conserved quantity. Interactions between the particles must be covariantly mediated by fields – which also carry 4-momentum (→ *Chapter 6*):

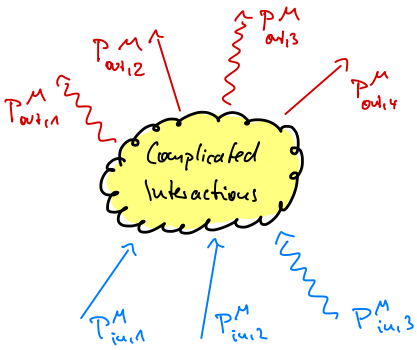
Conserved Noether charge:

⁂ *Total 4-momentum*: $$P^\mu := \sum_i p_i^\mu + p_{\text{Fields}}^\mu \tag{5.60}$$

with

- p_i^μ the 4-momentum of particle i, and
- p_{Fields}^μ the total 4-momentum of the fields mediating the interations.

7 | ⊲ Scattering process:



Long before and after the interactions play a role we can approximate the system by non-interacting particles and set $p_{\text{Fields}}^\mu = 0 \rightarrow$

$$\boxed{\sum_i p_{\text{in},i}^\mu = \sum_j p_{\text{out},j}^\mu} \tag{5.61}$$

$\rightarrow$ Conservation of *energy* ($\mu = 0$) and *momentum* ($\mu = 1, 2, 3$)

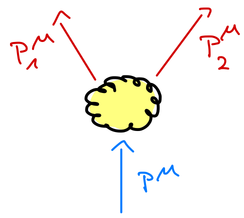
- In RELATIVITY, conservation of total energy and total momentum is combined into the conservation of 4-momentum.
- We will denote the 4-momenta of *massive* particles (solid lines) with p^μ and the 4-momenta of *massless* particles with q^μ (wiggly lines).

Examples:

i | Particle decay: $\sphericalangle$ Radioactive Nucleus → Nucleus 1 & Nucleus 2

→ Energy-momentum conservation:

$$\underbrace{p^\mu}_{\text{in}} = \underbrace{p_1^\mu + p_2^\mu}_{\text{out}} \tag{5.62}$$



$\sphericalangle$ Center-of-mass frame where $\vec{p} = \vec{p}_1 + \vec{p}_2 = 0 \overset{\circ}{\rightarrow}$

$$mc^2 = m_1c^2 + E_{\text{kin},1} + m_2c^2 + E_{\text{kin},2} \tag{5.63}$$

→ Decay only possible if

$$m \geq m_1 + m_2 \tag{5.64}$$

If $E_{\text{kin},1} \neq 0$ or $E_{\text{kin},2} \neq 0$, it is $m \neq m_1 + m_2$.

→ The rest mass of composite objects is *not additive*.

Composite objects also contain binding energy (potential energy) which contributes to the rest mass of the object.

$\overset{\circ}{\rightarrow}$

$$E_{\text{kin},1} = \frac{(m-m_1)^2c^2 - m_2^2c^2}{2m} \tag{5.65}$$

In the COM frame, the kinetic energy of the two decay products is constant and depends only on the masses of the particles. So if you find a non-trivial energy distribution for the products of a decay process, there must at least three particles be produced (of which you might not be able to detect all). This is how the neutrino was predicted by Pauli from the decay of the neutron: $n \to p + e^- + \bar{\nu}_e$.

ii | Particle creation:

Note that a single massless (light-like) particle (like a photon) cannot decay into two massive (time-like) particles because $(p_1 + p_2)^2 = q^2 = 0$ cannot be solved if $p_i^2 = m_i^2c^2 > 0$.

Indeed (we set $c = 1$): With the ↓ *Cauchy-Schwarz inequality* we find

$$m_1m_2 + \vec{p}_1 \cdot \vec{p}_2 \leq \sqrt{m_1^2 + \vec{p}_1^2}\sqrt{m_2^2 + \vec{p}_2^2} = p_1^0p_2^0 \tag{5.66a}$$

$$\Rightarrow \quad 0 < m_1m_2 \leq p_1 \cdot p_2 \tag{5.66b}$$

so that for arbitrary m_1 and m_2 (particle creation: $q^\mu = p_1^\mu + p_2^\mu$)

$$(p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2 > 0 \quad \Rightarrow \quad \text{Time-like} \tag{5.67}$$

Furthermore, for $m_1 = m_2$ (scattering: $p_1^\mu - p_2^\mu = q^\mu$):

$$(p_1 - p_2)^2 = m_1^2 + m_2^2 - 2p_1 \cdot p_2 \tag{5.68a}$$

$$\leq m_1^2 + m_2^2 - 2m_1m_2 \overset{m_1=m_2}{=} 0 \quad \xRightarrow{p_1 \neq p_2} \quad \text{Space-like} \tag{5.68b}$$

(For the Cauchy-Schwarz inequality, *equality* holds iff the two vectors are linearly dependent; for $m_1 = m_2$ this is only possible if $p_1 = p_2$, i.e., in the trivial case of no scattering.)

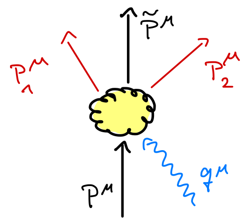
Eq. (5.67) shows that two particles (of arbitrary masses) can never annihilate into a single photon, and, vice versa, a single photon can never create a pair of massive particles. This is reason why we need an additional (heavy) nucleus for the creation of a particle & antiparticle pair from a photon.

By contrast, Eq. (5.68) tells us that a single massive particle cannot emit or absorb a single photon *if it cannot change its mass* (i.e., has no different energy states). This is true for free elementary particles like electrons (an electron cannot be excited, it always has the same mass). Thus a free electron cannot emit a single photon. If the massive particle in question *has* different internal energy states (and therefore the two masses m_1 and m_2 can be different), this argument does not hold. This is why atoms can spontaneously emit or absorb single photons.

$\sphericalangle$ Photon (+Nucleus) → Electron & Positron (+Nucleus)

→ Energy-momentum conservation:

$$P^\mu_{\text{in}} \equiv \underbrace{q^\mu + p^\mu}_{\text{in}} = \underbrace{p^\mu_1 + p^\mu_2 + \tilde{p}^\mu}_{\text{out}} \equiv P^\mu_{\text{out}} \tag{5.69}$$



With the mass M of the nucleus and the momentum/energy $|\vec{q}| = E_\gamma/c$ of the incoming photon, we find

$$\underbrace{\left(\frac{E_\gamma + Mc^2}{c}\right)^2 - \left(\frac{E_\gamma}{c}\right)^2}_{\text{Rest frame of nucleus}} = P^2_{\text{in}} \overset{!}{=} P^2_{\text{out}} = \underbrace{\left(\frac{E_{\text{Nuc}} + E_{e^-} + E_{e^+}}{c}\right)^2}_{\text{COM frame of system}} \tag{5.70}$$

where the right hand side was evaluated in the COM frame with $\vec{P}_{\text{out}} = \vec{0}$ and the left hand side in the rest frame of the nucleus (which is allowed since $P^2 = P^\mu P_\mu$ is a Lorentz scalar).

Please appreciate the subtlety of this evaluation: The 4-momentum conservation Eq. (5.69) is Lorentz *covariant*. Therefore you *cannot* evaluate the left hand side P^μ_{in} in one inertial system and the right hand side P^μ_{out} in another. However, in any inertial system Eq. (5.69) implies $P^2_{\text{in}} = P^2_{\text{out}}$ where left and right hand side are now Lorentz *invariant*; hence you can evaluate the two sides in *different* inertial systems.

$\overset{\circ}{\rightarrow}$ Threshold for particle creation:

$$E_{\gamma,\min} = 2m_e^2c^2\left(1 + \frac{m_e}{M}\right) > 2m_e c^2 \tag{5.71}$$

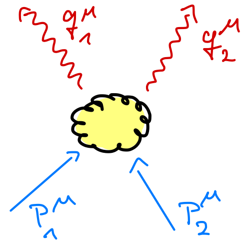
The threshold follows for vanishing kinetic energy of the products in the COM frame.

The threshold energy is larger than twice the rest energy of the electron $2m_e c^2$ (the positron has the same mass as the electron) because the scattering products necessarily aquire kinetic energy in the initial rest frame of the nucleus (to carry the momentum of the photon).

iii | Annihilation: $\sphericalangle$ Electron & Positron → Photon & Photon

→ Energy-momentum conservation:

$$P^\mu_{\text{in}} \equiv \underbrace{p_1^\mu + p_2^\mu}_{\text{in}} = \underbrace{q_1^\mu + q_2^\mu}_{\text{out}} \equiv P^\mu_{\text{out}} \tag{5.72}$$



⊲ COM frame:

$$P^\mu_{\text{in}} = \begin{pmatrix} E_{e^-}/c \\ \vec{p} \end{pmatrix} + \begin{pmatrix} E_{e^+}/c \\ -\vec{p} \end{pmatrix} = \begin{pmatrix} |\vec{q}| \\ \vec{q} \end{pmatrix} + \begin{pmatrix} |\vec{q}| \\ -\vec{q} \end{pmatrix} = P^\mu_{\text{out}} \tag{5.73}$$

Using that electron and positron have the same mass m_e, we find for the energy of the emitted photons:

$$E_\gamma = c\sqrt{\vec{p}^2 + m_e^2 c^2} \tag{5.74}$$

Note that the individual rest massess of particles in scattering processess are not conserved: $p_1^2 = p_2^2 = m_e^2 c^2 > 0$ for the incoming electron and the positron, but $q_1^2 = q_2^2 = 0$ for the outgoing photons. The rest mass of the *composite system* remains the same, though. In particular, the two photons together have the same rest mass as the electron-positron system before: $P_{\text{out}}^2 = P_{\text{in}}^2 = 4(\vec{p}^2 + m_e^2 c^2) > 0$.

→ The rest masses of individual particles are *not conserved*.

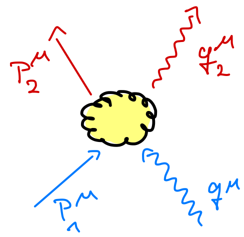
iv | Compton scattering: ⊲ Electon & Photon → Electron & Photon

Details: ➲ Problemset 6

Compton scattering is an example of ↓ *elastic scattering* where the total kinetic energy is conserved and the rest energies of in- and outgoing particles remains the same.

→ Energy-momentum conservation:

$$\underbrace{q_1^\mu + p_1^\mu}_{\text{in}} = \underbrace{q_2^\mu + p_2^\mu}_{\text{out}} \tag{5.75}$$

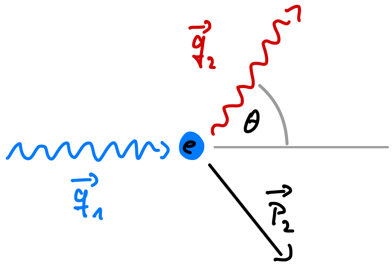


With $q_1^2 = q_2^2 = 0$ and $p_1^2 = p_2^2 = m_e^2 c^2$ one finds:

$$\underbrace{E_1 E_2/c^2(1-\cos\theta)}_{\text{Rest frame of } e^-} \overset{\circ}{=} \underbrace{q_1 \cdot q_2 = p\cdot(q_1 - q_2)}_{\text{Lorentz invariant}} \overset{\circ}{=} \underbrace{m_e c(E_1/c - E_2/c)}_{\text{Rest frame of } e^-} \tag{5.76a}$$

$$\Rightarrow \quad \frac{1}{E_2} - \frac{1}{E_1} = \frac{1}{m_e c^2}(1-\cos\theta) \tag{5.76b}$$

Here the left and right hand sides are evaluated in the rest frame of the electron: $p_1^\mu = (m_e c, \vec{0})^T$; θ is the angle between incoming and outgoing photon (scattering angle):



With the photon energy $E_i = hc/\lambda_i$ we find the change in wavelength due to scattering:

$$\boxed{\Delta\lambda = \lambda_2 - \lambda_1 = \underbrace{\frac{h}{m_e c}}_{\lambda_e}(1-\cos\theta)} \tag{5.77}$$

with ⁂ *Compton wavelength* λ_e of the electron.

- With Compton scattering one can measure the Compton wavelength of the electron and thereby determine the Planck constant h.
- Because the Compton wavelength is the natural length scale associated to a massive quantum particle, it appears in many field equations of relativistic quantum mechanics (Klein-Gordon equation, Dirac equation, …).

5.4. ‡ Reparametrization invariance

The action of the free relativistic particle Eq. (5.41) has the peculiar property of "reparametrization invariance", a feature that plays an important role in GENERAL RELATIVITY, and is also crucial for the quantization of the relativistic string in string theory (↑ *Nambu-Goto action*).

1 | $\sphericalangle$ Trajectory γ parametrized by $x^\mu(\lambda)$ for $\lambda \in [\lambda_a, \lambda_b]$.

$\sphericalangle$ Diffeomorphism $\varphi : [\lambda_a, \lambda_b] \to [\lambda_a, \lambda_b]$ with $\lambda_{a/b} = \varphi(\lambda_{a/b})$ and write $\tilde{\lambda} = \varphi(\lambda)$.

Diffeomorphism = Bijective map where both the map and its inverse are continuously differentiable.

$\to$ Define new trajectory $\tilde{\gamma}$ via $\tilde{x}^\mu(\tilde{\lambda}) := x^\mu(\varphi^{-1}(\tilde{\lambda})) = x^\mu(\lambda)$ with $\tilde{\lambda} \in [\lambda_a, \lambda_b]$.

$\tilde{x}^\mu(\tilde{\lambda})$ is a *reparametrization* of $x^\mu(\lambda)$: $\tilde{x}^\mu$ and x^μ are *different* functions on $[\lambda_a, \lambda_b]$ that parametrize the *same* trajectory in spacetime $\mathbb{R}^{1,3}$.

→ Action of new trajectory:

$$S[\tilde{\gamma}] \stackrel{\text{def}}{=} -mc\int_{\lambda_a}^{\lambda_b}\sqrt{\dot{\tilde{x}}_\mu(\lambda)\dot{\tilde{x}}^\mu(\lambda)}\,\mathrm{d}\lambda \tag{5.78a}$$

Rename the dummy variable: $\lambda \to \tilde{\lambda}$

$$= -mc\int_{\lambda_a}^{\lambda_b}\sqrt{\dot{\tilde{x}}_\mu(\tilde{\lambda})\dot{\tilde{x}}^\mu(\tilde{\lambda})}\,\mathrm{d}\tilde{\lambda} \tag{5.78b}$$

Use $\tilde{x}^\mu(\tilde{\lambda}) = x^\mu(\lambda)$ and the chain rule

$$= -mc\int_{\lambda_a}^{\lambda_b}\sqrt{\dot{x}_\mu(\lambda)\frac{\mathrm{d}\lambda}{\mathrm{d}\tilde{\lambda}}\dot{x}^\mu(\lambda)\frac{\mathrm{d}\lambda}{\mathrm{d}\tilde{\lambda}}}\,\mathrm{d}\tilde{\lambda} \tag{5.78c}$$

Substitution in the integral: $\tilde{\lambda} = \varphi(\lambda)$

$$= -mc\int_{\lambda_a}^{\lambda_b}\sqrt{\dot{x}_\mu(\lambda)\dot{x}^\mu(\lambda)}\,\mathrm{d}\lambda \tag{5.78d}$$

$$\stackrel{\text{def}}{=} S[\gamma] \tag{5.78e}$$

→ S is invariant under diffeomorphisms on parameter space.

→ ⁂ *Reparametrization invariance (RI)*

2 | Infinitesimal generators:

i | Consider infinitesimal deformations $\varepsilon(\lambda)$ of the parametrization (i.e., $|\varepsilon(\lambda)| \ll 1$ for all λ):

$$\tilde{\lambda} = \varphi(\lambda) \equiv \lambda + \varepsilon(\lambda) \tag{5.79}$$

With this we find:

$$x^\mu(\lambda) \stackrel{\text{def}}{=} \tilde{x}^\mu(\tilde{\lambda}) = \tilde{x}^\mu(\lambda + \varepsilon(\lambda)) = \tilde{x}^\mu(\lambda) + \varepsilon(\lambda)\partial_\lambda\tilde{x}^\mu(\lambda) + \mathcal{O}(\varepsilon^2) \tag{5.80}$$

ii | The infinitesimal variation of the trajectory is:

$$\delta_\varepsilon x^\mu := \tilde{x}^\mu(\lambda) - x^\mu(\lambda) \tag{5.81a}$$

$$= -\varepsilon(\lambda)\partial_\lambda x^\mu(\lambda) + \mathcal{O}(\varepsilon^2) \tag{5.81b}$$

$$\equiv G_\varepsilon x^\mu + \mathcal{O}(\varepsilon^2) \tag{5.81c}$$

Note that we can replace $\tilde{x}^\mu$ by x^μ in linear order of ε.

→ ⁂ *Generators* of one-dimensional diffeomorphisms:

$$\boxed{G_\varepsilon = -\varepsilon(\lambda)\partial_\lambda \quad \text{for arbitrary (infinitesimal) } \varepsilon(\lambda).} \tag{5.82}$$

iii | We can expand $\varepsilon(\lambda)$ into a Taylor series $\varepsilon(\lambda) = \sum_n \frac{\varepsilon_n}{n!}\lambda^n$ to write

$$G_\varepsilon = \sum_n \frac{\varepsilon_n}{n!}\left(-\lambda^n\partial_\lambda\right) \equiv \sum_n \frac{\varepsilon_n}{n!}G_n\,. \tag{5.83}$$

→ Basis of generators that generate infinitesimal reparametrizations is given by

$$\boxed{G_n = -\lambda^n\partial_\lambda \quad \text{for } n \in \mathbb{N}_0.} \tag{5.84}$$

→ RI = Infinite-dimensional continuous symmetry group

Note that in particular $\varepsilon(\lambda)$ can be chosen such that it is non-zero only for a compact subinterval of $[\lambda_a, \lambda_b]$, i.e., reparametrization invariance is a *local* symmetry (local in parameter space).

→ RI is a gauge symmetry

3 | Conserved quantities:

You know from your course on classical mechanics that Noether's theorem assigns a conserved quantity to each continuous symmetry of an action. What are these quantities for the infinitely many symmetry transformations G_ε associated to RI?

i | ∢ Variation of the Lagrangian $L = -mc\sqrt{\dot{x}_\mu \dot{x}^\mu}$ under G_ε:

$$\delta_\varepsilon L = \frac{\partial L}{\partial \dot{x}^\mu}\delta_\varepsilon \dot{x}^\mu \tag{5.85a}$$

Use $\delta_\varepsilon \dot{x}^\mu := \dot{\tilde{x}}^\mu - \dot{x}^\mu = \partial_\lambda(\delta_\varepsilon x^\mu)$:

$$\overset{5.81}{=} -\frac{mc\dot{x}_\mu}{\sqrt{\dot{x}_\sigma \dot{x}^\sigma}}\partial_\lambda\left[-\varepsilon(\lambda)\dot{x}^\mu\right] \tag{5.85b}$$

$$= \frac{mc}{\sqrt{\dot{x}_\sigma \dot{x}^\sigma}}\left[\dot{x}_\mu \dot{\varepsilon}(\lambda)\dot{x}^\mu + \dot{x}_\mu \varepsilon(\lambda)\ddot{x}^\mu\right] \tag{5.85c}$$

$$= mc\sqrt{\dot{x}_\mu \dot{x}^\mu}\,\dot{\varepsilon}(\lambda) + mc\varepsilon(\lambda)\partial_\lambda\sqrt{\dot{x}_\mu \dot{x}^\mu} \tag{5.85d}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda}\underbrace{\left[mc\varepsilon(\lambda)\sqrt{\dot{x}_\mu \dot{x}^\mu}\right]}_{=:K_\varepsilon(\lambda,\dot{x}^\mu)} = \frac{\mathrm{d}K_\varepsilon}{\mathrm{d}\lambda} \tag{5.85e}$$

→ $\delta_\varepsilon L$ is a total derivative → G_ε is a continuous symmetry of S

Note that in Eq. (5.78) we assumed $\lambda_{a/b} = \varphi(\lambda_{a/b})$ which corresponds to $\varepsilon(\lambda_{a/b}) = 0 = K_\varepsilon(\lambda_{a/b}, \dot{x}^\mu)$ such the boundary terms vanish and the action is completely invariant.

ii | ↓ *Noether's (first) theorem* →

For each continuous symmetry $\delta_\varepsilon x^\mu = G_\varepsilon x^\mu$ there is a conserved Noether charge:

$$Q_\varepsilon \overset{*}{=} \delta_\varepsilon x^\mu \frac{\partial L}{\partial \dot{x}^\mu} - K_\varepsilon \overset{5.85e}{=} \varepsilon(\lambda)mc\frac{\dot{x}^\mu \dot{x}_\mu}{\sqrt{\dot{x}_\sigma \dot{x}^\sigma}} - \varepsilon(\lambda)mc\sqrt{\dot{x}_\mu \dot{x}^\mu} = 0 \tag{5.86}$$

→ The Noether charge corresponding to G_ε vanishes identically!

"Vanishing identically" means that $Q_\varepsilon(\lambda, x^\mu, \dot{x}^\mu) \equiv 0$ for *all* functions $x^\mu(\lambda)$, and not just those that satisfy the equations of motion.

- Naïvely, we expected infinitely many conserved quantities from the infinitely many symmetry generators G_n. We found them, but quite surprisingly, they turned out to be trivially zero. This is a general feature of *local* or *gauge* symmetries; here we use the reparametrization invariance of the relativistic free particle only as an example.
- So while the conserved charges of local symmetries are trivial, such symmetries have other non-trivial implications: they enforce *constraints* on the equations of motion, so that they are no longer independent. Mathematically, this is described by ↑ *Noether's* second *theorem*.

4 | We can illustrate the implications of Noether's second theorem for the relativistic free particle:

i | The Lagrangian

$$L = -mc\sqrt{\dot{x}_\mu \dot{x}^\mu} \tag{5.87}$$

leads to the conjugate momenta

$$p_\sigma = \frac{\partial L}{\partial \dot{x}^\sigma} = -\frac{mc\dot{x}_\sigma}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} \tag{5.88}$$

which satisfy the identity

$$p^2 = p^\mu p_\mu = m^2 c^2 \tag{5.89}$$

- Eq. (5.89) is an *identity*, i.e., it holds for arbitrary trajectories $x^\mu(\lambda)$. In particular, $x^\mu(\lambda)$ does not need to satisfy the equations of motion for Eq. (5.89) to be valid. In Hamiltonian mechanics, such constraints are called ⁂ *primary constraints*. So our four canonical momenta p^μ are not independent!
- Eq. (5.89) is equivalent to:

$$\frac{\mathrm{d}p^2}{\mathrm{d}\lambda} = 0 \quad \Leftrightarrow \quad \left(\frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda}\right) p_\mu = 0 \tag{5.90}$$

ii | ⊲ Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\sigma} - \frac{\partial L}{\partial x^\sigma} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\sigma} = \frac{\mathrm{d}p_\sigma}{\mathrm{d}\lambda} = 0 \tag{5.91}$$

→ *Four* differential equations ($\sigma = 0, 1, 2, 3$) for *four* undetermined functions $x^\mu(\lambda)$.

However: Eq. (5.91) not independent:

$$p^\mu \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = p_\mu \frac{\mathrm{d}p^\mu}{\mathrm{d}\lambda} \overset{5.90}{=} 0 \tag{5.92}$$

- Eq. (5.92) is again an *identity*, i.e., valid for *all* functions x^μ, and not only those that satisfy the equations of motion.
- As a consequence, the system of equations of motion Eq. (5.91) effectively looses one of the four equations, and is therefore *underdetermined*.

  Put differently, if you specify a spacetime position $x^\mu(\lambda = 0)$ and its first derivative $\dot{x}^\mu(\lambda = 0)$ (note that the Euler-Lagrange equations are second-order differential equations), the equations of motion do *not* determine a unique solution $x^\mu(\lambda)$. Mathematically speaking, the initial value problem is ill-posed. This is the characteristic property of a *gauge theory*.
- This makes sense in the light of reparametrization invariance: If $x^\mu(\lambda)$ solves the equations of motion, you can construct a new solution $\tilde{x}^\mu(\lambda) = x^\mu(\varphi(\lambda))$ where φ is some diffeomorphism that is the identity except for a compact subinterval somewhere in the interior of $[\lambda_a, \lambda_b]$. In particular, $\tilde{x}^\mu(\lambda) = x^\mu(\lambda)$ in the neighborhood of λ_a, such that the two solutions cannot be distinguished by their initial value and derivative. Note how important the *locality* of the symmetry is for this argument to hold!
- This is a special case of ↑ *Noether's* second *theorem* [75, 76].

iii | The fact that our theory is a gauge theory has another, at first glance surprising, consequence:

$$H = p_\mu \dot{x}^\mu - L = -\frac{mc\dot{x}_\mu \dot{x}^\mu}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} + mc\sqrt{\dot{x}_\mu \dot{x}^\mu} = 0 \tag{5.93}$$

→ The (canonical) Hamiltonian vanishes identically

- ¡! This does *not* mean that there is no time-evolution in our system. The Hamiltonian Eq. (5.93) describes the "parameter evolution" in λ – which, as we have seen, can be modified arbitrarily by gauge transformations; λ has therefore *no physical significance*.

  This phenomenon will become important for the interpretation of the Einstein field equations in GENERAL RELATIVITY.

- If one fixes a gauge, the Hamiltonian that describes evolution in this parameter is non-zero in general. E.g., for the "static gauge" $\lambda = t = x^0/c$ one finds the Hamiltonian Eq. (5.49) which coincides with the relativistic energy of the particle.

↓Lecture 12 [16.01.24]

6. Relativistic Field Theories I: Electrodynamics

6.1. A primer on classical field theories

We start with a general discussion of classical field theories on Minkowski space; Maxwell's electrodynamics is the prime example for such theories (→ *next section*).

Details: Chapter 1 of my QFT script [19]

6.1.1. Remember: Classical mechanics of "points"

With "points" we mean a discrete set of degrees of freedom.

1 | $\triangleleft$ Degrees of freedom q_i labeled by $i = 1, \dots, N$

2 | Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t) = T - V$
We write q for $\{q_i\} = \{q_1, \dots, q_N\}$. T is the kinetic, V the potential energy.

3 | Action $S[q] = \int \mathrm{d}t\; L(q(t), \dot{q}(t), t) \;\in \mathbb{R}$
This is a *functional* of trajectories $q = q(t)$.

4 | Hamilton's principle of least action:

$$\frac{\delta S[q]}{\delta q} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \delta S = \int \mathrm{d}t\, \delta L \stackrel{!}{=} 0 \tag{6.1}$$

δ denotes functional derivatives/variations.

5 | Euler-Lagrange equations ($i = 1, \dots, N$):

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0 \tag{6.2}$$

6.1.2. Analogous: Lagrangian Field Theory

Now we consider a *continuous* set of degrees of freedom:

6 | $\triangleleft$ One or more fields $\phi(x)$ on spacetime $x \in \mathbb{R}^{1,3}$ with derivatives $\partial_\mu \phi(x)$
If there are multiple fields we label them by indices: $\phi_k(x)$.
In the following we suppress these indices for the sake of simplicity.

7 | ⁂ *Lagrangian density* $\mathcal{L}(\phi, \partial\phi, x)$
Most general form: $\mathcal{L}(\{\phi_k\}, \{\partial_\mu \phi_k\}, \{x^\mu\})$. (No explicit x^μ-dependence in the following!)
→ Lagrangian $L = \int_{\text{Space}} \mathrm{d}^3x\; \mathcal{L}(\phi, \partial\phi)$
(We omit the "density" in the following.)

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8 | Action:

$$S[\phi] = \int \mathrm{d}t L = \int \mathrm{d}t\, \mathrm{d}^3x\ \mathcal{L}(\phi, \partial\phi) = \frac{1}{c}\int_{\text{Spacetime}} \mathrm{d}^4x\ \mathcal{L}(\phi, \partial\phi) \tag{6.3}$$

$S[\phi]$ is a functional of "field trajectories" in $\mathbb{R}^{1,3}$.

9 | Action principle:

The classical field evolutions of the system extremize the action:

$$\delta S[\phi] \stackrel{!}{=} 0 \tag{6.4}$$

This variation can be evaluated along the same lines as for the classical mechanics of points:

$$0 \stackrel{!}{=} \delta S[\phi] = \int \mathrm{d}^4x\, \delta\mathcal{L} \tag{6.5a}$$

$$= \int \mathrm{d}^4x \left\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\right\} \tag{6.5b}$$

Add zero and use $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$

$$= \int \mathrm{d}^4x \left\{\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)\right\} \tag{6.5c}$$

Gauss theorem

$$= \int_{\text{Boundary}} \mathrm{d}\sigma_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \underbrace{\delta\phi}_{=0} + \int \mathrm{d}^4x \underbrace{\left\{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right\}}_{=0}\delta\phi \tag{6.5d}$$

- Note that ϕ is fixed on the boundary and therefore $\delta\phi = 0$.
- The second term vanishes because the integral must vanish for arbitrary variations $\delta\phi$.

10 | Euler-Lagrange equations (one for each field):

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0} \tag{6.6}$$

- Note the *Einstein summation* over repeated indices.
- These equations are manifestly Lorentz covariant if $\mathcal{L}$ is a Lorentz scalar; such field theories are called ⁂ *relativistic field theories*.
- If there are multiple fields ϕ_k, there is one Euler-Lagrange equation per field (it is straightforward to generalize the derivation above).

11 | Hamiltonian formalism:

Just like for the mechanics of points, we can define:

$$\boxed{\pi := \frac{\partial\mathcal{L}}{\partial\dot\phi} \qquad ⁂\ \textit{Momentum density}\text{ conjugate to }\phi} \tag{6.7}$$

Like $\phi(x)$, the momentum is a *field*: $\pi(x)$. Here it is $\dot\phi(x) \equiv \partial_0\phi(x)$.

→

$$\boxed{\mathcal{H}(\pi, \phi, \nabla\phi) := \pi\dot\phi - \mathcal{L}(\phi, \partial\phi) \qquad ⁂\ \textit{Hamiltonian density}} \tag{6.8}$$

- Here $\dot{\phi}$ is to be expressed as a function of the conjugate momentum via Eq. (6.7).
- The argument $\partial\phi$ of $\mathcal{L}$ is short for $\{\partial_\mu \phi\}$ or $\{\nabla\phi, \dot{\phi}\}$.

→

$$\boxed{H := \int \mathrm{d}^3x\, \mathcal{H} \qquad ⁂\ \textit{Hamiltonian}} \tag{6.9}$$

For given fields $\pi(x)$ and $\phi(x)$, H is a (potentially constant) function of time. By contrast, the Hamiltonian *density* $\mathcal{H}$ is a function of space $\vec{x}$ and time t.

6.2. Electrodynamics: Covariant formulation and Lagrange function

We now want to reformulate Maxwell's electrodyamics in this formalism, i.e., we want to find a Lagrangian density (and an associated action) such that the Euler-Lagrange equations are the Maxwell equations.

1 | Remember:

i | ↓ *Maxwell equations* (in cgs units):

$$\begin{aligned}
\text{Magnetic Gauss's law:} &\quad \nabla\cdot\vec{B} = 0 && (6.10a)\\
\text{Maxwell-Faraday law:} &\quad \nabla\times\vec{E} + \tfrac{1}{c}\partial_t\vec{B} = 0 && (6.10b)\\
\text{Electric Gauss's law:} &\quad \nabla\cdot\vec{E} = 4\pi\rho && (6.10c)\\
\text{Ampère's law:} &\quad \nabla\times\vec{B} - \tfrac{1}{c}\partial_t\vec{E} = \tfrac{4\pi}{c}\vec{j} && (6.10d)
\end{aligned}$$

with charge density $\rho(x)$ and current density $\vec{j}(x)$ that satisfy the ⁂ *continuity equation*

$$\boxed{\partial_t\rho + \nabla\cdot\vec{j} = 0\,.} \tag{6.11}$$

This follows from the two inhomogeneous Maxwell equations Eqs. (6.10c) and (6.10d). Note that here ρ and $\vec{j}$ are external fields and not dynamic degrees of freedom. The statement is therefore that only for external fields that satisfy Eq. (6.11) the Maxwell equations yield solutions for $\vec{E}$ and $\vec{B}$.

ii | Homogeneous Maxwell equations (HME) Eq. (6.10a) & Eq. (6.10b)

$\overset{\circ}{\rightarrow}$ $\exists$ "Scalar" potential φ and "Vector" potential $\vec{A}$:

$$\vec{E} = -\nabla\varphi - \tfrac{1}{c}\partial_t\vec{A} \quad \text{and} \quad \vec{B} = \nabla\times\vec{A} \tag{6.12}$$

- Constraining the fields $\vec{E}$ and $\vec{B}$ to this form satisfies the *homogeneous Maxwell equations* Eqs. (6.10a) and (6.10b) automatically.
- Because of the two homogeneous Maxwell equations, the *six* fields $\{E_x, E_y, E_z, B_x, B_y, B_z\}$ are not independent so that all degrees of freedom can be encoded in the *four* fields $\{\varphi, A_x, A_y, A_z\}$. This suggests a reformulation of Maxwell's theory in terms of these "potentials".

iii | Gauge transformation:

$\sphericalangle$ Arbitrary function $\lambda : \mathbb{R}^{1,3} \to \mathbb{R}$ and define

$$\vec{A}' := \vec{A} + \nabla\lambda \quad \text{and} \quad \varphi' := \varphi - \tfrac{1}{c}\partial_t\lambda \tag{6.13}$$

This transformation of fields is called a ⁂ *gauge transformation* (→ *below*).

$\overset{\circ}{\to}$ $\vec{E} = \vec{E}'$ and $\vec{B} = \vec{B}'$

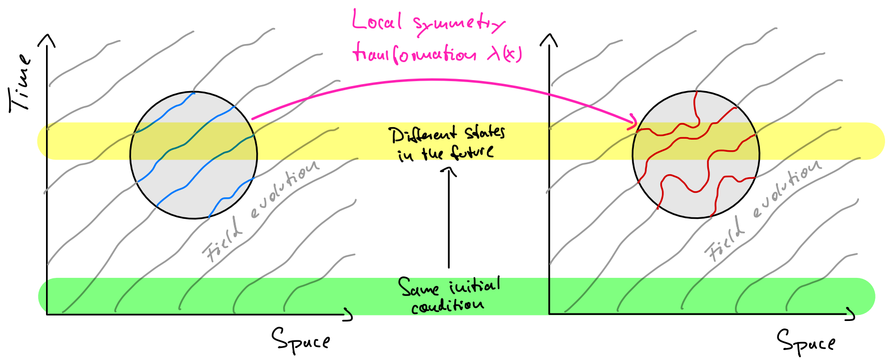
$\to$ The potentials φ and $\vec{A}$ are not unique.

iv | Inhomogeneous Maxwell equations (IME) Eqs. (6.10c) and (6.10d) in terms of the potentials:

$$\text{Eq. (6.10c)} \quad \Leftrightarrow \quad \nabla^2\varphi + \tfrac{1}{c}\partial_t(\nabla\cdot\vec{A}) = -4\pi\rho \tag{6.14a}$$

$$\text{Eq. (6.10d)} \quad \Leftrightarrow \quad \nabla^2\vec{A} - \tfrac{1}{c^2}\partial_t^2\vec{A} = -\tfrac{4\pi}{c}\vec{j} + \nabla\left(\nabla\cdot\vec{A} + \tfrac{1}{c}\partial_t\varphi\right) \tag{6.14b}$$

In this form, electrodynamics is a *gauge theory* because it has a *local* symmetry, namely the transformation Eq. (6.13). Indeed, it is straightforward to show that if $(\varphi, \vec{A})$ is a solution of Eq. (6.14), then $(\varphi', \vec{A}')$ given by Eq. (6.13) is another solution. Since $\lambda(x)$ is arbitrary, one can choose continuously differentiable $\lambda(x)$ that vanish everywhere except for a compact region of spacetime. This makes Eq. (6.13) a *local* symmetry transformation of the PDE system Eq. (6.14); such local symmetries are called ⁂ *gauge transformations*, and models that feature such symmetries are refered to as ⁂ *gauge theories*. The locality of the symmetry has profound implications:



Thus, if we want a deterministic theory (meaning: a theory with predictive power), we *cannot* interpret the gauge fields $(\varphi, \vec{A})$ as physical (= observable) degrees of freedom. Our only choice (to save predictability) is to identify the *equivalence classes* $[(\varphi, \vec{A})]$ of field configurations that are related by (local) gauge transformations as physical states; this is the defining property of a gauge theory. In a nutshell: local symmetries must be interpreted as gauge symmetries and fields related by such transformations are mathematically redundant descriptions of the *same* physical state.

v | Eq. (6.14) Gauge theory $\to$ Fix a gauge:

$$\boxed{\nabla\cdot\vec{A} + \tfrac{1}{c}\partial_t\varphi \overset{!}{=} 0 \quad ⁂\ \textit{Lorenz gauge (LG)}} \tag{6.15}$$

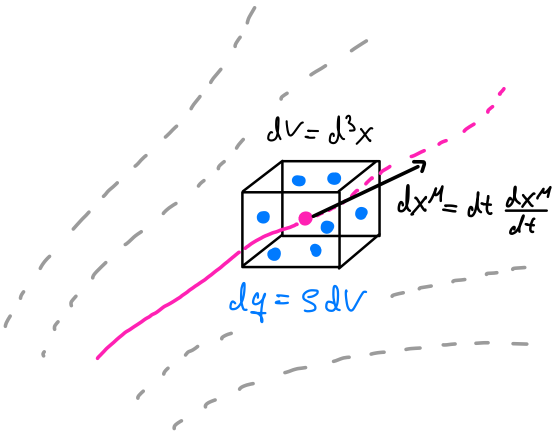
It is straightforward to show that for any given $(\varphi, \vec{A})$ there is a gauge transformation λ such that $(\varphi', \vec{A}')$ satisfies Eq. (6.15).

→

$$\boxed{\begin{aligned} &\text{Eq. (6.10c)} \quad \Leftrightarrow \quad \left(\tfrac{1}{c^2}\partial_t^2 - \nabla^2\right)\varphi = \tfrac{4\pi}{c}\, c\rho & (6.16a)\\ &\text{Eq. (6.10d)} \quad \Leftrightarrow \quad \left(\tfrac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{A} = \tfrac{4\pi}{c}\,\vec{j} & (6.16b)\end{aligned}}$$

- Expressed in potentials in the Lorenz gauge, the inhomogeneous Maxwell equations become a set of four decoupled wave equations.
- We do not have to consider the *homogeneous* Maxwell equations in the gauge field representation because Eq. (6.12) ensures that Eqs. (6.10a) and (6.10b) are automatically satisfied.

2 | <u>Observation:</u> Charge $dq = \rho\, \mathrm{d}^3x$ in volume $\mathrm{d}V = \mathrm{d}^3x$ independent of inertial system:



$$\rho\, \mathrm{d}^3x = \bar{\rho}\, \mathrm{d}^3\bar{x} \quad \Rightarrow \quad \underbrace{\underbrace{\rho\, \mathrm{d}^3x}_{\text{Scalar}}\ \underbrace{\mathrm{d}x^\mu}_{4\text{—vector}}}_{4\text{—vector}} = \rho\, \mathrm{d}^3x\, \mathrm{d}t \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \underbrace{\frac{1}{c}\,\mathrm{d}^4x}_{\substack{\text{Eq. (4.23)}\\ \Downarrow \\ \text{Scalar}}}\ \underbrace{\rho \frac{\mathrm{d}x^\mu}{\mathrm{d}t}}_{\substack{\Downarrow \\ 4\text{—vector}}} \tag{6.17}$$

This suggests that charge and current density are actually components of a Lorentz 4-vector:

$$\boxed{j^\mu := \rho \frac{\mathrm{d}x^\mu}{\mathrm{d}t} = \begin{pmatrix} c\rho \\ \rho\vec{v}\end{pmatrix} = \begin{pmatrix} c\rho \\ \vec{j}\end{pmatrix} \quad ⁂\ \textit{4-current (density)}} \tag{6.18}$$

with ⁂ *charge density* $\rho = \rho(x)$ and ⁂ *current density* $\vec{j} = \vec{j}(x) = \rho(x)\vec{v}(x)$.

- In the argument above, the trajectory $\vec{x}(t)$ in $x^\mu = (ct, \vec{x}(t))$ parametrizes the movement of the infinitesimal volume $\mathrm{d}V = \mathrm{d}^3x$ with charge $\mathrm{d}q = \rho \mathrm{d}V$; the coordinate velocity $\vec{v}(t) = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$ is therefore the velocity of the charge distribution at position $\vec{x}(t)$ at time t: $\vec{v}(x)$. Thus, in general, the current density $\vec{j}(x) = \rho(x)\vec{v}(x)$ depends on position and time via the charge density $\rho(x)$ and the velocity field $\vec{v}(x)$.
- That the charge density ρ is *not* a Lorentz scalar is intuitively clear as it is defined as charge per volume. Volumes, however, are clearly not Lorentz invariant because they are Lorentz contracted. Since the charge (not the charge density!) *is* Lorentz invariant (this is an observational fact), the ratio of charge by volume must change under boosts.

3 | Eq. (6.18) and Eq. (6.16) suggest the compact notation

$$\boxed{\left.\begin{array}{r}\text{Eq. (6.16a)}\\ \text{Eq. (6.16b)}\end{array}\right\} \quad \partial^2 A^\mu = \frac{4\pi}{c} j^\mu \quad \text{(IME in LG)}} \tag{6.19}$$

Remember that $\partial^2 = \Box = \frac{1}{c^2}\partial_t^2 - \nabla^2$.

with

$$\boxed{A^\mu := \begin{pmatrix}\varphi \\ \vec{A}\end{pmatrix} \quad ⁂\ \textit{4-potential}} \tag{6.20}$$

The covariant components of the gauge field are $A_\mu = (\varphi, -\vec{A})$.

The transformation of the 4-potential must be that of a Lorentz 4-vector:

$$\left.\begin{array}{rl}\bar{\partial}^2 = \partial^2 & : \text{Scalar [Eq. (4.36b)]}\\ \bar{j}^\mu = \Lambda^\mu{}_\nu j^\nu & : \text{4-vector [Eq. (6.18)]}\end{array}\right\} \quad \rightarrow \quad \bar{A}^\mu = \Lambda^\mu{}_\nu A^\nu \ : \ \text{4-vector} \tag{6.21}$$

With this transformation, the Maxwell equations in their simple formulation Eq. (6.19) are *manifestly* Lorentz covariant:

$$\partial^2 A^\mu = \frac{4\pi}{c} j^\mu \quad \xrightarrow{K \xrightarrow{R,\vec{v},s,\vec{b}} \bar{K}} \quad \bar{\partial}^2 \bar{A}^\mu = \frac{4\pi}{c} \bar{j}^\mu \tag{6.22}$$

4 | We can now rewrite our previous equations in tensor notation:

i | The Lorenz gauge condition can be compactly written as:

$$\boxed{\partial A \equiv \partial_\mu A^\mu = 0 \quad \text{(Lorenz gauge)}} \tag{6.23}$$

→ The Lorenz gauge is Lorentz invariant

Note: The Lorenz gauge is named after ↑ *Ludvig Lorenz*; by contrast, the Lorentz transformation is named after ↑ *Hendrik Lorentz*. Thus: *The Lorenz gauge* (no "t") *is Lorentz invariant.*

ii | The continuity equation also becomes very simple (and Lorentz covariant):

$$\boxed{\partial j \equiv \partial_\mu j^\mu = 0 \quad \text{(Continuity equation)}} \tag{6.24}$$

iii | The gauge transformation can be written as follows:

$$\boxed{A'^\mu = A^\mu - \partial^\mu \lambda \quad \text{(Gauge transformation)}} \tag{6.25}$$

Recall that $\partial^\mu = (\frac{1}{c}\partial_t, -\nabla)$.

5 | Let us summarize our findings so far:

$$\left.\begin{array}{rr} \text{Maxwell equations}: & \partial^2 A^\mu = \frac{4\pi}{c} j^\mu \\ \text{Lorenz gauge}: & \partial A = 0 \\ \text{Continuity equation}: & \partial j = 0 \end{array}\right\} \xrightarrow{K \xrightarrow{\Lambda} \bar{K}} \left\{\begin{array}{r} \bar\partial^2 \bar{A}^\mu = \frac{4\pi}{c} \bar{j}^\mu \\ \bar\partial \bar{A} = 0 \\ \bar\partial \bar{j} = 0 \end{array}\right. \tag{6.26}$$

→ Electrodynamics satisfies Einstein's principle of Special Relativity **SR**

- In contrast to Newtonian mechanics, electrodynamics was a relativistic theory all along and there was no need to modify it. It's Lorentz covariance was simply not manifest and required a bit of work to unveil.
- The treatment above relies on (1) expressing the Maxwell equations in terms of the gauge fields and (2) choosing a particular gauge (the Lorenz gauge). While this is mathematically legit (and not restrictive), it would be nice to have manifestly Lorentz covariant expressions (1) withou fixing a gauge and (2) in terms of the physically observable fields $\vec{E}$ and $\vec{B}$.

To achieve both goals, we first need a new tensorial quantity:

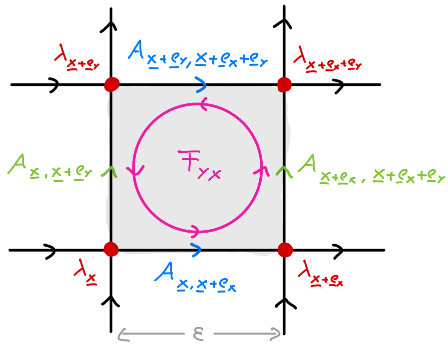
6 | Field strength tensor:

i | Motivation: We are looking for the simplest field that …

- …is gauge-invariant (i.e., has a physical interpretation).
- …is Lorentz covariant (i.e., can be used to construct Lorentz covariant equations).

ii | ∢ Discretized spacetime on a (hypercubic) lattice (here we consider the xy-plane):

- The gauge field A^μ lives on *edges* in μ-direction.
- The gauge transformation λ lives on *vertices* of the lattice.



→ Discretized gauge transformation:

$$A'_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_\mu} = A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_\mu} + \underbrace{\tfrac{1}{\varepsilon}\left(\lambda_{\boldsymbol{x}+\boldsymbol{e}_\mu} - \lambda_{\boldsymbol{x}}\right)}_{\sim\partial_\mu\lambda} \tag{6.27}$$

→ Sums along paths P transform non-trivially only at their "start site" $\boldsymbol{s}$ and "end site" $\boldsymbol{e}$:

$$\sum_{e\in P} A'_e = \sum_{e\in P} A_e + \tfrac{1}{\varepsilon}\left(\lambda_{\boldsymbol{e}} - \lambda_{\boldsymbol{s}}\right) \tag{6.28}$$

Edges e are pairs of adjacent lattice sites, e.g., $e = (\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{e}_x)$ with lattice vector $|\boldsymbol{e}_x| = \varepsilon$.

→ Sums $\sum_{e\in L} A_e$ along *closed loops* L are *gauge-invariant* (because $\boldsymbol{s} = \boldsymbol{e}$)!

→ Smallest gauge-invariant loop (= loop around a single face $f = yx$):

$$F_{yx} := A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_x} + A_{\boldsymbol{x}+\boldsymbol{e}_x,\boldsymbol{x}+\boldsymbol{e}_x+\boldsymbol{e}_y} - A_{\boldsymbol{x}+\boldsymbol{e}_y,\boldsymbol{x}+\boldsymbol{e}_x+\boldsymbol{e}_y} - A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_y} \tag{6.29a}$$

$$= \left(A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_x} - A_{\boldsymbol{x}+\boldsymbol{e}_y,\boldsymbol{x}+\boldsymbol{e}_x+\boldsymbol{e}_y}\right) - \left(A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}_y} - A_{\boldsymbol{x}+\boldsymbol{e}_x,\boldsymbol{x}+\boldsymbol{e}_y+\boldsymbol{e}_x}\right)$$

$$\xrightarrow{\varepsilon\to 0} \partial_y A_x - \partial_x A_y \tag{6.29b}$$

iii | This motivates the definition:

$$\boxed{F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \quad \text{⁂ } \textit{Field strength tensor (FST)}} \tag{6.30a}$$

$$\overset{6.12}{\underset{6.20}{=}} \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}_{\mu\nu} \tag{6.30b}$$

Details: ➔ Problemset 7

→ $F_{\mu\nu}$ is a $(0,2)$ Lorentz tensor

- The FST is gauge-invariant by construction. You can also check this by applying the gauge transformation Eq. (6.25).
- It is easy to see that the FST has the following properties:

$$\text{Antisymmetry:} \quad F^{\mu\nu} = -F^{\nu\mu} \tag{6.31a}$$

$$\text{Tracelessness:} \quad F^{\mu}{}_{\mu} = g_{\mu\nu} F^{\mu\nu} = 0 \tag{6.31b}$$

- ¡! When we write "E_x", we refer to the x-component of the original electric field $\vec{E}$ as it occurs in the Maxwell equations Eq. (6.10). In this context, an expression like E^x does not make sense since $\vec{E}$ is not a 4-vector but the component of a rank-2 tensor.

iv | Using that $\varepsilon^{\mu\nu\alpha\beta}$ is a Lorentz pseudo-tensor [recall Eq. (4.41)], we can define:

$$\boxed{\tilde{F}^{\mu\nu} := \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \quad \text{⁂ } \textit{Dual field strength tensor (DFST)}} \tag{6.32a}$$

$$\overset{3.41}{\underset{6.30}{=}} \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}_{\mu\nu} \tag{6.32b}$$

Details: ➔ Problemset 7

→ $\tilde{F}^{\mu\nu}$ is a $(2,0)$ *pseudo* Lorentz tensor

- The dual field-strength tensor will be useful below.
- $\tilde{F}^{\mu\nu}$ is obtained from $F^{\mu\nu}$ (*contravariant*!) by the substitution $\vec{E} \mapsto \vec{B}$ and $\vec{B} \mapsto -\vec{E}$. [Just like in vacuum the homogeneous Maxwell equations Eqs. (6.10a) and (6.10b) transform into the "inhomogeneous" ones Eqs. (6.10c) and (6.10d)].

7 | Transformation of the electromagnetic field:

The field strength tensor Eq. (6.30) has the useful properties that (1) we know how it transforms under Lorentz transformations, and (2) we know how it relates to the observable fields $\vec{E}$ and $\vec{B}$. Hence we can use it to derive the transformation of the electromagnetic field when transitioning from one inertial system to another.

i | The (contravariant) FST transforms under a Lorentz transformation Λ as follows:

$$\underbrace{\bar{F}^{\mu\nu}(\bar{x})}_{\{\bar{E}_i(\bar{x}),\,\bar{B}_i(\bar{x})\}} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\underbrace{F^{\alpha\beta}(x)}_{\{E_i(x),\,B_i(x)\}} \tag{6.33}$$

Here it is $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$ as usual.

ii | $\triangleleft$ Boost $\Lambda_{\vec{v}}$ [Eq. (4.10)]:

$$\boxed{\begin{aligned}\bar{\vec{E}}(\bar{x}) &\stackrel{\circ}{=} \gamma\left[\vec{E}(x) + \tfrac{1}{c}\vec{v}\times\vec{B}(x)\right] - (\gamma-1)\frac{\vec{v}\cdot\vec{E}(x)}{v^2}\vec{v} \qquad (6.34a)\\ \bar{\vec{B}}(\bar{x}) &\stackrel{\circ}{=} \gamma\left[\vec{B}(x) - \tfrac{1}{c}\vec{v}\times\vec{E}(x)\right] - (\gamma-1)\frac{\vec{v}\cdot\vec{B}(x)}{v^2}\vec{v} \qquad (6.34b)\end{aligned}}$$

with $x^{\mu} = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu}\,\bar{x}^{\nu}$.

¡! Note that on the left-hand side the arguments are $\bar{x}$ and on the right-hand side x!

→ Electric and magnetic fields "mix" under boosts!

- Please appreciate what we showed: If you start from Maxwell Eq. (6.10) and perform an arbitrary Lorentz boost $\bar{x}^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$, transforming the derivatives as $\bar{\partial}_{\mu} = \Lambda_{\mu}{}^{\nu}\partial_{\nu}$, you obtain a set of horribly looking PDEs. But if you recombine the equations appropriately, group the terms according to Eq. (6.34) and define the new fields $\bar{\vec{E}}(\bar{x})$, $\bar{\vec{B}}(\bar{x})$, the equations look again like Eq. (6.10), only with bars over coordinates and fields.

  You *could* show this directly, without ever introducing the gauge field A^{μ} and without using the machinery of tensor calculus (this is what Einstein did for a boost in z-direction in his 1905 paper "Zur Elektrodynamik bewegter Körper" [9]); but hopefully you agree that our more advanced route (using the gauge field and tensor calculus) is a more elegant approach.

- Because of our motivation from Einstein's principle of Special Relativity **SR**, we frame our discussion in the terminology of *passive* transformations (= coordinate transformation): The same electromagnetic field that looks like $\vec{E}(x)$, $\vec{B}(x)$ in an inertial system K looks like $\bar{\vec{E}}(\bar{x})$, $\bar{\vec{B}}(\bar{x})$ in another system $\bar{K}$.

  Because we showed that the Maxwell equations satisfy **SR**, they have exactly the same form in $\bar{K}$ as in K. This, however, allows you to interpret the transformation *actively*: If you are given a solution of Maxwell equations $\vec{E}(x)$, $\vec{B}(x)$, then, for any $\vec{v}$, the *new* functions $\bar{\vec{E}}(\bar{x})$, $\bar{\vec{B}}(\bar{x})$ defined by Eq. (6.34) and $x^{\mu} = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu}\,\bar{x}^{\nu}$ are again solutions (in the same coordinates). This shows that the Lorentz group is (part of) the invariance group of the PDE system Eq. (6.10) we call Maxwell equations (just like the Galiei group was an invariance group of Newton's equation, recall Section 1.2).

↓Lecture 13 [23.01.24]

iii | ⊲ Non-relativistic limit:

$$\text{Eq. (6.34)} \xrightarrow{\gamma\approx 1} \begin{cases} \bar{\vec{E}}(\bar{x}) \approx \vec{E}(x) + \frac{1}{c}\vec{v}\times\vec{B}(x) \\ \bar{\vec{B}}(\bar{x}) \approx \vec{B}(x) - \frac{1}{c}\vec{v}\times\vec{E}(x) \end{cases} \tag{6.35}$$

- The interconversion between magnetic and electric fields happens already in linear order of v/c.
- The separation of the electromagnetic field into "electric" and "magnetic" components is observer dependent!
- Example: A charge at rest has a non-zero electric field, but a vanishing magnetic field. The *same* charge as seen from an inertial system in relative motion gives rise to a current that is accompanied by a non-vanishing magnetic field perpendicular to the direction of motion and the electric field. This is a direct consequence of Eq. (6.35): $\bar{\vec{B}}(\bar{x}) \approx -\frac{1}{c}\vec{v}\times\vec{E}(x) \neq \vec{0}$.

iv | ⊲ Special case: Boost Λ_{v_x} in x-direction: Eq. (6.34) $\xrightarrow{\vec{v}=(v_x,0,0)}$

$$\bar{E}_x = E_x\,,\quad \bar{E}_y = \gamma\left[E_y - \tfrac{v}{c}B_z\right],\quad \bar{E}_z = \gamma\left[E_z + \tfrac{v}{c}B_y\right], \tag{6.36a}$$

$$\bar{B}_x = B_x\,,\quad \bar{B}_y = \gamma\left[B_y + \tfrac{v}{c}E_z\right],\quad \bar{B}_z = \gamma\left[B_z - \tfrac{v}{c}E_y\right], \tag{6.36b}$$

(Here the fields in $\bar{K}$ on the left-hand side are functions of $\bar{x}$ whereas the fields in K on the right-hand side are functions of x.)

→

- The field components *parallel* to the boost remain unchanged.
- The *perpendicular* components mix and get enhanced by a Lorentz factor $\gamma > 1$.
- Einstein derived this transformation directly (without using gauge fields and tensor notation) in his 1905 paper "Zur Elektrodynamik bewegter Körper" [9]; you follow this path in ➔ Problemset 7.

v | Lorentz scalars:

The electric and magnetic field components transform in a complicated way under Lorentz transformations. Is it possible to combine them into scalar quantities? Thanks to our knowledge of tensor calculus and the field strength tensor, this question is easy to answer:

a | We can construct a scalar by contracting the FST with itself:

$$\boxed{F^{\mu\nu}F_{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} \mathrel{\hat{=}} 2(\vec{B}^2 - \vec{E}^2)} \tag{6.37}$$

→ If $|\vec{E}| \gtreqless |\vec{B}|$ is true in one IS, it is true in all IS.

b | We can construct a *pseudo* scalar by contracting the FST with the DFST:

$$\boxed{\tilde{F}^{\mu\nu}F_{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}F_{\mu\nu} \mathrel{\hat{=}} -4(\vec{E}\cdot\vec{B})} \tag{6.38}$$

→ If $\vec{E}\perp\vec{B}$ is true in one IS, it is true in all IS.

Some comments:

- Note that $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} \stackrel{\circ}{=} -F^{\mu\nu}F_{\mu\nu}$ (use contraction identities for Levi-Civita symbols to show this, ➔ Problemset 7); i.e., the two quantities above exhaust all elementary gauge-invariant scalar fields that we can construct ($A^\mu A_\mu$ is of course also a scalar, but not a gauge-invariant one).
- The combination of Eq. (6.37) and Eq. (6.38) can be used to infer whether inertial systems exist in which either the electric or magnetic field vanishes. For example: If $\tilde{F}^{\mu\nu}F_{\mu\nu} = 0$ and $F^{\mu\nu}F_{\mu\nu} > 0$, it is possible to find an inertial system where $\vec{E} = 0$ and $\vec{B} \neq 0$ (but not the other way around). If $\tilde{F}^{\mu\nu}F_{\mu\nu} \neq 0$ there is no inertial system in which one of the fields vanishes.

8 | Manifest covariant form of the Maxwell equations:

Using the FST and the DFST, we can write the Maxwell equations manifestly covariant without using the gauge field and/or fixing a gauge (cf. Eq. (6.19)):

i | The equations we look for must be …

- …manifestly covariant ($\to$ tensor equations).
- …linear in the FST or the DFST (the ME are linear in $\vec{E}$ and $\vec{B}$).
- …use one 4-divergence ∂_μ (the ME are first-order PDEs).

$\to \sphericalangle$

$$\partial_\nu \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \varepsilon^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma = 0 \tag{6.39a}$$

$$\partial_\nu F^{\mu\nu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu(\partial A) - \partial^2 A^\mu \tag{6.39b}$$

ii | The homogeneous ME Eqs. (6.10a) and (6.10b) must be *identically* true if the fields are given in terms of gauge fields. Eq. (6.39a) then suggests that the homogeneous ME are:

$$\partial_\nu \tilde{F}^{\mu\nu} = 0 \qquad \text{Homogeneous ME (?)} \tag{6.40}$$

To check this evaluate:

$$\partial_\nu \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} \tag{6.41a}$$

$$= \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\nu F_{\rho\sigma} + \partial_\sigma F_{\nu\rho} + \partial_\rho F_{\sigma\nu}\right) \tag{6.41b}$$

$$= \frac{1}{2}\sum_{\nu<\rho<\sigma}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho}\right) \tag{6.41c}$$

Here we used that the Levi-Civita symbol is invariant under cyclic permutations of (subsets) of indices and that the FST (and the Levi-Civita symbol) is antisymmetric in its indices. Note that for every fixed μ there are $3! = 6$ non-vanishing assignments of indices $\nu\rho\sigma$. However, pairs of terms like $\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = \varepsilon^{\mu\nu\sigma\rho}\partial_\nu F_{\sigma\rho}$ are identical, so that only 3 distinct terms remain. These can be w.l.o.g. written like cyclic permutations as in Eq. (6.41c). Note that for a fixed index μ, the sum contains only one non-vanishing summand.

$\to$

$$\underbrace{\forall_{\nu<\rho<\sigma}:\ \partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0}_{\uparrow\ \textit{Bianchi identity}\ \text{(4 equations)}} \quad \Leftrightarrow \quad \underbrace{\forall_\mu:\ \partial_\nu \tilde{F}^{\mu\nu} = 0}_{\text{(4 equations)}} \tag{6.42}$$

It is straightforward to check by hand, using Eq. (6.30), that the four Bianchi identities correspond to the four homogeneous Maxwell Eqs. (6.10a) and (6.10b). For example:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} \mathrel{\hat{=}} -\nabla\cdot\vec{B} = 0 \quad \Leftrightarrow \quad \text{Eq. (6.10a)} \tag{6.43}$$

Details: Problemset 7

- As shown in Eq. (6.39a), the homogeneous ME are *identities* if the FST is expressed in terms of gauge fields.
- By contrast, if the FST is expressed in terms of physical fields $\vec{E}$ and $\vec{B}$ [as given in Eq. (6.30)], the equation $\partial_\nu \tilde{F}^{\mu\nu} = 0$ becomes a non-trivial constraint on the field configurations.

iii | ◁ Lorenz gauge Eq. (6.23) →

$$\text{Eq. (6.39b)} \quad \Rightarrow \quad \partial_\nu F^{\mu\nu} = -\partial^2 A^\mu \tag{6.44}$$

Compare Eq. (6.19) (inhomogeneous ME in Lorenz gauge):

$$-\partial^2 A^\mu = -\frac{4\pi}{c} j^\mu \tag{6.45}$$

This suggests that the inhomogeneous ME are:

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \qquad \text{Inhomogeneous ME (?)} \tag{6.46}$$

It is straightforward to check by hand that these four equations are equivalent to the four inhomogeneous ME Eqs. (6.10c) and (6.10d) using Eq. (6.30). For example for $\mu = 0$:

$$\partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} \mathrel{\hat{=}} -\nabla\cdot\vec{E} = -\frac{4\pi}{c} j^0 = -4\pi\rho \quad \Leftrightarrow \quad \text{Eq. (6.10c)} \tag{6.47}$$

Details: Problemset 7

- In this form, the continuity equation Eq. (6.24) follows trivially from the antisymmetry of the FST:

$$\partial_\mu j^\mu = -\frac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0 \tag{6.48}$$

- If you express the FST in terms of the gauge field, the inhomogeneous ME read (without fixing a gauge!):

$$\partial^2 A^\mu - \partial^\mu(\partial A) = \frac{4\pi}{c} j^\mu \tag{6.49}$$

  This equation becomes Eq. (6.19) in the Lorenz gauge Eq. (6.23). It is easy to check that this equation is still gauge symmetric under the transformation Eq. (6.25).

iv | In summary, the 8 (=1+3+1+3=4+4) Maxwell equations can be written in the covariant form:

$$\boxed{\begin{aligned} &\text{Homogeneous ME:} && \partial_\nu \tilde{F}^{\mu\nu} = 0 && (6.50a) \\ &\text{Inhomogeneous ME:} && \partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu && (6.50b) \end{aligned}}$$

- ¡! Using Eqs. (6.30) and (6.32), these equations make sense without introducing the gauge field.
- Note that these equations show that under Lorentz transformations the four homogeneous (inhomogeneous) Maxwell equations mix among each other. You show this explicitly in ➔ Problemset 7 for a boost in z-direction.
- In particular, this means that the Maxwell equations written in their conventional form Eq. (6.10) (i.e., as two scalar and two vector equations) remain *not* invariant under Lorentz transformations for each equation separately, rather the magnetic Gauss law mixes with the Maxwell-Faraday law, and the electric Gauss law mixes with Ampère's law. This explains why showing the Lorentz covariance of the PDE system Eq. (6.10) is quite messy and complicated without using the tensor formalism. This is why we say that its Lorentz covariance *is not manifest*. By contrast, the Lorentz covariance of the formulation Eq. (6.50) *is manifest* as these are tensor equations.

9 | Lagrangian formulation:

Our final goal is to make a connection to the formalism introduced in Section 6.1 and obtain the Lorentz covariant Maxwell equations as the Euler-Lagrange equations of some action/Lagrangian:

i | It is convenient to construct the Lagrangian as a function of the gauge fields A^μ because in this formulation the HME are identically satisfied:

$$\partial_\nu \tilde{F}^{\mu\nu} \equiv 0 \quad \Rightarrow \quad \mathcal{L} = \mathcal{L}(A, \partial A) \tag{6.51}$$

→ Only the inhomogeneous ME must follow as Euler-Lagrange equations

Note that the counting matches: We have four fields A^μ and thus four Euler-Lagrange equations – just as we have four IME: $\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu$.

ii | We have the following hints to construct a reasonable Lagrangian density:

- The IME are Lorentz covariant. This can be ensured by a Lagrangian density that is a Lorentz (pseudo) scalar.
- The Maxwell equations are linear (superposition principle!); thus the Lagrangian must be quadratic in the fields.
- The IME are gauge invariant. This can be ensured by a Lagrangian density that is gauge invariant up to a total derivative (here: surface term) which does not affect the equations of motion.

→ Most general form:

$$\mathcal{L}(A, \partial A) = a_1\, F^{\mu\nu} F_{\mu\nu} + a_2 \underbrace{\cancel{\tilde{F}^{\mu\nu} F_{\mu\nu}}}_{\text{Surface term}} + a_3 \underbrace{\cancel{\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu}}}_{\propto F^{\mu\nu} F_{\mu\nu}} + a_4 \underbrace{A_\mu j^\mu}_{\text{Gauge inv. up to surface term}} \tag{6.52}$$

Details: ➔ Problemset 7

- It is straightforward to check that

$$\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} \mathrel{\hat{=}} -F^{\mu\nu} F_{\mu\nu} \tag{6.53}$$

so that we can drop the a_3-term without loss of generality.

- One can also check that

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \tag{6.54a}$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu \partial_\rho A_\sigma + \partial_\nu A_\mu \partial_\sigma A_\rho - \partial_\nu A_\mu \partial_\rho A_\sigma - \partial_\mu A_\nu \partial_\sigma A_\rho) \tag{6.54b}$$

$$= 2\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)(\partial_\rho A_\sigma) \tag{6.54c}$$

$$= \underbrace{2\,\partial_\mu\, \overbrace{\varepsilon^{\mu\nu\rho\sigma}(A_\nu \partial_\rho A_\sigma)}^{\uparrow\text{ Chern-Simons 3-form}}}_{\text{Surface term}} \tag{6.54d}$$

  so that the a_2-term has no effect on the equations of motion and we can drop it as well.

  Note: The a_2-term is known as the ↑ *θ-term* and is special because it is *topological* (it does not "feel" the *geometry* of spacetime). This is easy to see: One does not need a metric tensor to construct it because the contravariant indices of the DFST stem from the Levi-Civita symbol! Despite being a surface term, such terms are important when one quantizes the theory and/or when the gauge theory is non-Abelian (like the SU(3) gauge theory of the strong interaction). Note also that this term is a *pseudo* scalar, i.e., it breaks parity symmetry (which we know electrodynamics does not).

- The a_4-term is *not* gauge invariant. However, the continuity equation ensures that it modifies the Lagrangian only by a surface term under gauge transformations:

$$\tilde{A}_\mu j^\mu = (A_\mu - \partial_\mu\lambda)j^\mu = A_\mu j^\mu - (\partial_\mu\lambda)j^\mu = A_\mu j^\mu - \underbrace{\partial_\mu(\lambda j^\mu)}_{\text{Surface term}} \tag{6.55}$$

  (Here we used the continuity equation $\partial_\mu j^\mu = 0$.)

  Consequently, the equations of motion must be gauge invariant despite the a_4-term.

- It is easy to check that the quadratic Lorentz scalar $A_\mu A^\mu$ is not gauge invariant (not even up to a surface term); thus it is forbidden.

  Note: Coincidentally, it is this term that would give the quantized excitations of the A-field a mass. Thus if you want massive gauge excitations (like the $W^\pm$- and Z-bosons of the weak interaction), you must find a way to smuggle the term $A_\mu A^\mu$ into your Lagrangian. This is what the ↑ *Higgs mechanism* achieves.

iii | Thus we propose the
Lagrangian density for Maxwell theory:

$$\boxed{\mathcal{L} \equiv \mathcal{L}_{\text{Maxwell}}(A, \partial A) = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_\mu j^\mu} \tag{6.56}$$

The prefactors have been chosen such that the Euler-Lagrange equations match the IME (→ *next step*).

iv | Euler-Lagrange equations:

Details: ➔ Problemset 7

There are four ($\mu = 0, 1, 2, 3$) Euler-Lagrange equations:

$$\frac{\partial\mathcal{L}}{\partial A_\mu} - \partial_\nu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0 \tag{6.57}$$

Straightforward calculations yield:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -\frac{1}{c} j^\mu \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \mathrel{\hat{=}} \frac{1}{4\pi} F^{\mu\nu} \tag{6.58}$$

Hence the Euler-Lagrange equations are exactly the inhomogeneous Maxwell equations:

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \tag{6.59}$$

→ Eq. (6.56) is the correct Lagrangian density for Maxwell theory.

10 | Coordinate-free notation:

Remember the coordinate-free concepts introduced in Chapter 3: All tensor fields $T^I{}_J$ are the chart-dependent components of chart-independent objects T (the actual tensor fields). This formalism allows us to reformulate the Maxwell equations in the language of differential geometry, without using coordinates altogether:

i | First, write gauge field

$$A := A_\mu \mathrm{d}x^\mu \tag{6.60}$$

and the field strength coordinate-free:

$$F := F_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \frac{1}{2} F_{\mu\nu}\, \underbrace{[\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu]}_{=:\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \text{ (“wedge product”)}} . \tag{6.61}$$

We say that A is a 1-form and F is a 2-form.

ii | We can evaluate ↑ *exterior derivative* of the gauge field:

$$\mathrm{d}A \stackrel{\text{def}}{=} \mathrm{d}A_\nu \wedge \mathrm{d}x^\nu = \partial_\mu A_\nu\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = F \tag{6.62}$$

The exterior derivative d maps k-forms onto $k+1$-forms.

iii | Now evaluate the exterior derivative of the field strength:

$$\mathrm{d}F \stackrel{\text{def}}{=} \frac{1}{2} \partial_\sigma F_{\mu\nu}\, \mathrm{d}x^\sigma \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \tag{6.63a}$$

$$= \frac{1}{6} \left(\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma}\right) \mathrm{d}x^\sigma \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \tag{6.63b}$$

$$= \frac{1}{2} \sum_{\sigma<\nu<\mu} \left(\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma}\right) \mathrm{d}x^\sigma \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \tag{6.63c}$$

(Here we used the antisymmetry of the wedge product in all factors.)

Thus we find:

$$\mathrm{d}F = 0 \quad \Leftrightarrow \quad \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0 \quad \stackrel{6.42}{\Leftrightarrow} \quad \partial_\nu \tilde{F}^{\mu\nu} = 0 \tag{6.64}$$

If the field strength is expressed in terms of the gauge field, the homogeneous Maxwell equations $\partial_\nu \tilde{F}^{\mu\nu} = 0$ are identities. In the coordinate-free notation of differential geometry, this identity follows from the fact that applying an exterior derivative twice produces the zero field:

$$\mathrm{d}F = \mathrm{dd}A = 0 \quad \text{since} \quad d^2 = 0 \tag{6.65}$$

The relation $\mathrm{d}F = 0$ is known as a ↑ *Bianchi identity*.

iv | Define the linear ↑ *Hodge star operator* (here for a 4-dimensional Minkowski manifold):

$$\star(\mathrm{d}x^\mu) := \frac{1}{3!}\varepsilon^\mu{}_{\nu\rho\sigma}\,(\mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma) \tag{6.66a}$$

$$\star(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) := \frac{1}{2!}\varepsilon^{\mu\nu}{}_{\rho\sigma}\,(\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma) \tag{6.66b}$$

$$\star(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho) := \frac{1}{1!}\varepsilon^{\mu\nu\rho}{}_\sigma\,(\mathrm{d}x^\sigma) \tag{6.66c}$$

Note that the definition makes use of the metric tensor via pulling up/down indices of the Levi-Civita symbols. This implies in particular that any equation that uses the Hodge star depends on the geometry of spacetime (here flat Minkowski space).

v | The dual field-strength tensor (DFST) is the Hodge dual of the field-strength tensor (FST):

$$\star F = \frac{1}{2}F_{\mu\nu}\,\star\,(\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu) \tag{6.67a}$$

$$= \frac{1}{4}F_{\mu\nu}\varepsilon^{\mu\nu}{}_{\rho\sigma}\,(\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma) \tag{6.67b}$$

$$= \frac{1}{2}\tilde{F}_{\rho\sigma}\,(\mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma) \equiv \tilde{F} \tag{6.67c}$$

Beware: The Hodge star $\star$ is not a multiplication symbol (as the notation on the right-hand side might suggest) but a linear operator that acts on the differential form to the right.

vi | The Hodge dual of the exterior derivative of the DFST yields:

$$\star\mathrm{d}(\star F) = \frac{1}{4}\varepsilon^{\mu\nu}{}_{\rho\sigma}\partial_\pi F_{\mu\nu}\,\star\,(\mathrm{d}x^\pi \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma) \tag{6.68a}$$

$$= \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\pi\rho\sigma}{}_\alpha\partial_\pi F^{\mu\nu}\,(\mathrm{d}x^\alpha) \tag{6.68b}$$

$$= \frac{1}{2}(\delta^\pi_\mu\eta_{\nu\alpha} - \delta^\pi_\nu\eta_{\mu\alpha})\partial_\pi F^{\mu\nu}\,(\mathrm{d}x^\alpha) \tag{6.68c}$$

$$= \eta_{\nu\alpha}\partial_\mu F^{\mu\nu}\,(\mathrm{d}x^\alpha) \tag{6.68d}$$

$$\overset{6.50\mathrm{b}}{=} \frac{4\pi}{c}j_\alpha\,(\mathrm{d}x^\alpha) \tag{6.68e}$$

Here we used a contraction identity for Levi-Civita symbols (over the two red pairs of indices).

vii | This motivates the definition of the coordinate-free current:

$$J := \frac{4\pi}{c}j_\mu\,\mathrm{d}x^\mu \tag{6.69}$$

viii | In conclusion, the Maxwell equations can be written without using a coordinate system as:

$$\boxed{\begin{aligned} &\text{Homogeneous ME:} & \mathrm{d}F &= 0 \qquad (6.70a)\\ &\text{Inhomogeneous ME:} & \star d(\star F) &= J \qquad (6.70b)\end{aligned}}$$

- If one uses that $(\star)^2 = +1\cdot\mathbb{1}$ on odd differential forms ($\mathrm{d}(\star F)$ is a 3-form), Eq. (6.70b) can alternatively be written as $d(\star F) = \star J$. If one then defines the current not as a 1-form but as the dual 3-form, $J := \frac{4\pi}{c}j_\mu\,\star\,\mathrm{d}x^\mu$, the inhomogeneous Maxwell equations take their simplest form: $d(\star F) = J$.
- Eq. (6.70) is the most general and elegant formulation of the Maxwell equations. In this form, the equations remain valid even in GENERAL RELATIVITY on curved space times. Then the Minkowski metric used in the definition of the Hodge star $\star$ (to pull the indices of the Levi-Civita symbols up/down) must be replaced by the dynamic, potentially curved metric of GENERAL RELATIVITY.

6.3. Noether theorem and the energy-momentum tensor

In the following, we consider first a generic (classical, relativistic) field theory, and specialize to electrodynamics later. This is to emphasize that most of the results in this chapter are not specific to electrodynamics.

Details: Chapter 1 of my QFT script [19]

1 | $\sphericalangle$ General transformation of field $\phi \mapsto \phi'$:

$$x \mapsto x' = x'(x) \quad \text{and} \quad \phi(x) \mapsto \phi'(x') = \mathcal{F}(\phi(x)) \tag{6.71}$$

Two effects: coordinates and (values of the) field transformed

These are *active transformations* that change physics. $x' = x'(x)$ is *not* a (passive) coordinate transformation; the frame of reference remains fixed in the following!

> **Example 6.1: Homogeneous Lorentz transformations**
>
> The (active) homogeneous Lorentz transformation of a *vector field* A^μ reads
>
> $$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \text{and} \quad A_\nu(x) \mapsto A'_\mu(x') = \underbrace{\Lambda_\mu{}^\nu A_\nu(x)}_{\mathcal{F}(A_\mu(x))} \tag{6.72}$$
>
> whereas the Lorentz transformation of a *scalar field* ϕ reads
>
> $$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \text{and} \quad \phi(x) \mapsto \phi'(x') = \underbrace{\phi(x)}_{\mathcal{F}(\phi(x))} \,. \tag{6.73}$$

2 | $\sphericalangle$ Infinitesimal transformations (IT) ($|w_a| \ll 1$):

$$x'^\mu = x^\mu + w_a\, \delta^a x^\mu(x) \quad \text{and} \quad \phi'(x') = \phi(x) + w_a\, \delta^a \phi(x) \tag{6.74}$$

Here, w_a denotes infinitesimal parameters of the transformation (sum over a implied!) and we label different transformations by the labels a.

> **Example 6.2: Homogeneous Lorentz transformations**
>
> Infinitesimal homogeneous Lorentz transformations take the form (Problemset 4)
>
> $$\Lambda_w = \exp\left(-\frac{i}{2} w_{\alpha\beta}\, \mathcal{J}^{\alpha\beta}\right) \overset{|w_{\alpha\beta}|\ll 1}{\approx} \mathbb{1} - \frac{i}{2} w_{\alpha\beta}\, \mathcal{J}^{\alpha\beta} \tag{6.75}$$
>
> (note that the $a = \alpha\beta$ are labels of generators that are not required to be tensor indices) with generators
>
> $$(\mathcal{J}^{\alpha\beta})^\mu{}_\nu = i\left(\eta^{\alpha\mu}\delta^\beta_\nu - \delta^\alpha_\nu \eta^{\beta\mu}\right) \,. \tag{6.76}$$
>
> With this it follows for the coordinates
>
> $$w_{\alpha\beta}\, \delta^{\alpha\beta} x^\mu = x'^\mu - x^\mu = -\frac{i}{2} w_{\alpha\beta} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu x^\nu = w_{\alpha\beta} \underbrace{\frac{1}{2}\left(\eta^{\alpha\mu}\delta^\beta_\nu - \delta^\alpha_\nu \eta^{\beta\mu}\right) x^\nu}_{\delta^{\alpha\beta} x^\mu} \tag{6.77}$$

so that

$$\delta^{\alpha\beta} x^\mu = \frac{1}{2}\left(\eta^{\alpha\mu} x^\beta - \eta^{\beta\mu} x^\alpha\right) . \tag{6.78}$$

Similar arguments yield $\delta^{\alpha\beta} A^\mu = \frac{1}{2}\left(\eta^{\alpha\mu} A^\beta - \eta^{\beta\mu} A^\alpha\right)$ for a vector field and $\delta^{\alpha\beta}\phi = 0$ for a scalar field.

3 | Generator of IT:

$$\delta_w \phi(x) := \phi'(x) - \phi(x) \equiv -i w^a\, G_a \phi(x) \tag{6.79}$$

With (omit first line and refer to previous equation)

$$\phi'(x') = \phi(x) + w^a\, \delta_a \phi(x) \tag{6.80a}$$
$$= \phi(x') - w^a (\delta_a x^\mu)\partial_\mu \phi(x') + w^a \delta_a \phi(x') + \mathcal{O}(w^2) \tag{6.80b}$$

(Here we replaced x by x' in the last term because this is a $\mathcal{O}(w^2)$ modification.)

it follows (replace x' by x; these are just labels!)

$$\boxed{i G_a \phi = (\delta_a x^\mu)\partial_\mu \phi - \delta_a \phi} \tag{6.81}$$

This function describes the infinitesimal change of the field at the same point.

Example 6.3: Translations

i | $x'^\mu := x^\mu + w^\mu \equiv x^\mu + w^\nu \delta_\nu x^\mu$ with $\delta_\nu x^\mu = \delta^\mu_\nu$

ii | $\delta_\nu \phi = 0$ (This is true for scalar and vector fields.)

iii | $i G_\mu \phi = \delta^\nu_\mu \partial_\nu \phi - 0$ and therefore

$$G_\mu = -i\partial_\mu \equiv P_\mu \tag{6.82}$$

→ The "momentum operator" generates translations.

4 | So far the continuous transformations $\phi \mapsto \phi'$ were arbitrary.

$\triangleleft$ Continuous transformation [with infinitesimal form Eq. (6.74)] which is a

$$\boxed{\text{Symmetry of the action} \quad :\Leftrightarrow \quad S[\phi] = S[\phi']} \tag{6.83}$$

In principle, the action can vary by a surface term – equivalently, the Lagrangian density $\mathcal{L}$ can vary by a 4-divergence $\partial_\mu K^\mu(\phi, x)$ – under the symmetry transformation (because such modifications do not affect the equations of motion). Here we consider for simplicity only the case where no such terms exist and the action is strictly invariant.

Then one can prove (see Chapter 1 of my QFT script [19] or Refs. [1,77]): $\overset{*}{\rightarrow}$

5 | ⁂ *Noether's (first) theorem:*

For solutions ϕ of the equations of motion, the ⁂ *(canonical) (Noether) currents*

$$\boxed{j_a^\mu \stackrel{*}{=} \left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta_\nu^\mu\mathcal{L}\right\}\delta_a x^\nu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_a\phi} \tag{6.84}$$

(associated to the infinitesimal transformations of coordinates $\delta_a x^\nu$ and fields $\delta_a\phi$)

satisfy the continuity equations

$$\boxed{\forall_a: \qquad \partial_\mu j_a^\mu = 0\,.} \tag{6.85}$$

This means there is one conserved current j_a^μ for each generator a of the continuous symmetry.

6 | Conserved charge:

The currents Eq. (6.84) are called "conserved" because they describe the flow of a conserved ...

$$\boxed{Q_a := \int_{\text{Space}} \mathrm{d}^{D-1}x\, j_a^0 \qquad \text{⁂ \textit{(Noether) charge}}} \tag{6.86}$$

There is one conserved charge Q_a for each generator a of the continuous symmetry.

Indeed:

$$\frac{1}{c}\frac{\mathrm{d}Q_a}{\mathrm{d}t} = \int_{\text{Space}} \mathrm{d}^{D-1}x\,\partial_0 j_a^0 \stackrel{6.85}{=} -\int_{\text{Space}} \mathrm{d}^{D-1}x\,\partial_k j_a^k \stackrel{\text{Gauss}}{=} -\int_{\text{Surface}} \mathrm{d}\sigma_k\, j_a^k = 0 \tag{6.87}$$

Here we assume that $j_a^k \equiv 0$ on the spatial boundaries—typically at infinity, i.e., the universe is closed. $k = 1, 2, 3$ denotes the spatial coordinates.

Note 6.1

The current Eq. (6.84) is called canonical current because it is not unique:

$$\tilde{j}_a^\mu := j_a^\mu + \partial_\nu B_a^{\mu\nu} \quad \text{with} \quad B_a^{\mu\nu} = -B_a^{\nu\mu} \quad \text{arbitrary} \quad \Rightarrow \quad \partial_\mu \tilde{j}_a^\mu = 0 \tag{6.88}$$

This is particularly important for the energy-momentum tensor (→ *below*).

6.3.1. Application: The Energy-Momentum Tensor (EMT)

Details: Problemset 7

7 | ⊲ Infinitesimal spacetime translations:

$$x'^\mu = x^\mu + w^\mu \quad \Rightarrow \quad \delta_\nu x^\mu = \delta_\nu^\mu \quad \text{and} \quad \delta_\nu\phi = 0 \tag{6.89}$$

& Translation-invariant action: $S' = S$ (This includes translations in time!)

8 | Conserved currents: Eq. (6.84) →

$$\Theta^\mu{}_\nu := \left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\rho\phi - \delta^\mu_\rho\mathcal{L}\right\}\underbrace{\delta_\nu x^\rho}_{\delta^\rho_\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L} \tag{6.90}$$

Note that the generator index a is in this case a proper Lorentz index ν so that we can pull it up, $\Theta^{\mu\nu} = \eta^{\nu\rho}\Theta^\mu{}_\rho$, and obtain:

⁂ *(Canonical) Energy-Momentum Tensor:*

$$\boxed{\Theta^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}} \tag{6.91}$$

with

$$\boxed{\partial_\mu\Theta^{\mu\nu} = 0 \quad \text{and four conserved charges} \quad P^\nu := \frac{1}{c}\int \mathrm{d}^3x\,\Theta^{0\nu}\,.} \tag{6.92}$$

- Note that these quantities are only conserved for *solutions* of the Euler-Lagrange equations.
- P^ν is a 4-vector (show this!). Note that this is a non-trivial statement because d^3x is not a Lorentz scalar and $\Theta^{0\nu}$ not a 4-vector.
- The prefactor $1/c$ ensures that P^0 has the same dimension as a conventional 4-momentum with $p^0 = E/c$; note that Θ^{00} has the dimension of an *energy density* because $\mathcal{L}$ has this dimension.

9 | Interpretation:

i | *Energy* ($\nu = 0$):

$$cP^0 = \int \mathrm{d}^3x\,\Theta^{00} = \int \mathrm{d}^3x \underbrace{\left\{\frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\phi - \mathcal{L}\right\}}_{\text{Hamiltonian } density} = \underbrace{\int \mathrm{d}^3x\,\mathcal{H}(\phi,\pi)}_{\text{Hamiltonian}} = H \tag{6.93}$$

→ The Hamiltonian is the component of a 4-vector and not Lorentz invariant!
By contrast, the Lagrangian *is* Lorentz invariant (for relativistic field theories).

ii | *Kinetic momentum* ($\nu = i$):

$$P^i = \int \mathrm{d}^3x\,\Theta^{0i} = \int \mathrm{d}^3x\,\frac{\partial\mathcal{L}}{\partial\dot\phi}(-\partial_i\phi) = -\int \mathrm{d}^3x\,\pi\,\partial_i\phi \tag{6.94}$$

π is the *canonical* momentum conjugate to the field ϕ.

10 | The canonical EMT of electrodynamics:

i | ⊲ Free ($j^\mu = 0$) electromagnetic field: $\mathcal{L}_{\text{em}} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}$

→ Invariant under spacetime translations

Indeed, with $x'^\mu = x^\mu + w^\mu$ and the field transformation $A'_\mu(x) := A_\mu(x-w)$ it is

$$S_{\text{em}}[A'] = \int \mathrm{d}^4x\,\mathcal{L}_{\text{em}}(A'(x),\partial A'(x)) = \int \mathrm{d}^4x\,\mathcal{L}_{\text{em}}(A(x-w),\partial A(x-w)) \tag{6.95a}$$

$$= \int \mathrm{d}^4y\,\mathcal{L}_{\text{em}}(A(y),\partial A(y)) = S_{\text{em}}[A] \tag{6.95b}$$

where we integrate over the full Minkowski spacetime $\mathbb{R}^{1,3}$, substituted $y^\mu = x^\mu - w^\mu$ and used $\mathrm{d}^4x = \mathrm{d}^4y$.

ii | → Canonical EMT conserved: $\partial_\mu \Theta^{\mu\nu}_{\text{em}} = 0$ with

$$\Theta^{\mu\nu}_{\text{em}} = \frac{\partial \mathcal{L}_{\text{em}}}{\partial(\partial_\mu A_\sigma)}\partial^\nu A_\sigma - \eta^{\mu\nu}\mathcal{L}_{\text{em}} \overset{6.58}{=} \frac{1}{4\pi}F^{\sigma\mu}\partial^\nu A_\sigma + \frac{\eta^{\mu\nu}}{16\pi}F_{\sigma\rho}F^{\sigma\rho} \tag{6.96}$$

Note that because the gauge field has multiple components A_μ, there is now an additional summation in the first term over these components (marked indices). This follows directly from a generalization of the proof of Noether's theorem for fields with multiple components.

Details: Problemset 7

iii | Problems:

The canonical EMT $\Theta^{\mu\nu}_{\text{em}}$ has two problematic properties:

- Because of the term $\partial^\nu A_\sigma$, $\Theta^{\mu\nu}_{\text{em}}$ is gauge-dependent!

  This is problematic because it means that we cannot hope to identify physical quantities like the energy density or the momentum density of the electromagnetic field with (the components) of this tensor.

- The canonical EMT is non-symmetric: $\Theta^{\mu\nu}_{\text{em}} \neq \Theta^{\nu\mu}_{\text{em}}$!

  In GENERAL RELATIVITY, we will find that the right-hand side of the → *Einstein field equations* (which determine how spacetime curves and evolves)

  $$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{6.97}$$

  is given by the → *Hilbert energy-momentum tensor*

  $$T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\delta(\mathcal{L}_{\text{Matter}})}{\delta g_{\mu\nu}} \tag{6.98}$$

  where $\mathcal{L}_{\text{Matter}}$ describes the Lagrangian density of all fields in the universe (except the metric tensor field). For example, $\mathcal{L}_{\text{Matter}}$ contains the Maxwell Lagrangian $\mathcal{L}_{\text{em}}$ ("matter" here includes every degree of freedom that has energy & momentum, i.e., also electromagnetic radiation).

  Note that $T^{\mu\nu}$ is *symmetric* because the metric $g_{\mu\nu}$ is. Hence it cannot be identified with the canonical EMT $\Theta^{\mu\nu}$ in general (here for the example of Maxwell theory).

¡! These problems are not specific to electrodynamics but typically affect all theories that are gauge theories and/or include non-scalar fields.

→ How to solve these issues?

6.3.2. The Belinfante-Rosenfeld energy-momentum tensor (BRT)

We consider again first a generic field theory, and specialize to electrodynamics later.

Details: Problemset 7

11 | Remember (Note 6.1) that the canonical EMT is not the only conserved EMT because

$$\tilde{\Theta}^{\mu\nu} := \Theta^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} \quad \text{with} \quad K^{\rho\mu\nu} = -K^{\mu\rho\nu} \tag{6.99}$$

yields another EMT $\tilde{\Theta}^{\mu\nu}$ for any suitable tensor $K^{\rho\mu\nu}$.

→ *Idea:* Find $K^{\rho\mu\nu}$ such that $\tilde{\Theta}^{\mu\nu} = \tilde{\Theta}^{\nu\mu}$ is symmetric (and hopefully gauge-invariant).

12 | Let us assume that our theory is also invariant under homogeneous Lorentz transformations (in addition to the spacetime translations needed for the conservation of the EMT).

$\sphericalangle$ Generators of homogeneous LTs for coordinates:

$$\text{Eq. (6.78)} \to \quad \delta^{\alpha\beta} x^\mu = \frac{1}{2}\left(\eta^{\alpha\mu} x^\beta - \eta^{\beta\mu} x^\alpha\right) . \tag{6.100}$$

Assume that fields transform as $\delta^{\alpha\beta}\phi$.

For the following arguments, we do not need to fix whether our fields transform as scalar, vector, or even → *spinor fields*.

Eq. (6.84) & Eq. (6.91) & Eq. (6.100) →

Noether currents for homogeneous LTs:

$$\boxed{L^{\mu\alpha\beta} \mathrel{\hat{=}} \frac{1}{2}\left(\Theta^{\mu\alpha} x^\beta - \Theta^{\mu\beta} x^\alpha\right) + \frac{1}{2} S^{\mu\alpha\beta}} \tag{6.101}$$

with

$$\boxed{\text{⁂ } \textit{Spin current}: \quad S^{\mu\alpha\beta} := -2 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta^{\alpha\beta}\phi} \tag{6.102}$$

which satisfies $S^{\mu\alpha\beta} = -S^{\mu\beta\alpha}$.

(This follows because $\delta^{\alpha\beta}\phi = -\delta^{\beta\alpha}\phi$ as the generators of homogeneous LTs are antisymmetric.)

The continuity equation reads

$$\partial_\mu L^{\mu\alpha\beta} = 0 . \tag{6.103}$$

Because homogeneous LTs describe *rotations* in space and time, the conserved current $L^{\mu\alpha\beta}$ can be identified as ↑ *(canonical) angular momentum current*. The first part in Eq. (6.101) corresponds to the (canonical) *orbital angular momentum* while the second part $S^{\mu\alpha\beta}$ encodes the *intrinsic angular momentum* of the field (= its ↓ *spin*). This immediately explains why for a scalar field with $\delta^{\alpha\beta}\phi = 0$, the spin current vanishes $S^{\mu\alpha\beta} = 0$.

13 | Eq. (6.92) & Eq. (6.103) →

$$\boxed{\partial_\mu S^{\mu\alpha\beta} \mathrel{\hat{=}} \Theta^{\alpha\beta} - \Theta^{\beta\alpha}} \tag{6.104}$$

This means that a non-vanishing divergence in the spin current is responsible for the "non-symmetry" of the canonical EMT!

14 | Now define

$$K^{\rho\mu\nu} := -\frac{1}{2}\left(S^{\mu\nu\rho} + S^{\nu\mu\rho} - S^{\rho\nu\mu}\right) \tag{6.105}$$

$\to K^{\rho\mu\nu} = -K^{\mu\rho\nu}$ (This follows from $S^{\mu\alpha\beta} = -S^{\mu\beta\alpha}$.)

With this we can finally define the …

⁂ *Belinfante-Rosenfeld energy-momentum tensor (BRT)*:

$$\boxed{T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} := \Theta^{\mu\nu} - \tfrac{1}{2}\partial_\rho\left(S^{\mu\nu\rho} + S^{\nu\mu\rho} - S^{\rho\nu\mu}\right)} \tag{6.106}$$

15 | It remains to be shown that $T^{\mu\nu}$ is always symmetric:

$$T^{\mu\nu} - T^{\nu\mu} \overset{6.104}{=} 0 \quad ☺ \tag{6.107}$$

It can be rigorously shown that the BRT is identical to the Hilbert EMT that shows up in GENERAL RELATIVITY as the source of gravity [78]. This is why the BRT gets its own symbol $T^{\mu\nu}$.

↓**Lecture 14** [30.01.24]

16 | The BRT of electrodynamics:

Details: ➔ Problemset 7

i | Using $\mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$ and the transformation of a vector field (= spin-1)

$$\delta^{\alpha\beta} A_\mu = \frac{1}{2}\left(\delta^\alpha_\mu A^\beta - \delta^\beta_\mu A^\alpha\right) \tag{6.108}$$

in Eq. (6.102) yields the spin current:

$$S^{\mu\alpha\beta}_{\text{em}} \stackrel{\circ}{=} \frac{1}{4\pi}\left(F^{\mu\alpha} A^\beta - F^{\mu\beta} A^\alpha\right) \tag{6.109}$$

ii | Eq. (6.96) & Eq. (6.106) & Eq. (6.109) →

$$\boxed{\begin{aligned} T^{\mu\nu}_{\text{em}} &\stackrel{\circ}{=} \frac{1}{4\pi} F^\mu{}_\rho F^{\rho\nu} - \eta^{\mu\nu}\mathcal{L}_{\text{em}} && (6.110\text{a}) \\ &= \frac{1}{4\pi}\left[F^\mu{}_\rho F^{\rho\nu} + \frac{\eta^{\mu\nu}}{4} F^{\rho\sigma} F_{\rho\sigma}\right] && (6.110\text{b}) \\ &\stackrel{\circ}{=} \left(\begin{array}{c|c} \mathcal{E} & c\vec{\Pi}^T \\ \hline c\vec{\Pi} & \mathbf{\Sigma} \end{array}\right)_{\mu\nu} && (6.110\text{c}) \end{aligned}}$$

To show this you have to use the Maxwell equations in vacuum: $\partial_\nu F^\nu{}_\mu = 0$.

Components:

$$\text{Energy density:} \quad \mathcal{E} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2) \tag{6.111a}$$

$$\text{Momentum density:} \quad \vec{\Pi} = \frac{1}{4\pi c}(\vec{E} \times \vec{B}) \tag{6.111b}$$

$$\text{⁂ \textit{Maxwell stress tensor}:} \quad \mathbf{\Sigma}_{ij} = \frac{1}{4\pi}\left[\frac{\delta_{ij}}{2}(\vec{E}^2 + \vec{B}^2) - E_i E_j - B_i B_j\right] \tag{6.111c}$$

¡! Convince yourself that $T^{\mu\nu}_{\text{em}}$ is *symmetric* and *gauge invariant*. Note that we did not construct it to be gauge invariant, only to be symmetric! We got this as a bonus.

iii | The conservation $\partial_\mu T^{\mu\nu} = 0$ of the BRT implies the following physical interpretations:

- $\nu = 0$:

$$\partial_\mu T^{\mu 0} = \frac{1}{c}\frac{\partial\mathcal{E}}{\partial t} + c\nabla\cdot\vec{\Pi} = 0 \tag{6.112}$$

→ ↓ *Poynting's theorem* (in vacuum)

$$\boxed{\frac{\partial\mathcal{E}}{\partial t} + \nabla\cdot\vec{S} = 0} \tag{6.113}$$

with

$$\boxed{\downarrow \textit{Poynting vector}: \quad \vec{S} = c^2\vec{\Pi} = \frac{c}{4\pi}(\vec{E}\times\vec{B})} \tag{6.114}$$

Eq. (6.113) → Poynting vector = Energy current density

This is simply the formal statement of *energy conservation* for the free electromagnetic field. As energy is the Noether charge for translations in time, it is of course no coincidence that the Poynting theorem follows from the time-component $\nu = 0$.

- $\nu = i$:

$$\partial_\mu T^{\mu i} = \frac{\partial \Pi_i}{\partial t} + \partial_k \mathbf{\Sigma}_{ki} = 0 \tag{6.115}$$

→ Conservation of momentum with ...

  - Π_i: i-momentum density
  - $\mathbf{\Sigma}_{ki}$: i-momentum current density

→ Maxwell stress tensor = Momentum current density

Note that are three momentum densities and corresponding current densities because there are three spatial momenta: $i = x, y, z$.

iv | Some final remarks:

- With the symmetric BRT one can define a gauge-invariant and conserved *angular momentum tensor*

$$M^{\rho\mu\nu} := T^{\rho\mu}x^\nu - T^{\rho\nu}x^\mu \tag{6.116}$$

with $\partial_\rho M^{\rho\mu\nu} = 0$ (show this!). The conserved Noether charges are

$$J^{\mu\nu} := \frac{1}{c}\int \mathrm{d}^3x\, M^{0\mu\nu} = \frac{1}{c}\int \mathrm{d}^3x\, \left(T^{0\mu}x^\nu - T^{0\nu}x^\mu\right) \tag{6.117}$$

which encodes the *total angular momentum* of the field. Indeed, for the spatial components one finds

$$J_{ij} := \int \mathrm{d}^3x\, \left(\Pi_i x_j - \Pi_j x_i\right)\,. \tag{6.118}$$

Since Π_i is the momentum density, the three components $J_x \equiv J_{32}$, $J_y \equiv J_{13}$ and $J_z \equiv J_{21}$ can be identified as the total angular momentum $\vec{J}$ of the field.

- If the electric current j^μ does *not* vanish (i.e., the field is not in vacuum), the BRT derived above is no longer conserved. Rather one finds

$$\partial_\mu T^{\mu\nu}_{\text{em}} = -\frac{1}{c}F^{\nu\rho}j_\rho \tag{6.119}$$

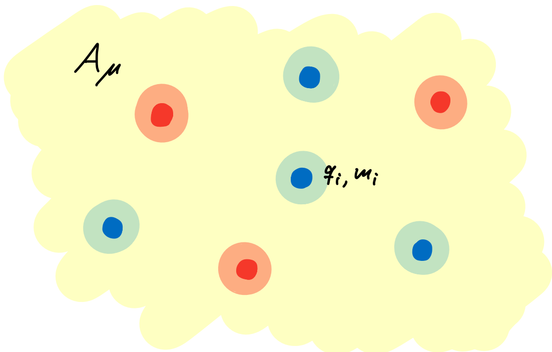
which can be identified as the *Lorentz force density*. This is perfectly reasonable as an external (non-dynamic) current j^μ breaks the translation symmetry of the system in space and time on which the conservation of the BRT relies. Physically, the electromagnetic field is no longer a closed system because it can exchange momentum and energy with the charges described by j^μ. Only if one describes the charges as dynamic degrees of freedom (→ *next section*) and considers the total BRT

$$T^{\mu\nu} = T^{\mu\nu}_{\text{em}} + T^{\mu\nu}_{\text{charges}} \tag{6.120}$$

one would recover the conservation $\partial_\mu T^{\mu\nu} = 0$; this is then a statement about total energy and momentum conservation, including the energy and momentum of the charges.

6.4. Charged point particles in an electromagnetic field

1 | $\sphericalangle$ N charged point particles with charge q_i and mass m_i in a EM field A_μ:



Eq. (6.56) & Eq. (5.41) → Relativistic action of the complete system:

$$\boxed{S[\{x_k\},A] = \underbrace{\int \mathrm{d}^4x\,\Big[\underbrace{-\frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}}_{\substack{S_{\mathrm{em}}[A]\\ \text{EM field}}}\;\underbrace{-\frac{1}{c^2}A_\mu j^\mu}_{\substack{S_{\mathrm{c}}[\{x_k\},A]\\ \text{Coupling}}}\Big]}_{S_j[A]:\ \text{EM field with static current } j^\mu}\overbrace{-\sum_{i=1}^{N}\underbrace{m_i c\int \mathrm{d}s_i}_{\substack{S_{\mathrm{p}}[x_i]\\ \text{Particle } i}}}^{S_A[\{x_k\}]:\ N\text{ particles in static field } A_\mu}} \tag{6.121}$$

Note that the Lagrangian is a Lorentz scalar! $S[\{x_k\},A]$ is short for $S[x_1,\dots,x_N,A]$.

with current density

$$\boxed{j^\mu(x) \overset{6.18}{=} \sum_i \rho_i(x)\frac{\mathrm{d}x_i^\mu}{\mathrm{d}t} = \sum_i \underbrace{q_i\delta(\vec{x}-\vec{x}_i)}_{\text{Point particle}}\frac{\mathrm{d}x_i^\mu}{\mathrm{d}t}\,.} \tag{6.122}$$

2 | $\sphericalangle$ Coupling:

$$S_{\mathrm{c}}[\{x_k\},A] = -\frac{1}{c^2}\int \mathrm{d}^4x\, A_\mu(x)j^\mu(x) \stackrel{\circ}{=} \sum_i \underbrace{\left\{-\frac{q_i}{c}\int A_\mu(ct,\vec{x}_i)\,\mathrm{d}x_i^\mu\right\}}_{S_{\mathrm{c}}[x_i,A]} \tag{6.123}$$

Here we used $\frac{\mathrm{d}x_i^\mu}{\mathrm{d}t}\mathrm{d}t = \mathrm{d}x_i^\mu$; the last integral is therefore a four-dimensional ↓ *line integral* of the 4-vectorfield A_μ along the trajectory of particle i.

3 | Hamilton's principle:

$$\delta S[x_k,A] = 0 \quad\Leftrightarrow\quad \begin{cases} \dfrac{\delta S_{\mathrm{em}}[A]}{\delta A} + \dfrac{\delta S_{\mathrm{c}}[\{x_k\},A]}{\delta A} = \dfrac{\delta S_j[A]}{\delta A} & = 0 \\ \forall_i: \quad \dfrac{\delta S_{\mathrm{c}}[x_i,A]}{\delta x_i} + \dfrac{\delta S_{\mathrm{p}}[x_i]}{\delta x_i} = \dfrac{\delta S_A[\{x_k\}]}{\delta x_i} & = 0 \end{cases} \tag{6.124}$$

4 | ⊲ Gauge field variations δA:

Here we don't have to do anything because we already computed the Euler-Lagrange equations:

$$\boxed{\frac{\delta S_j[A]}{\delta A}=0 \overset{6.56\,\&\,6.58}{\Longleftrightarrow} \partial_\nu F^{\nu\mu} \overset{6.122}{=} \frac{4\pi}{c}\sum_i q_i\,\delta(\vec{x}-\vec{x}_i)\frac{\mathrm{d}x_i^\mu}{\mathrm{d}t}} \tag{6.125}$$

These are the inhomogeneous Maxwell equations with the N point particles as sources of the EM field. Note that this PDE system couples the particle coordinates $\{x_k^\mu\}$ to the EM field A^μ.

5 | ⊲ Particle trajectory variations δx_i:

i | Eqs. (6.121) and (6.123) →

$$S_A[\{x_k\}] = -\sum_i \int \left[m_i c\sqrt{\dot{x}_{i\mu}\dot{x}_i^\mu} + \frac{q_i}{c}A_\mu(x_i)\,\dot{x}_i^\mu\right]\mathrm{d}\lambda \tag{6.126}$$

Note that this action is again reparametrization invariant.

$\overset{\circ}{\to}$ Euler-Lagrange equation for particle i:

$$\frac{\delta S_A[\{x_k\}]}{\delta x_i}=0 \overset{x^\mu\equiv x_i^\mu}{\Longleftrightarrow} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left[\frac{m_i c\,\dot{x}_\mu}{\sqrt{\dot{x}_\mu\dot{x}^\mu}}\right]+\frac{q_i}{c}\left[\dot{A}_\mu(x)-\dot{x}^\nu\frac{\partial A_\nu(x)}{\partial x^\mu}\right]=0 \tag{6.127}$$

ii | Choose proper-time parametrization $\lambda=\tau$:

$$m_i\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}+\frac{q_i}{c}\underbrace{\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau}}_{\frac{\partial A_\mu}{\partial x^\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}}-\frac{\partial A_\nu}{\partial x^\mu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}=0 \tag{6.128}$$

Thus we find as the EOM for particle i:

$$m_i\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}=\frac{q_i}{c}\underbrace{\left(\frac{\partial A_\nu}{\partial x^\mu}-\frac{\partial A_\mu}{\partial x^\nu}\right)}_{F_{\mu\nu}}u^\nu \tag{6.129}$$

Or in the form discussed previously in Chapter 5 (we restore the particle index i):

$$\boxed{\frac{\mathrm{d}p_i^\mu}{\mathrm{d}\tau}=\frac{q_i}{c}F^\mu{}_\nu(x_i)\,u_i^\nu} \tag{6.130}$$

with 4-momentum $p_i^\mu=m_iu_i^\mu$.

¡! The field strength tensor is evaluated at the position of the particle at a given time.

iii | Compare Eqs. (5.6) and (6.130) → 4-force:

$$K^\mu=\begin{pmatrix}\gamma_v\frac{\vec{F}\cdot\vec{v}}{c}\\ \gamma_v\vec{F}\end{pmatrix}=\frac{q_i}{c}F^\mu{}_\nu\,\gamma_v\frac{\mathrm{d}x^\nu}{\mathrm{d}t} \tag{6.131}$$

→ 3-force (we restore the particle index i):

$$\boxed{\vec{F}_i \overset{\circ}{=} q_i \vec{E}_i + \frac{q_i}{c}(\vec{v}_i \times \vec{B}_i) \quad \downarrow \textit{Lorentz force}} \tag{6.132}$$

with $\vec{E}_i = \vec{E}(x_i)$, $\vec{B}_i = \vec{B}(x_i)$ and $\vec{v}_i = \frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t}$.

- This result demonstrates that our concept of the relativistic 3-force introduced in Eq. (5.11) was reasonable: for a force due to an electromagnetic field, it exactly matches the Lorentz force.
- It also demonstrates that the common expression for the Lorentz force is already fully relativistic. However, note that the 3-force determines the change rate of the *relativistic* 3-momentum $\vec{p} = \gamma_v m\vec{v}$, recall Eq. (5.16).

6 | Comments:

- Eqs. (6.125) and (6.130) together are the equations of motion of the composite system, i.e., the EM field and the N particles. Note that the system of differential equations is coupled: The dynamical positions of the particles determine the evolution of the EM field via Eq. (6.125), and the dynamical EM field determines the trajectories of the charged particles via Eq. (6.130).
- This model of N charged particles interacting with and via an electromagnetic field is the culmination of our discussion of *relativistic mechanics* in Chapter 5 and *electrodynamics* in Chapter 6.
- The theory Eqs. (6.125) and (6.130) is fully relativistic as the EOMs are manifestly Lorentz covariant (they are tensor equations).
- Note that this model describes interactions between the N particles not directly via forces (as one would in Newtonian mechanics), but via coupling to the dynamic EM field. Thus a particle can *locally* affect the EM field due to its motion, the EM field then can propagate with the speed of light through space and affect the trajectory of any other particle within the lightcone of the first. There is no instantaneous interaction between the particles!
- One can also consider the $\mu = 0$ component of Eq. (6.130). Then one finds with $p_i^0 = E_i/c$:

$$\frac{\mathrm{d}E_i}{\mathrm{d}t} \overset{\circ}{=} q_i\, \vec{E}_i \cdot \vec{v}_i\,. \tag{6.133}$$

This is just the statement that the change of energy for particle i is given by the distance it travels collinear with the electric field per time. This is no surprise: The Lorentz force Eq. (6.132) tells us that the force due to the magnetic field is always perpendicular to the direction of motion and therefore cannot not perform work on the particle.

7 | Corollary: Single particle in a static electromagnetic field:

i | The action follows from Eq. (6.126) with $N = 1$ as:

$$S_A[x] = \int \mathrm{d}\lambda\, L(x^\mu, \dot{x}^\mu) = -\int \left[mc\sqrt{\dot{x}_\mu \dot{x}^\mu} + \frac{q}{c} A_\mu(x_i)\, \dot{x}^\mu\right] \mathrm{d}\lambda \tag{6.134}$$

where A_μ is a fixed parameter (the static field configuration).

ii | $\sphericalangle$ Parametrization in coordinate time $\lambda = t$:

$$L(\vec{x}, \vec{v}) = -mc^2\sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{c}\vec{A} \cdot \vec{v} - q\,\varphi \tag{6.135}$$

with $A_\mu = (\varphi, -\vec{A})$ (covariant!) and $\dot{\vec{x}} = \vec{v}$.

iii | Canonical momentum:

$$\vec{\pi} := \frac{\partial L}{\partial \vec{v}} \stackrel{\circ}{=} m\gamma_v \vec{v} + \frac{q}{c}\vec{A} \tag{6.136}$$

with *mechanical momentum* $\vec{p} = m\gamma_v\vec{v} \rightarrow$

$$\vec{p} = \vec{\pi} - \frac{q}{c}\vec{A} \tag{6.137}$$

$\vec{p}$: Measurable momentum
→ Mechanical momentum $\vec{p}$ gauge-invariant
→ Canonical momentum $\vec{\pi}$ *not* gauge-invariant

iv | Hamiltonian:

$$H = \vec{\pi}\cdot\vec{v} - L \stackrel{\circ}{=} \underbrace{\overbrace{\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}}^{\text{Mechanical energy}}}_{E} + q\varphi \stackrel{5.26}{=} c\sqrt{\left(\pi - \frac{q}{c}\vec{A}\right)^2 + m^2c^2} + q\varphi \tag{6.138}$$

so that

$$E = H - q\varphi \tag{6.139}$$

E is gauge invariant → H is *not* gauge invariant

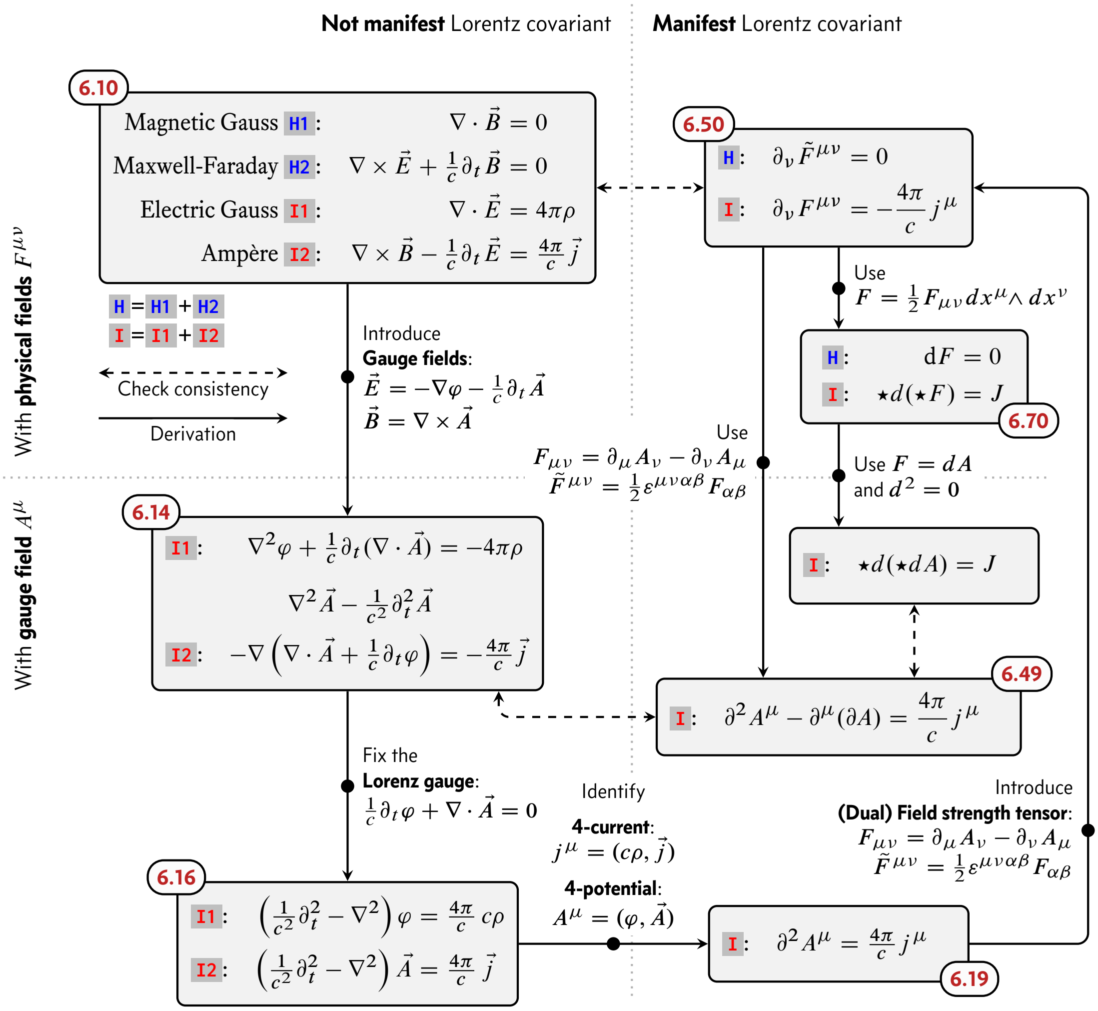
v | Summary:

$$\text{Gauge invariant} \left\{\begin{aligned} E &= H - q\varphi \\ \vec{p} &= \vec{\pi} - \frac{q}{c}\vec{A} \end{aligned}\right. \Leftrightarrow \left.\begin{aligned} E + q\varphi &= H \\ \vec{p} + \frac{q}{c}\vec{A} &= \vec{\pi} \end{aligned}\right\} \text{Gauge dependent} \tag{6.140}$$

For more details on the aspect of the gauge-(in)variance of certain quantities, see Ref. [79]. Note that these subtleties are not specific to a relativistic treatment, they already appear in Newtonian mechanics (only the specific dependency of the Hamiltonian on the mechanical/-canonical momentum and the functional form of the Lagrangian are relativistic).

6.5. Summary: The many faces of Maxwell's equations

Here is a compact overview over the many (physically equivalent) forms of Maxwell's equations that we encountered in this chapter:



7. Relativistic Field Theories II: Relativistic Quantum Mechanics

Reminder

1 | The ↓ *Schrödinger equation* (SE)

$$i\hbar\partial_t\psi(t,\vec{x}) = \hat{H}\psi(t,\vec{x}) \tag{7.1}$$

is a linear field equation with ↓ *Hamilton operator*

$$\hat{H} = \frac{\vec{p}^2}{2m} + V(\vec{x}) = -\frac{\hbar^2}{2m}\Delta + V(\vec{x}) \tag{7.2}$$

and the complex-valued field $\psi : \mathbb{R}^{1,3} \to \mathbb{C}$.

It describes the time evolution of a single quantum particle with mass m in a potential $V(\vec{x})$ that is initially described by the wavefunction $\psi_0(\vec{x}) = \psi(0,\vec{x})$ at $t = 0$.

2 | The wavefunction has the interpretation

$$|\psi(t,\vec{x})|^2 = \langle\text{Probability to find particle at time } t \text{ at position } \vec{x}\rangle \tag{7.3}$$

which necessitates the normalization condition

$$\forall_t: \quad \|\psi(t)\|_2 := \int \mathrm{d}^3x\, |\psi(t,\vec{x})|^2 = 1\,. \tag{7.4}$$

Thus the wavefunction is an element of the Hilbert space $\psi \in L^2 \equiv L^2(\mathbb{R}^3,\mathbb{C})$ of square-integrable functions.

The Hermiticity $\hat{H} = \hat{H}^\dagger$ of the Hamiltonian implies a unitary time evolution and thereby guarantees a conserved norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|_2 = \int \mathrm{d}^3x\, [\psi^*\partial_t\psi + \psi\partial_t\psi^*] \overset{7.1}{=} \frac{1}{i\hbar}\int \mathrm{d}^3x\, \left[\psi^*(\hat{H}\psi) - \psi(\hat{H}\psi)^*\right] \overset{7.6}{=} 0\,, \tag{7.5}$$

where we used that for $\psi,\phi \in L^2$ and a Hermitian Hamiltonian

$$\int \mathrm{d}^3x\, \phi^*(\hat{H}\psi) \overset{\text{def}}{=} \langle\phi|\hat{H}\psi\rangle \overset{\text{def}}{=} \langle\hat{H}^\dagger\phi|\psi\rangle \overset{\text{def}}{=} \int \mathrm{d}^3x\, (\hat{H}^\dagger\phi)^*\psi \overset{\hat{H}=\hat{H}^\dagger}{=} \int \mathrm{d}^3x\, \psi(\hat{H}\phi)^*\,. \tag{7.6}$$

3 | Problem: The SE is Galilei covariant but *not* Lorentz covariant! (recall ➔ Problemset 1)

- The SE is of first order in time but of second order in the spatial derivatives. This asymmetry already suggests that the equation cannot be Lorentz covariant: Time is treated differently than space in (non-relativistic) quantum mechanics.
- We would like quantum mechanics to be described by a Lorentz covariant equation because we subscribed to ← *Einstein's principle of special relativity* **SR** at the beginning of this course: All laws of physics must take the same form in all inertial systems (which are related by Lorentz transformations). This certainly includes quantum mechanics.

  However, **SR** is just a (empirically motivated) *principle*, it is neither a law nor a theorem; there may be conceivable universes in which **SR** simply does not apply to the quantum realm – in which case the Schrödinger Eq. (7.1) would be a perfectly valid model.

  As good physicists, we should seek for empirical evidence to settle the matter …

4 | Evidence:

- First: The Schrödinger equation, published and studied by Erwin Schrödinger in a sequence of papers in 1926 [80–83] (so RELATIVITY was already known at the time), was (and is) a highly successful theory that describes a plethora of microscopic phenomena remarkably well. Examples are the ↓ *double-slit experiment*, ↓ *quantum tunneling effects*, and, of course, the ↓ *spectrum of the hydrogen atom*:

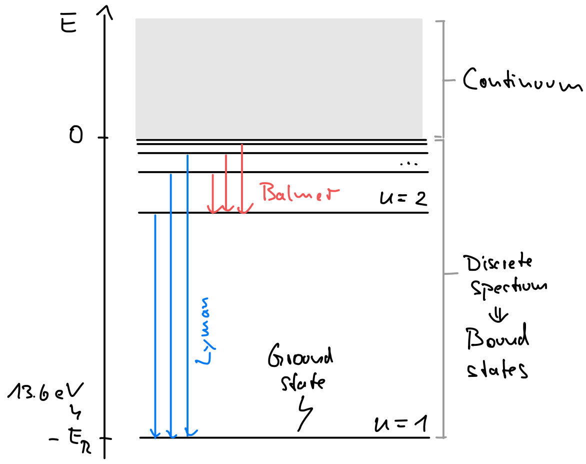
  The Hamilton operator for the relative electron-proton system of the hydrogen atom is

  $$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{e^2}{|\vec{x}|} \tag{7.7}$$

  with reduced mass $\mu = m_e m_p/(m_e + m_p)$. The discrete part of the spectrum of the operator $\hat{H}$ can be computed exactly (E_R is the ↓ *Rydberg energy*),

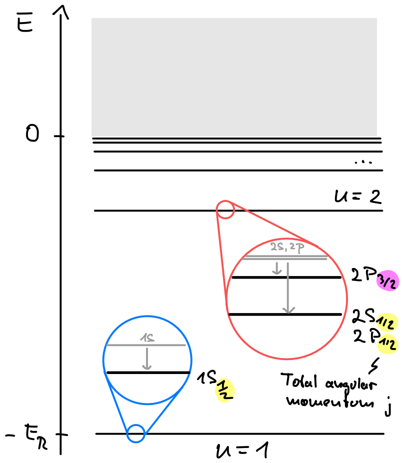
  $$E_n = -\frac{E_R}{n^2} \quad \text{with principal quantum number } n \in \{1, 2, \dots\}, \tag{7.8}$$

  and determines the hydrogen spectrum:

  The transitions between the levels of the hydrogen spectrum can be measured by spectroscopy (↓ *Lyman series* [84], ↓ *Balmer series* [85], …; these observations have been made around 1900). The explanation of these spectral lines by the non-relativistic Schrödinger equation is the crown jewel of quantum mechanics, and one of the most remarkable advances of 20th century physics.

- However, it's not all sunshine and roses. It was already known at the end of the 19th century (due to advances in spectroscopy [86]) that the spectral lines of various atomic species (including hydrogen) had a ↓ *fine-structure*. Expressed in terms of the energy levels of the hydrogen atom, this means that some of the degenerate eigenstates of Eq. (7.7) are actually *not exactly* degenerate:



Note that this was known to Schrödinger when he published his equation in 1926; he writes in Ref. [83] (p. 132–133):

> *Im Anschluß an die zuletzt erwähnten physikalischen Probleme, [..], möchte ich nun doch die vermutliche relativistisch-magnetische Verallgemeinerung der Grundgleichungen [..] hier ganz kurz mitteilen, wenn ich es auch vorerst nur fur das Einelektronenproblem und nur mit der allergrößten Reserve tun kann. Letzteres aus zwei Grunden. Erstens beruht die Verallgemeinerung vorläufig auf rein formaler Analogie. Zweitens führt sie, wie schon in der ersten Mitteilung erwähnt wurde, im Falle des Keplerproblems zwar formal auf die Sommerfeldsche Feinstrukturformel und zwar mit „halbzahligem" Azimutal- und Radialquant, was heute allgemein als korrekt angesehen wird; allein es fehlt noch die zur Herstellung numerisch richtiger Aufspaltungsbilder der Wasserstofflinien notwendige Ergänzung, die im Bohrschen Bilde durch den Goudsmit-Uhlenbeckschen Elektronendrall geliefert wird.*

Note that Schrödinger was very much aware that his equation lacked Lorentz covariance and viewed (and constructed) it as a *non-relativistic approximation* of a truly "relativistic quantum mechanics" (which he didn't know how to formulate consistently).

He also makes this clear in the introduction of Ref. [82] (p. 439):

> *Wesentlich größeres Interesse wird natürlich die (hier noch nicht darchgeführte) Anwendung auf den Zeemaneffekt bieten. Diese erscheint mir unlöslich geknüpft an eine korrekte Formulierung des relativistischen Problems in der Sprache der Wellenmechanik, weil bei vierdimensionaler Formulierung das Vektorpotential von selbst dem skalaren ebenbürtig an die Seite tritt. Schon in der ersten Mitteilung wurde erwähnt, daß das relativistische Wasserstoffatom sich zwar ohne weiteres behandeln läßt, aber zu "halbzahligen" Azimutalquanten, also zu einem Widerspruch mit der Erfahrung führt. Es mußte also noch "etwas fehlen". Seither habe ich [..] gelernt, was fehlt: in der Sprache der Elektronenbahnentheorie der Drehimpuls des Elektrons um seine Achse, der ihm ein magnetisches Moment verleiht.*

- We can also make a back-of-the-envelope calculation to estimate whether relativistic effects could be the root cause for the discrepancy between the non-relativistic Schrödinger equation and the observed fine-structure:

  In a *classical* approximation, kinetic and potential energy are of the same order:

$$\text{Kinetic energy}\quad \frac{1}{2}mv^2 \sim \frac{e^2}{r} \quad \text{Potential energy}\,. \tag{7.9}$$

  Because the system is quantum, momentum and position obey the ↓ *Heisenberg uncertainty relation* $\Delta p \Delta r \sim \hbar$. In the energy eigenstates of an interacting quantum system (like an atom) we typically have $\Delta p \sim p$ and $\Delta r \sim r$, and in our semi-classical approximation it is

$p \sim mv$, so that

$$v \sim \frac{e^2}{mvr} \sim \frac{e^2}{\hbar c} c = \alpha c = \text{Fine-structure constant} \times c \approx \frac{c}{137}\,. \tag{7.10}$$

The semi-classical velocity of the electron v is therefore much smaller than the speed of light c; this explains why the non-relativistic Schrödinger equation is so successful (and your course non non-relativistic quantum mechanics is no waste of time). However, the observed fine-structure splitting of spectral lines is indeed very small, so it is reasonable that relativistic effects can have small but measurable effects in atomic physics.

The situation is therefore similar to that of Newtonian mechanics before we made it relativistic: We have a very successful Galilei covariant theory that, however, shows signs of being the low-velocity/energy approximation of another, presumably relativistic theory.

(Note that *historically* the situation is very different, though: While Newtonian mechanics, born in the 17th century, had to wait more than 200 years to be "made relativistic", the development of relativistic quantum mechanics was very fast: Non-relativistic quantum mechanics was established in 1925/26 – and just two years later, in 1928, Paul Dirac published the correct equation describing relativistic electrons: the → *Dirac equation* [87].)

→ Are there relativistic field equations which allow for a probabilistic interpretation?

7.1. The Klein-Gordon equation

The Klein-Gordon equation has been studied by Klein [88] and Gordon [89] in 1926 as a possible relativistic version of the Schrödinger equation. Schrödinger and Fock found the equation independently as well.

1 | $\sphericalangle$ *Complex* scalar field: $\phi : \mathbb{R}^{1,3} \to \mathbb{C}$

→ Most general *quadratic* (superposition principle!) and *Lorentz covariant* Lagrangian density:

$$\boxed{\mathcal{L}_{\text{KG}}(\phi, \partial\phi) = (\partial^\mu \phi)(\partial_\mu \phi^*) - M^2 \phi\phi^*} \tag{7.11}$$

$M = \frac{mc}{\hbar} \in \mathbb{R}$: arbitrary parameter ($m$ will be the mass of the particle)

- Note that $M = \frac{mc}{\hbar} = \frac{2\pi}{\lambda}$ has the dimension of an inverse length; here $\lambda = \frac{h}{mc}$ is the ← *Compton wavelength* Eq. (5.77).
- One can also derive the non-relativistic Schrödinger equation from a Lagrangian density (→ *below*):

$$\mathcal{L}_{\text{SE}}(\psi, \partial\psi) = i\hbar\psi^* \partial_t \psi - \frac{\hbar^2}{2m}(\nabla\psi^*)(\nabla\psi) - V(x)\psi^*\psi \tag{7.12}$$

This is of course not a Lorentz scalar (you cannot write this combining only tensors).

2 | Euler-Lagrange equations:

Trick: Consider ϕ and ϕ^* as *independent* fields; let ϕ^* be the complex conjugate of ϕ at the end.

$$\frac{\partial \mathcal{L}_{\text{KG}}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}_{\text{KG}}}{\partial(\partial_\mu \phi^*)} = 0 \quad \Rightarrow \quad -M^2\phi - \partial_\mu \partial^\mu \phi = 0 \tag{7.13}$$

The Euler-Lagrange equations for the field ϕ yield the complex conjugate Klein-Gordon equation.

→

$$\boxed{(\partial^2 + M^2)\phi(x) = 0 \qquad ⁂\ \textit{Klein-Gordon equation}} \tag{7.14}$$

The Klein-Gordon equation (KGE) is the simplest relativistic wave equation.

The non-relativistic Schrödinger equation follows along the same lines from Eq. (7.12):

$$\frac{\partial \mathcal{L}_{\text{SE}}}{\partial \psi^*} - \partial_\mu \frac{\partial \mathcal{L}_{\text{SE}}}{\partial(\partial_\mu \psi^*)} = 0 \quad \Rightarrow \quad i\hbar\partial_t\psi - V\psi + \frac{\hbar^2}{2m}\nabla^2\psi = 0 \tag{7.15}$$

The Euler-Lagrange equations for ψ^* yield the complex conjugate of the Schrödinger equation.

3 | Lorentz symmetry of the KGE:

The KGE is manifest Lorentz covariant. However, it is instructive (and useful for our derivation of the Dirac equation → *later*) to check its invariance manually. To this end, we view Lorentz transformations as *active* transformations, mapping solutions to different solutions. This is equivalent to the *passive* viewpoint where the coordinate system is transformed instead:

i | $\sphericalangle$ Coordinate transformation: $\bar{x} = \Lambda x$ & Field transformation: $\bar{\phi}(\bar{x}) = \phi(x)$
We write $\bar{x} = \Lambda x$ for $\bar{x}^\mu = \Lambda^\mu{}_\nu x^\nu$.

ii | $\sphericalangle$ $\phi(x)$ with $(\partial^2 + M^2)\phi(x) = 0$ for all x
That is, $\phi(x)$ is a solution of the KGE.

iii | $\overset{\circ}{\rightarrow}$ $\bar{\phi}(x) := \phi(\Lambda^{-1}x)$ is a *new solution*:
Use the chain rule in the first step twice:

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + M^2)\bar{\phi}(x) = [\eta^{\mu\nu}(\Lambda^{-1})^\sigma{}_\mu\partial_\sigma(\Lambda^{-1})^\rho{}_\nu\partial_\rho + M^2]\phi(\Lambda^{-1}x) \tag{7.16a}$$

Use invariance of the metric Eq. (4.21) (7.16b)

$$= (\eta^{\sigma\rho}\partial_\sigma\partial_\rho + M^2)\phi(\Lambda^{-1}x) \tag{7.16c}$$

$$= (\partial^2 + M^2)\phi(\Lambda^{-1}x) \overset{\phi \text{ solution}}{=} 0 \tag{7.16d}$$

Here $\partial_\sigma\phi(\Lambda^{-1}x)$ must be read as $\partial_\sigma\phi(y)|_{y=\Lambda^{-1}x}$, i.e., we compute the derivative of the function ϕ with respect to its argument y and then plug in the value $\Lambda^{-1}x$.

4 | Conserved current:

i | $\sphericalangle$ Global phase rotations:

$$\phi'(x) = e^{i\alpha}\phi(x) \quad \text{for} \quad \alpha \in [0, 2\pi) \tag{7.17}$$

with infinitesimal generator $|\alpha| = |w| \ll 1$

$$\phi'(x) = \phi(x) + iw\phi(x) \equiv \phi(x) + w\,\delta\phi(x) \quad \Rightarrow \quad \delta\phi = i\phi \tag{7.18}$$

Note that this is an "internal symmetry" that has nothing to do with spacetime; thus $\delta x = 0$. For the complex conjugate field ϕ^* one finds analogously $\delta\phi^* = -i\phi^*$.

→ Continuous symmetry:

$$\mathcal{L}_{\text{KG}}(\phi, \partial\phi) = \mathcal{L}_{\text{KG}}(\phi', \partial\phi') \tag{7.19}$$

If the Lagrangian density is invariant, the action is trivially invariant!

ii | Noether theorem Eq. (6.85) → Conserved Noether current density Eq. (6.84):

$$\boxed{j^\mu_{\mathrm{KG}} \mathrel{\hat{=}} i(\partial^\mu\phi)\phi^* - i(\partial^\mu\phi^*)\phi} \tag{7.20}$$

Note that if one treats ϕ and ϕ^* independent fields, one has to sum over the two fields in the evaluation of the Noether current; this then yields the real-valued current density above.

→ Noether charge density:

$$\rho_{\mathrm{KG}}(x) := j^0_{\mathrm{KG}}(x) = \frac{i}{c}\left(\dot\phi\phi^* - \dot\phi^*\phi\right) \quad \text{with} \quad \rho_{\mathrm{KG}}(x) \in \mathbb{R} \tag{7.21}$$

→ Conserved Noether charge:

$$Q = \int \mathrm{d}^3x\, \rho_{\mathrm{KG}}(x) = \frac{i}{c}\int \mathrm{d}^3x\, \left(\dot\phi\phi^* - \dot\phi^*\phi\right) \tag{7.22}$$

Important: $\rho_{\mathrm{KG}}(x) \lesseqgtr 0$ is *not* positive-definit! →

$$\boxed{\rho_{\mathrm{KG}}(x) \text{ \textit{cannot} be interpreted as a probability density!}} \tag{7.23}$$

- To sum up:
  - The inner product (= positive-definite, symmetric sesquilinear form) on $L^2(\mathbb{R}^{1,3}, \mathbb{C})$

$$\langle\phi|\psi\rangle_{L^2} := \int \mathrm{d}^3x\, \phi^*\psi \tag{7.24}$$

    is *not* conserved under the time-evolution of the KGE.
  - The *indefinite* symmetric sesquilinear form (which is not an inner product!)

$$\langle\phi|\psi\rangle_{\mathrm{KG}} := \frac{i\hbar}{2mc^2}\int \mathrm{d}^3x\, \left(\phi^*\dot\psi - \dot\phi^*\psi\right) \tag{7.25}$$

    is conserved under the time-evolution of the KGE. But because it is not positive-(semi)definite, we cannot interpret the induced "norm" as a probability.

    The prefactor $\frac{\hbar}{2mc^2}$ is chosen such that it has the dimension of a time (because $\frac{\hbar}{mc} \propto \lambda$ has the dimension of a length). Then the square of the fields (= wave-functions) has the dimension of one over a volume – which is the conventional dimension of wavefunctions. The factor $\frac{1}{2}$ is chosen to simplify expressions later.

- Compare this to the conserved current for the same phase rotation symmetry that follows for the Schrödinger field Eq. (7.12) with $\delta\psi = i\psi$ and $\delta\psi^* = -i\psi^*$:

$$j^\mu_{\mathrm{SE}} \equiv \begin{pmatrix} \hbar c\rho_{\mathrm{SE}} \\ \hbar\vec{j}_{\mathrm{SE}} \end{pmatrix} \overset{7.12}{\underset{6.84}{=}} \begin{cases} \hbar c\psi^*\psi\,, & \mu = 0 \\ \frac{i\hbar^2}{2m}\left[(\nabla\psi^*)\psi - (\nabla\psi)\psi^*\right] & \mu = i = 1,2,3 \end{cases} \tag{7.26}$$

  (Recall that you must sum over the fields ψ and ψ^*.)

  This is the *positive-definite probability density* you already know from quantum mechanics,

$$\rho_{\mathrm{SE}}(x) = \psi^*(x)\psi(x) = |\psi(x)|^2 \;\geq 0\,, \tag{7.27}$$

and the ↓ *probability current density*

$$\vec{j}_{\text{SE}} = \frac{i\hbar}{2m}\left[(\nabla\psi^*)\psi - (\nabla\psi)\psi^*\right]. \tag{7.28}$$

In this context, Noether's theorem ensures probability conservation:

$$\partial_\mu j^\mu_{\text{SE}} = 0 \quad \Leftrightarrow \quad \partial_t \rho_{\text{SE}} + \nabla\cdot\vec{j}_{\text{SE}} = 0\,. \tag{7.29}$$

5 | Solutions: (for the free Klein-Gordon field)

i | The KG Eq. (7.14) is a wave equation:

$$\left[\frac{1}{c^2}\partial_t^2 - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right]\phi(t,\vec{x}) = 0 \tag{7.30}$$

→ Solution space spanned by plane waves:

$$\phi(t,\vec{x}) = e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{x}-Et)} \tag{7.31}$$

Plug this ansatz into Eq. (7.30) → Dispersion relation:

$$-\frac{E^2}{c^2\hbar^2} + \frac{\vec{p}^2}{\hbar^2} + \frac{m^2c^2}{\hbar^2} \stackrel{!}{=} 0 \tag{7.32}$$

$$\boxed{E = \pm\sqrt{\vec{p}^2c^2 + m^2c^4}} \tag{7.33}$$

- This is the relativistic energy-momentum relation Eq. (5.26).
- ¡! There are *two* solutions for each 3-momentum $\vec{p}$, one of which has *negative* energy $E < 0$ (if we interpret the prefactor of t as the energy as usual). This is a consequence of the quadratic nature of the KGE (as compared to the SE), and therefore a direct consequence of its relativistic covariance.
- At the time of its inception, the negative energy solutions of the KGE could not be interpreted properly. This (together with the fact that its conserved "norm" cannot be interpreted as a probability and it fails to predict the fine-structure of the hydrogen atom correctly, → *below*) lead to its dismissal as a relativistic wave equation for quantum wave functions. It only became clear later that the negative energy solutions herald the existence of *antiparticles*. Only in modern ↑ *relativistic quantum field theories* [where the KGE reappears as the equation of motion of (free scalar) quantum fields, see Chapter 2 of my script on QFT [19]] this "feature" can be cast into a consistent framework: The negative energy solutions are interpreted as eigenmodes of antiparticles with *positive* energies (and norms). If the particles are charged, their antiparticles have opposite charge; then the conserved Noether charge Eq. (7.22) is interpreted as *charge conservation* (and not probability conservation).

ii | As usual, one can "normalize" the plane wave solutions Eq. (7.31) if one considers a finite system with volume $V = L^3$. Then one finds the "orthonormal" solution basis of the KGE:

$\overset{\circ}{\rightarrow}$

$$\boxed{\begin{aligned} &\phi_{\vec{k}}^{(\pm)}(t,\vec{x}) = N_{\vec{k}}\, e^{i(\vec{k}\cdot\vec{x}\mp\omega_k t)} \quad \text{with} \dots \\ &\text{Dispersion:} \quad \omega_k = \sqrt{\vec{k}^2c^2 + m^2c^4/\hbar^2} \\ &\text{Momentum:} \quad \vec{p} = \hbar\vec{k} \in \hbar\frac{2\pi}{L}\mathbb{Z}^3 \\ &\text{Normalization:} \quad N_k = \sqrt{\frac{mc^2}{V\hbar\omega_k}} \end{aligned}} \tag{7.34a, 7.34b}$$

- It is straightforward to check that these states are "orthonormal" with respect to the Klein-Gordon sesquilinear form Eq. (7.25):

$$\langle\psi_{\vec{k}}^{(\alpha)}|\psi_{\vec{k}'}^{(\beta)}\rangle_{\mathrm{KG}} \overset{\circ}{=} \alpha\,\delta_{\alpha,\beta}\delta_{\vec{k},\vec{k}'} \quad \text{with} \quad \alpha,\beta\in\{\pm\}\,. \tag{7.35}$$

  Note that the $(-)$ states have negative "norm".

- The fact that there are "twice as many" linearly independent solutions (two for each momentum) means that you need "twice as many" parameters to specify a particular solution (i.e., a linear combination of the plane waves). This corresponds to the fact that the KGE is of *second order* in the time derivative, so that you need to provide both $\phi(t=0,\vec{x})$ and $\dot{\phi}(t=0,\vec{x})$ to specify a unique solution.

↓Lecture 15 [06.02.24]

6 | Coupling to a static EM field:

The KGE can be coupled to the gauge field of electrodynamics. This is necessary to described charged particles (in particular: the hydrogen atom). Note that in the following the gauge field is a *parameter* and not a dynamic degree of freedom.

i | Goal: Construct Lagrangian density that is …

- … a Lorentz scalar.
- … quadratic in ϕ.
- … gauge invariant under the gauge transformation $A'_\mu = A_\mu - \partial_\mu\lambda$.
- … couples ϕ and A_μ in a non-trivial way.

Without additional tools, this is a tough job!

ii | $\sphericalangle$ Gauge transformation $A'_\mu = A_\mu - \partial_\mu\lambda(x)$

Let us assume that the KG field transforms under the gauge transformation as follows:

$$\phi'(x) := e^{iQ\lambda(x)}\phi(x) \quad \text{with the ⁂ } U(1) \textit{ charge } Q = \frac{q}{\hbar c} \in \mathbb{R}\,. \tag{7.36}$$

q: electric charge of the particle described by the wavefunction ϕ

- It is reasonable to assume that the KG field must transform via phase factors because we already know [recall Eq. (7.19)] that the KG Lagrangian is invariant under *global* phase transformations $\lambda(x) = \text{const}$. Our hope is that we can "extend" this symmetry for arbitrary non-constant $\lambda(x)$.
- The charge Q is a property of the field and quantifies how it transforms under gauge transformations; it essentially plays the role of the electric charge of the particle described by ϕ; e.g., for an electron we would set $Q = \frac{-e}{\hbar c} < 0$.

  The additional division by $\hbar c$ is necessary for dimensional reasons: $[\lambda] = L[\varphi]$ with $A^\mu = (\varphi, \vec{A})$; therefore $[\lambda q] = L[\varphi q] = L[E] = \frac{ML^3}{T^2}$ and it is $[\hbar c] = \frac{ML^3}{T^2}$ as well. In natural units (where $\hbar = 1 = c$, $Q = q$ is simply the electric charge.
- The term "$U(1)$ charge" highlights that the gauge transformation $e^{iQ\lambda(x)} \in \mathrm{U}(1)$ is a $U(1)$ gauge transformation; the charge is the generator of this Lie group.

iii | Problem:

Derivatives transform complicated under gauge transformations:

$$\partial_\mu\phi'(x) \stackrel{\circ}{=} e^{iQ\lambda(x)}\left[iQ(\partial_\mu\lambda)\phi(x) + \partial_\mu\phi(x)\right] \tag{7.37}$$

→ It is hard to combine derivatives of fields to construct gauge-invariant terms!

Solution:

Define the …

$$\boxed{\text{⁂ } \textit{(Gauge) Covariant derivative:} \quad D_\mu := \partial_\mu + iQA_\mu} \tag{7.38}$$

→ Lorentz vector (thus we can it use to construct Lorentz scalars!)

The covariant derivative has the following useful property:

$$D'_\mu\phi' = \left[\partial_\mu + iQA_\mu - iQ(\partial_\mu\lambda)\right]e^{iQ\lambda}\phi \overset{\circ}{=} e^{iQ\lambda}D_\mu\phi \tag{7.39}$$

→ $D_\mu\phi$ transforms like ϕ under gauge transformations. [and not as ugly as Eq. (7.37)!]

This is useful because it allows us to combine derivatives into gauge-invariant terms.

iv | Using the covariant derivative, we can now construct the following general Lagrangian density that satisfies our four requirements above:

$$\boxed{\mathcal{L}_A(\phi,\partial\phi) = (D^\mu\phi)(D_\mu\phi)^* - M^2\phi\phi^*} \tag{7.40}$$

Please appreciate the ingenuity of the term $(D^\mu\phi)(D_\mu\phi)^*$: It is *Lorentz invariant* because we pair the indices correctly, and it is *gauge invariant* because we pair $(D^\mu\phi)$ with its complex conjugate $(D_\mu\phi)^*$ (which is sufficient because $D^\mu\phi$ gauge-transforms like ϕ).

This Lagrangian density is gauge-invariant by construction in the sense that

$$\mathcal{L}_A(\phi,\partial\phi) = \mathcal{L}_{A'}(\phi',\partial\phi') \quad \text{or} \quad \mathcal{L}(\phi,D\phi) = \mathcal{L}(\phi',D'\phi')\,. \tag{7.41}$$

- A comparison of the free Klein-Gordon Lagrangian Eq. (7.11) and the new one Eq. (7.40) reveals that we simply made the substitution $\partial_\mu \mapsto D_\mu$, i.e., we replaced partial derivatives by covariant derivatives (which depend on the gauge field). This trick is not specific to the Klein-Gordon field and yields gauge-invariant theories in general. This procedure is called ↑ *minimal coupling*.
- Note that the transformation Eq. (7.36) is a *local* phase rotation of the KG-field. In Eq. (7.17) we considered a *global* phase rotation and identified it as a continuous symmetry of the KG Lagrangian $\mathcal{L}_{\text{KG}}$. You can check that the new *local* transformation does not leave $\mathcal{L}_{\text{KG}}$ invariant, but it does leave $\mathcal{L}_A$ invariant if A^μ transforms together with ϕ as defined above. The transition from $\mathcal{L}_{\text{KG}}$ (with a global symmetry) to $\mathcal{L}_A$ (with a local version of the same symmetry) is called *gauging the symmetry*. You can use this line of reasoning to "invent" the electromagnetic gauge field: If you start from a global continuous symmetry and demand that it becomes a local symmetry, you have to pay for it by introducing a new field: the gauge field.

v | Klein-Gordon equation in a static EM field:

The Euler-Lagrange equations of $\mathcal{L}_A$ yield: Eq. (6.6) $\xrightarrow{\text{Eq. (7.40)}}$

$$\boxed{(D^2 + M^2)\phi(x) = 0} \tag{7.42}$$

with $D^2 = D_\mu D^\mu$ and $M = \frac{mc}{\hbar}$.

In the form Eq. (7.42) both Lorentz covariance and gauge invariance are manifest (because we use the covariant derivative). If we expand everything, we loose these features but obtain a less abstract (but more complicated) form of the PDE:

$$\left[\frac{1}{c^2}\left(\partial_t + iQc\varphi\right)^2 - \left(\nabla - iQ\vec{A}\right)^2 + \frac{m^2c^2}{\hbar^2}\right]\phi(t,\vec{x}) = 0 \tag{7.43}$$

Here we used $A_\mu = (\varphi, -\vec{A})$ (covariant!).

vi | Example: Hydrogen atom

Goal: Describe the electron of the hydrogen atom in the static EM field generated by the proton in terms of the KGE; i.e., we interpret the KG field ϕ naïvely as the wavefunction of the electron. Our hope is that the energy spectrum of this relativistic theory explains the observed fine-structure splitting.

a | $\sphericalangle$ Coulomb potential (of proton with charge $e > 0$) $\rightarrow$

$$\text{Choose a gauge where} \quad \varphi(x) = \frac{e}{|\vec{x}|} \quad \text{and} \quad \vec{A} = \vec{0} \tag{7.44}$$

$\overset{\circ}{\rightarrow}$ With electron charge $Q = \frac{-e}{\hbar c} < 0$ one finds:

$$\left[\frac{1}{c^2}\left(i\partial_t + \frac{e^2}{\hbar|\vec{x}|}\right)^2 + \nabla^2 - \frac{m^2c^2}{\hbar^2}\right]\phi(t,\vec{x}) = 0 \tag{7.45}$$

b | $\sphericalangle$ Ansatz $\phi(t,\vec{x}) = \tilde{\phi}(\vec{x})e^{-\frac{i}{\hbar}Et}$ $\rightarrow$ "Stationary" Klein-Gordon equation:

$$\left[c^2\hbar^2\Delta + \left(E + \frac{e^2}{|\vec{x}|}\right)^2 - m^2c^4\right]\tilde{\phi}(\vec{x}) = 0 \tag{7.46}$$

Note that this PDE is *quadratic* in the energy E (and not linear, like the time-independent Schrödinger equation).

c | One can use a clever mapping to the non-relativistic Schrödinger equation to solve for $\tilde{\phi}(\vec{x})$ and determine the energies E for which solutions exist:

$$\overset{*}{\rightarrow} \quad E_{n,l} = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2}{(n-\delta_l)^2}}} \quad \text{with} \quad \delta_l = l + \tfrac{1}{2} - \sqrt{\left(l + \tfrac{1}{2}\right)^2 - \alpha^2}\,. \tag{7.47}$$

Here $n = 1, 2, \dots$ is the ↓ *principal quantum number* and $l = 0, 1, 2, \dots$ is the ↓ *orbital angular momentum quantum number*. $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the fine-structure constant.

d | Comments:

- The spectrum Eq. (7.47) predicts a splitting of the l-degeneracy; recall that this degeneracy is perfect in the non-relativistic hydrogen atom [cf. Eq. (7.8)]. Unfortunately, the spectrum Eq. (7.47) does *not* match observations! The reason is that the Klein-Gordon equation does not know about the *electron spin*. Schrödinger and his contemporaries were aware of this solution and its problems (this shines through in the quotes at the beginning of this chapter). This failure to predict the fine-structure correctly led to the dismissal of the Klein-Gordon equation and motivated Paul Dirac to search for another equation (→ *next section*).
- Today we know that the Klein-Gordon equation is *not wrong*: It simply does not apply to particles with non-zero spin (and the electron in the hydrogen atom happens to have spin $s = \frac{1}{2}$). However, it *does* apply to spin-0 particles like ↑ *kaons* (K mesons, bound states of two quarks), ↑ *pions* (pi mesons), and the ↑ *Higgs boson* (the latter being the only *elementary* particle with zero spin). But since we cannot build hydrogen atoms out of these particles, the significance of the above solution remains limited.

7 | First-order formulation:

Here we consider again the free KGE (without EM field) for simplicity.

i | KGE = *Second-order* PDE in time

Problem: $\phi(t = 0, \vec{x})$ does *not* specify the state of the system completely [unlike for the Schrödinger equation one also needs $\dot{\phi}(t = 0, \vec{x})$ to pick out a unique solution $\phi(t, \vec{x})$].

Recall: Every higher-order differential equation can be recast as a first-order differential equation with multiple components.

→ Goal: Rewrite the KGE in the first-order form

$$i\hbar\partial_t \Phi = \hat{H}_{\mathrm{KG}}\Phi \quad \text{with} \quad \Phi = \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} . \tag{7.48}$$

Downside: In this form, the KGE is no longer manifest Lorentz covariant.

ii | Define

$$\phi_\pm := \frac{1}{2}\left(\phi \pm \frac{i\hbar}{mc^2}\partial_t\phi\right) \tag{7.49}$$

so that

$$\phi = \phi_+ + \phi_- \quad \text{and} \quad \frac{i\hbar}{mc^2}\partial_t\phi = \phi_+ - \phi_- . \tag{7.50}$$

iii | Define the 2×2 differential operator

$$\hat{H}_{\mathrm{KG}} := \begin{pmatrix} \hat{H}_0 + mc^2 & \hat{H}_0 \\ -\hat{H}_0 & -\hat{H}_0 - mc^2 \end{pmatrix} = \hat{H}_0 \otimes (\sigma^z + i\sigma^y) + mc^2\sigma^z \tag{7.51}$$

with $\hat{H}_0 = -\frac{\hbar^2}{2m}\nabla^2$ the free particle Hamiltonian and the Pauli matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{7.52}$$

$\hat{H}_{\mathrm{KG}}$ is a linear operator on the Hilbert space $L^2 \otimes \mathbb{C}^2$ of two-component square-integrable functions. Note that $\hat{H}_{\mathrm{KG}}^\dagger = \hat{H}_0 \otimes (\sigma^z - i\sigma^y) + mc^2\sigma^z \neq \hat{H}_{\mathrm{KG}}$ is non-Hermitian with respect to the conventional inner product on $L^2 \otimes \mathbb{C}^2$:

$$\langle\Phi|\Psi\rangle_{L^2\otimes\mathbb{C}^2} = \int \mathrm{d}^3x\, \Phi^\dagger(x)\Psi(x) = \int \mathrm{d}^3x\, \left(\phi_+^*\psi_+ + \phi_-^*\psi_-\right) . \tag{7.53}$$

iv | Check that the differential equation in first-order Schrödinger form

$$i\hbar\partial_t\Phi = \hat{H}_{\mathrm{KG}}\Phi \quad \Leftrightarrow \quad \begin{cases} i\hbar\partial_t\phi_+ = (\hat{H}_0 + mc^2)\phi_+ + \hat{H}_0\phi_- \\ i\hbar\partial_t\phi_- = -\hat{H}_0\phi_+ - (\hat{H}_0 + mc^2)\phi_- \end{cases} \tag{7.54}$$

is equivalent to the KGE:

a | Indeed, the *difference* of the two equations yields

$$-\frac{\hbar^2}{mc^2}\partial_t\chi = (\hat{H}_0 + mc^2)\phi + \hat{H}_0\phi \quad \Leftrightarrow \quad \frac{1}{c^2}\partial_t\chi - \nabla^2\phi + \frac{m^2c^2}{\hbar^2}\phi = 0 \tag{7.55}$$

where we defined $\phi := \phi_+ + \phi_-$ and $\chi := \frac{mc^2}{i\hbar}(\phi_+ - \phi_-)$.

b | By contrast, the *sum* of the two equation yields

$$mc^2\partial_t\phi = (\hat{H}_0 + mc^2)\chi - \hat{H}_0\chi \quad \Leftrightarrow \quad \partial_t\phi = \chi . \tag{7.56}$$

c | Combining Eq. (7.55) and Eq. (7.56) returns the KGE:

$$\frac{1}{c^2}\partial_t^2\phi - \nabla^2\phi + \frac{m^2c^2}{\hbar^2}\phi = 0 . \tag{7.57}$$

v | If one defines the

$$\text{⁂ } \textit{Klein-Gordon adjoint} \quad \bar{\Phi} := \Phi^\dagger \sigma^z = (\phi_+^*, -\phi_-^*) \,, \tag{7.58}$$

one can express the Klein-Gordon sesquilinear form Eq. (7.25) as

$$\langle \Phi | \Psi \rangle_{\mathrm{KG}} := \int \mathrm{d}^3x \, \bar{\Phi}(x)\Psi(x) \overset{7.49}{=} \frac{i\hbar}{2mc^2} \int \mathrm{d}^3x \, (\phi^* \dot{\psi} - \dot{\phi}^* \psi) \overset{7.25}{=} \langle \phi | \psi \rangle_{\mathrm{KG}} \,. \tag{7.59}$$

Remember that this is not a proper inner product because it is not positive-definite.

vi | If one defines additionally for an operator A on $L^2 \otimes \mathbb{C}^2$ the

$$\text{⁂ } \textit{Klein-Gordon adjoint} \quad \bar{A} := \sigma^z A^\dagger \sigma^z \,, \tag{7.60}$$

it follows $\overline{A\Phi} = \bar{\Phi}\bar{A}$ and $\bar{\bar{A}} = A$, and thereby

$$\langle \Phi | A \Psi \rangle \overset{7.59}{=} \langle \bar{A} \Phi | \Psi \rangle \,. \tag{7.61}$$

vii | It is easy to verify that the KG Hamiltonian is "Klein-Gordon Hermitian", namely

$$\bar{\hat{H}}_{\mathrm{KG}} \overset{7.51}{=} \hat{H}_{\mathrm{KG}} \tag{7.62}$$

because $\sigma^z \sigma^y \sigma^z = -\sigma^y$.

viii | With this machinery, we have now a new method to check that the time-evolution generated by the KGE leaves the KG sesquilinear form invariant:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \phi | \psi \rangle_{\mathrm{KG}} \overset{7.59}{=} \frac{\mathrm{d}}{\mathrm{d}t} \langle \Phi | \Psi \rangle_{\mathrm{KG}} \tag{7.63a}$$

$$= \langle \Phi | \dot{\Psi} \rangle_{\mathrm{KG}} + \langle \dot{\Phi} | \Psi \rangle_{\mathrm{KG}} \tag{7.63b}$$

$$\overset{7.54}{=} \frac{1}{i\hbar} \langle \Phi | \hat{H}_{\mathrm{KG}} \Psi \rangle_{\mathrm{KG}} - \frac{1}{i\hbar} \langle \hat{H}_{\mathrm{KG}} \Phi | \Psi \rangle_{\mathrm{KG}} \tag{7.63c}$$

$$\overset{7.61,\,7.62}{=} \frac{1}{i\hbar} \left(\langle \Phi | \hat{H}_{\mathrm{KG}} \Psi \rangle_{\mathrm{KG}} - \langle \Phi | \hat{H}_{\mathrm{KG}} \Psi \rangle_{\mathrm{KG}} \right) = 0 \tag{7.63d}$$

We already knew this from Noether's theorem, but it is always nice to derive such statements in various ways.

8 | Non-relativistic limit:

i | Goal: Derive a non-relativistic approximation of the Klein-Gordon equation

$$\left[\frac{1}{c^2} \partial_t^2 - \nabla^2 + \frac{m^2c^2}{\hbar^2} \right] \phi(t, \vec{x}) = 0 \,. \tag{7.64}$$

ii | $\triangleleft$ Kinetic energy: $E_{\mathrm{kin}} = E - mc^2 = \sqrt{\vec{p}^2c^2 + m^2c^4} - mc^2 \approx \frac{1}{2}mv^2 + \mathcal{O}(\beta^4)$

(Note that both E_{kin} and E are non-negative!)

$\triangleleft$ Ansatz:

$$\phi_\pm(t, \vec{x}) = \tilde{\phi}_\pm(\vec{x}) e^{\mp \frac{i}{\hbar} Et} = \underbrace{\tilde{\phi}_\pm(\vec{x}) e^{\mp \frac{i}{\hbar} E_{\mathrm{kin}} t}}_{=:\hat{\phi}_\pm(t,\vec{x})} e^{\mp \frac{i}{\hbar} mc^2 t} \tag{7.65}$$

$\hat{\phi}(t, \vec{x})$ contains only the time evolution due to the kinetic energy, excluding the rest energy.

iii | If we use that

$$\partial_t^2 \hat{\phi}_\pm = -\frac{E_{\text{kin}}^2}{\hbar^2} \hat{\phi} \,, \tag{7.66}$$

we can make the following approximation in the non-relativistic limit $E_{\text{kin}} \ll mc^2$:

$$\partial_t^2 \phi_\pm = e^{\mp \frac{i}{\hbar} mc^2 t} \left\{ \partial_t^2 \hat{\phi}_\pm \mp \frac{2imc^2}{\hbar} \partial_t \hat{\phi}_\pm - \left(\frac{mc^2}{\hbar}\right)^2 \hat{\phi}_\pm \right\} \tag{7.67a}$$

$$= -e^{\mp \frac{i}{\hbar} mc^2 t} \left\{ \pm \frac{2imc^2}{\hbar} \partial_t \hat{\phi}_\pm + \left(\frac{mc^2}{\hbar}\right)^2 \left[1 + \left(\frac{E_{\text{kin}}}{mc^2}\right)^2\right] \hat{\phi}_\pm \right\} \tag{7.67b}$$

$$\approx -e^{\mp \frac{i}{\hbar} mc^2 t} \left\{ \pm \frac{2imc^2}{\hbar} \partial_t \hat{\phi}_\pm + \left(\frac{mc^2}{\hbar}\right)^2 \hat{\phi}_\pm \right\} \tag{7.67c}$$

iv | Eq. (7.67c) in Eq. (7.64) yields:

$$e^{\mp \frac{i}{\hbar} mc^2 t} \left[\pm \frac{2im}{\hbar} \partial_t + \frac{m^2c^2}{\hbar^2} + \nabla^2 - \frac{m^2c^2}{\hbar^2} \right] \hat{\phi}_\pm(t, \vec{x}) = 0 \tag{7.68}$$

And finally:

$$\boxed{\pm i\hbar \partial_t \hat{\phi}_\pm(t, \vec{x}) = -\frac{\hbar^2}{2m} \nabla^2 \hat{\phi}_\pm(t, \vec{x})} \tag{7.69}$$

This is the Schrödinger equation for a free particle.

Note that the "negative energy solutions" ϕ_- lead to the *time-inverted* Schrödinger equation.

7.2. The Dirac equation

The Dirac equation was published by Paul Dirac in [87], only two years after Schrödinger published the Schrödinger equation.

1 | Goal:

The Klein-Gordon equation has a few undesirable quirks:

- *It's conserved* U(1) *current has no positive-definite density and therefore cannot be interpreted as a probability current. Conversely, the conventional norm on* L^2 *is not conserved. In the first-order formulation, this corresponds to a non-Hermitian Hamiltonian.*

  → Can we construct a relativistic field equation with a conserved positive-definite density that gives rise to a norm and a Hermitian Hamiltonian?

- *In its manifest Lorentz covariant formulation, the KGE is of second order in time, so that we must provide both the wavefunction and its time derivative as initial data.*

  → Can we construct a relativistic field equation which is first order in time (just like the Schrödinger equation)?

- *For each momentum there is are two solutions: one with positive and one with negative energy.*

  → Can we get rid of the negative energy solutions?

The Dirac equation succeeds in solving the first two issues – but not the last one, i.e., there will still be negative energy solutions.

2 | Observation:

To reach our goals we must equip our "toolbox" of tensor calculus with additional building blocks. As it turns out, there is another type of field (besides the tensor fields we introduced in Chapter 3) that plays an important role in quantum mechanics: ↑ *spinor fields*.

Remember: Vector fields under rotations: $\vec{\phi}'(\vec{x}) = R\vec{\phi}(R^{-1}\vec{x})$

→ In general, a field $\phi(x) \in \mathbb{C}^n$ can transform under homogeneous Lorentz transformations as

$$\phi_a'(x) = M_{ab}(\Lambda)\phi_b(\Lambda^{-1}x) \qquad a = 1,\dots,n \tag{7.70}$$

where

$$M(\Lambda')M(\Lambda)\phi(\Lambda^{-1}\Lambda'^{-1}x) \stackrel{!}{=} M(\Lambda'\Lambda)\phi((\Lambda'\Lambda)^{-1}x) \tag{7.71}$$

is a n-dimensional representation of the (proper orthochronous) Lorentz group $\mathrm{SO}^+(1,3)$.

- Regarding groups and their representations: ➔ Problemset 1.
- More explicitly: The tensor fields (of various rank) we know so far allow only for the description of particles with *integer spin* $S = 0, 1, 2, \cdots$ (spin = internal angular momentum). What we are missing are fields that can describe particles with *half-integer spin* $S = \frac{1}{2}, \frac{3}{2}, \cdots$; these are the spinor fields.

  The reason why this is crucial for relativistic *quantum mechanics* in particular has to do with the fact that multiplying wave functions by a global phase does not change the state. In mathematical parlance we are dealing with ↑ *projective Hilbert spaces* and ↑ *projective representations* of symmetries. Thus if you are interested what rotations SO(3) do to the quantum state of your system, you must study all *projective* representations of SO(3). It turns out that these can be identified with the "conventional" (= linear) representations of another group: SU(2) (the so called ↑ *double cover* of SO(3)). And you know that the irreducible representations of SU(2) are labeled by "spin quantum numbers" $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. In general, the double covers of $\mathrm{SO}(n)$ are called ↑ *spin groups* Spin(n), and similarly, the double cover of the proper orthochronous Lorentz group $\mathrm{SO}^+(1,3)$ is the group $\mathrm{Spin}(1,3) \simeq \mathrm{SL}(2,\mathbb{C})$ (the group of complex 2×2 matrices with determinant one). It turns out that the irreducible representations of this group can be labeled by *two* numbers (m,n) with $m,n = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. The spinor representations we are interested in (the ones missing from our discussion of tensor fields) are the ones for which $m+n$ is half-integer. Conversely, the $(\frac{1}{2},\frac{1}{2})$ representation is our well-known 4-vector representation A^μ and the $(0,0)$ representation is that of a scalar like ϕ.

3 | We want a first-order relativistic field equation → Ansatz:

$$(\partial^\mu\partial_\mu + \text{const})\phi = 0 \quad \Rightarrow \quad (i\blacksquare^\mu\partial_\mu + \text{const})\phi = 0 \tag{7.72}$$

We do not yet know what ■ is (only that it cannot be a derivative).
The i anticipates wave-like solutions for real ■.

A covariant equation of the form $\partial_\mu\phi = 0$ or $\partial_\mu A^\mu = 0$ would of course also be possible; their solutions, however, are either too simple or do not match observations.

4 | Then (combine 2 & 3)

- **i** | ⊲ Coordinate transformation $x' = \Lambda x$ & Field transformation $\phi'(x') = M(\Lambda)\phi(x)$
- **ii** | ⊲ ϕ with $(i\blacksquare^\mu\partial_\mu + \text{const})\phi(x) = 0$ for all x

  That is, $\phi(x)$ is a solution of the equation we want to construct.

iii | When is $\phi'(x) = M(\Lambda)\phi(\Lambda^{-1}x)$ is a new solution?

We want the equation to be Lorentz covariant; this means that the Lorentz group must be (part of) its invariance group: Lorentz transformations map solutions to new solutions.

$$(i\blacksquare^\mu\partial_\mu + \text{const})\phi'(x) = [i\blacksquare^\mu(\Lambda^{-1})^\nu{}_\mu\partial_\nu + \text{const}]\, M(\Lambda)\phi(\Lambda^{-1}x) \stackrel{!}{=} 0 \quad (7.73)$$

Multiply with $M^{-1}(\Lambda)$:

$$\Leftrightarrow \quad [i\, \underbrace{M^{-1}(\Lambda)\blacksquare^\mu M(\Lambda)(\Lambda^{-1})^\nu{}_\mu}_{\stackrel{!}{=}\blacksquare^\nu}\partial_\nu + \text{const}]\,\phi(\Lambda^{-1}x) \stackrel{!}{=} 0 \quad (7.74)$$

$\rightarrow \blacksquare^\mu \equiv \gamma^\mu$ must be $n \times n$-matrices with

$$\boxed{M^{-1}(\Lambda)\gamma^\mu M(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu} \quad (7.75)$$

The γ-matrices "translate" the "spinor"-representation $M(\Lambda)$ into the "vector"-representation Λ and vice versa.

5 | *Question:* How to find appropriate γ^μ and $M(\Lambda)$ that satisfy Eq. (7.75)?

Remember: $\mathrm{SO}^+(1,3)$ is a Lie group (Recall ➲ Problemset 4):

$$\Lambda = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\right] \stackrel{\omega\ll1}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta} \quad (7.76a)$$

$$M(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}\mathcal{S}^{\alpha\beta}\right] \stackrel{\omega\ll1}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\mathcal{S}^{\alpha\beta} \quad (7.76b)$$

$\omega_{\alpha\beta}$ antisymmetric tensor $\rightarrow$ 3 rotations (angles) + 3 boosts (rapidities)

It is $(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu)$; these 4×4 matrices $\mathcal{J}^{\alpha\beta}$ generate the 4-vector representation $(\frac{1}{2},\frac{1}{2})$, i.e., the 4×4-matrices Λ. By contrast, the $n\times n$-matrices $\mathcal{S}^{\alpha\beta}$ generate the spinor-representation $M(\Lambda)$ [we will find $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$]. The generators are antisymmetric in the spacetime indices.

- Infinitesimal form of Eq. (7.75):

$$\left[\gamma^\mu, \mathcal{S}^{\alpha\beta}\right] \stackrel{\circ}{=} (\mathcal{J}^{\alpha\beta})^\mu{}_\nu\gamma^\nu \stackrel{\circ}{=} i(\eta^{\alpha\mu}\gamma^\beta - \eta^{\beta\mu}\gamma^\alpha) \quad (7.77)$$

- $\lessdot\ \mathcal{J}^{\alpha\beta}$ (➲ Problemset 4) $\rightarrow$ Lie-algebra of Lorentz group ($J = \mathcal{S}, \mathcal{J}$):

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] \stackrel{\circ}{=} i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \quad (7.78)$$

The Lie algebra defines the structure of the Lie group by exponentiation and is therefore the same for all representations, recall Eq. (4.63).

6 | Solution to Eq. (7.75) via Dirac's trick [87]: $\lessdot\ \gamma^\mu$ such that

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}_{n\times n} \quad ⁂\ \textit{Dirac algebra}} \quad (7.79)$$

with the ↓ *anticommutator* $\{X, Y\} = XY + YX$.

- Matrices $\gamma^\mu = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ that satisfy Eq. (7.79) are called ⁂ *Dirac matrices* or ⁂ *Gamma matrices*.
- This is the 16-dimensional *Clifford algebra* $C\ell_{1,3}(\mathbb{C})$.

Then

$$\boxed{\mathcal{J}^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]} \tag{7.80}$$

satisfies the Lorentz algebra Eq. (7.78) and Eq. (7.77).

Check this by plugging Eq. (7.80) into Eq. (7.78) and Eq. (7.77) and using Eq. (7.79)!

→ Problem of solving Eq. (7.75) has been reduced to finding 4 matrices γ^μ that satisfy Eq. (7.79).

7 | Representations of Eq. (7.79):

- At least $n = 4$-dimensional
  (Think of the γ^μ as Majorana modes and construct ladder operators → 2 modes.)
- All 4-dimensional representations are unitarily equivalent
  (Actually, they constitute the *unique* irrep of the Dirac algebra which is 4-dimensional.)
- We use the Weyl representation (sometimes called *chiral representation*):

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad \text{and} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad i = 1, 2, 3} \tag{7.81}$$

  - Recall the Pauli matrices Eq. (7.52):

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.82}$$

  - Other common choices are the ↑ *Dirac representation* and the ↑ *Majorana representation*.

- Henceforth: $\Lambda_{\frac{1}{2}} \equiv M(\Lambda)$
  It turns out that these are two "copies" of a spin-$\frac{1}{2}$ projective representation: $\Lambda_{\frac{1}{2}}$ corresponds to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of $SL(2, \mathbb{C})$. Since $n + m = \frac{1}{2}$, this is a spinor representation, i.e., a projective representation of the Lorentz group $SO^+(1, 3)$. The fact that it is the sum of two such representations makes it *reducible*. The wavefunction $\Psi(x)$ has therefore $n = 4$ components and is a *spinor field* (and not a tensor field).

8 | Setting const $= -M = -\frac{mc}{\hbar}$ (which has dimension of an inverse length), we find:

$$\boxed{(i\gamma^\mu \partial_\mu - M)\Psi = 0 \quad ⁂\ \textit{Dirac equation}} \tag{7.83}$$

Here, $\Psi(x)$ is a ⁂ *(bi)spinor*-field:

$$\Psi : \mathbb{R}^{1,3} \to \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2 . \tag{7.84}$$

Introduce the ⁂ *Feynman slash notation*: $\not{O} := \gamma^\mu O_\mu$

(Here, O_μ stands for any object with a 4-vector index.)

With the slash notation, the Dirac equation can be written as:

$$\boxed{(i\partial\!\!\!/ - M)\Psi = 0} \tag{7.85}$$

The Dirac equation is engraved in a plaque on the floor of Westminster Abbey next to Isaac Newton's tomb (they abbreviate $\gamma \cdot \partial = \gamma^\mu \partial_\mu$ and are in natural units $\hbar = 1 = c$ where $M = m$):

(Photograph from https://cerncourier.com/a/paul-dirac-a-genius-in-the-history-of-physics.)

9 | The components $\Psi_a(x)$ $(a = 1, 2, 3, 4)$ satisfy the KGE:

$$0 = (-i\gamma^\mu\partial_\mu - M)(i\gamma^\nu\partial_\nu - M)\Psi \overset{7.79}{=} (\partial^2 + M^2)\Psi \tag{7.86}$$

On the right hand side of Eq. (7.86) there is an identity $\mathbb{1}_{4\times 4}$ that we omit.

- The Dirac differential operator is the "square root" of the Klein-Gordon differential operator.
- ¡! Although Ψ has as many components as the EM gauge field A^μ, we do *not* write these components as Ψ^μ, but either simply as Ψ (and think of it as a four-dimensional column vector), or as Ψ_a with *spinor index* $a = 1, 2, 3, 4$. The purpose of this notational difference is to denote the different ways the fields transform under Lorentz transformations:

$$A'^\mu = \Lambda^\mu{}_\nu A^\nu \quad \text{versus} \quad \Psi'_a = (\Lambda_{\frac{1}{2}})_{ab}\Psi_b \quad \text{or simply} \quad \Psi' = \Lambda_{\frac{1}{2}}\Psi\,. \tag{7.87}$$

  Note that $\Lambda \equiv \Lambda^\mu{}_\nu$ and $\Lambda_{\frac{1}{2}} = M(\Lambda)$ are *not* the same 4×4 matrices!

10 | Dirac adjoint:

We would like to find a Lagrangian density for the Dirac equation; since this must be a Lorentz scalar, we ask the question:

How to form Lorentz scalars from Dirac spinors?

i | First try: $\Psi^\dagger\Psi$

$$\Psi'^\dagger\Psi' = \Psi^\dagger \underbrace{\Lambda^\dagger_{\frac{1}{2}}\Lambda_{\frac{1}{2}}}_{\neq\mathbb{1}} \Psi \neq \Psi^\dagger\Psi \tag{7.88}$$

$\Lambda_{\frac{1}{2}}$ is *not* unitary because $\mathcal{S}^{\mu\nu}$ is not Hermitian for boosts ($\mu = 0$ and $\nu = 1, 2, 3$).

This is a consequence of the ↑ *non-compactness* of the Lorentz group due to boosts.

ii | Define

$$\boxed{\bar\Psi := \Psi^\dagger\gamma^0 \quad ⁂\ \textit{Dirac adjoint}} \tag{7.89}$$

$\overset{\circ}{\rightarrow}$ $\bar{\Psi}'\Psi' = \bar{\Psi}\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\Psi = \bar{\Psi}\Psi \Rightarrow$ Lorentz scalar

Use Eq. (7.80) and Eq. (7.76b) and the Dirac algebra to show this!

11 | Lagrangian:

With these tools, it is reasonable to propose the following Lagrangian density:

$$\boxed{\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(i\gamma^\mu\partial_\mu - M)\Psi = \bar{\Psi}(i\not{\partial} - M)\Psi} \tag{7.90}$$

$\overset{\circ}{\rightarrow}$ Euler-Lagrange equations = Dirac equation

- Note that in explicit index notation, the Lagrangian density reads

$$\mathcal{L}_{\text{Dirac}} = i\bar{\Psi}_a\gamma^\mu_{ab}(\partial_\mu\Psi_b) - M\bar{\Psi}_a\Psi_a \tag{7.91}$$

where sums over pairs of spinor indices are implied.

The Euler-Lagrange equations follow again by treating Ψ_a and $\bar{\Psi}_a$ as independent fields:

$$0 \overset{!}{=} \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\bar{\Psi}_a} - 0 = i\gamma^\mu_{ab}(\partial_\mu\Psi_b) - M\Psi_a = \left[(i\not{\partial} - M)\Psi\right]_a \tag{7.92a}$$

$$0 \overset{!}{=} \frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial\Psi_a} - \partial_\mu\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\Psi_a)} = -M\bar{\Psi}_a - i(\partial_\mu\bar{\Psi}_b)\gamma^\mu_{ba} \overset{\circ}{=} \left[\overline{(i\not{\partial} - M)\Psi}\right]_a \tag{7.92b}$$

Note that the two equations are Dirac adjoints of each other.

- Let us check that $\mathcal{L}_{\text{Dirac}}$ is a Lorentz scalar:

$$\begin{aligned}
\mathcal{L}'_{\text{Dirac}} &= \bar{\Psi}'\left(i\gamma^\mu\partial'_\mu - M\right)\Psi' && (7.93a)\\
&= \bar{\Psi}\Lambda_{\frac{1}{2}}^{-1}\left(i\gamma^\mu\Lambda_\mu{}^\nu\partial_\nu - M\right)\Lambda_{\frac{1}{2}}\Psi && (7.93b)\\
&= \bar{\Psi}\left(i\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}\Lambda_\mu{}^\nu\partial_\nu - M\right)\Psi && (7.93c)\\
&\overset{7.75}{=} \bar{\Psi}\left(i\Lambda^\mu{}_\rho\gamma^\rho\Lambda_\mu{}^\nu\partial_\nu - M\right)\Psi && (7.93d)\\
&= \bar{\Psi}\left(i\gamma^\nu\partial_\nu - M\right)\Psi = \mathcal{L}_{\text{Dirac}} && (7.93e)
\end{aligned}$$

Here we used the following fact:

$$\boxed{\text{The gamma matrices transform } \textit{not} \text{ like Lorentz vectors: } \gamma'^\mu = \gamma^\mu.} \tag{7.94}$$

This is good because otherwise the Dirac equation would be different in different inertial systems.

This also means that slashed quantities (like $\not{\partial} = \gamma^\mu\partial_\mu$) are *not* Lorentz scalars. Think of it like this: they do *not* have a Lorentz index, but they *do* have a two spinor indices (which we don't write) because they are *matrices*. To get rid of these indices, you must pair them with the indices of spinor fields. That is, slashed quantities become Lorentz scalars if put between two Dirac spinors like in the Dirac Lagrangian: $\bar{\Psi}\not{\partial}\Psi$ *is* a scalar field.

12 | Conserved current:

Now that we have a Lagrangian, it is just a straightforward application of Noether's theorem to obtain the conserved current associated to global phase rotations:

i | $\sphericalangle$ Global phase rotations:

Eq. (7.90) is clearly invariant under global phase rotations of the spinors:

$$\Psi'(x) = e^{i\alpha}\Psi(x) \quad \text{for} \quad \alpha \in [0, 2\pi) \tag{7.95}$$

with infinitesimal generator $|\alpha| = |w| \ll 1$

$$\Psi'(x) = \Psi(x) + iw\Psi(x) \equiv \Psi(x) + w\,\delta\Psi(x) \quad \Rightarrow \quad \delta\Psi = i\Psi \tag{7.96}$$

→ Continuous symmetry:

$$\mathcal{L}_{\text{Dirac}}(\Psi, \partial\Psi) = \mathcal{L}_{\text{Dirac}}(\Psi', \partial\Psi') \tag{7.97}$$

ii | Noether theorem 6.85 → Conserved current density:

A straightforward calculation yields:

$$j^\mu_{\text{Dirac}} \stackrel{6.84}{=} -\frac{\partial\mathcal{L}_{\text{Dirac}}}{\partial(\partial_\mu\Psi_a)}\delta\Psi_a \stackrel{7.91}{=} \bar{\Psi}_b\gamma^\mu_{ba}\Psi_a = \bar{\Psi}\gamma^\mu\Psi\,. \tag{7.98}$$

$$\boxed{j^\mu_{\text{Dirac}} = \bar{\Psi}\gamma^\mu\Psi \quad \text{with} \quad \partial_\mu j^\mu_{\text{Dirac}} = 0} \tag{7.99}$$

Since the Lagrangian density $\mathcal{L}_{\text{Dirac}}$ is a Lorentz scalar, this Noether current must be a 4-vector. We can check this explicitly:

$$j'^\mu_{\text{Dirac}} = \bar{\Psi}'\gamma^\mu\Psi' = \bar{\Psi}\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}\Psi \stackrel{7.75}{=} \Lambda^\mu{}_\nu\bar{\Psi}\gamma^\nu\Psi = \Lambda^\mu{}_\nu j^\mu_{\text{Dirac}}\,. \tag{7.100}$$

iii | Conserved Noether charge:

$$Q = \int \mathrm{d}^3x\,\bar{\Psi}\gamma^0\Psi = \int \mathrm{d}^3x\,\underbrace{\Psi^\dagger\Psi}_{\geq 0} \geq 0 \tag{7.101}$$

→

$$\boxed{\text{Conserved norm on } L^2 \otimes \mathbb{C}^4\text{:} \quad \|\Psi\|^2 := \int \mathrm{d}^3x\,\Psi^\dagger\Psi} \tag{7.102}$$

- ¡! The positive-definite density $\Psi^\dagger\Psi = \bar{\Psi}\gamma^0\Psi$ is the time-component of a 4-vector and therefore *not* Lorentz invariant. However, the Noether charge Q is a Lorentz scalar so that the norm is Lorentz invariant: $\|\Psi'\| \stackrel{*}{=} \|\Psi\|$.

  Note that not all Noether charges are Lorentz scalars. The total field momentum Eq. (6.92), for example, is a 4-vector; similarly, the total field angular momentum Eq. (6.117) is a tensor of rank 2. However, it can be shown that the Noether charges of *internal* symmetries (like the $U(1)$ symmetry considered here) are necessarily Lorentz scalars (↑ *Coleman-Mandula theorem* [90]).

  Let us prove $Q'_a = Q_a$ in the case where the Noether current j^μ_a has no other Lorentz index (and the internal group generators commute with the generators of Lorentz transformations):

a | We consider an infinitesimal Lorentz transformation.

Coordinates transform according to Eq. (6.78),

$$\delta^{\alpha\beta} x^\mu = \frac{1}{2}\left(\eta^{\alpha\mu} x^\beta - \eta^{\beta\mu} x^\alpha\right) , \tag{7.103}$$

and, as a 4-vector, the *components* of the current transform in the same way:

$$\delta^{\alpha\beta} j_a^\mu = \frac{1}{2}\left(\eta^{\alpha\mu} j_a^\beta - \eta^{\beta\mu} j_a^\alpha\right) \,\hat{=}\, j_a^\nu \left(\partial_\nu \delta^{\alpha\beta} x^\mu\right) . \tag{7.104}$$

(The labels a of the internal symmetry do not mix under this transformation because the internal symmetry is assumed to commute with Lorentz transformations.)

The generator of Lorentz transformations acts then according to Eq. (6.81) on the current *field*

$$-iG^{\alpha\beta} j_a^\mu(x) = \delta^{\alpha\beta} j_a^\mu - (\partial_\nu j_a^\mu)\delta^{\alpha\beta} x^\nu . \tag{7.105}$$

In the following we suppress the indices $\alpha\beta$ whenever possible.

b | It is easy to check that $\partial_\nu \delta x^\nu = 0$; furthermore, we know that $\partial_\nu j_a^\nu = 0$ from the Noether theorem. Together, this allows us to write the action of infinitesimal Lorentz transformations on the current as a 4-divergence:

$$-iGj_a^\mu(x) = \underbrace{(\partial_\nu j_a^\nu)}_{=0}\delta x^\mu + j_a^\nu(\partial_\nu \delta x^\mu) - (\partial_\nu j_a^\mu)\delta x^\nu - j_a^\mu \underbrace{(\partial_\nu \delta x^\nu)}_{=0} \tag{7.106a}$$

$$= \partial_\nu \left(j_a^\nu \delta x^\mu - j_a^\mu \delta x^\nu\right) . \tag{7.106b}$$

Here we used that $\delta j_a^\mu = j_a^\nu(\partial_\nu \delta x^\mu)$.

c | We finally obtain for the infinitesimal Lorentz transformation of the Noether charge:

$$-iGQ_a = \int \mathrm{d}^3x\,(-iGj_a^0) \tag{7.107a}$$

$$= \int \mathrm{d}^3x\,\partial_\nu\left(j_a^\nu \delta x^0 - j_a^0 \delta x^\nu\right) \tag{7.107b}$$

$$= \int \mathrm{d}^3x\,\partial_i\left(j_a^i \delta x^0 - j_a^0 \delta x^i\right) \tag{7.107c}$$

Gauss's theorem

$$= \int_\partial \mathrm{d}\sigma_i\left(j_a^i \delta x^0 - j_a^0 \delta x^i\right) = 0 \tag{7.107d}$$

In the last step we used that on the surface ∂ (typically spatial infinity) all fields vanish (for wavefunctions in L^2 this is clearly true).

Thus any Noether charge derived from internal symmetries transforms as a Lorentz scalar. In particular, the Dirac norm $\|\Psi\|$ is invariant under Lorentz transformations of the bispinor fields $\Psi(x)$.

13 | Hamiltonian:

i | Since the Dirac equation is first order in time, we can easily bring it into Schrödinger form and identify the Hamiltonian as the generator of time translations:

$$\text{Eq. (7.83)} \quad \Leftrightarrow \quad \left[i\hbar\gamma^0\partial_t + i\hbar c\gamma^i\partial_i - mc^2\right]\Psi = 0 \tag{7.108}$$

Use $(\gamma^0)^2 = \mathbb{1} \rightarrow$

$$i\hbar\partial_t\Psi = \left[-i\hbar c\gamma^0\gamma^i\partial_i + \gamma^0 mc^2\right]\Psi \tag{7.109}$$

ii | Let us define the new matrices:

$$\boxed{\beta := \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \alpha_i := \gamma^0\gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \quad i = 1,2,3} \tag{7.110}$$

with $\beta^2 = \mathbb{1} = \alpha_i^2$ and $\{\alpha_i, \alpha_j\} = 0 = \{\alpha_i, \beta\}$ for $i \neq j$, and in particular

$$\beta^\dagger = \beta \quad \text{and} \quad \alpha_i^\dagger = \alpha_i \,. \tag{7.111}$$

¡! Note that the spatial gamma matrices are *anti-Hermitian*: $(\gamma^i)^\dagger = -\gamma^i$.

iii | With these matrices we can define the …

$$\boxed{\begin{array}{l} \text{⁂ \textit{Dirac Hamiltonian}:} \\ \hat{H}_{\text{Dirac}} = -i\hbar c\,\vec{\alpha}\cdot\nabla + \beta\,mc^2 = c\,\vec{\alpha}\cdot\vec{p} + \beta\,mc^2 \end{array}} \tag{7.112}$$

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and the ↓ *momentum operator* $\vec{p} = -i\hbar\nabla$.

→ The Dirac Hamiltonian is Hermitian:
(With respect to the standard inner product on $L^2 \otimes \mathbb{C}^4$):

$$\hat{H}^\dagger_{\text{Dirac}} = c\,\vec{\alpha}^\dagger \cdot \vec{p}^\dagger + \beta^\dagger\,mc^2 = c\,\vec{\alpha}\cdot\vec{p} + \beta\,mc^2 = \hat{H}_{\text{Dirac}} \tag{7.113}$$

Here we use that the momentum operator is self-adjoint (Hermitian) for (a dense subset of) functions in $L^2(\mathbb{R}^3, \mathbb{C})$:

$$\langle\psi|\vec{p}\phi\rangle = \int \mathrm{d}^3x\,\psi^*(-i\hbar\nabla\phi) = \int \mathrm{d}^3x\,(-i\hbar\nabla\psi)^*\phi = \langle\vec{p}\psi|\phi\rangle\,. \tag{7.114}$$

We used partial integration and $\lim_{|\vec{x}|\to\infty}\phi(\vec{x}) = 0 = \lim_{|\vec{x}|\to\infty}\psi(\vec{x})$ for admissible functions.

iv | The Dirac equation then takes the Schrödinger form

$$\boxed{i\hbar\partial_t\Psi(x) = \hat{H}_{\text{Dirac}}\Psi(x)} \tag{7.115}$$

In this form its Lorentz covariance is no longer manifest.

v | Eq. (7.102) conserved → ∢ Inner product on $L^2 \otimes \mathbb{C}^4$:

$$\boxed{\langle\Psi|\Phi\rangle := \int \mathrm{d}^3x\,\Psi^\dagger(t,\vec{x})\Phi(t,\vec{x}) \quad \text{with} \quad \|\Psi\| = \sqrt{\langle\Psi|\Psi\rangle}} \tag{7.116}$$

This inner product is constant under the evolution of the Dirac equation:

$$\text{Eq. (7.113) \& Eq. (7.115)} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi|\Phi\rangle \stackrel{\circ}{=} 0 \tag{7.117}$$

- This generalizes our previous finding in Eq. (7.102) about the conserved norm.

- That the inner product is constant is straightforward to show:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi|\Phi\rangle = \int \mathrm{d}^3x\,\left[\Psi^\dagger\dot{\Phi} + \dot{\Psi}^\dagger\Phi\right] \tag{7.118a}$$

$$\overset{7.115}{=} \frac{1}{i\hbar}\int \mathrm{d}^3x\,\left[\Psi^\dagger\left(\hat{H}_{\mathrm{Dirac}}\Phi\right) - \left(\hat{H}_{\mathrm{Dirac}}\Psi\right)^\dagger\Phi\right] \tag{7.118b}$$

$$\overset{7.113}{=} \frac{1}{i\hbar}\int \mathrm{d}^3x\,\left[\Psi^\dagger\left(\hat{H}_{\mathrm{Dirac}}\Phi\right) - \Psi^\dagger\left(\hat{H}_{\mathrm{Dirac}}\Phi\right)\right] = 0 \tag{7.118c}$$

14 | Conclusion:

Let us summarize our findings and compare them to the Klein-Gordon equation:

	Klein-Gordon Equation $(\partial^2+M^2)\phi=0$	**Dirac Equation** $(i\partial\!\!\!/-M)\Psi=0$
Time derivative	second order	first order
Function space	$L^2(\mathbb{R}^{1,3},\mathbb{C})$	$L^2(\mathbb{R}^{1,3},\mathbb{C}^2\oplus\mathbb{C}^2)$
Wavefunction	Complex scalar field $\phi(x)$	Complex bispinor field $\Psi(x)$
Conserved form	$i\int \mathrm{d}^3x\,(\phi_1^*\dot{\phi}_2-\dot{\phi}_1^*\phi_2)$	$\int \mathrm{d}^3x\,\Psi_1^\dagger\Psi_2$
Positive definite?	✗	✓
Hermitian Hamiltonian?	✗	✓

→ What about the *eigenenergies* and *eigenstates* of $\hat{H}_{\mathrm{Dirac}}$?

7.2.1. Free-particle solutions of the Dirac equation

15 | Eq. (7.86): Solutions of the Dirac equation satisfy the Klein-Gordon equation component-wise:

$\xrightarrow{\text{Eq. (7.34)}}$ Ansatz:

$$\Psi^\pm(x) = \psi^\pm(p)e^{\mp\frac{i}{\hbar}px} \quad\text{with}\quad p^0 = \frac{E}{c} = \sqrt{\vec{p}^2+m^2c^2} > 0 \tag{7.119}$$

with complex-valued four-component ⁂ *bispinor*

$$\psi^\pm(p) \equiv \begin{pmatrix}\psi_L^\pm\\ \psi_R^\pm\end{pmatrix} \in \mathbb{C}^4 \simeq \mathbb{C}^2\oplus\mathbb{C}^2\,. \tag{7.120}$$

- We set $p^0>0$ for both positive (+) and negative (−) energy/frequency solutions and change the sign of p in the exponent (to simplify the discussion below).
- Note that $px = p_\mu x^\mu = Et - \vec{p}\cdot\vec{x}$.

16 | Eq. (7.119) in Eq. (7.83) yields:

$$(\pm\gamma^\mu p_\mu - mc)\psi^\pm(p) = \begin{pmatrix}-mc & \pm p\sigma\\ \pm p\bar{\sigma} & -mc\end{pmatrix}\begin{pmatrix}\psi_L^\pm\\ \psi_R^\pm\end{pmatrix} = 0 \tag{7.121}$$

with $p\sigma = p_\mu\sigma^\mu$ and $\sigma^\mu = (\mathbb{1},\sigma^x,\sigma^y,\sigma^z)$ and $\bar{\sigma}^\mu = (\mathbb{1},-\sigma^x,-\sigma^y,-\sigma^z)$.

17 | Mathematical facts (check these!):

- $(p\sigma)(p\bar{\sigma}) \stackrel{\circ}{=} p^2 = m^2c^2$
- Eigenvalues of $p\sigma$ and $p\bar{\sigma}$: $p^0 \pm |\vec{p}| \to$ for $p^0 > 0$ and $m \neq 0$ *positive spectrum*

  → $p\sigma$ and $p\bar{\sigma}$ are *invertible* and the positive square roots $\sqrt{p\sigma}$ and $\sqrt{p\bar{\sigma}}$ are Hermitian.

18 | $\sphericalangle\ \psi_L^\pm \equiv \sqrt{p\sigma}\,\xi^\pm$ with arbitrary, normalized $[(\xi^\pm)^\dagger\xi^\pm = 1]$ ⁂ *spinor* $\xi^\pm \in \mathbb{C}^2$:

$$\text{Eq. (7.121)} \quad \Rightarrow \quad -mc\sqrt{p\sigma}\,\xi^\pm \pm p\sigma\,\psi_R^\pm = 0 \tag{7.122}$$

Use $\sqrt{p\sigma}\sqrt{p\bar{\sigma}} = mc$:

$$\psi_R^\pm = \pm\frac{mc}{\sqrt{p\sigma}}\xi^\pm = \pm\sqrt{p\bar{\sigma}}\xi^\pm \tag{7.123}$$

→ $\psi_L^\pm$ and $\psi_R^\pm$ are now parametrized by the spinor $\xi^\pm \in \mathbb{C}^2$ (which is unconstrained!).

The second equation in Eq. (7.121) yields the same solution.

19 | Solutions:

Let us adopt the more conventional notation

$$\begin{aligned}\xi^+ &\mapsto \xi \\ \xi^- &\mapsto \eta\end{aligned} \quad \text{and} \quad \begin{aligned}\psi^+ &\mapsto u \\ \psi^- &\mapsto v\end{aligned} \tag{7.124}$$

and choose the spinor basis ξ^s, η^s $(s = \uparrow, \downarrow)$ with

$$\xi^\uparrow, \eta^\uparrow = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \xi^\downarrow, \eta^\downarrow = \begin{pmatrix}0\\1\end{pmatrix}. \tag{7.125}$$

Then linearly independent solutions of the free Dirac equation can be written as:

$$\boxed{\begin{aligned}\Psi^+_{\vec{p},s}(x) &= \underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\xi^s\\ \sqrt{p\bar{\sigma}}\,\xi^s\end{pmatrix}}_{u^s(\vec{p})} e^{-\frac{i}{\hbar}px} \quad \text{(positive energy solutions)} & \text{(7.126a)}\\ \Psi^-_{\vec{p},s}(x) &= \underbrace{\begin{pmatrix}\sqrt{p\sigma}\,\eta^s\\ -\sqrt{p\bar{\sigma}}\,\eta^s\end{pmatrix}}_{v^s(\vec{p})} e^{+\frac{i}{\hbar}px} \quad \text{(negative energy solutions)} & \text{(7.126b)}\end{aligned}}$$

with $p^\mu = (p^0, \vec{p})$, $p^0 = \sqrt{\vec{p}^2 + m^2c^2} > 0$ and $s = \uparrow, \downarrow$.

→ *Four* linearly independent solutions for each 3-momentum $\vec{p}$ ($\pm$ and $s = 1, 2$).

You can easily check that Eq. (7.126) form an orthogonal eigenbasis of the Dirac Hamiltonian:

$$\hat{H}_{\text{Dirac}}\Psi^\pm_{\vec{p},s} \stackrel{\circ}{=} \pm E_{\vec{p}}\Psi^\pm_{\vec{p},s} \quad \text{with spectrum} \quad E_{\vec{p}} = \sqrt{p^2c^2 + m^2c^4}\,. \tag{7.127}$$

Their orthogonality follows with the identities

$$[u^r(\vec{p})]^\dagger u^s(\vec{p}) \stackrel{\circ}{=} \tfrac{2}{c}E_{\vec{p}}\delta^{rs}\,, \quad [v^r(\vec{p})]^\dagger v^s(\vec{p}) \stackrel{\circ}{=} \tfrac{2}{c}E_{\vec{p}}\delta^{rs}\,, \quad [u^r(\vec{p})]^\dagger v^s(-\vec{p}) \stackrel{\circ}{=} 0\,. \tag{7.128}$$

$$\boxed{\to \text{The Dirac equation still has } \textit{negative-energy solutions}.} \tag{7.129}$$

20 | Interpretation:

- The negative energy solutions are not problematic as long as we consider a single particle (electron) without interactions (this is also why we can apply the Dirac equation to describe the hydrogen atom, → *below*). However, in reality the electron couples to a dynamic electromagnetic field and therefore could emit a photon (thereby lowering its energy). If the negative energy eigenstates really exist, there is no reason why this process should terminate; as a consequence, no stable electrons should exist.

  Dirac writes in Ref. [91]:

  > *It is true that in the case of a steady electromagnetic field we can draw a distinction between those solutions [..] with E positive and those with E negative and may assert that only the former have a physical meaning (as was actually done when the theory was applied to the determination of the energy levels of the hydrogen atom), but if a perturbation is applied to the system it may cause transitions from one kind of state to the other. In the general case of an arbitrarily varying electromagnetic field we can make no hard-and-fast separation of the solutions of the wave equation into those referring to positive and those to negative kinetic energy. Further, in the accurate quantum theory in which the electromagnetic field also is subjected to quantum laws, transitions can take place in which the energy of the electron changes from a positive to a negative value even in the absence of any external field, the surplus energy [..] being spontaneously emitted in the form of radiation. [..] Thus we cannot ignore the negative-energy states without giving rise to ambiguity in the interpretation of the theory.*

  Dirac suggested a "fix" for this problem [91]: Because the electron is a fermion, it obeys the Pauli exclusion principle. Thus one can imagine that (for some reason) all the negative energy states *are already occupied* by electrons. The electrons we see can then only occupy the positive energy states and cannot decay to states of arbitrarily low energy. This construct is know as the ↑ *hole theory* because creating a "hole" in this ↑ *Dirac sea* of electrons with negative energy can be viewed as an excitation with *positive* energy. Dirac's holes are of course a precursor to what we know today as ↑ *antiparticles*. (Dirac didn't think of it this way, he conjectured that the holes in his sea of electrons are the *protons*!)

- However, Dirac's interpretation is not how we deal with the negative-energy solutions today: Within the modern framework of ↑ *relativistic quantum field theories*, the four single-particle wave functions are associated (through "second" quantization of the Dirac field and the construction of a fermionic ↑ *Fock space*) to two particle types, both with *positive* energy and two internal spin-$\frac{1}{2}$ states:

	Type	Momentum	Spin	Energy	Charge
$\Psi^+_{\vec{p},\uparrow}$:	fermion	$+\vec{p}$	$+\frac{1}{2}$	$+E_{\vec{p}}$	$+1$
$\Psi^+_{\vec{p},\downarrow}$:	fermion	$+\vec{p}$	$-\frac{1}{2}$	$+E_{\vec{p}}$	$+1$
$\Psi^-_{\vec{p},\uparrow}$:	antifermion	$+\vec{p}$	$-\frac{1}{2}$	$+E_{\vec{p}}$	-1
$\Psi^-_{\vec{p},\downarrow}$:	antifermion	$+\vec{p}$	$+\frac{1}{2}$	$+E_{\vec{p}}$	-1

(7.130)

Here "Spin" refers to the ↓ *spin-polarization quantum number* $m_z = \pm\frac{1}{2}$.

→ *Take home message:*

$$\boxed{\text{Relativistic quantum mechanics predicts } \textit{spin} \text{ and } \textit{antiparticles}.} \tag{7.131}$$

The negative energy solutions (and therefore the existence of antiparticles) are a necessary feature of relativistic quantum mechanics (more precisely: relativistic quantum field theories, via the ↑ *CPT-theorem*).

By contrast, the fact that particles can have an internal angular momentum (spin), and that this angular momentum can take half-integer values $S = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ is not a relativistic feature *per se*: Spin enters quantum mechanics the moment one considers *spatial rotations* and its representations on the Hilbert space. Because these can be ↑ *projective*, one is forced to study the irreducible linear representations of SU(2) – the double cover of the rotation group SO(3) – which happen to be labeled by the "spin quantum numbers" $S = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. Now, since the rotation group is a subgroup of the homogeneous Lorentz group, $\mathrm{SO}(3) \subset \mathrm{SO}^+(1,3)$, the moment a quantum theory is relativistic [i.e., features a representation of $\mathrm{SO}^+(1,3)$], spin enters the stage automatically. However, you can describe quantum particles with spin *without* making quantum mechanics relativistic.

- The Dirac equation applies to all spin-$\frac{1}{2}$ fermions. The most prominent example is of course the electron e^- and its associated antiparticle, the positron e^+. However, all other elementary fermions, namely leptons (like the muon/antimuon, the tau/antitau and the neutrinos) and the six quark/antiquark pairs, are described by the Dirac equation as well.

7.2.2. The relativistic hydrogen atom

21 | Dirac equation with a static EM field:

To couple the Dirac field Ψ in a gauge- and covariant way to a static EM field A^μ, we use the same trick as for the Klein-Gordon equation:

← *Minimal coupling* Eq. (7.38) →

$$\partial_\mu \mapsto D_\mu = \partial_\mu + iQA_\mu \quad \Rightarrow \quad \not{\partial} \mapsto \not{D} = \not{\partial} + iQ\not{A} = \gamma^\mu\partial_\mu + iQ\gamma^\mu A_\mu \tag{7.132}$$

For an electron it is $Q = -\frac{e}{\hbar c}$ with $e > 0$.

→

$$\boxed{(i\not{D} - M)\Psi = 0} \tag{7.133}$$

In this form, the Dirac equation is manifest Lorentz- and gauge invariant.

We can expand Eq. (7.133) to obtain a less abstract (but more convoluted) expression:

$$\left[i\gamma^\mu\partial_\mu - Q\gamma^\mu A_\mu - M\right]\Psi = 0 \tag{7.134a}$$

$$\Leftrightarrow \quad \left[i\hbar\gamma^0\partial_t + i\hbar c\gamma^i\partial_i - q\gamma^0\varphi + q\gamma^i A_i - mc^2\right]\Psi = 0 \tag{7.134b}$$

$$\Leftrightarrow \quad \left[i\hbar\partial_t + i\hbar c\,\vec{\alpha}\cdot\nabla - q\,\varphi + q\,\vec{\alpha}\cdot\vec{A} - \beta\,mc^2\right]\Psi = 0 \tag{7.134c}$$

Here we used $Q = \frac{q}{\hbar c}$, $M = \frac{mc}{\hbar}$, and $A_\mu = (\varphi, -\vec{A})$; q is the charge of the particle.

In Schrödinger form the Dirac equation reads then:

$$i\hbar\partial_t\Psi = \left[-i\hbar c\,\vec{\alpha}\cdot\nabla + q\,\varphi - q\,\vec{\alpha}\cdot\vec{A} + \beta\,mc^2\right]\Psi \tag{7.135a}$$

$$i\hbar\partial_t\Psi = \underbrace{\left[c\,\vec{\alpha}\cdot\left(\vec{p} - \tfrac{q}{c}\vec{A}\right) + q\,\varphi + \beta\,mc^2\right]}_{\hat{H}_{\mathrm{Dirac},A}}\Psi \tag{7.135b}$$

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22 | Choose the Coulomb potential (of the proton)

$$\varphi(x) = \frac{e}{|\vec{x}|} \quad \text{and} \quad \vec{A} = \vec{0} \tag{7.136}$$

and set $q = -e$ (charge of the electron) $\rightarrow$

$$i\hbar\partial_t\Psi = \left[-i\hbar c\,\vec{\alpha}\cdot\nabla - \frac{e^2}{|\vec{x}|} + \beta\,mc^2\right]\Psi \tag{7.137}$$

With the ansatz $\Psi(t,\vec{x}) = \psi(\vec{x})e^{-\frac{i}{\hbar}Et}$ one obtains the time-independent eigenvalue problem

$$\left[-i\hbar c\,\vec{\alpha}\cdot\nabla - \frac{e^2}{|\vec{x}|} + \beta\,mc^2 - E\right]\psi(\vec{x}) = 0 \quad \text{with} \quad \psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} : \mathbb{R}^3 \to \mathbb{C}^4 . \tag{7.138}$$

Note that β (unlike α_i) is an off-diagonal block matrix that mixes the two spinors ψ_+ and ψ_-; this complicates the solution. However, one can solve Eq. (7.138) *exactly* and compute the eigenvalues E and eigenstates $\psi(\vec{x})$.

23 | Solution: $\overset{*}{\rightarrow}$ Eigenenergies (including the rest energy of the electron):

$$\boxed{E_{n,j} = mc^2\left\{1 + \frac{\alpha^2}{\left[n - j - \frac{1}{2} + \sqrt{\left(j+\frac{1}{2}\right)^2 - \alpha^2}\right]^2}\right\}^{-\frac{1}{2}}} \tag{7.139}$$

with

- ↓ *principal quantum number* $n = 1, 2, \dots$
- ↓ *total angular momentum quantum number* $j = \frac{1}{2}, \frac{3}{2}, \dots, n - \frac{1}{2}$
- ↓ *fine-structure constant* $\alpha \approx \frac{1}{137}$

The principal quantum number $n = 1, 2, \dots$ constrains the allowed *orbital* angular momentum to $l = 0, 1, \dots, n-1$. The allowed *total* angular momentum is then given by the usual rules of angular momentum addition: $|l - \frac{1}{2}| \leq j \leq |l + \frac{1}{2}|$ (in integer steps, $s = \frac{1}{2}$ is the electron spin).

So for example $n = 1$ allows only for $l = 0$ and therefore $j = \frac{1}{2}$; this is the $1S_{1/2}$ orbital and the ground state of the hydrogen atom. For $n = 2$ one finds again $l = 0$ with $j = \frac{1}{2}$ (the $2S_{1/2}$ orbital) but also $l = 1$ with $j = \frac{1}{2}$ and $j = \frac{3}{2}$ (the $2P_{1/2}$ and $2P_{3/2}$ orbitals – which are no longer degenerate because $E_{2,1/2} \neq E_{2,3/2}$).

This result explains why in the hydrogen spectrum the degeneracy of the $2S_{1/2}$ and $2P_{3/2}$ orbitals is lifted whereas the $2S_{1/2}$ orbital remains degenerate with the $2P_{1/2}$ orbital (← *fine-structure*).

$\rightarrow$

$$\boxed{\text{The Dirac equation explains the fine-structure of the hydrogen atom ☺.}} \tag{7.140}$$

Note: You may have encountered the following Hamiltonian for the hydrogen atom with added *relativistic corrections*:

$$\hat{H}_{\text{rel}} = \underbrace{\frac{\vec{p}^2}{2m} - \frac{e^2}{r}}_{\substack{\text{Non-rel.}\\ \text{hydrogen atom}}} \underbrace{\underbrace{- \frac{1}{2mc^2}\left(\frac{p^2}{2m}\right)^2}_{\text{Rel. kinetic energy}} + \underbrace{\frac{e^2}{2m^2c^2}\frac{\vec{L}\cdot\vec{S}}{r^3}}_{\text{Spin-orbit coupling}} - \underbrace{\frac{e^2\hbar^2}{8m^2c^2}\Delta\left(\frac{1}{r}\right)}_{\text{Darwin term}}}_{\text{Relativistic corrections}} . \tag{7.141}$$

This Hamiltonian can reproduce the fine-structure as well. It has several drawbacks, though:

- It is only an *approximation*.
- It is hard to solve (perturbation theory!).
- The Schrödinger equation $i\hbar\partial_t\psi = \hat{H}_{\text{rel}}\psi$ is not manifestly Lorentz covariant.
- The relativistic corrections are *ad hoc* and seemingly independent of each other.

Luckily, Eq. (7.141) does not have to appear out of thin air; one can show via a complicated derivation (↑ *Foldy-Wouthuysen transformation*) that it is indeed the non-relativistic limit [with corrections in order $(v/c)^2$] of the Dirac equation Eq. (7.138) in a Coulomb potential (without the rest energy mc^2 of the electron).

7.2.3. The electron g-factor

Besides the fine structure, there is one other "mystery" that the relativistic treatment of the electron in terms of the Dirac equation finally explains: The non-classical ratio between the electrons internal magnetic moment and its spin.

24 | $\sphericalangle$ Dirac electron in homogeneous magnetic field $\vec{B} = \nabla \times \vec{A}$ ($\varphi = 0$):

$$\text{Eq. (7.135b)} \xrightarrow{\Psi=\psi e^{-\frac{i}{\hbar}Et}} \left[c\,\vec{\alpha}\cdot\left(\vec{p}+\tfrac{e}{c}\vec{A}\right)+\beta\,mc^2-E\right]\psi = 0 \tag{7.142}$$

with bispinor

$$\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} : \mathbb{R}^3 \to \mathbb{C}^4 . \tag{7.143}$$

Using Eq. (7.110) we can write this equation in terms of the two spinors:

$$(-c\vec{\sigma}\vec{\pi}-E)\,\psi_L + mc^2\psi_R = 0 \tag{7.144a}$$
$$(+c\vec{\sigma}\vec{\pi}-E)\,\psi_R + mc^2\psi_L = 0 \tag{7.144b}$$

Here we used $\vec{\pi} = \vec{p} + \frac{e}{c}\vec{A}$ and introduced $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$.

We can now use one of the two equations to decouple the system:

$$(c\vec{\sigma}\vec{\pi}+E)\,(c\vec{\sigma}\vec{\pi}-E)\,\psi_R + (mc^2)^2\psi_R = 0 \tag{7.145a}$$
$$\Leftrightarrow \quad c^2(\vec{\sigma}\vec{\pi})^2\psi_R - \left[E^2-(mc^2)^2\right]\psi_R = 0 \tag{7.145b}$$

25 | $\overset{\circ}{\to}$ Non-relativistic approximation:

We can use $E^2-(mc^2)^2 = (E-mc^2)(E+mc^2) \approx 2mc^2\tilde{E}$ with $\tilde{E} = E - mc^2$ to find a non-relativistic approximation of Eq. (7.145b):

$$\frac{1}{2m}(\vec{\sigma}\vec{\pi})^2\psi_R = \tilde{E}\psi_R \tag{7.146}$$

Last, use the Pauli algebra $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$ and $B_k = \varepsilon_{ijk}(\partial_i A_j)$ to show that $(\vec{\sigma}\vec{\pi})^2 \overset{\circ}{=} \vec{\pi}^2 + \frac{\hbar e}{c}\vec{\sigma}\cdot\vec{B}$. We end up with the non-relativistic, time-independent Schrödinger equation of a charged particle in a magnetic field with a spin-dependent Zeeman term:

$$\Bigg[\underbrace{\frac{1}{2m}\left(\vec{p}+\frac{e}{c}\vec{A}\right)^2}_{\text{Particle in mag. field}} + \underbrace{\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}}_{\text{Zeeman effect}}\Bigg]\psi_R = \tilde{E}\psi_R \tag{7.147}$$

→ Potential energy of electron in magnetic field:

$$E_{\text{mag}} \overset{\text{def}}{=} -\vec{\mu}\cdot\vec{B} \overset{7.147}{=} \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B} \tag{7.148}$$

26 | → Magnetic moment (operator) of the electron:

$$\vec{\mu}_e = -\frac{e\hbar}{2mc}\vec{\sigma} = g_e\frac{\mu_B}{\hbar}\vec{S} \tag{7.149}$$

with ↓ *spin operator* $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ and ↓ *Bohr magneton* $\mu_B = \frac{e\hbar}{2mc}$ and

$$\boxed{⁂\ \textit{Electron g-factor}\quad g_e = -2\,.} \tag{7.150}$$

27 | Comments:

- What makes Eq. (7.149) with $g_e = -2$ remarkable is that it is *not* what one would expect if the magnetic moment would be caused by a charge flying along a tiny orbit with angular momentum $\vec{S}$. Indeed, a straightforward classical calculation yields for the relation between magnetic moment and (orbital) angular momentum $\vec{L}$:

$$\vec{\mu}_L = g_L\frac{\mu_B}{\hbar}\vec{L} \quad \text{with} \quad g_L = -1\,. \tag{7.151}$$

  So, quite surprisingly, the Dirac equation predicts that the internal angular momentum (= spin) produces "twice as much" magnetic moment as one would naïvely expect.

  That this really is the case can be easily measured: Just apply a magnetic field to hydrogen atoms and observe how strongly their spectral lines split as a function of the magnetic field strength (↑ *anomalous Zeeman effect*). This effect had already been experimentally observed at the end of the 19th century [92, 93]. Since it was unknown at the time that electrons had spin, certain line splittings could not be explained (therefore "anomalous"). The fact that the Dirac equation explains both – the electron spin and its "non-classical" g-factor – is therefore a remarkable feature of relativistic quantum mechanics.

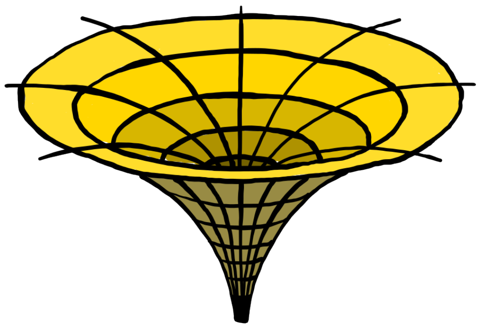
- If one measures the electron g-factor *really, really* precisely, one finds [94]

$$g_e = -2.00231930436118(27)\,. \tag{7.152}$$

  You may notice that this is not *exactly* -2 but a tiny bit off. One cannot explain this deviation with the Dirac equation because it stems from "virtual particles" that modify how the electron interacts with the EM field (and the Dirac equation is a single-particle wave equation). It is therefore remarkable that modern theories *can* explain this deviation perfectly (up to error bars), but for this one needs the machinery of ↑ *relativistic quantum field theory*.

Part II.

General Relativity



↓Lecture 16 [09.04.24]

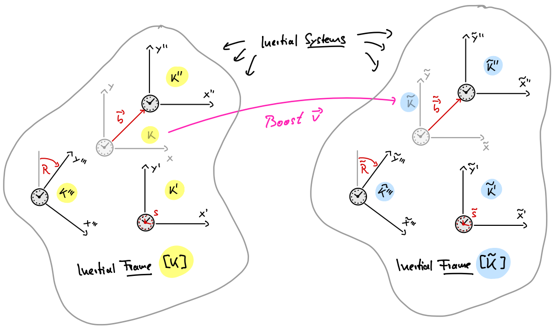
8. Limitations of SPECIAL RELATIVITY

8.1. Reminder: SPECIAL RELATIVITY

1 | SPECIAL RELATIVITY in a nutshell:

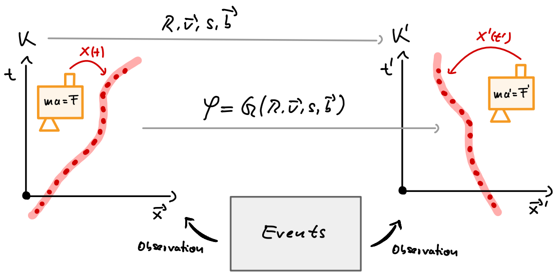
- ← *Inertial frames* [Section 1.1]

  There exists a special class of infinitely extended reference frames (equipped with Cartesian coordinates) in which the law of inertia holds (IN = the trajectories of free particles are straight lines that are traversed with constant velocity). All inertial frames move relative to each other with constant velocities:



- ← *Einstein's principle of (special) relativity* SR [Section 1.3]

  The laws of physics (orange boxes in the sketch below) have the same form in all inertial systems. This extends Galilei's principle of relativity which makes this claim only for the realm of mechanics. The modifier "special" emphasizes that the principle makes only claims about the special class of inertial systems:



We characterized SR previously as follows: *No experiment can distinguish between inertial frames.* This description can be misleading, so let me prevent any misconceptions: What we mean is that there is no local physical experiment that you can perform in a sealed box (at rest in some inertial system) that allows you to figure out in which inertial system your box is at rest. In this way you probe the *form of physical laws* (e.g., whether there is an additional Coriolis term in your equation of motion or not) and thus probe the validity of SR as formulated above.

The statement above does *not* mean that there is no operational way to label specific inertial systems. For example, we can define the (approximate) inertial frame in which the center of Earth is at rest and, for comparison, another inertial frame in which the ↑ *cosmic microwave background (CMB)* has no dipole structure (the latter has a velocity of roughly $360\,\mathrm{km\,s^{-1}}$ wrt. the former). Clearly there are experiments to decide whether you are in one or the other (measure the CMB dipole and/or the velocity of Earth relative to you). This does *not* violate SR though, because all phenomena you observe in these frames of reference are described correctly *by the same equations* (e.g. you can use the same Maxwell equations to describe the CMB radiation in both inertial frames). This is also why the existence of the global inertial frame labeled by a CMB without dipole is not in conflict with SPECIAL RELATIVITY. There is a difference between *physical states* and *physical laws*; SR only makes claims about the latter.

- ← *Lorentz transformations* [Section 1.5]

  The coordinate transformations that map the record of physical events from one inertial system to another are given by Lorentz transformations (more generally: Poincaré transformations). (Proper orthochronous) Lorentz transformations form a group $\mathrm{SO}^+(1,3)$ and are parametrized by a three-dimensional rotation and a three-dimensional boost velocity. They linearly map the spacetime coordinates $(t,\vec{x})$ of an event in one inertial system K to the spacetime coordinates $(t',\vec{x}')$ of the same event in another inertial system K'.

  A pure boost in x-direction has the form:

$$\Lambda(K \xrightarrow{v_x} K'):\quad \begin{cases} ct' = \gamma\left(ct - \frac{v_x}{c}x\right) \\ x' = \gamma(x - v_x t) \\ y' = y \\ z' = z \end{cases} \quad \text{with Lorentz factor} \quad \gamma = \frac{1}{\sqrt{1 - v_x^2/c^2}} \tag{8.1}$$

- ← *Constancy of the speed of light* SL [Section 1.5]

  Lorentz transformations are characterized (and differ from Galilei transformations) by the existence of a *finite maximum velocity* $v_{\max}$. Experience tells us to identify this velocity with the speed of light c. Lorentz transformations then imply that this maximum velocity is the same for all initial observers (← *relativistic addition of velocities*):

$$\text{Experiments} \;\rightarrow\; v_{\max} < \infty \;\Leftrightarrow\; \text{Lorentz transformations} \tag{8.2a}$$

$$v_{\max} = \infty \;\Leftrightarrow\; \text{Galilei transformations} \tag{8.2b}$$

- ← *Tensor calculus* [Chapter 3 and Chapter 4]

  Combining the principle of relativity with the assertion that Lorentz transformations translate between inertial systems implies that the laws of nature must be expressed as equations that are forminvariant under Lorentz transformations (← *Lorentz covariance*). The Lorentz covariance of a theory can be quite tedious to show and even more tedious to ensure when constructing it from scratch. (Recall Maxwell equations in their conventional form!) This is why we prefer equations in which the Lorentz covariance is *manifest*. To achieve this, we developed tensor calculus as a "toolbox" to construct Lorentz covariant equations from Lorentz scalars, vectors, tensors, ….

  For example, the equation of motion for a charged particle in an electromagnetic field reads and transforms as follows:

$$\overbrace{\frac{\mathrm{d}p_i^\mu}{\mathrm{d}\tau} = \frac{q_i}{c} F^\mu{}_\nu\, u_i^\nu}^{\text{Inertial system } K} \quad \xrightarrow[\bar{F}^\mu{}_\nu \,=\, \Lambda^\mu{}_\alpha \Lambda_\nu{}^\beta \bar{F}^\alpha{}_\beta]{\bar{p}^\mu \,=\, \Lambda^\mu{}_\nu p^\nu \qquad \bar{u}^\mu \,=\, \Lambda^\mu{}_\nu u^\nu} \quad \overbrace{\frac{\mathrm{d}\bar{p}_i^\mu}{\mathrm{d}\tau} = \frac{q_i}{c} \bar{F}^\mu{}_\nu\, \bar{u}_i^\nu}^{\text{Inertial system } \bar{K}} \tag{8.3}$$

2 | Problems:

Despite the undeniable success of SPECIAL RELATIVITY, it's not just sunshine and roses:

- What about gravitation?

  In our discussion of SPECIAL RELATIVITY we explicitly avoided the phenomenon of gravitation (we will see below why). This makes SPECIAL RELATIVITY clearly incomplete (and *special*) as a description of nature (which, on very large scales, is dominated by the gravitational force) and asks for a more *general* theory.

- Why are inertial coordinate systems special?

  SPECIAL RELATIVITY describes physics with respect to a particular class of reference frames (inertial frames) in a particular class of coordinates (Cartesian coordinates). Only in these coordinate frames the laws of nature take their "simplest" form and the Lorentz transformation only translates between these special coordinate systems. However, in our very general discussion of differential geometry (Chapter 3) we established the notion of "geometric objects" that are independent of coordinates. We also interpreted coordinates as mathematical auxiliary structures to label events, and denied their physical existence ("coordinates do not exist"). SPECIAL RELATIVITY does not live up to this rather fundamental claim with its focus on inertial coordinate systems. Shouldn't there be a formulation of physics in which coordinates play no role at all?

- What is the origin of inertia?

  Remember Newton's bucket (p. 13)? It's punchline was to argue for the existence of an entity ("absolute space") which determines whether an object is accelerated or not. SPECIAL RELATIVITY, of course, disposes of Newton's absolute space wrt. to which position and velocity can be measured (no *ether*!). The existence of such, however, was never implied by the bucket experiment anyway, which asks about the absolute notion of *acceleration*. And SPECIAL RELATIVITY is silent about the origin of inertia and what determines whether the water in Newton's bucket is concave or flat (we simply assumed that inertial frames exist, we neither asked where they come from nor what makes them inertial in the first place). This situation is clearly unsatisfactory.

3 | Non-problems:

Sometimes one hears that *acceleration* is a problem for SPECIAL RELATIVITY. This is not so:

- Accelerated motion ✓

  SPECIAL RELATIVITY of course describes accelerated objects perfectly well. Recall our concept of 4-acceleration in Section 5.1, the relativistic equation of motion in Eq. (5.6), and the validity of the proper time integral for arbitrary time-like trajectories in Eq. (2.25). Note that these equations are only valid in inertial frames, though.

- Accelerated observers ✓

  While our equations were given in inertial systems (where, according to Einstein's principle of relativity, the laws of physics take the same *and simplest* form), SPECIAL RELATIVITY *can* describe the physics in accelerated non-inertial frames as well (e.g. using the concept of instantaneous rest frames). In such non-inertial coordinate systems the physical laws do *not* take their simplest forms and can look messy (in particular, one cannot Lorentz transform into these frames). This, however, does not mean that we cannot describe what happens in such systems. As an example recall the relativistic rocket of ➔ Problemset 6. (For details see Chapter 6 in MISNER *et al.* [2] and also Einstein's original work [95].)

8.2. The special role of gravity

Let us now focus on the problem of incorporating gravity into SPECIAL RELATIVITY. It is important to understand why the gravitational force poses a fundamental problem for the framework of SPECIAL RELATIVITY (and is not just a technical inconvenience).

Note on nomenclature:

In English, there are two terms with slightly different meaning (if we take *Merriam-Webster* as a reference):

> ***Gravity:*** *the gravitational attraction of the mass of the Earth, the moon, or a planet for bodies at or near its surface*
>
> ***Gravitation:*** *a force manifested by acceleration toward each other of two free material particles or bodies or of radiant-energy quanta*

This distinction has no counterpart in German as far as I can tell (perhaps "Schwerkraft" vs. "Gravitation"?). Given that even the English literature does not seem to be consistent, I will use these two terms interchangeably. Their context will suffice to establish semantic clarity.

4 | Recall Newton's law of universal gravitation:

i | $\sphericalangle$ Mass distribution $\rho(\vec{x}) \rightarrow$

$$\nabla^2\phi(\vec{x}) = 4\pi G\rho(\vec{x}) \quad\rightarrow\quad \text{Gravitational potential } \phi(\vec{x}) \tag{8.4}$$

$G \approx 6.674\,301 \times 10^{-11}\,\mathrm{m^3s^{-2}/kg}$: Gravitational constant

Please appreciate the smallness of G (and therefore the weakness of gravity) as compared to the human-scale units in which it is given. Gravity is, if compared to the other three fundamental forces, by far (really really really far) the weakest force. It is a fundamental unsolved problem of physics why this is so.

→ Equation of motion of test mass (e.g. a satellite):

$$m_{\mathrm{I}}\ddot{\vec{r}} = -m_{\mathrm{G}}\nabla\phi(\vec{r}) \tag{8.5}$$

m_{I}: inertial mass
m_{G}: gravitational mass

We will discuss the relation of m_{I} and m_{G} later (Section 9.1).

ii | Example: Static point mass $M_{\mathrm{G}} \gg m_{\mathrm{G}}$ as source in origin (e.g. Earth):

$$\phi(\vec{x}) = -\frac{GM_{\mathrm{G}}}{|\vec{x}|} \quad\Rightarrow\quad m_{\mathrm{I}}\ddot{\vec{r}} = \vec{F}_{\mathrm{G}} = -G\frac{m_{\mathrm{G}}M_{\mathrm{G}}}{r^2}\hat{r} \tag{8.6}$$

If the source is dynamic as well, $\vec{r}$ is the relative distance vector between the two masses and m_{I} must be replaced by the reduced mass of the two bodies.

iii | Observation: Equations [especially Eq. (8.4)] are not Lorentz covariant!

You can check that they are ← *Galilei invariant*, recall Eq. (1.18).

This is no surprise: We already know from our discussions in Section 6.4 that in RELATIVITY, classical forces can only act *locally*, and not at a distance. Interactions between distant objects must be mediated by dynamical degrees of freedom (a "field") to obey the speed limit for information propagation imposed by Lorentz symmetry. But Newton's gravitational potential ϕ is static and not dynamic!

5 | Problem: "Action at a distance" (Gravitational force acts instantaneously and has no dynamics.)

Isaac Newton writes in a letter to Bentley in 1692 [96]:

> *It is inconceivable, that inanimate brute Matter should, without the Mediation of something else, which is not material, operate upon, and affect other Matter without mutual Contact, as it must be, if Gravitation in the Sense of Epicurus, be essential and inherent in it. And this is one Reason why I desired you should not ascribe innate Gravity to me. That Gravity should be innate, inherent and essential to Matter, so that one Body may act upon another at a Distance through a Vacuum, without the Mediation of any thing else, by and through which their Action and Force may be conveyed from one to another, is to me so great an Absurdity, that I believe no Man who has in philosophical Matters a competent Faculty of thinking, can ever fall into it. Gravity must be caused by an Agent acting constantly according to certain Laws; but whether this Agent be material or immaterial, I have left to the Confederation of my Readers.*

Thus even Newton himself was not entirely satisfied with his law of universal gravitation (which describes an action at a distance) and anticipated some entity that mediates the force.

6 | First try: Make gravitational potential a *dynamic* field:

$$\text{Poisson equation Eq. (8.4)} \quad \rightarrow$$

$$\text{Wave equation:} \quad \partial^2\phi(t,\vec{x}) = \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\phi = -4\pi G\,\rho(t,\vec{x}) \tag{8.7}$$

→ Gravity propagates with the speed of light ☺

$\stackrel{*}{\rightarrow}$ Problems:

For a detailed study of a fully specified scalar theory of gravity: ➔ Problemset 1 (also Exercise 7.1 in MISNER *et al.* [2]). See also Ref. [97].

- Electromagnetic field cannot couple to gravity → No bending of light ☹

  Today it is a well tested fact that light follows a curved trajectory in strong gravitational fields (→ *later*). Thus any theory that does not couple the EM field to gravity must be incorrect.

  Here is a quick-and-dirty explanation why a theory of the form Eq. (8.7) fails to couple to the electromagnetic field in a relativistic setting:

  Since ϕ is assumed to be a scalar field, for Eq. (8.7) to be Lorentz covariant, ρ must be a scalar as well. In a relativistic theory, energy and (inertial) mass are equivalent ($E_0 = mc^2$). If we assume that gravitational mass and inertial mass are equivalent (→ *later*), this implies that energy (density) must be a source of gravity. The problem is that the energy (density) (of any theory) is the 00-component of the ← *energy-momentum tensor* T^{00} (this is the charge density associated to the Noether current that comes from translation symmetry in time); in particular, the energy density is *not* a scalar and therefore cannot be used as a source on the right-hand side of Eq. (8.7). The only scalar we can construct from the energy-momentum tensor is the ⁂ *Laue scalar* $T = T^\mu{}_\mu = \eta_{\mu\nu}T^{\mu\nu}$, i.e., the trace of the EMT. Thus a simplistic but fully Lorentz covariant form of scalar gravity is

$$\partial^2\phi = -\tfrac{4\pi G}{c^2}\,T\,. \tag{8.8}$$

(For a complete theory one also needs a Lorentz covariant analog of Eq. (8.5) which determines the motion of matter in dependence of the gravitational field ϕ. This equation is not relevant for the following argument.)

If the EM field couples to gravity, it must also be a source of gravity. This coupling is then described by the EMT of Maxwell theory Eq. (6.110) (in its symmetric, gauge-invariant form). The problem is that the trace of this particular EMT vanishes identically, $T_{\text{em}} = (T_{\text{em}})^{\mu}{}_{\mu} = 0$ (check this!), so that the scalar gravitational field and the EM field do not "feel" each other. In particular, there is no bending of light in the vicinity of massive bodies.

- Wrong value for perihelion precession (even with a wrong sign) ☹

  The ↑ *perihelion precession* of Mercury deviates measurably from its Newtonian value (which is caused by perturbations by other planets). For Einstein, this anomaly served as a "litmus test" on his quest to generalize SPECIAL RELATIVITY, and his first application of GENERAL RELATIVITY was the successful explanation of Mercury's anomalous perihelion precession [13] (→ *later*). Thus any theory that does not predict the correct value for the perihelion precession cannot be correct.

Historically, this first approach [Eq. (8.7)] to patch up Newton's theory and make it consistent with SPECIAL RELATIVITY goes back to the Finnish physicist GUNNAR NORDSTRÖM. He quickly dismissed Eq. (8.7) because of fundamental problems (especially its linearity, → *below*). He then proposed another (non-linear) scalar theory of gravity (↑ *Nordström's theory of gravitation*) which circumvented the most glaring issues but still failed to predict the bending of light (for the same fundamental reason sketched above) and produced the wrong value for the perihelion precession (even with a wrong sign!). Nonetheless, the theory merits consideration because it led EINSTEIN and ADRIAAN FOKKER to a groundbreaking realization [98]: Properly reformulated, the scalar field could be interpreted as a local "stretching" of the Minkowski metric. For the first time there was a clear formal link between a relativistic theory of gravity and a geometric deformation of spacetime, where the shape of the latter is determined by the distribution of mass and energy.

For a historical account of Nordström's gravity and its role in the genesis of GENERAL RELATIVITY see Refs. [99, 100].

7 | Second try: Make potential a *vector field*:

Since scalar gravity fails to match observations, a natural next step would be to consider a vector field and treat gravity analogous to Maxwell theory. This is also reasonable insofar as the gravitational potential of a point mass in Newton's theory and the Coulomb potential of a point charge in Maxwell's theory share the same form. For example, we can take Eq. (6.121) as a blueprint and propose an analogous Lagrangian for a vector gravitational field:

Eq. (6.121) → ◁ Vector field ϕ^μ & particle with trajectory $y^\mu(\lambda)$:

$$S_G[y,\phi] \overset{?}{=} \underbrace{\frac{+1}{16\pi G}\int \mathrm{d}^4x\, G_{\mu\nu}G^{\mu\nu}}_{\text{Gravitational field}} \underbrace{-mc\int \sqrt{\dot y^\mu \dot y_\mu}\,\mathrm{d}\lambda}_{\text{Relativistic particle}} \underbrace{-\frac{m}{c}\int \phi_\mu \dot y^\mu \mathrm{d}\lambda}_{\substack{\text{Coupling}\\ \text{with "charge" } m}} \tag{8.9}$$

with $\dot y^\mu = \frac{\mathrm{d}y^\mu}{\mathrm{d}\lambda}$ and the "gravitational field strength tensor" $G_{\mu\nu} := \partial_\mu\phi_\nu - \partial_\nu\phi_\mu$.

- Note the sign difference compared to Eq. (6.121)!
  This ensures that equal charges (= masses) *attract* each other.

- The Lagrangian for the relativistic particle differs from the one given in Exercise 7.2 in MISNER *et al.* [2]; the two are equivalent and lead to the same equations of motion.

$\overset{*}{\rightarrow}$ Results:

For details see Exercise 7.2 in MISNER *et al.* [2]; see also Ref. [97].

- No bending of light ☹
- Wrong perihelion precession ☹
- Gravitational waves have negative energy ☹

8 | Third try: Make potential a *tensor field*:

At that point, desperation starts to kick in. But since scalar and vector fields failed miserably, we have no other choice: add another index and consider a tensor field. Interestingly, this makes it rather straightforward to write down a Lorentz-covariant modification of Eq. (8.8) [or Eq. (8.4)] where we no longer must butcher the EMT by taking a trace:

$\sphericalangle$ Symmetric tensor field $\phi^{\mu\nu} = \phi^{\nu\mu}$:

$$\partial^2 \phi^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu} \tag{8.10}$$

The EMT on the right is the symmetric BRT of whatever matter occupies space (Section 6.3.2).

$\overset{*}{\rightarrow}$ Results:

For details: ➔ Problemset 1 (also Exercise 7.3 in MISNER *et al.* [2])

- Light is bent around gravitational potentials ☺
- Gravitational waves have positive energy ☺
- Describes perihelion precession not correctly ☹
- Theory not self-consistent ☹

Notes:

- Eq. (8.10) will describe the *linearized* version of the correct field equations of GENERAL RELATIVITY (the → *Einstein field equations*) with $\phi^{\mu\nu}$ essentially the (small) deviation of the metric tensor from flat Minkowski space.
- That the linear tensor theory of gravity Eq. (8.10) is not self-consistent follows if one completes the theory with dynamic matter (which is the source of the gravitational field, but also influenced by the latter). Then one can show that this system of differential equations as no solution.
- As we will discuss below, the deficiency of this theory is its *linearity* (in the gravitational field); this is the root cause for its inconsistency and wrong predictions. And here comes a fascinating insight: One can show [101] that if one systematically fixes the inconsistencies of this theory, it becomes inevitably non-linear and one eventually ends up with the *correct* equations of GENERAL RELATIVITY (which we will find much later via a different route)!

9 | So far, all our tentative theories of relativistic gravity failed (none of them describe observations correctly and they even suffer from intrinsic inconsistencies). There is a simple argument why this must be so, and why the correct theory *must* be more complicated:

i | The source (= charge) of gravity is, by definition, the *gravitational mass* m_G.

This is a physically vacuous statement.

ii | A relativistic theory of gravity must be a field theory with a dynamical field.

This is necessary so that gravity does not propagate faster than the speed of light.

iii | Since the field is dynamical, it has a non-vanishing energy density.

Recall that energy is the Noether charge of time translations and therefore generates the time evolution (think of the Hamiltonian in quantum mechanics).

iv | As a relativistic theory it must obey the mass-energy equivalence: $E_0 = m_\mathrm{I} c^2$.

We write m_I to emphasize that SPECIAL RELATIVITY only knows about *inertial* mass.

v | Experiments tell us that inertial and gravitational mass are the same: $m_\mathrm{I} = m_\mathrm{G}$.

We will discuss this, and the closely related → *equivalence principle*, in detail below.

vi | Thus a gravitational field has a non-vanishing density of gravitational mass m_G.

Please appreciate how strange this is! If an analog statement were true for Maxwell theory (which it is not), electromagnetic waves would be *electrically charged*, and other electromagnetic waves could scatter off them!

vii | Excitations of the gravitational field are sources of the gravitational field.

This means that a relativistic theory of gravity must allow for self-interaction. In particular, it *cannot* feature a ↓ *superposition principle* and the field equations must be *non-linear*.

$\rightarrow$

The field theory of gravity must be *non-linear* and allow for self-interactions.

- All of the above theories are linear in the gravitational field; hence they are bound to fail!
- This argument also clarifies the fundamental difference between relativistic theories of *gravity* and *electrodynamics* (both of which are classical field theories that mediate forces): The EM field is also dynamical and carries energy, hence, via the mass-energy equivalence and the equivalence of inertial and heavy mass, it is a source of *gravity*. But the mass/energy carried by the EM field is not the source of the *EM field* (electrical charge is). Thus Maxwell theory does not close the "vicious circle" from above and can be both *relativistic* and *linear*.
- If you want an even more boiled down version:

  Gravity is special in RELATIVITY, because SPECIAL RELATIVITY has something to say about (inertial) mass ($E_0 = m_\mathrm{I} c^2$) and the latter is – via the → *equivalence principle* – the source of gravity.

↓Lecture 17 [16.04.24]

10 | Yet another problem:

Besides the formal complications encountered above, there are also less formal yet fundamental lines of reasoning that suggest that the phenomenon of gravitation and the premises of SPECIAL RELATIVITY are incompatible:

i | Experimental facts:

- Gravity cannot be shielded.

  Contrary to all other forces (which have negative charges), there is no negative mass.

- Gravity is typically *inhomogeneous*.

  In a gravitationally homogeneous universe there are no planets and we wouldn't exist.

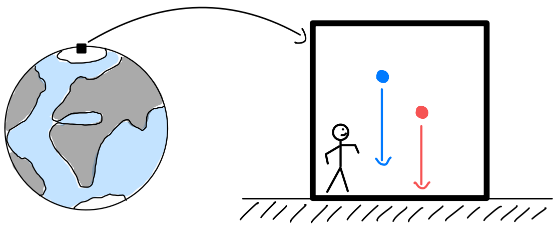
- In free fall, gravity is exactly countered by the inertial force.

  We will discuss this in more detail later (→ *equivalence principle*).

ii | For the machinery of SPECIAL RELATIVITY to work, we need inertial frames. Can we find inertial frames in the presence of gravity?

Thought experiment:

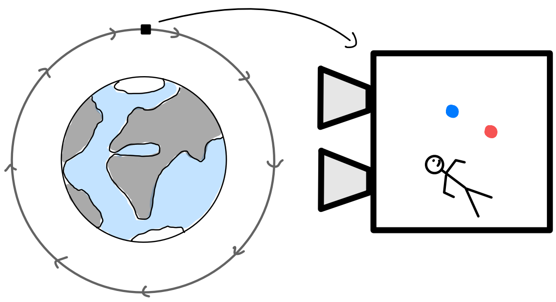
a | ⊲ Laboratory on the surface of Earth:



→ Not an inertial system ☹

The problem is that we cannot simply shroud our lab by some magic material that shields the gravitational force. By contrast, this *can* be done for the electromagnetic field (↓ *Faraday cage*, ↑ *Mu-metal*). Note that this is not a technical problem, it is a fundamental one!

b | ⊲ Interior of orbital space station:

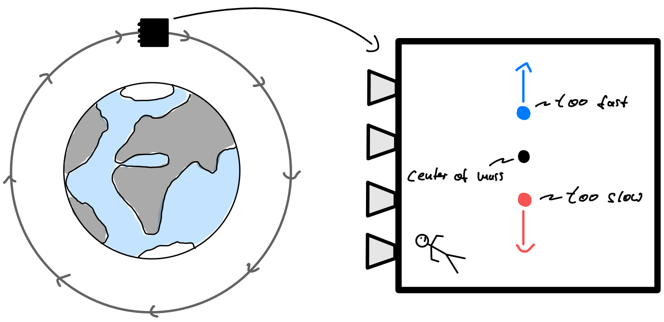


→ Approximate inertial system ☺

The space station is equivalent to a free falling laboratory, where the gravitational force is canceled exactly by inertia. What makes a space station so convenient is that it also has orbital velocity so that it "falls around Earth" and therefore can be used much longer than a free falling lab that eventually crashes on the surface.

This is a situation where it actually makes sense to use the terms "gravity" and "gravitation" differently: In the space station, there is no *gravity* (the astronauts float), but there is a *gravitational field*! (The latter is just canceled by the inertial force due to free fall.) This situation is different from being in a spaceship far away from Earth in interstellar space where no gravitational forces can be measured. (Although these two situations cannot be distinguished from within a small space station/spaceship, → *later*.)

c | ◁ Very large orbital space station:

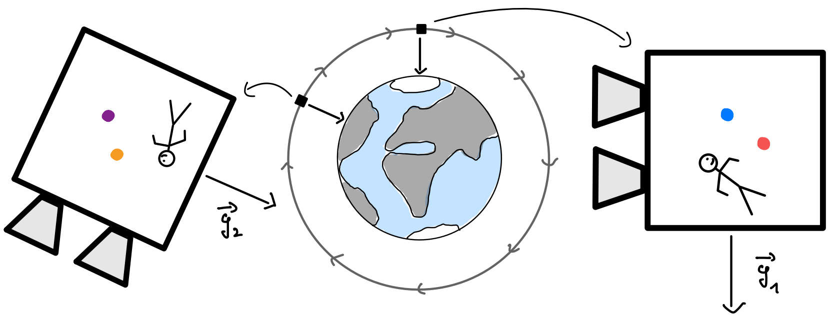


↑ *Tidal forces* → Not an inertial system ☹

When we extend the size of the space station, the inhomogeneity of the gravitational acceleration becomes noticeable and the "inertial test" `IN` fails. Inhomogeneous gravitational fields therefore constrain the size (both in space and time) of reference systems that satisfy the properties of an inertial system. Hence our assumption that inertial systems cover all of spacetime (and therefore can describe arbitrary physical phenomena, not just local ones) is invalidated by the presence of inhomogeneous gravitational fields.

Note that there is another way to detect the inhomogeneous gravitational field and make the system non-inertial: Stay in the small spacecraft and wait longer. At some point you will notice that the two balls drift apart—even when they are only centimeters apart. This shows that the approximate inertial system really must be small in space *and time*.

d | ◁ Two small orbital space stations:



→ Local inertial systems accelerated wrt. each other ☹

If you imagine that these small inertial systems overlap on their boundaries, you could ask how to transform the coordinates of an event in this overlap from one of these systems into the other. Because these systems are accelerated wrt. each other, this transformation cannot be linear, in particular it cannot be a Lorentz transformation! There seems to be something missing; what determines this transformation?

iii | This thought experiment leads us to the following (troubling) conclusion:

> Inertial systems can only exist *locally* in an inhomogeneous gravitational field. How to transform between these local inertial systems is unclear (→ *later*).

→ Extended phenomena cannot be described by SPECIAL RELATIVITY!

"Extended" here means "on the scale of gravitational inhomogeneities."

11 | In a nutshell:

> SPECIAL RELATIVITY cannot …
>
> - … describe the gravitational field itself.
> - … describe physics in inhomogeneous gravitational fields.
>
> How to fix this? → GENERAL RELATIVITY

8.3. ‡ The gravitational redshift and curved spacetime

Quite surprisingly, one can derive some predictions of GENERAL RELATIVITY without knowledge of the detailed theory. One is the → *gravitational redshift* of light, which, again without the usage of heavy math, implies that GENERAL RELATIVITY must describe a *curved* spacetime.

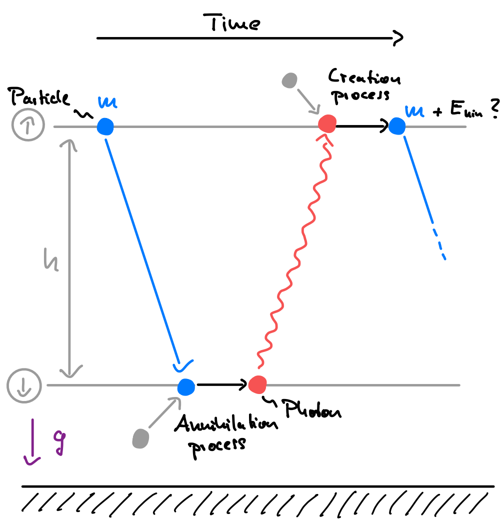
The following is based on Sections 7.2 and 7.3 of MISNER *et al.* [2] and Section 2.1 of CARROLL [102].

1 | Gravitational redshift:

EINSTEIN already concluded in 1908 that light leaving a gravitational potential must be redshifted [95]. The following he showed in 1911 [103], that is, years before he finalized GENERAL RELATIVITY.

i | ⊲ Particle of rest mass m in (homogeneous) gravitational potential:

In the following, we assume that *inertial* and *gravitational* mass are equal: $m_{\mathrm{I}} = m = m_{\mathrm{G}}$.



ii | Step 1: Drop the particle by h → New total energy:

$$E_{\text{tot}} = m_{\text{I}}c^2 + m_{\text{G}}gh = m(c^2 + gh) \tag{8.11}$$

iii | Step 2: Assume electron annihilates into a photon with energy:

$$E_\downarrow = E_{\text{tot}} \tag{8.12}$$

iv | Step 3: Let the photon propagate upwards by h → New photon energy $E_\uparrow$:

- Possibility 1: Photon is *not* affected by gravity → $E_\uparrow = E_\downarrow > mc^2$ ✗

  This immediately leads to a violation of energy conservation because the photon can now be used to recreate the particle *plus some kinetic energy* that wasn't there before.

- Possibility 2: Photon is *redshifted* by gravitational field such that $E_\uparrow = mc^2$ ✓

  This is the only possibility consistent with energy conservation, i.e., the photon must loose energy just as a particle would when climbing the potential.

v | Thus we find for the photon energies:

$$E_\downarrow = \frac{E_\uparrow}{c^2}(c^2 + gh) = E_\uparrow\left(1 + \frac{gh}{c^2}\right) \tag{8.13}$$

vi | ⊲ ⁂ *Redshift parameter* $z := \Delta\lambda/\lambda = (\lambda_\uparrow - \lambda_\downarrow)/\lambda_\downarrow$

The redshift z measures the relative change in wavelength $\Delta\lambda$ wrt. a reference wavelength λ.

→ Gravitational redshift: (Use the photon energy $E = h\nu = hc/\lambda$.)

$$\boxed{1 + z = \frac{\lambda_\uparrow}{\lambda_\downarrow} = \frac{E_\downarrow}{E_\uparrow} = 1 + \frac{gh}{c^2}} \tag{8.14}$$

Using the ↑ *Mößbauer effect*, Robert Pound and Glen Rebka verified this prediction in 1960 with their famous ↑ *Pound-Rebka experiment* [104, 105].

2 | Schild's argument:

The following reasoning goes back to Alfred Schild (see references in [2]) and demonstrates that a relativistic theory of gravity cannot be formulated on a flat Minkowski spacetime:

i | Assumptions:

- There exists an extended inertial frame K attached to Earth's center.
  (We relax our definition and allow particles to be accelerated near earth.)
- In this frame, proper time and lengths are given by the Minkowski metric.
- There is some gravitational field (of unspecified nature) that matches observations.
  (This implies the gravitational redshift derived above.)

ii | Thought experiment:

a | ⊲ Two observers $O_{\downarrow,\uparrow}$ at height $z_{\downarrow,\uparrow}$ with $z_\uparrow = z_\downarrow + h$ at rest in K

b | Observer $O_\downarrow$ emits a light signal with wavelength $\lambda_\downarrow$

→ Time for one wavelength: $\Delta t_\downarrow = \lambda_\downarrow/c$

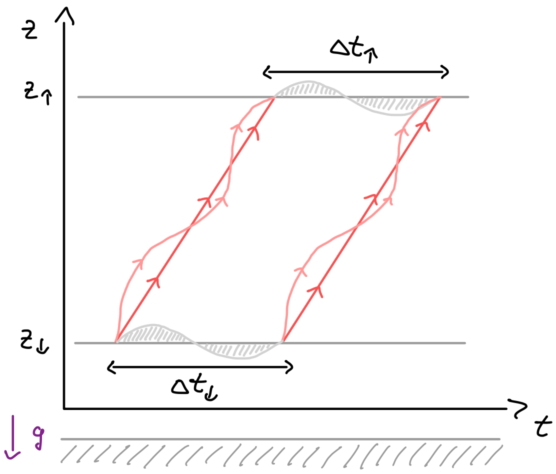
Note that since both observers are at rest in K, their proper times and the coordinate time of K coincide.

c | Observer $O_\uparrow$ receives the signal with wavelength $\lambda_\uparrow$

$\rightarrow$ Time for one wavelength: $\Delta t_\uparrow = \lambda_\uparrow / c$

d | Redshift $\rightarrow \lambda_\uparrow > \lambda_\downarrow \rightarrow \Delta t_\uparrow > \Delta t_\downarrow$

e | But in the Minkowski diagram of the (imagined) global inertial frame, the experiment looks as follows:

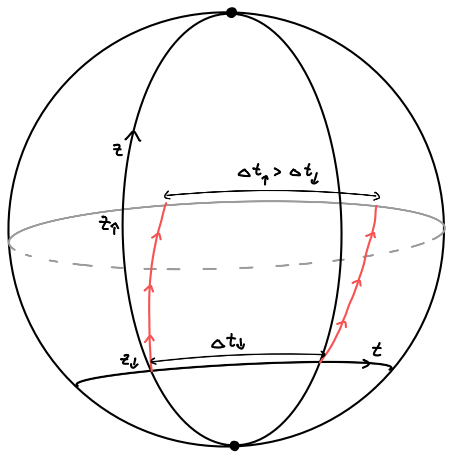


$\rightarrow \Delta t_\uparrow = \Delta t_\downarrow$ ϟ

Note that the only important aspect for the contradiction is that the two world lines of the start and end of one wavelength are *congruent* in Minkowski space. That is, we do not need to know how gravity affects the trajectory of light (maybe it is bent). The only important thing is that both trajectories are bent in the same way, which is to be expected in a static scenario where the gravitational field does not change.

iii | Conclusion:

In the presence of gravity, the trajectories of light signals in spacetime must be congruent (if they are straight: parallel)—but at the same time their distance in time direction must change! This is impossible in the flat (pseudo-)Euclidean geometry of Minkowski space; but it *is* possible in a curved spacetime. As we will see → *later*, the tendency of initially parallel "straight lines" (→ *geodesics*) to approach or recede from another is exactly what characterizes a curved space(time):



$\rightarrow$ Spacetime must be curved!

3 | We can summarize:

A Lorentz covariant theory of gravity cannot be formulated on Minkowski space.

This already suggests that we will need the more general machinery of differential geometry, introduced in Chapter 3, to model spacetime not as flat Minkowski space, but as a more general, curved pseudo-Riemannian manifold.

9. Conceptual Foundations

9.1. Einstein's equivalence principle

The wording of the equivalence priciples are paraphrased from CARROLL [102].

1 | Remember:

There are two concepts of mass in Newtonian physics:

$$\text{Inertial mass } m_\mathrm{I}: \quad \vec{F} = m_\mathrm{I}\vec{a} \tag{9.1a}$$

$$\text{Gravitational mass } m_\mathrm{G}: \quad \vec{F} = -m_\mathrm{G}\nabla\phi \tag{9.1b}$$

Strictly speaking, two gravitational masses must be conceptually distinguished: The *passive* gravitational mass is the charge that couples to the gravitational field via Eq. (9.1b). The *active* gravitational mass is the source of the gravitational potential $\phi = -GM_\mathrm{G}/r$. However, given that Newton's third law is valid (action equals reaction), the situation is completely symmetric and these two masses can be identified. Thus, in the following we only distinguish between inertial and gravitational mass.

→ Gravitational acceleration:

$$a_\mathrm{G} = \frac{m_\mathrm{G}}{m_\mathrm{I}}\underbrace{\frac{GM_\oplus}{r^2}}_{g} \tag{9.2}$$

It has been long known (since Galileo Galilei) that the gravitational acceleration is *independent* of the material of the body (if one can ignore air resistance): *All bodies fall at the same rate.*

Experience:

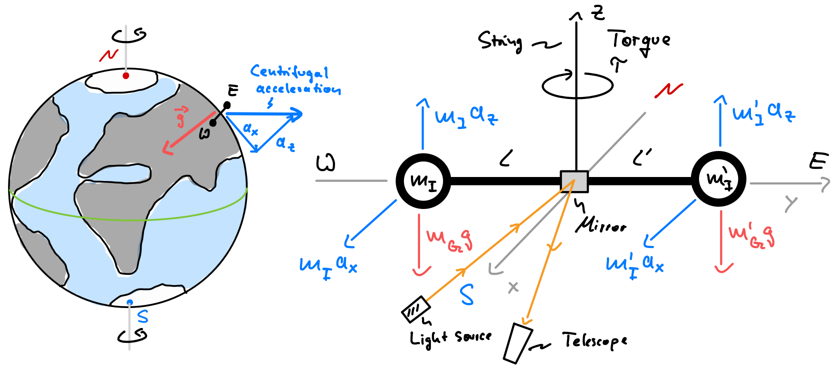
$$\frac{m_\mathrm{G}}{m_\mathrm{I}} = \text{const} \quad \rightarrow \quad \text{Choose units appropriately:} \quad \frac{m_\mathrm{G}}{m_\mathrm{I}} = 1 \tag{9.3}$$

In classical mechanics, this is just an observation; it is neither explained nor necessary for its consistency.

2 | The Eötvös experiment [106]: (See also the later publication [107].)

While there have been earlier experiments that quantitatively tested the equivalence of inertial and gravitational mass, the experiment by Hungarian physicist ROLAND EÖTVÖS made huge improvements in precision. The experiment was used by Einstein as an argument for his → *equivalence principle*.

$\sphericalangle$ Torsion balance with two different test bodies: (Details: ➔ Problemset 1)



➔ Problemset 1 $\overset{\circ}{\rightarrow}$

$$\text{Torque:} \quad \tau \approx \underbrace{m_G a_x l}_{\neq 0} \left(\frac{m_I}{m_G} - \frac{m_I'}{m_G'}\right) \tag{9.4a}$$

$$\tau = 0 \quad \Leftrightarrow \quad \underbrace{\frac{m_I}{m_G}}_{\text{Material A}} = \underbrace{\frac{m_I'}{m_G'}}_{\text{Material B}} = \text{const} \tag{9.4b}$$

→ Result by Eötvös [106]:

$$\frac{\delta m}{m} = \frac{m_I - m_G}{m_I} < 3 \times 10^{-9} \tag{9.5}$$

The latest (2022) and most precise results testing the equivalence of inertial and gravitational mass come frome the satellite-based MICROSCOPE experiment [108]; it improved the upper bound for a violation of the equivalence to $\delta m/m < 10^{-15}$. Recent experiments also demonstrated the equivalence for antimatter [109].

→ Experimental fact:

$$\boxed{\text{Inertial mass and gravitational mass are proportional } (\textit{w.l.o.g. } m_I = m_G).} \tag{9.6}$$

This trivial sounding assertion (when have you ever distinguished between these two masses?) has profound consequences: Recall that SPECIAL RELATIVITY is concerned with the concept of *inertia* (e.g. by using inertial systems); in particular, the (rest) mass that shows up in the mass-energy equivalence $E_0 = mc^2$ is the inertial mass m_I of the system. The equivalence above now links this mass to the gravitational mass, and therefore asserts the the inert bodies of SPECIAL RELATIVITY must be affected by (and be sources of) gravity. But SPECIAL RELATIVITY had nothing to say about gravity! Quite to the contrary: as discussed in Section 8.2, the theory cannot accomodate gravity in a consistent way.

3 | Classical mechanics does not explain Eq. (9.6). However, if we take Eq. (9.6) for granted, a homogeneous gravitational field vanishes in an accelerated frame:

$\sphericalangle$ N particles of (inertial = gravitational) mass m_k in gravitational field:

$$\underbrace{m_k \frac{\mathrm{d}^2\vec{x}_k}{\mathrm{d}t^2}}_{\text{Inertia}} = \underbrace{m_k \vec{g}}_{\text{Gravity}} + \sum_{l\neq k} \vec{F}_{kl}(\vec{x}_k - \vec{x}_l) \quad \text{with } k = 1, \dots, N. \tag{9.7}$$

⊲ Coordinate transformation into free-falling frame:

$$t' = t \quad \text{and} \quad \vec{x}'_k = \vec{x}_i - \tfrac{1}{2}\vec{g}t^2\,. \tag{9.8}$$

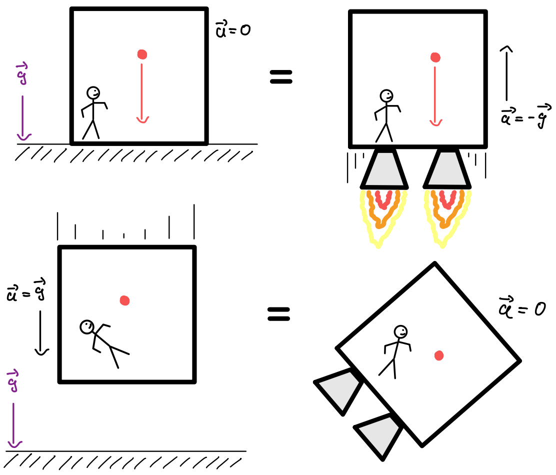
This coordinate transformation is *non-linear*, in particular, it is not a Galilei transformation!

$\overset{\circ}{\rightarrow}$ Equations of motion in the free-falling coordinate system:

$$m_k \frac{\mathrm{d}^2\vec{x}'_k}{\mathrm{d}t'^2} = \sum_{l\neq k} \vec{F}_{kl}(\vec{x}'_k - \vec{x}'_l) \quad \text{with } k = 1, \dots, N\,. \tag{9.9}$$

→ No graviy in the free-falling frame!

This matches our experience and can be illustrated with the following though experiment:

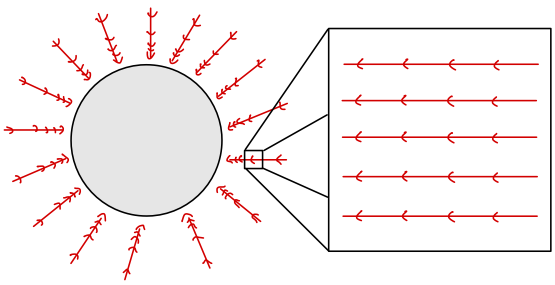


4 | Caveat: Only true for *homogeneous* gravitational fields: $m_k\vec{g}$.

What about *inhomogeneous* gravitational fields?

Note that the inhomogeneity of gravity is essential for planets and stars to form; it is the root cause for complexity in the world that is neccessary for life to exist. This is not a slight inconvenience we can sweep under the rug!

→ Small enough regions look homogeneous:



→ Gravity can be compensated *locally* in an accelerated frame.

¡! This implies that there is no transformation to a *global* free-falling reference frame in which inhomogeneous gravitational fields vanish. Thus acceleration and gravity are only equivalent *locally*; globally, they are physically distinct. In particular, this means that the phenomenon of gravity is not just "acceleration in disguise." As mentioned previously, accelerated coordinate systems (and bodies) are something that SPECIAL RELATIVITY can handle. If gravity and acceleration were equivalent globally, SPECIAL RELATIVITY would be sufficient to describe gravity and there was no need for GENERAL RELATIVITY.

5 | Given all these facts, it is reasonable to proclaim the following principle:

§ Postulate 1: Weak equivalence principle `WEP` **(*Univesality of free fall*)**

In small enough regions of spacetime, the motion of *freely-falling particles in a gravitational field* and *free particles in a uniformly accelerated frame* are the same.

(This formulation formalizes the idea sketched in the upper panel of the sketch above.)

Equivalently:

For every event, there is a local reference frame, covering small enough regions of spacetime in its vicinity, such that gravity has no effect on the motion of arbitrary particles in this frame and the law of inertia holds.

(This formulation formalizes the idea sketched in the lower panel of the sketch above.)

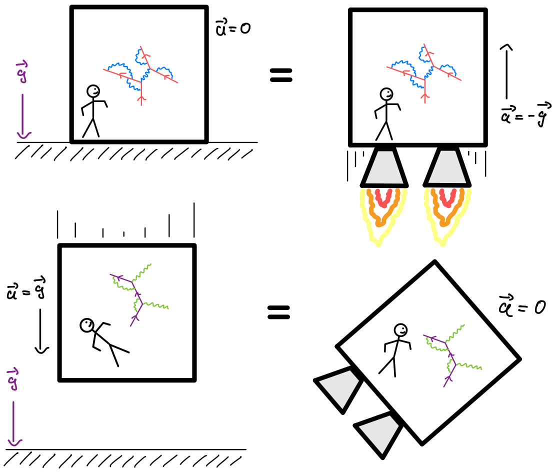
6 | Einstein's generalization:

The local equivalence of gravity and accelerated frames is true for *all* physical phenomena (and not only classical mechanics).

Einstein was aware of the Eötvös experiment and was convinced that the equivalence of inertial and gravitational mass hinted at a deep relationship between inertia (acceleration) and gravitation. He wrote in 1907 [95] (highlights are mine):

> *Bisher haben wir das Prinzip der Relativität, [..], nur auf beschleunigungsfreie Bezugssysteme angewendet. Ist es denkbar, daß das Prinzip der Relativität auch für Systeme gilt, welche relativ zueinander beschleunigt sind? [..]*
>
> *Wir betrachten zwei Bewegungssysteme Σ_1 und Σ_2. Σ_1 sei in Richtung seiner X-Achse beschleunigt, und es sei γ die (zeitlich konstante) Größe dieser Beschleunigung. Σ_2 sei ruhend; es befinde sich aber in einem homogenen Gravitationsfelde, das allen Gegenständen die Beschleunigung $-\gamma$ in Richtung der X-Achse erteilt.*
>
> *Soweit wir wissen, unterscheiden sich die physikalischen Gesetze in bezug auf Σ_1 nicht von denjenigen in bezug auf Σ_2;* es liegt dies daran, daß alle Körper im Gravitationsfelde gleich beschleunigt werden. *Wir haben daher bei dem gegenwärtigen Stande unserer Erfahrung keinen Anlaß zu der Annahme, daß sich die Systeme Σ_1 und Σ_2 in irgendeiner Beziehung voneinander unterscheiden,* und wollen daher im folgenden die völlige physikalische Gleichwertigkeit von Gravitationsfeld und entsprechender Beschleunigung des Bezugssystems annehmen.

Pictorially, Einstein claims that *any type* of local experiment cannot distinguish gravity from acceleration (here, for example, some quantum mechanical scattering process):



7 | We can formalize this as follows:

> **§ Postulate 2: Einstein's equivalence principle** `EEP`
>
> In small enough regions of spacetime, the laws of physics reduce to those of SPECIAL RELATIVITY: It is impossible to detect the existence of a gravitational field by means of local (non-gravitational) experiments.
>
> *Equivalently:*
>
> For every event, there is a local reference frame, covering small enough regions of spacetime in its vicinity, such that gravity has no effect *on any (non-gravitational) experiment* in this frame and the law of inertia holds.

- The `EEP` implies the `WEP`.
- Excluding non-gravitational experiments means that the intrinsic gravitational energy of our experiment does not contribute significantly to its mass (see `SEP` below). Note that we do not require that gravitational experiments (using large masses) *can* locally distinguish between gravity and acceleration; the `WEP` does simply not constraint such experiments.
- It is important to appreciate the profound implications of this principle for doing physics in a gravitational field: It asserts that as long as your laboratory is small (compared to the inhomogeneties of the gravitational field) and free-falling (e.g. a space station in orbit), SPECIAL RELATIVITY is sufficient to describe *all experiments* that you can conduct in this lab. In particular, the fact that SPECIAL RELATIVITY cannot describe gravity is not important because in the free-falling lab there is none. This means that everything we discussed last term remains valid—and therefore useful—*locally*. Thus gravity does not completely invalidate SPECIAL RELATIVITY, it only restricts its domain of validity to local, free-falling inertial frames. I hope you are happy to hear that!
- More precisely, for every event (point in spacetime) there is an *equivalence class* of local inertial frames (related by boosts), equiped with inertial coordinate systems (related by translations and rotations), in all of which SPECIAL RELATIVITY holds good. The coordinate transformations between these systems are given by Lorentz transformations. (You can check

the existence of such frames in our Newtonian calculation above by adding a term $\vec{v}t$ to the transformation of the position coordinates.)

- In our mathematical framework of differential geometry, the equivalence class of local inertial systems at a spacetime point will be identified with the ← *tangent space* of the spacetime manifold at that point.

8 | Concerning *gravitational* laws of physics:

In our defintion of the EEP, we excluded experiments that depend on the gravitational interaction itself (e.g., use objects with considerable intrinsic gravitational energy). This exclusion follows Schröder [3], whereas other authors like Carroll [102] include the (unkown) gravitational laws of physics in the EEP.

For us, it makes then sense to define an extension of the EEP as follows:

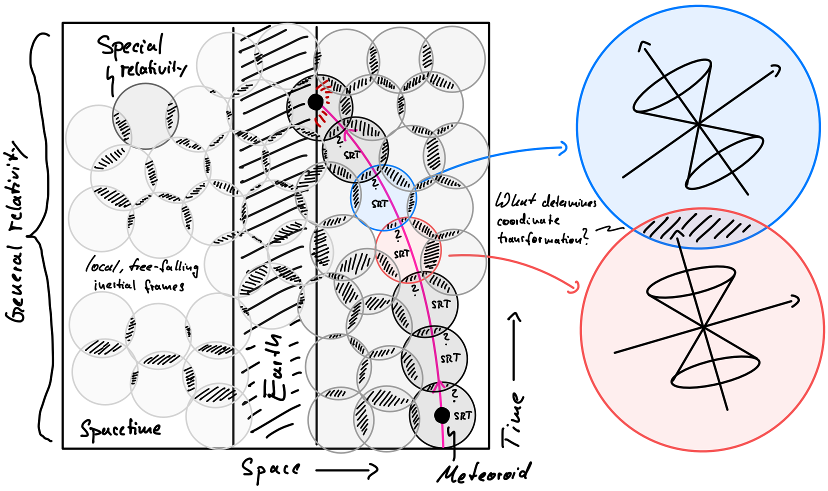
> **§ Postulate 3: Strong equivalence principle SEP**
>
> The EEP is valid for all laws of physics, including the gravitational laws.

- General Relativity satisfies the SEP (and thereby the EEP and the WEP).
- The reason to separate the EEP from the SEP is that alternatives to General Relativity can *satisfy* the EEP (and the WEP) but *violate* the SEP. This alternatives can be metric theories like General Relativity with additional fields; see Ref. [110] for details.
- In particular, the SEP requires that the universality of free fall (WEP) also holds for large bodies like planets (not just small test particles) with significant amounts of gravitational self-energy. More precisely, the SEP demands that the rest mass E_{grav}/c^2 that comes from the gravitational self-energy E_{grav} accelerates just like any other rest mass in an external gravitational field.

  Note that the validity of the SEP cannot be deduced from typical experiments that test the WEP because these experiments use small test masses with a gravitational self-energy that is way too small to detect any violation of the SEP (because gravity is such a weak force). One needs to use planet-sized objects to draw conclusions about the SEP (→ *next*).
- Using reflectors on the moon (left by Apollo 11 in 1969), lunar laser ranging (LLR) can be used to experimentally test the SEP since the fractions of gravitational self-energy of moon and earth are different enough to modify moon's orbit measurably if the SEP was violated. To date there is no evidence of such a violation to high precision [111, 112], hence we will assume that the SEP holds.

9 | If Special Relativity explains *everything* you can do in a local, free-falling laboratory, at which point does gravity enter the picture? Well, the hitch is that not all physical processes *can* be restricted to a single, local inertial frame:

⊲ Meteroid traversing the inhomogeneous gravitational field of earth:



How to model the trajectory?

Imagine you start in a local inertial system where you know the initial data (position, velocity) of the meteroid. Since SPECIAL RELATIVITY is valid in this small patch, you can use the known equations of relativistic mechanics to compute the trajectory of the meteroid. However, at some point, the meteroid will leave the local inertial system and enter another one. To proceed with your application of relativistic mechanics, you need to know the coordinate transformation that maps the coordinates of the final position and velocity in the first inertial system to the coordinates of the next.

But a priori these inertial system are unrelated, in particular, they can be accelerated with respect to one another (recall the two small space stations in Section 8.2). To proceed with your application of relativistic mechanics, you need this coordinate transformation! And this is where gravity hides: The gravitational field (here generated by earth) and the pattern of local coordinate transformations are one and the same thing! This is what is meant by *gravity becoming a geometric property of spacetime*.

→

> The gravitational field is the (dynamical) structure that determines which local frames of reference are inertial or, equivalently, how to transform from one local inertial frame to the next.

↓Lecture 18 [23.04.24]

9.2. General relativity and covariance, background independence

The equivalence principle EEP is the foundation of GENERAL RELATIVITY; it motivates both the metrization of gravity (Section 9.4 and ??) and the minimal coupling of matter to gravity (??). However, there are additional principles that are conceptually important to understand and were historically important for the genesis of GENERAL RELATIVITY as well:

10 | Motivation:

- No global inertial systems anymore → More general coordinate charts needed!

  As we have seen, the description of gravity forces us to give up the restriction to formulate physical models within the distinguished family of infinitely extended inertial coordinate systems. Hence we must formulate our physical theories in a way that is valid for arbitrary coordinate charts, and allows for arbitrary coordinate transformations between them.

  Einstein was not satisfied with the distinguished role of inertial systems in SPECIAL RELATIVITY. After all, RELATIVITY is all about the *relativity of states of motion*, i.e., only motion of systems with respect to one another are of physical significance – and no class of states of motion should be distinguished. SPECIAL RELATIVITY clearly does not live up to this rigorous form of relativity as it singles out inertial frames as special. Einstein's ultimate goal was to make all states of motion (including accelerated motion) "equivalent." GENERAL RELATIVITY *does not achieve this goal!* Even in GENERAL RELATIVITY, inertial motion is physically distinct from accelerated motion; the new thing is that mass and energy determine which states of motion *are* inertial.

  We are therefore in the strange (and confusing) situation, that Einstein's original motivation to seek out equations that have "the same form" in all coordinate systems does not achieve its goal, but nevertheless is the correct way forward (see → *next* point). We will also see that "having the same form" means something different in SPECIAL RELATIVITY than in GENERAL RELATIVITY because the former is formulated on a fixed background (Minkowski space) and the latter not (→ *background independence*) – and this changes what it means for two equations to have "the same form." The situation is quite convoluted and we can disentangle it not until the end of this course.

- Chapter 3: *Physics describes relations of geometric entities* ("Coordinates don't exist.")

  → Coordinates should play no role in the formulation of physical models!

  Recall our motivation in Chapter 3 for the introduction of tensor fields: We realized that coordinates are mathematical artifacts that we use to label events in spacetime. The essence of physical laws should clearly be independent of the labeling scheme we choose. Thus we should strive for a formulation of physical models (which, hopefully, capture physical laws) that is independent of coordinates, or at least makes it manifest that physical predictions do not depend on the choice of coordinates.

  Note that this argument is very different from Einstein's hope to extend the principle of special relativity SR by "equalizing" more states of motion. The argument is way more fundamental, has nothing to say about states of motion, and, in some sense, is almost tautological. It's only physical content is the rather uncontroversial statement that "coordinates do not exist as physical entities."

11 | This motivates the following definition:

★ **Definition 1: General covariance** GC **(*Coordinates don't exist*)**

An equation is said to be ⁂ *generally covariant* if it is forminvariant under arbitrary (differentiable) coordinate transformations.

→ Tensor equations are automatically generally covariant.

Generally covariant equations have an alternative ⁂ *coordinate-free* formulation in terms of geometric objects on a manifold (↑ *differential forms*):

Examples:

- $\sphericalangle$ Two vector fields $A = A^\mu \partial_\mu$ and $B = B^\mu \partial_\mu$:

$$\underbrace{\left\{\begin{array}{c|l} \phi_A = \phi_B & \bar\phi_A = \frac{\partial \bar x^0}{\partial x^0}\phi_A + \frac{\partial \bar x^0}{\partial x^i}A^i \\ \vec A = \vec B & \bar A^i = \dots \\ & \bar\phi_B = \dots \\ & \bar B^i = \dots \end{array}\right\}}_{\text{Generally covariant}} \Leftrightarrow \underbrace{A^\mu = B^\mu}_{\substack{\textit{Manifestly}\\ \text{(generally)}\\ \text{covariant}}} \Leftrightarrow \underbrace{A = B}_{\substack{\text{Coordinate}\\ \text{free}}} \tag{9.10}$$

  - While mathematicians often prefer the coordinate-free notation, in physics, the coordinate-dependent, manifestly covariant notation is more widespread. This has to do with how physics is done: While coordinates do not exist *a priori*, physicists typically make them exist in their labs because measurements always use some form of reference system. The generally covariant equations are more useful in that regard because they can be specialized to any coordinate system most convenient for an experiment.

    → In this course we will only use the manifestly covariant notation.

  - To decided whether an equation remains form invariant under arbitrary coordinate transformations, you must first know how the elementary fields of the equation transform. This is why the *non*-manifest notation is so cumbersome: In addition to the equation(s), you must figure out (or specify) how the different fields transform. (Recall the non-tensorial form of the Maxwell equations and how cumbersome it was to check their Lorentz covariance [see Eq. (6.34)]!)

    This makes the benefit of the *manifest* notation clear: First, *by convention*, the tensor notation A^μ implies that the transformation of the field is $\bar A^\mu = \frac{\partial \bar x^\mu}{\partial x^\nu} A^\nu$, and second, because of the rules of tensor calculus, checking the general covariance of a (valid) tensor equation is trivial.

- $\sphericalangle$ Inhomogeneous Maxwell equations on arbitrary spacetime (??):

$$\underbrace{F^{\mu\nu}{}_{;\nu} = -\frac{4\pi}{c} J^\mu}_{\text{Manifestly covariant}} \quad \Leftrightarrow \quad \underbrace{d(\star F) = \star J}_{\text{Coordinate-free}} \tag{9.11}$$

  Remember that $;\nu$ denotes the ← *covariant derivative* Eq. (3.79) which implicitly depends on the metric of spacetime. In the coordinate-free notation, the metric is hidden in the definition of the ↑ *Hodge star operator* ⋆.

12 | We can now use this definition to formulate our physical insight that the equations that describe physical laws must not single out specific coordinate systems:

In Einstein's words [20]:

> *Die Gesetze der Physik müssen so beschaffen sein, daß sie in bezug auf beliebig bewegte Bezugssysteme gelten. (p. 772)*
>
> *Die allgemeinen Naturgesetze sind durch Gleichungen auszudrücken, die für alle Koordinatensysteme gelten, d.h. die beliebigen Substitutionen gegenüber kovariant (allgemein kovariant) sind. (p. 776)*

→

§ Postulate 4: General relativity GR

Models of laws of nature must take the same form in *all* coordinate systems; i.e., they must be expressed in terms of generally covariant equations.

Here the "must take the form" means that it must be *possible* to formulate them in a coordinate-independent way; if this were not the case, the theory (and its prediction) would implicitly depend on (and single out) a specific coordinate system. Note that there is nothing wrong in formulating such a theory in a way that is not *generally covariant*.

For example, Maxwell equations in their conventional (non-tensorial) form are not generally covariant, they are only Lorentz covariant. This is not a problem, though, because these equations are just a specialization of Eq. (9.11) to a particular class of coordinate systems (namely inertial systems). If you (naïvely) apply these specialized equations in a non-inertial frame (such as a laboratory on the surface of earth!), you can get seemingly paradoxical results (➔ Problemset 3 and Ref. [113]).

13 | What is the physical content of GR?

GR, while being important for the formulation of physical models in general and being strictly satisfied in GENERAL RELATIVITY, is neither specific nor fundamental to and for GENERAL RELATIVITY. For example, the Maxwell equations in the manifestly covariant form of Eq. (9.11) satisfy the GR on the fixed background of Minkowski space and have nothing to do with GENERAL RELATIVITY.

→

The principle of general relativity GR has (almost) no physical content.

The "almost" referes to the fact that the principle asserts that there are no distinguished coordinate systems that exist as physically independent structures.

- The relativity postulate GR, and its mathematical manifestation as general covariance GC have been criticized already in 1917 by KRETSCHMANN [114]:

  > *[Man] vergegenwärtigt sich, daß alle physikalischen Beobachtungen letzten Endes in der Feststellung rein topologischer Beziehungen ("Koinzidenzen") zwischen räumlich-zeitlichen Wahrnehmungsgegenständen besteht und daher durch sie unmittelbar kein Koordinatensystem vor irgend einem anderen bevorrechtigt ist, so wird man zu dem Schlusse gezwungen, daß* jede physikalische Theorie *ohne Änderung ihres–beliebigen–durch Beobachtungen prüfbaren Inhaltes mittels einer rein mathematischen und mit*

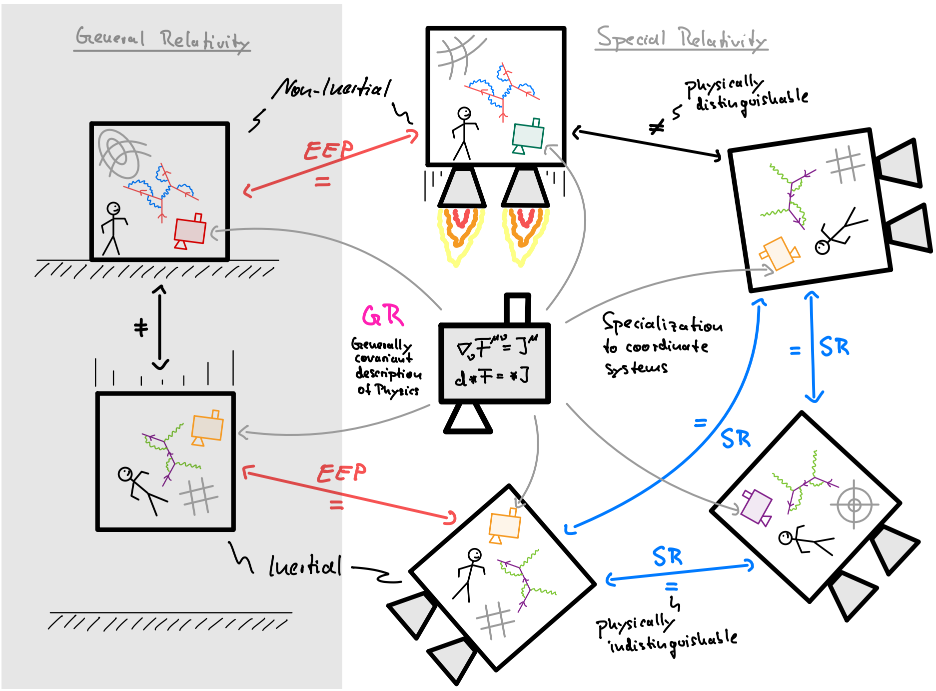
höchstens mathematischen Schwierigkeiten verbundenen Umformung der sie darstellenden Gleichungen mit jedem beliebigen–auch dem allgemeinsten–Relativitätspostulate in Einklang gebracht werden kann.

- To drive the point home: One can also formulate good old non-relativistic Newtonian mechanics in a generally covariant form (it's quite ugly, though)! See the original literature [115] and MISNER *et al.* [2] (Box 12.4 and §12.5):

  Any physical theory originally written in a special coordinate system can be recast in geometric, coordinate-free language. Newtonian theory is a good example [..]. Hence, as a sieve for separating viable theories from nonviable theories, the principle of general covariance is useless.

- For a detailed account on the role general covariance plays in GENERAL RELATIVITY (and historically played in its inception), see Ref. [116].

14 | We can summarize the relation of **SR**, **EEP**, and **GR** as follows:



- Both **SR** and **EEP** make claims about the equivalence (indistinguishability) of certain states of motion. These are physical claims about reality that can be assessed by experiments. Note that to check whether they are false or true you do not even know how to work with mathematical equations. It's a simple matter of collecting the results of experiments (recall the ← *Michelson-Morley experiment*). It is this physical content that makes **SR** and **EEP** the foundations of SPECIAL RELATIVITY and GENERAL RELATIVITY, respectively.
- By contrast, **GR** makes no such claims about reality. **GR** does *not* claim that all states of motion are indistinguishable (they are not, even in GENERAL RELATIVITY you can tell local inertial frames and accelerated frames apart); the principle only claims that all fundamental theories of physics should have a formulation that can be applied by all possible observers. **GR** is therefore more a statement about physical models than about reality.
- The sketch makes it clear that the **EEP** is actually more similar to the **SR** (in the role it plays for GENERAL RELATIVITY) than the **GR** is. In that sense "principle of general relativity" is kind of a misnomer.

15 | There is another important concept that (contrary to **GC** / **GR**) distinguishes GENERAL RELATIVITY from other theories and must itself be distinguished from **GC** / **GR**:

★ **Definition 2: Background independence BI (*No prior geometry*)**

Physical models that do not contain the geometry of spacetime as an absolute element are called ⁂ *background independent*. This implies that the geometry of spacetime emerges dynamically as *solutions* of the theory.

(Counter-)examples:

- ✓ GENERAL RELATIVITY is (and historically was the first example of) a background independent theory (→ *below*):

$$S_{\text{Einstein-Hilbert}}[g] = \frac{c^4}{16\pi G}\int d^4x\,\sqrt{g}\,R \tag{9.12}$$

  Here R is the → *Ricci scalar* that depends in a complicated way on the metric tensor field $g_{\mu\nu}(x)$. $\sqrt{g}$ is short for $\sqrt{|\det[g_{\mu\nu}(x)]|}$ (Minkowski metric: $\sqrt{\eta}=1$).

- ✗ Maxwell theory is *not* background independent (recall Eq. (6.56)):

$$S_{\text{Maxwell}}[A] = \int d^4x\,\sqrt{g}\left(-\frac{1}{16\pi}g_{\mu\alpha}g_{\nu\beta}F^{\alpha\beta}F^{\mu\nu}\right) \tag{9.13}$$

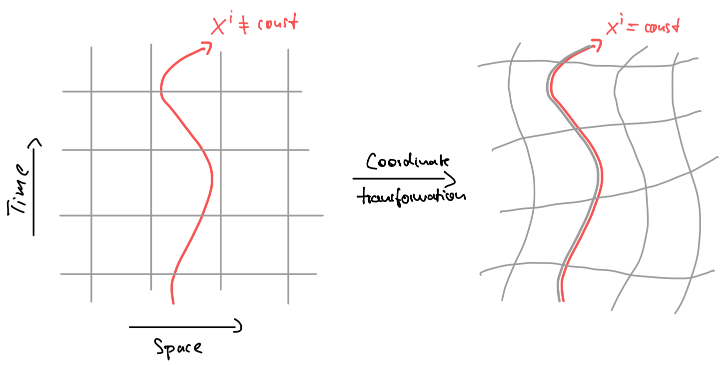
  with field-strength tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

  - Note that you do *not* extremize this action wrt. the metric g; the metric (e.g. Minkowski metric $g=\eta$) is given as a *parameter* (absolute element) of the theory. Hence it plays the role of a static background.
  - Note also that the Maxwell equations (and their Lagrangian) *are* generally covariant: they are tensorial expressions that descibe geometric objects on a manifold. That does not prevent them to have a tensor field (the metric) as an absolute element.

16 | Beware:

GENERAL RELATIVITY is most likely the first (and possibly last) generally covariant and background-independent theory that you will encounter in your university courses. Thus it is important to mention a peculiarity that, if ignored, can lead to lots of confusion when studying such theories:

i | ⊲ Trajectory of particle in spacetime:

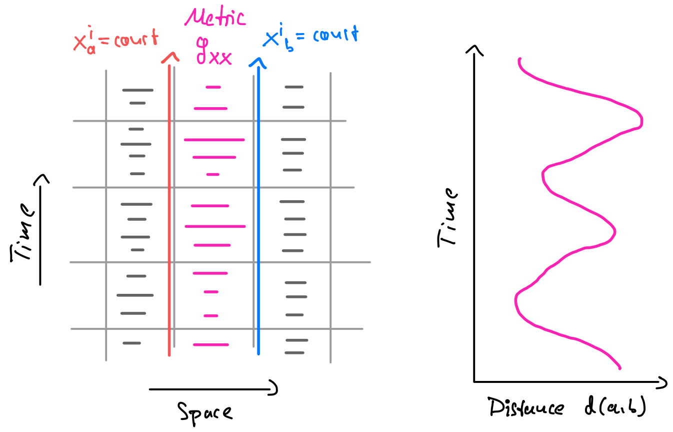


Because of general covariance (GC) there is always a coordinate system in which the (spatial) coordinates of an object are constant in time.

→ One cannot infer from coordinates whether an object is moving!

(There is no absolute notion of "motion" in GENERAL RELATIVITY; the only motion that makes sense is motion wrt. some reference object, → *next step*.)

ii | ⊲ Two objects at rest in some coordinate system:



In a background-independent theory (**BI**) the metric is a dynamic degree of freedom. Therefore two objects a and b can have *constant coordinates* in space while their distance varies over time! Note that the coordinates are completely independent of the metric in general.

→ One cannot infer from coordinates whether distances of objects change!

We can sum this up as follows:

> Coordinates have no physical meaning in GENERAL RELATIVITY; they are simply (arbitrary) labels of events.

- Please make sure you grasp this statement fully (we will see explicit examples later when we understand GENERAL RELATIVITY better):

  If I tell you in SPECIAL RELATIVITY that (in some inertial frame) two test particles have constant spatial coordinates x^i and y^i, you immediately know their relative velocity and distance: $\vec{v}_{\text{rel}} = \dot{\vec{x}} - \dot{\vec{y}} = \vec{0}$ and $\Delta l^2 = -(x^\mu - y^\nu)^2 = |\vec{x} - \vec{y}|^2$.

  By contrast, if I tell you in GENERAL RELATIVITY that two test particles have (in some coordinate system) constant spatial coordinates x^i and y^i, this tells you *nothing* about their distance; not even whether it is constant or varies in time! This information is hidden in the values of the metric field, not in the coordinates.

- This is important, for example, when studying the effects of gravitational waves (→ *later*).

17 | Summary:

> - Every reasonable fundamental theory has a generally covariant formulation.
> - A generally covariant theory does not need to be background independent.
> - GENERAL RELATIVITY is background independent and generally covariant.

- We will return to the question of general covariance, background independence (and diffeomorphism invariance) → *later* when we know more about GENERAL RELATIVITY. At this point it is only important that you know the conceptual difference between the terms *general* *covariance* GC and *background independence* BI and the requirement of the *principle of general relativity* GR.

- One sometimes hears that general covariance GC is a distinguishing feature of GENERAL RELATIVITY (as explained above, it is not). Sometimes even the very concepts of general covariance GC and background independence BI are confused. This confusion is partially rooted in history because Einstein himself didn't separate the two concepts clearly. MISNER *et al.* explain [2] (p. 431):

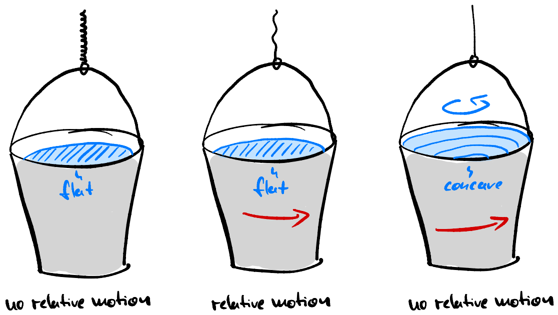
  > *Mathematics was not sufficiently refined in 1917 to cleave apart the demands for "no prior geometry" and for a "geometric, coordinate-independent formulation of physics." Einstein described both demands by a single phrase, "general covariance." The "no-prior-geometry" demand actually fathered general relativity, but by doing so anonymously, disguised as "general covariance," it also fathered half a century of confusion.*

- For more details on the relation between the concepts of background independence, general covariance, and diffeomorphism invariance see Ref. [117] (and references therein).

9.3. Mach's principle (an its failure in GENERAL RELATIVITY)

Mach's principle is not a logical postulate of GENERAL RELATIVITY and mostly of historical (and perhaps philosophical) importance. However, it is also conceptually interesting because at a fist glance once might conclude (as Einstein did), that GENERAL RELATIVITY actually satisfies the principle. It is rather subtle (and instructive) why this is not so:

18 | Recall ← *Newton's bucket*:



Question: Rotation with respect to *what* determines the shape of the water surface?

NEWTON: Absoute space!

Note that the experiment already suggests that this "absolute space" must have certain symmetries since the experiment cannot distinguish specific points nor specific states of uniform motion. SPECIAL RELATIVITY tells us that the correct symmetry group of spacetime is the Poincaré group. So from our modern perspective, Newton's answer must be read as follows: *The experiment demonstrates the independent existence of an entity which determines the local inertial systems.* We may call this entity "spacetime."

19 | The austrian physicist ERNST MACH fervently disagreed with Newton [118]:

> *Der Versuch Newton's mit dem rotirenden Wassergefäss lehrt nur, dass die Relativdrehung des Wassers gegen die Gefässwände keine merklichen Centrifugalkräfte weckt, dass dieselben aber durch die Relativdrehung gegen die Masse der Erde und die übrigen Himmelskörper geweckt werden. Niemand kann sagen, wie der Versuch verlaufen würde, wenn die Gefässwände immer dicker und massiger, zuletzt mehrere Meilen dick würden. Es liegt nur der eine Versuch*

vor, und wir haben denselben mit den übrigen uns bekannten Thatsachen, nicht aber mit unsern willkürlichen Dichtungen in Einklang zu bringen.

MACH denied NEWTON's notion of an independent entity responsible for inertia and proposed that the large-scale structure of matter in the cosmos determines the local inertial systems instead (↑ *relationalism*):

§ Principle: Mach's principle `MP`

Local inertial frames are determined by the cosmic motion and distribution of matter.

- MACH never formulated his principle precisely, which leaves room for interpretation and personal taste. This is why there are various readings of `MP` in the literature, not all equivalent. The above phrasing is a rather strict version of the principle.
- Here is an alternative way to illustrate the point by STEVEN WEINBERG [119] (p. 17):

  > *There is a simple experiment that anyone can perform on a starry night, to clarify the issues raised by Mach's principle.*
  >
  > *First stand still, and let your arms hang loose at your sides. Observe that the stars are more or less unmoving, and that your arms hang more or less straight down. Then pirouette. The stars will seem to rotate around the zenith, and at the same time your arms will be drawn upward by centrifugal force. It would surely be a remarkable coincidence if the inertial frame, in which your arms hung freely, just happened to be the reference frame in which typical stars are at rest, unless there were some interaction between the stars and you that determined your inertial frame.*

  Put this way, the situation is quite puzzling indeed and Mach's principle doesn't seem far fetched at all.
- EINSTEIN was responsible for coining the term "Mach's principle" and was influenced by it during his construction of GENERAL RELATIVITY. At first, he believed that in his new theory of gravity the principle was indeed satisfied. He writes in a letter to Mach in 1913 [120]:

  > *Dieser Tage haben Sie wohl meine neue Arbeit über Relativität und Gravitation erhalten, die nach unendlicher Mühe und quälendem Zweifel nun endlich fertig geworden ist. Nächstes Jahr bei der Sonnenfinsternis soll sich zeigen, ob die Lichtstrahlen an der Sonne gekrümmt werden, ob m. a. W. die zugrunde gelegte fundamentale Annahme von der Aequivalenz von Beschleunigung des Bezugssystems einerseits und Schwerefeld andererseits wirklich zutrifft.*
  >
  > *Wenn ja, so erfahren Ihre genialen Untersuchungen über die Grundlagen der Mechanik – Planck's ungerechtfertigter Kritik zum Trotz – eine glänzende Bestätigung. Denn es ergibt sich mit Notwendigkeit, dass die Trägheit in einer Art Wechselwirkung der Körper ihren Ursprung hat, ganz im Sinne Ihrer Überlegungen zum Newton'schen Eimer-Versuch.*

  (You may wonder how Einstein could write this letter in 1913 when he finalized GENERAL RELATIVITY in November of 1915. Einstein refers to his paper with Marcel Grossmann published in 1913 [121] in which they established the "Entwurftheorie", a precursor of GENERAL RELATIVITY that already included most of the pieces needed [but not yet the correct field equations].)

20 | So here is the case MACH (& early EINSTEIN) vs. NEWTON:

- NEWTON:

  Space exists as an independent entity and determines locally which frames are inertial.

- MACH:

  Space emerges from the relations between matter and does not exist independently. Hence the distribution of matter in the universe completely determines the local inertial systems.

Who is right according to GENERAL RELATIVITY?

21 | Answer: Both ... (in a sense, though Newton is more correct)

- NEWTON's conclusion was correct: Because of locality (constancy of the speed of light) the matter distribution of the cosmos (the fixed stars) cannot immediately influence the local inertial frame; there must be a mediator, some "background" that is here right now. GENERAL RELATIVITY tells us what this is: the metric tensor field that determines the geometry of spacetime.
- MACH was right insofar as it is indeed not a coincidence that the local inertial frame on earth is at rest with respect to the fixed stars. There *is* a relation, although not a direct and immediate one. GENERAL RELATIVITY tells us that the large-scale distribution of matter (and energy) in the universe determines the (large-scale) metric of spacetime, which, in turn, determines the local inertial systems everywhere. But there is a hitch: the metric is not uniquely determined by the mass distribution. Thus the metric (and therefore spacetime) carries independent degrees of freedom. There is more than matter in the world, spacetime is a real entity!

Notes:

- Today we know that there are solutions of the Einstein field equations (e.g. the ↑ *Gödel universe* [122]) that violate Mach's principle explicitly [123].
- MACH, in his critique of NEWTON's bucket experiment, asked (rhetorically) what would happen if the walls of the bucket would become very thick and massive. His point was that it is not excluded that at some point the rotation of the bucket *would* influence the shape of the water. GENERAL RELATIVITY tells us that this is so indeed, because a very massive bucket affects the geometry of spacetime. This is known as the → *Lense-Thirring effect* [124, 125] (also know as ↑ *frame-dagging*) and has been experimentally confirmed (not with a massive bucket, of course, but with earth) [126, 127]. However, for the reasons explained above, this effect does not make GENERAL RELATIVITY comply with Mach's principle in the strict sense.
- Because of the many different versions of MP floating around, for some the case is still not closed (at least for philosophers of science, it seems). For doing physics with GENERAL RELATIVITY, MP is irrelevant.
- MACH advocated a relational view of space(time): Only relations between the degrees of freedom of matter are observable. There is no independent meaning of, say, an electron being *here now*. It is interesting to realize that this relational view might very well be true (and in accordance with GENERAL RELATIVITY) if one accepts that the metric field is just another collection of degrees of freedom which can be in relations (coincide or interact) with other degrees of freedom. For example, an electron being *here now* might simply mean that an excitation of the field that describes the electron coincides/interacts with a particular degee of freedom of the metric field.

22 | Conclusion:

The controversy about the MP essentially boils down to the question whether spacetime has independent degrees of freedom (and therefore exists in a physical sense):

In its strict version, the MP denies spacetime this independent role. By contrast, GENERAL RELATIVITY grants spacetime independent degrees of freedom because the → *Einstein field equations* only constrain the → *Einstein tensor* but not the metric directly (→ *Gravitational waves*):

> GENERAL RELATIVITY *violates* Mach's principle MP because matter *influences* the geometry of spacetime but does not *determine* it uniquely.

This situation is exemplified by gravitational waves: When in 2015 the interferometers of LIGO detected a gravitational wave passing earth, the spacetime geometry in our vicinity changed by a very tiny bit. However, the mass distribution in the vicinity of earth didn't change at all. So while the geometry of spacetime certainly is *influenced* by earths mass, it is not uniquely *determined* by it. LIGO therefore measured directly the dynamics of the degrees of freedom the existence of which MP denies.

9.4. Overview and Outline

Now that we know the conceptual starting point of GENERAL RELATIVITY, and argued that more general spacetimes than flat Minkowski space are needed to accomodate gravity, we can reveal the gist of GENERAL RELATIVITY and sketch the plan for the remainder of this course:

¡! You are not required to fully grasp the *how* and *why* of the following statements. Understanding the details is the objective of this course. However, I think that it is useful to start off with a rough picture of what we want to accomplish because otherwise one is easily swamped by the details along the way.

GENERAL RELATIVITY in a Nutshell

- Ontology:

$$\boxed{\begin{aligned} \text{Spacetime} \quad &\equiv \quad \text{4D differentiable manifold } M \\ \text{Gravitational field} \quad &\equiv \quad \text{pseudo-Riemannian metric } g \text{ with signature } (1,3) \end{aligned}}$$

$\to$ Spacetime is a ← *4D Lorentzian manifold*

  - Note that we only fix the dimensionality of M (and thereby its *local* topology) but not its *global* topology (i.e., whether it is simply $\mathbb{R}^4$, a sphere, a torus, or something even more fancy). Thus, for example, GENERAL RELATIVITY makes no a priori statement about the finiteness of the universe. (Asking about the *local* topology is like asking where space and time come from—and GENERAL RELATIVITY is silent about that. A reasonable theory of quantum gravity must address this question!)
  - At this stage it is sufficient to interpet the points $E \in M$ of the manifold as points in spacetime and therefore as (equivalence classes of) events. However, we will see that this interpretation is problematic (→ *Hole argument*) because of the diffeomorphism invariance of GENERAL RELATIVITY. It is thus questionable whether points of the manifold (and thereby the manifold itself) can be associated to any existing entity. An entity that certainly does exist, however, is the metric field.

- Equivalence principle:

  The EEP is built right into the mathematical framework of GENERAL RELATIVITY:

  For every point with coordinates y, there exists a coordinate transformation φ_y with $\bar{x} = \varphi_y(x)$ such that:

$$\bar{g}^{\mu\nu}(\bar{x}) = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta} g^{\alpha\beta}(x) \overset{x\approx y}{\approx} \eta^{\mu\nu} \quad \text{and} \quad \bar{\partial}_\rho \bar{g}^{\mu\nu}(\bar{x}) \overset{x\approx y}{\approx} 0 \tag{9.14}$$

  with Minkowski metric

$$\eta^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu} . \tag{9.15}$$

  → *Locally* inertial coordinates ✓(➔ Problemset 2)

$$\boxed{\text{The EEP motivates the metrization of gravity.}} \tag{9.16}$$

  - ¡! In the presence of gravity, there is *no* coordinate transformation that brings the metric into Minkowski form *everywhere* on the spacetime manifold. Conversely, if this *is* possible, spacetime is flat Minkowski space and you were doing SPECIAL RELATIVITY all along (perhaps in curvilinear coordinates).
  - Note that metrization of gravity is not a *mathematical corollary* of the EEP (the latter is a physical principle, not a rigorous mathematical statement). However, the EEP is most naturally incorporated into a mathematical framework where gravity is described by a metric because the gist of the EEP is that all (local) physical phenomena are affected by gravity in the same way. This is exactly what happens if gravity is identified with the geometry of spacetime!
  - At every point, the basis $\{\bar{\partial}_\rho\}$ of the tangent space forms a so called ⁂ *local Lorentz frame*. You can choose such a basis for all points of spacetime. However, in general there is no coordinate system that induces this basis everywhere; you have to use multiple charts to patch together spacetime.

- Important fields:

  All degrees of freedom (some gauge, some physical) of GENERAL RELATIVITY are stored in the metric tensor field. From the metric, one can then derive other fields that play important roles in the formulation of the theory:

$$\begin{tikzcd}
 & & & & \overbrace{R_{\mu\nu}}^{\text{Ricci tensor}} \arrow[dd, "\mathrm{Tr}"] \arrow[dr, "\text{Ch. ??}"] & \\
\underbrace{\overbrace{g_{\mu\nu}(x)}^{\text{Metric tensor}}}_{\hat{=}\ \text{Gravitational potential}} \arrow[r, "\partial g", "\text{Sec. ??}"'] & \underbrace{\overbrace{\Gamma^\lambda{}_{\mu\nu}}^{\text{Connection (no tensor)}}}_{\hat{=}\ \text{Gravitational field strenght}} \arrow[r, "{\partial\Gamma,\, \Gamma^2}", "\text{Sec. ??}"'] & \underbrace{R_{\rho\sigma\mu\nu}}_{\text{Curvature tensor}} \arrow[urr, "\mathrm{Tr}"] & & & \underbrace{G_{\mu\nu}}_{\text{Einstein tensor}} \\
 & & & & \underbrace{R}_{\text{Ricci scalar}} \arrow[ur, "\text{Ch. ??}"'] &
\end{tikzcd}$$

Note that the Einstein tensor is non-linear in the metric an contains (up to) second-order derivatives.

- Einstein field equations (EFE): (here without cosmological constant)

  The centerpiece of GENERAL RELATIVITY is a tensorial partial differential equation that determines the metric tensor field in dependence of the energy momentum tensor of matter:

$$\boxed{\text{Einstein tensor} \rightarrow \underbrace{G_{\mu\nu}}_{\text{“Geometry”}} = \kappa \underbrace{T_{\mu\nu}}_{\text{“Matter”}} \leftarrow \text{Energy-momentum tensor}} \tag{9.17}$$

  ¡! “Matter” refers here to all degrees of freedom that carry energy and/or momentum. This includes bodies with rest mass but also electromagnetic radiation etc.

  ⊲ Eq. (9.17): Non-linear, second-order PDE for $g_{\mu\nu}$:

  GENERAL RELATIVITY describes the geometry of space as a dynamical field that evolves “in time” according to a highly nontrivial PDE: →

$$\boxed{\text{GENERAL RELATIVITY} = ⁂\ \textit{Geometrodynamics}} \tag{9.18}$$

  The nonlinearity makes Eq. (9.17) hard to solve, even in vacuum were the right-hand side vanishes.

  - Spacetime geometry is dynamical → Background independence ✓
  - Tensor equation → General covariance (no preferred coordinate system) ✓
  - *Mass distribution* determines *metric* determines *local inertial frames*
    However: Fixing $G_{\mu\nu}$ leaves some degrees of freedom of $g_{\mu\nu}$ unconstrained!
    → Boundary conditions required for unique solution
    → Mach's principle is not satisfied (but partially survives in spirit) ✗/✓
  - Recall Eq. (8.10) and our discussion that followed (also ➔ Problemset 1). Eq. (9.17) is structurally similar but fixes the problem of linearity because the Einstein tensor is a non-linear function of the metric.

- Physics with gravity:

  Once gravity is described by the metric, one must generalize the other relativistic theories (mechanics, electrodynamics, …) into a generally covariant form that couples to the metric. This generalization is a priori not unique because matter can couple in various ways to the fields derived from the metric.

  However, the **EEP** severely restricts the couplings that are allowed and leads to a “recipe” how the Lorentz covariant equations of SPECIAL RELATIVITY must be rewritten to match the principles of GENERAL RELATIVITY (→ *“Comma-Goes-to-Semicolon Rule”, Minimal coupling*):

  > The **GR** demands physical theories to be specified by *tensor equations*.
  > The **EEP** restricts the possible couplings of matter and metric.

  →

  - Mechanics: (with $u^\mu = \frac{dx^\mu}{d\tau}$ the 4-velocity)

$$m \underbrace{\frac{Du^\mu}{D\tau}}_{\substack{\text{Absolute}\\\text{derivative}}} = m \underbrace{\frac{d^2x^\mu}{d\tau^2}}_{\text{4-accel.}} + m \underbrace{\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}_{\text{New!}} = \underbrace{K^\mu}_{\text{4-force}} \tag{9.19}$$

→ Free particle: $K^\mu = 0$ → Geodesic equation ("straight lines" in spacetime)

- Electrodynamics: ⊲ Inhomogeneous Maxwell equations: (cf. Eq. (6.50))

$$\overbrace{F^{\mu\nu}{}_{,\nu} = -\frac{4\pi}{c} j^\mu}^{\text{SPECIAL RELATIVITY}} \quad \to \quad \overbrace{\underbrace{F^{\mu\nu}{}_{;\nu}}_{\text{Covariant derivative}} = -\frac{4\pi}{c} J^\mu}^{\text{GENERAL RELATIVITY}} \tag{9.20}$$

The covariant derivative contains the connection Γ and therefore the metric g. This is how the electromagnetic field is affected by the gravitational field (e.g., bent in the vicinity of heavy masses). Conversely, the EM field gives rise to the energy momentum tensor $T^{\mu\nu}_{\text{em}}$ [← *Eq.* (6.110)] and thereby contributes to the right-hand side of Eq. (9.17).

→ Energy-momentum tensor $T_{\mu\nu}$ is dynamical

A generally relativistic theory of matter, interacting with and via gravity, is then described by a *coupled, non-linear, higher-order system of partial differential equations* (where $T_{\mu\nu}$ depends on the dynamical variables of the matter theory).

→ Hard to solve in general → Approximations needed!

Outline of this course

Here is our approach for this course to study these various aspects of GENERAL RELATIVITY:

- Step 1 (Section 9.4): *How to describe non-Euclidean manifolds mathematically?*

  In Chapter 3 we introduced the concept of differentiable manifolds and introduced the concept of a (pseudo-)Riemannian metric to measure lengths of curves on the manifold. To formulate GENERAL RELATIVITY mathematically, we need to revisit and extend this toolbox of tensor calculus.

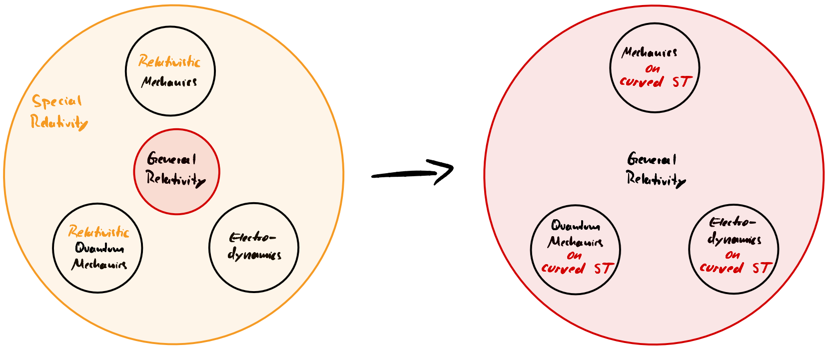
  In particular, we will study two (at first independent) structures that can be put on a differentiable manifold:

  → *Affine connection* → Determines parallel transport, straight lines, and curvature
  ← *Riemannian metric* → Determines lengths, shortest lines, and angles

  As it turns out, when you are given a Riemannian metric, there is a unique way to construct an affine connection. This means that once you are given a spacetime manifold with a (pseudo-) Riemannian (Lorentzian) metric, all concepts in the list above are well-defined. This is then the framework we will use: The spacetime of GENERAL RELATIVITY is a Lorentzian manifold and the degrees of freedom (field) of the theory is the Lorentzian metric itself.

- Step 2 (??): *How to formulate relativistic theories on non-Euclidean spacetimes?*

  In the first part of this course, we first established the tenets of SPECIAL RELATIVITY (Lorentz symmetry) and then incorporated them successively in known theories of physics (point mechanics in Chapter 5, electrodynamics in Chapter 6, quantum mechanics in Chapter 7). Now that we established the tenets of GENERAL RELATIVITY (the spacetime metric is not necessarily the Minkowski metric but an arbitrary Lorentzian metric), we must again reformulate our theories to comply with this new insight. Recall p. 16 in the introduction:



This will lead us to generally covariant formulations of relativistic mechanics in ?? and electrodynamics in ?? (we will skip quantum mechanics this time, but this is also possible). Thanks to the combination of **EEP** and **GR** (and tensor calculus), the recipe to go from the equations of SPECIAL RELATIVITY to that of GENERAL RELATIVITY will be very simple.

- Step 3 (??): *How to determine the geometry of spacetime dynamically?*

  Up until this point we simply declared that the metric of spacetime is an arbitrary Lorentzian metric and studied the effects on physics *given such a metric*. The core idea of GENERAL RELATIVITY (and maybe *the* most important insight of Albert Einstein) was that this metric was not part of the laws of nature but just another degree of freedom that had to be dynamically determined. This means that there is no a priori geometry of spacetime, a principle known as ← *background independence*. The equations that dynamically determine the metric are the → *Einstein field equations* Eq. (9.17); they are the centerpiece of GENERAL RELATIVITY and determine the geometry of spacetime, given the distribution of mass and energy and some boundary conditions. We will derive these equations via an action principle from a Lagrangian.

  Together with Step 2, this completes the framework of GENERAL RELATIVITY.

- Step 4 (??): *What does* GENERAL RELATIVITY *predict?*

  If we combine the results of Step 2 and Step 3 we obtain a self-contained, background independen framework to describe physics: Matter determines the geometry of spacetime (Step 3) and, conversely, this geometry determines how matter evolves (Step 2). This interplay makes for beautiful but mathematically complicated models. Thus, to study the predictions of GENERAL RELATIVITY, we typically resort to simplified approaches:

  - Consider a static, inhomogeneous distribution of large masses (e.g. the sun). Using the Einstein field equations from Step 3 (and reasonable boundary conditions), calculate the geometry of spacetime induced by this distribution. Then use the results of Step 2 to determine the evolution of small test particles on this curved spacetime (without taking their backaction on spacetime into account). This approach leads to a variety of phenomena, e.g., the slowing down of clocks close to large masses (??), the perihelion precession of Mercury (??), the bending of light (??), etc.
  - Consider the Einstein field equations in vacuum, i.e., without any matter (or energy). Because the EFEs are non-linear (recall Section 8.2) and the geometry of spacetime is not uniquely determined by the distribution of mass and energy, this situation is not as boring and trivial as it sounds (its actually very complicated). But even in the weak field limit (where one drops the self-interactions) one finds something interesting: gravitational waves (??).
  - Consider an idealized universe that is homogeneously filled with matter and energy (and potentially dark matter and dark energy). If one calculates the solutions of the EFEs in such a scenario, one obtains the (approximate) spacetime geometry of the whole universe. This leads into the field of ↑ *relativistic cosmology* and to the current standard model of cosmology, known as "ΛCDM". This is where one finds the possiblity of an expanding universe and its origin, the Big Bang; this is also where the cosmological constant becomes important (??).

↓Lecture 19 [30.04.24]

10. Mathematical Tools II: Curvature

Part III.

Excursions

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