Special and General Relativity

Lecture Notes • Winter Term 2023/24

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How Stable Diffusion imagined a light cone in 2023.



How DALL · E imagined a light cone in 2024.

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Preliminaries

Important

This script is in development and continously updated. To download the latest version:

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If you spot mistakes or have suggestions, send me an email:

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Requirements for this course

We assume that students are familiar with the following concepts:

- Classical mechanics (Lagrangian and Hamiltonian formalism ...)
- Non-relativistic quantum mechanics (Schrödinger equation ...)
- Classical electrodynamics (Maxwell equations ...)
- Basics of algebra & linear algebra (groups, linear maps, ...)
- Second quantization and path integrals ***** This is only required for the excursions on quantum gravity!

Literature recommendations

Special relativity

 Schröder: Spezielle Relativitätstheorie [1] ISBN 978-3-808-55653-5 Compact, pedagogic, mathematically precise introduction (in German).

General relativity

- Schröder: Gravitation: Einführung in die Allgemeine Relativitatstheorie [2] ISBN 978-3-817-11874-8 Compact, pedagogic, mathematically precise introduction (in German).
- Misner, Thorne, and Wheeler: Gravitation [3] ISBN 978-0-691-17779-3 Extensive standard textbook on special and general relativity (in English).
- Carroll: *Spacetime and Geometry* [4] ISBN 978-1-108-48839-6 Very accessible and mathematically clear textbook on general relativity (in English).



- Schutz: *A First Course in General Relativity* [5] ISBN 978-1-108-49267-6 Extensive, pedagogic, mathematically precise introduction.
- Rovelli: General Relativity: The Essentials [6] ISBN 978-1-009-01369-7
 Very high level and compact overview with links to quantum gravity.

Quantum gravity

- Zwiebach: *A First Course in String Theory* [7] ISBN 978-0-521-88032-9 Extensive, pedagogic introduction with many detailed calculations.
- Rovelli: Covariant Loop Quantum Gravity: An Elementary Introduction to Quantum Gravity and Spinfoam Theory [8] ISBN 978-1-108-81025-8 Compact, pedagogic introduction, omitting some technical details.

This course follows roughly the textbook *Spezielle Relativitätstheorie* by Ulrich Schröder [1] in the first part on special relativity (with admixtures from Schutz [5] and Straumann [9]). The second part on general relativity follows roughly the textbook *Gravitation* by Ulrich Schröder [2] (with admixtures from Misner [3], Carroll [4], and Rovelli [6]). The excursions on quantum gravity at the end draw from Barton Zwiebach's *A First Course in Sring Theory* [7] for the primer on bosonic string theory, and Carlo Rovelli's *Covariant Loop Quantum Gravity* [8] for the sneak peek at loop quantum gravity (*the latter is not yet written*).

Original literature

- A. Einstein: *Zur Elektrodynamik bewegter Körper* [10] Annalen der Physik, 17, p. 891-921, (1905) Einstein bootstraps SPECIAL RELATIVITY (in German).
- A. Einstein: *Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?* [11] Annalen der Physik, 18, p. 639-641, (1905) Einstein derives the famous mass-energy equivalence (in German).
- A. Einstein: Zur allgemeinen Relativitätstheorie [12]
 Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 778-786, 799-801, (1915)
 A. Einstein: Die Feldgleichungen der Gravitation [13]
 Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 844-847, (1915)
 Einstein bootstraps the field equations of general relativity (in German).
- A. Einstein:

Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie [14] Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 831-839, (1915) Einstein explains Mercury's apsidal precession (in German).

- A. Einstein: *Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie* [15] Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 142–152, (1917) Einstein kickstarts relativistic cosmology and introduces the cosmological constant (in German).
- A. Einstein: *Uber Gravitationswellen* [16] Sitzungsberichte der Preußischen Akademie der Wissenschaften, p. 154-167, (1918) Einstein predicts and studies gravitational waves (in German).



Goals of this course

The goal of this course is to gain a thorough understanding of relativity, our modern theory of space and time ("spacetime"). This includes both the *symmetries* and the *dynamics* of spacetime; the former being described by *special relativity*, the latter by *general relativity*. We close with an (optional) excursion into the quantization of gravity, and briefly discuss the two most prominent contenders: string theory and loop quantum gravity.

In particular (***** optional):

(Gray topics are not yet covered by the script.)

Special relativity

- · Conceptual foundations special relativity
- · Galileian and Einsteinian relativity principles
- Lorentz transformations and the principle of invariance
- Kinematical consequences of Lorentz transformations
- Tensor calculus and the metric tensor
- Special relativity in Minkowski space
- Lorentz- and Poincaré group
- Relativistic mechanics
- Lagrange function and principle of least action
- Electrodynamics as a relativistic field theory
- Noether theorem and the energy momentum tensor
- Relativistic quantum mechanics (Klein-Gordon- and Dirac equation)

General relativity

- Incompatibility of gravitation and special relativity
- Mathematical toolbox: Riemannian manifolds, metric tensor, Levi-Civita connection, curvature, ...
- Conceptual framework of general relativity: Metric field, general covariance vs. background independence, ...
- Classical mechanics in curved spacetime
- Electrodynamics in curved spacetime
- Dynamics of general relativity (Einstein field equations)
- Implications of the Einstein field equations: Newtonian limit, Gravitational time dilation, Apsidal precession, Light deflection ...
- Application: Gravitational waves (linearized Einstein equations)
- Application: Black holes (Schwarzschild solution)
- Application: The standard model of cosmology ***** (FLRW metric, Λ CDM, ...)
- Limitations of general relativity: ***** Einstein-Hilbert action, quantum field theory, (non-)renormalizability, ...



Quantum gravity (excursion)

- The bosonic string ***** : Quantization, Virasoro algebra, anomalies, Hilbert space, gravitons, tachyons, ...
- Concepts of quantum loop gravity \star : Discretized gravity, spin networks, vertex amplitude, transition amplitudes, ...

Notes on this document

- This document is not an extension of the material covered in the lectures but the script that I use to prepare them.
- Please have a look at the given literature for more comprehensive coverage. References to primary and secondary resources are also given in the text.
- The content of this script is color-coded as follows:
 - Text in black is written to the blackboard.
 - Notes in red should be mentioned in the lecture to prevent misconceptions.
 - Notes in blue can be mentioned/noted in the lecture if there is enough time.
 - Notes in green are hints for the lecturer.
- One page of the script corresponds roughly to one covered panel of the blackboard.
- Enumerated lists are used for more or less rigorous chains of thought:
 - 1 | This leads to ...
 - $\mathbf{2}$ | this. By the way:
 - i | This leads to ...
 - ii | this leads to ...
 - iii | this.
 - **3** | Let's proceed ...
- In the bibliography (p. 490 ff.) you can find links to download most papers referenced in this script (they look like this: **Download**). Because most of these papers are not freely available, you need a username & password to access them. These credentials are made available to students of my classes.
- This document has been composed in Vim on Arch Linux and is typeset by LuaLATEX and BIBTEX. Thanks to all contributors to free software!
- This document is typeset in Equity, Concourse and MathTimeBrofessional.

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- Manya Willberg translated some of my ugly sketches into nice TikZ figures.
- Thanks to Angelos Aslanidis, Manya Willberg and Niklas Buschmann who spotted mistakes and/or typos in the script.



Symbols & Scientific Abbreviations

The following abbreviations and glyphs are used in this document:

- cf confer ("compare") degree(s) of freedom dof exempli gratia ("for example") eg et cetera ("and so forth") etc et alii ("and others") et al id est ("that is") ie videlicet ("namely") viz versus ("against") vs without loss of generality wlog with respect to wrt if and only if iff "consider" \triangleleft "therefore" \rightarrow "Beware!" ;! non-obvious equality that may require lengthy, but straightforward calculations * non-trivial equality that cannot be derived without additional input $\stackrel{\circ}{\rightarrow}$ "it is easy to show" $\stackrel{*}{\rightarrow}$ "it is not easy to show" logical implication \Rightarrow logical conjunction \wedge logical disjunction \vee repeated expression anonymous reference "without" w/o "with" w/ internal forward reference ("see below/later") \rightarrow internal backward reference ("see above/before") ← external reference to advanced concepts ("have a look at an advanced textbook on...") ↑ external reference to basic concepts ("remember your basic course on...") $\mathbf{1}$ reference to previous or upcoming exercises € optional choice/item \star implicit or explicit definition of a new technical term ("so called ... ") ** ‡ Aside Synonymous terms \equiv
 - := Definition



BRT Belinfante-Rosenfeld tensor CERN European Organization for Nuclear Research COE Center of energy COM Center of mass | Center of momentum Continuity CO DFST Dual field-strength tensor EM Electromagnetic EMT Energy momentum tensor EOM Equation of motion ES Einstein synchronization FLRW Friedmann-Lemaître-Robertson-Walker (metric) FST Field-strength tensor GR GENERAL RELATIVITY HME Homogeneous Maxwell equations Homogeneity HO IC Invariance of coincidence Inhomogeneous Maxwell equations IME Inertial (test) IN Instantaneous rest frame IRF IRS Instantaneous rest system IS Inertial system | Isotropy ISS International space station IT Infinitesimal transformation KG Klein-Gordon KGE Klein-Gordon equation LT Lorentz transformation ME Maxwell equation(s) OC Orthonormal Cartesian (coordinates) PDE Partial differential equation QED Quantum electrodynamics OFT Quantum field theory RI Reparametrization invariance SE Schrödinger equation SI Système international d'unités Speed of light SL SR SPECIAL RELATIVITY

The following scientific abbreviations are used in this document:

UV Ultraviolett

↓ Lecture 1 [17.10.23]

0. Setting the Stage

0.1. Terminology

The most important terms in this course and their German correspondence:

RELATIVITY = *Relativitätstheorie* SPECIAL RELATIVITY = Spezielle Relativitätstheorie (SRT) GENERAL RELATIVITY = Allgemeine Relativitätstheorie (ART)

Relation of the theories:

RELATIVITY GENERAL RELATIVITY

0.2. Motivation

RELATIVITY is arguably the most popular of scientific theories, for it speaks about an entity of every day experience: space and time. This popularity comes with a caveat:

The "Mona Lisa perspective"

The popular status of RELATIVITY in physics parallels RELATIVITY is interesting because it describes some, but that of the Mona Lisa in arts: Einstein's magnum opus not all facets of reality. Its incompatibility with quantum inherits an aura of perfection and finality.

The "Puzzle Perspective"

mechanics hints at a reality even stranger than its pieces.

1! You should not view RELATIVITY as the "Mona Lisa of physics" but as the harbinger of quantum gravity¹ that, most likely, will come with a reformulation of reality so profound that the "strangeness" of quantum mechanics and RELATIVITY alike will pale in comparison (\rightarrow *Excursions*).

¹I use the term "quantum gravity" here very loosely and essentially synonymous with "theory of everything".







0.3. Ontology

1 The *** ontology of physics* is the collection of "things that exist" (*** entities*):

Ontology = { Leptons, Hadrons, Higgs, Gauge bosons }

Standard Model of Particle Physics

2 | Physical theories are *models* that describe how these entities behave. Examples:

Classical mechanics describes the dynamics *of* matter on macroscopic scales. Quantum mechanics describes the dynamics *of* matter on microscopic scales. Electrodynamics describes the dynamics *of* electromagnetic fields on macroscopic scales.

Note that these can be *effective* (approximate) descriptions that are restricted to finite scales of validity (length, energy, time).

- **3** What is RELATIVITY a theory of?
 - $i \mid \triangleleft Two notions of space and time:$



** Relational space & time

ii $| \triangleleft$ Delete all entities from the world:



** Newtonian space & time



Newtonian space & time left!

Question: Which notion describes reality?

Nothing!



<u>Question</u>: Rotation with respect to *what* determines the shape of the water surface? Tentative answer: Rotation with respect to Newtonian space!

i! Today, Newtonian space & time (sometimes called neo-Newtonian or Galilean spacetime) is *not* seen as a preferred ("absolute") coordinate system, with respect to which absolute *positions, times* and *velocities* can be measured; it is the entity that is responsible for the absolute notion of *acceleration* in Newtonian physics (which is also present in RELATIVITY). It is "the thing" that determines the reference frames that are *inertial* [6].

\rightarrow Space & time (Spacetime) is an independent "thing that exists."

The correct answer to the bucket experiment in RELATIVITY will be: The rotation with respect to the *local inertial frame*—which is determined by the local gravitational field— determines the shape of the water surface. This field is determined by the large-scale distribution of mass and energy in the universe, i.e., the fixed stars; the (rotating) mass of the earth has a non-zero but tiny effect as well (\rightarrow *Frame dragging*).

4 | Thus we should extend our ontology:

Extended Ontology = { $\underbrace{\text{Leptons, } H}_{\text{Stan}}$	Hadrons, Gauge bosons, Higgs, Spacetime } ndard Model of Particle Physics RELATIVITY
Core Theory	$\xleftarrow{\text{IR Energy scale UV}} \text{Theory of Everything (?)}$

The A Core Theory [17] (\Rightarrow below) is an effective (quantum) field theory that encompasses the standard model and RELATIVITY. It describes all entities know to us on our scales—but is expected to fail on the Planck scale (in the "UV limit"). The theory that the Core Theory renormalized to in this UV limit is the famous "Theory of Everything". This is uncharted territory and we do not know what this theory looks like.

The extended ontology above is known as * substantivalism in the philosophy of science, see [18] for a review and [19] for a supportive account of this ontology. Opposing substantivalism is * relationalism, which defends the view that spacetime is not an independent entity but an emergent description of relations between entities (\uparrow The Hole Argument). Relationalism is exemplified by \uparrow Mach's principle, which has been historically influential in the development of GENERAL RELATIVITY (though Einstein later changed his views). In the light of non-trivial solutions (of



the Einstein field equations) for "empty" universes in GENERAL RELATIVITY, and the (now experimentally confirmed) existence of gravitational waves, I take a substantivalist stance in this course.

5 | This extended ontology allows us to answers the question:

RELATIVITY is the theory *of* spacetime (on macroscopic scales), just as electrodynamics is the theory *of* the electromagnetic field.

i! Despite these conceptual similarities, there is a fundamental difference between RELATIVITY and electrodynamics (\rightarrow *below*): Whereas electrodynamics describes the dynamics of the electromagnetic field *on* spacetime, the gravitational field of RELATIVITY does *not* evolve *on* spacetime; it *is* spacetime!

0.4. **‡** The Core Theory

The $\stackrel{*}{\leftarrow}$ Core Theory S_* is the $\stackrel{*}{\leftarrow}$ effective field theory that describes all entities on the energy scales relevant for our everyday life [17]. As typical for a field theory, it is best expressed as a \uparrow path integral:



What makes this an *effective* theory is the momentum cutoff Λ : The theory describes the dynamics of the fields only up to some finite momentum/energy cutoff Λ . In [17] it is argued that $\Lambda \sim 10^{11}$ eV is a reasonable cutoff; since this is well below the Planck scale of 10^{28} eV, A_* does not describe the physics on these energy scales (e.g., what happens in black holes or near the Big Bang is *not* encoded in A_*). This reflects the lack of a consistent theory of quantum gravity.

The action S_* splits into two parts (plus one additional, technical term that we can savely ignore here):

$$S_*[g, G, \psi, \phi] = S_{\text{EH}}[g] + S_{\text{SM}}[g, G, \psi, \phi].$$

The first part is the famous ** Einstein-Hilbert action* and describes the gravitational field *g*:

$$S_{\rm EH}[g] = \frac{c^3}{16\pi G} \int \mathrm{d}^4 x \,\sqrt{g} R(g) \,.$$

Here, G in the denominator denotes the gravitational constant (not to be confused with the gauge fields G above). We will encounter this action in the second part of this course as it encodes the (source-free) $\stackrel{\text{\tiny \sc line trans}}{\stackrel{\text{\tiny \sc line trans}}}{\stackrel{\text{\tiny \sc line trans}}{\stackrel{\text{\tiny \sc line trans}}{\stackrel{\text{\tiny \sc line trans}}{\stackrel{\text{\tiny \sc line t$

The second part is the action of the * standard model of particle physics (coupled to gravity via g) and describes all the stuff in our world (matter and interactions) except gravity:





Here "ke&i" stands for *kinetic energy* and *interactions* (with gauge bosons). The standard model action $S_{\text{SM}}[G, \psi, \phi] \equiv S_{\text{SM}}[\eta, G, \psi, \phi]$ on a static, flat spacetime $g = \eta$ is typically discussed in a course on quantum field theory with focus on high energy physics (\uparrow Section 10.2 of my script on QFT[20]). In this course on RELATIVITY, the existence of S_{SM} will leave its (classical) mark on the Einstein field equations in form of the ** energy-momentum tensor.

0.5. Relation to other theories

1 | RELATIVITY is similar to other theories in that it is a theory *of* an entity that makes up reality. However, it is also different in that this very entity makes an appearance in most other theories:

Classical mechanics describes the macr. dynamics of matter *on* spacetime: $\vec{x}(t)$. **Quantum mechanics** describes the micr. dynamics of matter *on* spacetime: $\Psi(\vec{x}, t)$. **Electrodynamics** describes the macr. dynamics of **EM fields** *on* spacetime: $E(\vec{x}, t)$, $B(\vec{x}, t)$.

In the light of the extended ontology (where spacetime is an idependent entity described by RELA-TIVITY), it can be useful to reframe the objective of various theories as follows:

Classical mechanics describes the macr. dynamics of **matter** *interacting with* a (static) **spacetime**. **Quantum mechanics** describes the micr. dynamics of **matter** *interacting with* a (static) **spacetime**. **Electrodynamics** describes the macr. dynamics of **EM fields** *interacting with* a (static) **spacetime**.

Note that this reading is manifest in the background-independent formulation of the Core Theory $S_{\star}[g, G, \psi, \phi]$ where the metric g and the other fields are treated on the same footing.

 \rightarrow The properties of spacetime (as posited by RELATIVITY) must be reflected by these theories!

This means that we might have to *modify* known theories to be consistent with RELATIVITY. These modifications must adhere to the *correspondence priciple*: The "old" (non-relativistic) versions of the theories must be included in the "new" (relativistic) versions as limiting cases.

- 2 | Incorporating the tenets of SPECIAL RELATIVITY leads to ...
 - *Relativistic* mechanics
 - Relativistic quantum mechanics (Dirac equation, Klein-Gordon equation)
 - *Relativistic* electrodynamics (= classical electrodynamics)





Luckily, classical electrodynamics is already consistent with SPECIAL RELATIVITY and needs no modification. By constrast, both classical mechanics and the quantum mechanics you learned in your previous courses must be *modified* to reflect the symmetries of spacetime posited by SPECIAL RELATIVITY.

- **3** | Incorporating the tenets of GENERAL RELATIVITY leads to ...
 - (Relativistic) Mechanics on curved spacetimes
 - (Relativistic) Quantum mechanics on curved spacetimes
 - (Relativistic) Electrodynamics on curved spacetimes



In this course, we will discuss the modifications needed for *mechanics* and *electrodynamics* to fit the framework of GENERAL RELATIVITY. We won't discuss quantum mechanics on curved spacetimes.

i! Quantum mechanics (describing matter and gauge bosons) on a curved spacetime is *not* "quantum gravity!" Quantum gravity is a theory where the metric field g *itself* is quantized (which we do not know how to do).

0.6. Spoiler

The gist of RELATIVITY can be summarized as follows:

Spacetime \leftrightarrow Four dimensional Lorentzian manifold (M, g)Gravitational field \leftrightarrow Metric tensor field g

This is what is meant by the popular statement that gravity "is not a force" but a geometrical deformation ("curvature") of spacetime.

and

SPECIAL RELATIVITY : g has signature (1, 3) (Lorentz symmetry) GENERAL RELATIVITY : g is a dynamical field (Background independence)



You most likely do not understand these statements at this point. That's fine! To provide you with the background knowledge to do so is the purpose of this course.

So let's start ...

Part I. Special Relativity





1. Conceptual Foundations

♦ Concepts

- Events, Observations, Coincidences, Observers, Reference frames, Einstein synchronization, Cartesian coordinates, Inertial frames, Inertial coordinate systems, Coordinate transformations, Laws of nature, Physical models and theories
- Newtonian mechanics, Form-invariance and covariance, Invariance group, Active and passive transformations, Galilei transformations, Galilei group, Galilean principle of relativity
- Maxwell equations, Aether, Michelson Morley experiment, Principle of Special Relativity
- Isotropy, Homogeneity, Affine transformations
- Special Lorentz transformations, Lorentz Boosts, Lorentz group, Lorentz factor, Limiting velocity, Lorentz covariance, Addition of collinear velocities, Finite speed of causality
- Spacetime interval, Invariant interval, Time-like, Space-like, Light-like, Light cone, Invariant hyperbolae, Causality, Time-like trajectories, Partial order of events, Causal automorphism
- Relativity principles, Symmetries of spacetime, Simplicity of nature, Compressibility, Anthropic principle

1.1. Events, frames, laws, and models

1 Events:

i A. Einstein writes in his 1905 paper "Zur Elektrodynamik bewegter Körper" [10]:

Wir haben zu berücksichtigen, daß alle unserer Urteile, in welchen die Zeit eine Rolle spielt, immer Urteile über gleichzeitige Ereignisse sind. Wenn ich z. B. sage: "Jener Zug kommt hier um 7 Uhr an," so heißt dies etwa: "Das Zeigen des kleinen Zeigers meiner Uhr auf 7 und das Ankommen des Zuges sind gleichzeitige Ereignisse."

And in his 1916 review "Die Grundlage der allgemeinen Relativitätstheorie" [21]:

Alle unsere zeiträumlichen Konstatierungen laufen stets auf die Bestimmung zeiträumlicher Koinzidenzen hinaus. Bestände beispielsweise das Geschehen nur in der Bewegung materieller Punkte, so wäre letzten Endes nichts beobachtbar als die Begegnungen zweier oder mehrerer dieser Punkte. Auch die Ergebnisse unserer Messungen sind nichts anderes als die Konstatierung derartiger Begegnungen materieller Punkte unserer Maßstäbe mit anderen materiellen Punkten bzw. Koinzidenzen zwischen Uhrzeigern, Zifferblattpunkten und ins Auge gefaßten, am gleichen Orte und zur gleichen Zeit stattfindenden Punktereignissen.

We condense this into the following postulate:

§ Postulate 1: Invariance of coincidence IC

• Observations are *coincidences* of *events* local in space and time.



• Coincidences of events are absolute and observer independent.

ii | Example:

Event $e_1 = \langle \text{Clock A shows time 11:30} \rangle$ Event $e_2 = \langle \text{Detector B detects electron} \rangle$ Event $e_3 = \langle \text{Clock C shows time 9:45} \rangle$

If detector B and clock A are at the same location (spatial coincidence), and clock A shows 11:30 when detector B detects and electron (temporal coincidence), we say that the events e_1 and e_2 coincide: $e_1 \sim e_2$.

 \rightarrow Collect all events e_i that coincide into an equivalence class E:

 $e_1 \sim e_2 \sim e_3 \sim \ldots \rightarrow \underbrace{E = \{e_1, e_2, e_3, \ldots\}}_{\text{** Coincidence class}}$

In a slight abuse of nomenclature we call the coincidence class E also *event*. Sometimes we refer to E as also *equivalence class* (of events); we use the two terms "coincidence class" and "equivalence class" interchangeably when referring to classes of events.

Note that this abuse of nomenclature is also used in everyday life: What makes up an "event" (like a party) is the set of all "little events" (like you meeting your friend) that happen (roughly) at the same location and the same time.

iii | Assumption:

The set $\mathcal{E} = \{E_1, E_2, ...\}$ of all coincidence classes is a complete, observer independent record of reality.

We call the information stored in & absolute because all observers agree on it.

2 \ast ** *Observer* $\mathcal{O} \equiv \ast$ ** *(Reference) Frame* \mathcal{O} :

Goal: Systematic description of physical phenomena in terms of models.

Question: How to systematically observe reality and encode these observations?

:= Experimental setup to collect data about events in space & time:





↓ Lecture 2 [24.10.23]

Assumptions:

- The rods and clocks are conceptual: they do not affect physical experiments.
- All rods and clocks are identical (when brought together, the rods have the same, time-independent length and the clocks tick with the same rate).
- The lattice is "infinitely dense": there is a clock at every point in space.
- Each clock is assigned a unique position label \vec{x} and the reference frame label \mathcal{O} .

For example, a unique position label \vec{x} for a clock can be obtained by counting the rods in x-, y- and z-direction that one has to traverse to reach the clock from the origin. The origin O is, by definition, a "special" clock that is *assigned* the position label $\vec{x}_O = \vec{0}$.

i! Observers are *not* sitting at the origin, looking at their wristwatch, and observing the events with binoculars! They are simply collecting and processing the data that is accumulated by the contraption we call a reference frame.

Since we assume that (ideally) there is one clock at every point in space:

 \rightarrow For every observer \mathcal{O} and every coincidence class E there is a unique event $e_{\mathcal{O}}$

$$E \ni e_{\mathcal{O}} = \langle \text{Clock with frame label } \mathcal{O} \text{ and position label } \vec{x} \text{ shows time } t \rangle$$
 (1.1a)

$$=: (t, \vec{x})_{\mathcal{O}} \quad \Leftrightarrow \quad [E]_{\mathcal{O}} = (t, \vec{x}) \tag{1.1b}$$

for some position label \vec{x} and clock reading *t*.

We refer to the event $(t, \vec{x})_{\mathcal{O}}$ as the *spacetime coordinates* of *E* with respect to frame \mathcal{O} . A different observer \mathcal{O}' will use its own clocks and therefore other events ("coordinates") $(t', \vec{x}')_{\mathcal{O}'} \in E$ to refer to *E*.

In the real world, the \uparrow *tracking detectors* of particle colliders are reminiscent of this ideal setup: They are comprised of 3D arrangements of semiconductor-based particle detectors that all report to a central computer that then reconstructs the trajectories of scattering products from the combination of all detection events.

3 | *** Inertial (coordinate) systems:*

The setup of a reference frame \mathcal{O} above is incomplete and actually very hard to work with: Without additional constraints on the geometry of the lattice and the correlations of clocks (their "calibration"), the record of events is essentially arbitrary. Let us therefore impose some deterministic "calibration procedure" (the same for all frames) that determines how to lay out the rod lattice and how to synchronize the clocks. This procedure endows our reference frame with a specific *coordinate system*, a labeling scheme to describe events.

i | <u>Clock calibration:</u> ** (*Poincaré-*)Einstein synchronization ES

The conventional synchronization procedure (which is actually in practical use) is (Poincaré-)Einstein synchronization:

$$t_O \stackrel{?}{=} \frac{1}{2} \left(t_A + \tilde{t}_A \right) \tag{1.2}$$





You will study this particular procedure and its properties in *Problemset 1*.

In brief, the procedure goes as follows: Consider a reference clock O and some other clock A you wish to synchronize with O.

- (1) To do so, you send a light signal from A to O and note the time t_A your clock A reads when the signal is emitted.
- (2) When the signal arrives at O, it is immediately reflected back to A together with the reading t_O of clock O at this very moment.
- (3) When the signal arrives back at your clock A (together with the timestamp t_0), you note again the reading of your clock as \tilde{t}_A .
- (4) You are now in the possession of three timestamps: (t_A, t_O, \tilde{t}_A) . The idea of Einstein synchronization is to *postulate* the reciprocity of the speed of light: We *declare* that the speed of the signal from A to O is the same as on its way back from O to A (note that we cannot *measure* this reciprocity because we would need already synchronized clocks to do so!). Under this assumption, the readings of *synchronized clocks* must satisfy [10]

$$\Delta t_{A \to O} \equiv t_O - t_A \stackrel{!}{=} \tilde{t}_A - t_O \equiv \Delta t_{O \to A} \quad \Leftrightarrow t_O \stackrel{!}{=} \frac{1}{2} (t_A + \tilde{t}_A), \qquad (1.3)$$

which you can locally check with your data (t_A, t_O, \tilde{t}_A) . Note that you do not need to know the distance from O to A, nor the numerical value of the speed of light c for this procedure to work!

(5) Now if you just powered on your shiny new clock *A* for the first time, it is very unlikely that the condition Eq. (1.3) will be satisfied:

$$t_O = \frac{1}{2}(t_A + \tilde{t}_A) + \delta t = \frac{1}{2}[(t_A + \delta t) + (\tilde{t}_A + \delta t)].$$
(1.4)

Here δt is an offset that you might encounter. But then you can just recalibrate your clock A by δt such that the new readings are $t_A + \delta t$ and $\tilde{t}_A + \delta t$.

Repeating this procedure for all clocks of the frame \mathcal{O} allows you to establish a synchronization relation between arbitrary pairs of clocks. The fact that (under some reasonable and experimentally verified assumptions) the order in which you synchronize your clocks does not matter (the established relation is an *equivalence* relation, O Problemset 1 and Ref. [22]) makes Einstein synchronization a very useful and peculiar *convention* [23–25]. However, one can show that it is the only convention that yields a non-trivial equivalence relation of simultaneity that is consistent with the causal structure on \mathcal{E} (\rightarrow *later*) [26].

ii | <u>Lattice calibration:</u> ** Orthonormal Cartesian coordinates **OC**:

The layout of the lattice of rods assigns coordinates $\vec{x} = (x, y, z)$ to each clock. Depending on the actual shape of the lattice, we will denote events by different position labels. (Note that even with rigid rods connected in the topology of a cubic lattice the geometry is not fixed; for example, you can shear the lattice.) If we assume that space (not spacetime!) is a flat Euclidean space where all the facts of Euclidean geometry hold good (angles of triangles add up π , the Pythagorean theorem holds, the area of circles is πr^2 , etc.), we can parametrize it without loss of generality by *orthonormal Cartesian coordinates*. In these coordinates, distances can be calculated by the Pythagorean formula:



Spatial distance between clocks at \vec{x} and \vec{y} :

$$\underbrace{d(\vec{x}, \vec{y})}_{\text{"Physics"}} = \underbrace{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}}_{\text{"Mathematics"}}$$
(1.5)

The fact that the coordinates of a point (x, y, z) are *distances* along paths parallel to the coordinate axes makes the coordinates *Cartesian*. The fact that Eq. (1.5) holds makes them *orthonormal* (i.e., the axes are orthogonal and have the same scale, as suggested by the sketch above). Coordinates are an intrinsically mathematical concept, they are "labels" to identify points on a manifold of physical points (or *events*, if you consider space*time* coordinates). By contrast, distances carry physical significance: You can measure them with light signals or rods. The prevalence of Cartesian coordinates makes it easy to conflate these two concepts (this will become particularly important in GENERAL RELATIVITY).

Here is a way to check whether your lattice satisfies the **oc** condition using the clocks of \mathcal{O} (and the assumption of the isotropy of the two-way speed of light):



iii | "Inertial Test" (*** law of inertia*):

Once you have arranged your rods and synchronized your clocks and thereby established a Cartesian coordinate system and a (allegedly) well-defined notion of simultaneity, you can perform the following test and check whether your particular reference frame \mathcal{O} passes it or not:

IN Free particles move at constant velocity and in straight lines. (** *Homogeneity of Inertia*)

- It is implied that this statement is true everywhere, anytime, and in all directions.
- Velocities are computed as the time derivative of trajectories in the frame: $\frac{d\vec{x}(t)}{dt}$.
- The property **IN** implies a certain form of *homogeneity* in space and time (since free particles must move in straight lines *anywhere* and *anytime*) and *isotropy* in space (they must move in straight lines in *any direction*). Without additional empirical input, this does not automatically imply that *every experiment* yields the same result anywhere, anytime and in any direction. This more general form of homogeneity and isotropy will be introduced later as **HO** and **IS**. Empirical evidence shows that spacetime indeed is homogeneous **HO** and space isotropic **IS** (in the absence of gravity). With this additional input, the "Inertial Test" to establish **IN** can be simplified to only one particle moving in a straight line at one place for some finite time (which is actually doable). If you presupose homogeneity **HO** but *not* isotropy **IS**, you could observe multiple free particles starting at the same point but moving in different (linearly independent) directions.

Frames equipped with a coordinate system defined by **ES** + **OC** which satisfy **IN** are called ** inertial coordinate systems*.



To distinguish arbitrary frames \mathcal{O} (with arbitrary coordinates) from the special frames (equipped with Cartesian coordinates and synchronized clocks) that passed the inertial test, we label these coordinate systems by K, K', K'' etc. (if we refer to arbitrary inertial systems) and by A, B, C etc. (if we refer to specific inertial systems); the set of all inertial systems is denoted \mathcal{J} .

Alternative definitions:

There seem to be as many definitions of inertial systems as there are texts on SPECIAL RELATIVITY. Some are equivalent, some are not. Some more useful, others less so (none are "wrong", though, because *definitions* cannot be wrong). Some are operational in nature (like the one above), some purely mathematical. Here I only want to point out two ways one can modify the above definition without changing the concept of an inertial system:

- The "inertial test" is crucial to the concept of an inertial frame. It rules out accelerated frames (both linear or rotating). An alternative to throwing test masses in different directions and recording their trajectories is to *repeat* the **ES** procedure periodically to *test* whether the clocks stay in sync. That is, to setup the coordinate system one synchronizes the clocks *once* (by recalibrating the clocks) and then repeats the procedure periodically to check whether the Einstein-synchronization condition remains valid ($\delta t = 0$ in our description above). As it will turn out in GENERAL RELATIVITY, your clocks will not stay in sync in frames that do not pass **IN** (and vice versa). This is essentially the definition given by Schutz [5].
- Instead of "hiding" the law of inertia in the synchronization of clocks, one can do a somewhat reverse modification and "hide" the synchronization of clocks in (an extension of) the law of inertia. To this end one extends the "inertial test" by a second class of tests/experiments, namely:
 - **IN*** Two identical particles that are initially adjacent and at rest, and then interact to repel each other, fly apart with the same velocity in opposite directions. (*** Isotropy of Inertia*)

This statement about the isotropy of inertia implies an operational definition of simultaneity that is (empirically) equivalent to **ES**: You synchronize your clocks *such that* **IN*** is satisfied, for example by performing the experiment described by **IN*** equidistant between two clocks. When the particles reach the clocks, you reset both to t = 0. In this synchronization **IN*** is satisfied *by construction*; experiments show that clocks synchronized in this way are also synchronized according to **ES** (and vice versa).

4 | ****** Spacetime diagram

:= Data structure that encodes the collected data of an inertial coordinate system K:



• Often we draw only one dimension of space for the sake of simplicity.



• Because it will prove useful later, we measure time in units of length by multiplying t with the speed of light c. The choice of c is arbitrary at this point.

Notation: \triangleleft Two inertial systems *K* and *K*':

We use the following shorthand notations to refer to the coordinates of events in the spacetime diagrams of K and K', respectively:

$$(t, \vec{x})_K \equiv (x)_K \equiv x \equiv (t, \vec{x})$$
 and $(t', \vec{x}')_{K'} \equiv (x')_{K'} \equiv x' \equiv (t', \vec{x}')$ (1.6)

When it is clear to which inertial system the coordinates belong we drop the subscripts K and K'.

↓ Interlude: Reconstructing spacetime diagrams from &

If you are given the set \mathcal{E} of events you can reconstruct the spacetime diagram of an inertial system K by looking in each coincidence class $E \in \mathcal{E}$ for the clock event $(t, \vec{x})_K \in E$. You then place E (or some sub-event you are interested in) graphically at the coordinate (t, \vec{x}) on a sheet of paper. The resulting picture is the spacetime diagram of K. In a another inertial system K' the events are arranged differently because different clock events $(t', \vec{x}')_{K'} \in E$ and hence coordinates (t', \vec{x}') are used to draw the spacetime diagram. How (t, \vec{x}) and (t', \vec{x}') are related is unclear at this point.

5 | Empirical facts:

The following facts cannot be bootstrapped from logical thinking alone. They are facts about our physical reality that we have strong experimental evidence for.

• Inertial systems exist (at least in some approximation).

Examples would be an unaccelerated spaceship floating far away from the solar system or the interior of the international space station (if you do not measure too precisely). In SPECIAL RELATIVITY we assume that these systems can be extended to encompass all of spacetime.

- Constructing inertial systems (of arbitrary size) is not possible everywhere.
 - \rightarrow general relativity

We will find in our discussion of GENERAL RELATIVITY that in a gravitational field the construction of inertial systems is only possible locally. For example: If you extend the ISS inertial system rigidly beyond the ISS itself, at some point you will find the trajectories of free particles to deviate from straight lines due to the inhomogeneity of the gravitational field. We will also see that the synchronization procedure used to calibrate the clocks fails in gravitational fields (you cannot keep your clocks in sync). For our discussion of SPECIAL RELATIVITY we ignore this and assume that our inertial systems cover all of spacetime.

- **6** | Relations between inertial systems:
 - i | There are three *straightforward* ways to construct a new inertial system K' from a given one K. They have in common that the two observers do *not* move with respect to one another so that pairs of clocks from K and K' spatially coincide for all times (this implies in particular that you can check that these pairs of clock run at the same rate):
 - (1) Translation in time by $s \in \mathbb{R} (\rightarrow 1 \text{ parameter})$

Procedure:

Duplicate all clocks & rods in place. Label the new clocks with K' and the old position labels. Shift the reading of all clocks by a constant value -s:

$$(t', \vec{x}')_{K'} \sim (t, \vec{x})_K$$
 with $t' = t - s$ and $\vec{x}' = \vec{x}$. (1.7)



It is easy to see that this modification does not invalidate **ES**, **OC** or **IN**. In particular, the Einstein synchronization condition Eq. (1.2) remains valid:

$$t_O = \frac{1}{2} \left(t_A + \tilde{t}_A \right) \quad \Leftrightarrow \quad (t_O - s) = \frac{1}{2} \left[\left(t_A - s \right) + \left(\tilde{t}_A - s \right) \right]. \tag{1.8}$$

How to check from K:

At $(t)_K = 0$ the reading of the origin clock of K' is shifted by $-s \in \mathbb{R}$.

(2) Translation in space by $\vec{b} \in \mathbb{R}^3 (\rightarrow 3 \text{ parameters})$

Procedure:

Duplicate all clocks & rods and translate the whole lattice by \vec{b} (since all clocks are type-identical, you can also simply modify the position labels without moving anything). Label the new clocks with K' and keep their synchronization:

$$(t', \vec{x}')_{K'} \sim (t, \vec{x})_K$$
 with $t' = t$ and $\vec{x}' = \vec{x} - b$. (1.9)

i! If you move the lattice K' in direction \vec{b} , the origin clock of K with position label $\vec{x} = \vec{0}$ will spatially coincide with a clock of K' with position label translated in the *opposite* direction, namely $-\vec{b}$. The same happens for rotations (\rightarrow *below*) and translations in time (\leftarrow *above*).

It is easy to see that this modification does not invalidate ES, OC or IN. In particular, distances can still be computed with Eq. (1.5) since

$$d(\vec{x}, \vec{y}) = d(\vec{x} - \vec{b}, \vec{y} - \vec{b}) \text{ for } \vec{b} \in \mathbb{R}^3.$$
 (1.10)

How to check from K:

At $(t)_K = 0$ the origin of K' is translated by $\vec{b} \in \mathbb{R}^3$ wrt. the origin of K.

(3) Rotation in space by $R \in SO(3) (\rightarrow 3 \text{ parameters})$

Procedure:

Duplicate all clocks & rods and rotate the whole lattice by the axis and angle defined by the rotation matrix R (since all clocks are type-identical, you can again simply modify the position labels without moving anything). Label the new clocks with K' and keep their synchronization:

$$(t', \vec{x}')_{K'} \sim (t, \vec{x})_K$$
 with $t' = t$ and $\vec{x}' = R^{-1}\vec{x}$. (1.11)

It is easy to see that this modification does not invalidate ES, OC or IN. In particular, distances can still be computed with Eq. (1.5) since

$$d(\vec{x}, \vec{y}) = d(R^{-1}\vec{x}, R^{-1}\vec{y}) \quad \text{for } R^{-1} \in \text{SO}(3).$$
(1.12)

How to check from K:

The spatial axes of K' are rotated by $R \in SO(3)$ wrt. the spatial axes of K.

i! You can add spatial *reflections* to these transformations (\checkmark *improper rotations*), i.e., $R \in O(3)$ instead of $R \in SO(3)$. In our discussions we will omit these and only comment on them where necessary.

The combination of spatial rotations (proper and improper, i.e., including reflections) and spatial translations form the \uparrow *Euclidean group* E(3) = ISO(3).

However, experiments (and everyday experience) tell us that there is a fourth possibility how two inertial systems can be related:

Empirical fact:



(4) Uniform linear motion (** *Boost*) by $\vec{v} \in \mathbb{R}^3$ (\rightarrow 3 parameters)

You experience this fact whenever you have a very smooth flight: If you don't look out the window (and cover your ears) everything behaves just as if the airplane were standing still on the ground; there is no evidence that you move with several hundred kilometers per hour relative to the ground.

How to check from K:

The origin of K' moves with constant velocity $\left(\vec{v}\right)_{K} = \left(\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right)_{K} \in \mathbb{R}^{3}$.

Note that just from this observation one cannot distinguish between a *pure boost* and a boost combined with a spatial rotation of the axes (because one probes only for the trajectory of a single point). We will \rightarrow *later* be more precise about this distinction.

i! We cannot write down the coordinate transformations for this relation (yet). The fundamental difference to (1)-(3) is that now the clocks of K' move wrt. the clocks of K. We cannot interpret this as a simple relabeling of fixed clocks. We cannot even be sure that the K- and K'-clocks "run at the same rate" (even if they are type-identical) because to check this we would have to compare the reading of a pair of clocks (one in K and one in K') at *two* consecutive points in time. To do this, however, the two clocks must be at the same place (remember that we can only observe coincidences!). But this is not possible: Since the two frames move uniformly, two clocks can never meet twice! As it will turn out, it is this relation (4) [and its concatenations with (1)-(3)] that harbors the essence of SPECIAL RELATIVITY.

 $\mathbf{ii} \mid Empirical fact:$ The relations (1)-(4) are exhaustive.

With this we mean that whenever you encounter two inertial systems K and K' (i.e., both observers certify that they satisfy our definition of an inertial system, in particular, the "Inertial Test" **IN**), then you will find that the relation between the two is one of the four relations (1)-(4) or a combination of them.

 \rightarrow The relation of two inertial systems *K* and *K'* is given by 10 parameters:



Note that all these relations can be operationally defined and measured within the frame K.

i! The first three sketches can be taken at face value: For example, a translation in time really corresponds to the situation where all clocks are shifted by s an all spatial labels (in particular the axes) remain unaffected. However, for the boost (the last sketch on the right) we do *not* know (yet) how the coordinates transform (neither time nor space) except that the origin clock of K' follows a trajectory in K with uniform velocity \vec{v} . This implies that you should *not* take the sketch for a boost at face value: For example, we do not know whether the axes remain parallel as suggested by the sketch (spoiler: in general they will not).

Since the transformations (1)-(3) do not change the state of motion of the observer (and can therefore be interpreted as a simple relabeling of the position labels and clock readings), it makes sense to collect all inertial systems K that can be connected in this way into an equivalence class [K] which we call ...



* *Inertial frame* := Equivalence class [K] of all inertial coordinate systems K related by spacetime translations and spatial rotations.



Inertial *frames* [K] therefore correspond to the physical notion of a "state of motion." Physically, an inertial *frame* corresponds to the class of all freely moving particles in the universe that are mutually at rest. Given such a "state of motion" (e.g., by declaring one of the particles as reference point), you can then construct various Cartesian coordinate systems (e.g., using said reference particle as your origin) to describe events; these are the inertial *systems* that make up the equivalence class [K].

iv | Notation:

We denote these relations between two inertial systems with the following shorthand notations:

$$K \xrightarrow{R, \vec{v}, s, \vec{b}} K', \ K \xrightarrow{R, \vec{v}} K', \ K \xrightarrow{\vec{v}} K', \ K \xrightarrow{\vec{v}} K'$$
(1.13)

From left to right the relations become increasingly specialized.

i! These relations are *not* symmetric (as indicated by the arrow). For example, $K \xrightarrow{v_x} K'$ specifies the situation where the (origin of) system K' moves with velocity v_x in x-direction as measured in system K.

v | <u>Coordinate transformations:</u>

< Two descriptions of the *same* events:



 \rightarrow Transformation between these descriptions?

 $\varphi(K \to K'): (t, \vec{x})_K \mapsto (t', \vec{x}')_{K'} \quad \text{** Coordinate transformation}$



Finding the functional form of φ (for the non-trivial case $\vec{v} \neq 0$) will be our main goal and central result of this chapter. However, before we can tackle this problem, we first have to introduce a few more concepts.

↓ Interlude: Relative information

We called the data in \mathcal{E} *absolute* because all observers agree on the coincidence of events. However, this data cannot include arbitrary statements, e.g., the event "the particle has velocity \vec{v} " cannot be part of \mathcal{E} because we know from experience that different observers in general do not agree on the velocity of an object. However, following Einstein, we postulated that coincidences are all we can ever observe; thus all there is to know must be encoded in \mathcal{E} ! How is this consistent with the fact that velocities (for example) cannot show up in \mathcal{E} ?

To understand this, it is instructive to think about quantities that can be *derived* from the absolute data in \mathcal{E} by means of prescribed *algorithms*. An algorithm \mathcal{A} is simply a program using data from \mathcal{E} to compute other data (it can use potentially multiple events $E_1, E_2, \ldots, E_N \in \mathcal{E}$ to do so). Furthermore, we allow the algorithm to take the label of an inertial system $K \in \mathcal{J}$ as input:

$$\mathcal{A}: \mathcal{E}^N \times \mathcal{J} \to \text{Output data} \tag{1.14}$$

As a constraint, we require that the algorithm *must not* use any (static) labels $A, B, \ldots \in \mathcal{J}$ of inertial systems. The only reference to a frame it can use is the *variable* K. This somewhat arbitrary sounding restriction formalizes the notion that there are no inertial systems that are "special". Since all inertial systems must be treated equal, the algorithm cannot refer to any specific frame. (This \rightarrow principle of relativity will take the center stage later and turns out to be crucial for the derivation of the transformation φ .)

Let us now contrive two algorithms to compute two quantities that are clearly physically relevant but are *not* contained in \mathcal{E} :

• Example 1: Velocity

First think about how you would measure the velocity of a particle in the lab: You would detect the particle at two different (but nearby) locations, measure the time it requires to get from one to the other, and then compute the difference quotient of distance traveled by the time needed. Note that there is no way to measure the velocity at one point in space and time; you always need two points!

To formalize this, consider two events E_1 and E_2 that both contain the sub-event "particle detected". The algorithm $\mathcal{V}(E_1, E_2; K)$ computes the (average) velocity between the two events as follows:

- 1. Select the event $(t_1, \vec{x}_1)_K \in E_1$.
- 2. Select the event $(t_2, \vec{x}_2)_K \in E_2$.
- 3. Compute and return the value $\vec{v} = \frac{\vec{x}_2 \vec{x}_1}{t_2 t_1}$.

It is important that this algorithm can be used *without modifications* by all observers $K \in \mathcal{J}$. To do so, each observer K plugs into \mathcal{V} the two events (which are objective) an its *own* label K (since this is the only non-random choice possible).

But then two *different* observers K and K' will pick *different* coordinates (t_i, \vec{x}_i) (measured by different clocks) to compute their value of \vec{v} , which obviously can yield different outcomes (as expected for velocities). Note that for the velocities to be really different it must be $[K'] \neq [K]$, i.e., the two inertial *systems* must belong to different *frames*.

• Example 2: Duration & Simultaneity

A very natural question is how much time passed between two events E_1 and E_2 . The formal prescription how to answer this question is given by the algorithm $\mathcal{T}(E_1, E_2; K)$:

- 1. Select the event $(t_1, \vec{x}_1)_K \in E_1$.
- 2. Select the event $(t_2, \vec{x}_2)_K \in E_2$.



3. Compute and return the value $\Delta t = t_2 - t_1$.

For the very same reason as for the velocity algorithm above, the return value of course will depend on the chosen "clock events" (t_i, \vec{x}_i) . And so for the very same reason that velocities can be observer-dependent, time intervals can be as well. Since we define "simultaneity" as the property $\Delta t = 0$, this possibility for observer-dependent results directly transfers to our notion of simultaneity!

Note that we did not make *quantitative* statements about the outcomes for different observers. We neither showed *how* velocities depend on the frame nor whether simultaneity really *is* relative. (It could just be the case that in our world $t_2 - t_1$ always equals $t'_2 - t'_1$ for a fixed event.) This depends on the actual numbers of the coordinates. Such statements therefore require quantitative statements about the relation of $(t, \vec{x})_K \in E$ and $(t', \vec{x}')_{K'} \in E$, which we do not know at this point (this is exactly the question for the functional form of the coordinate transformation φ).

However, what we did show is the *possibility* that simultaneity is relative, just as we already expect velocities to be! So when we later find the correct transformation φ and (surprise!) that indeed simultaneity is not an observer independent fact, you should not be surprised.

Question: Can the values of the electric and magnetic fields \vec{E} and \vec{B} be included in \mathcal{E} ? If not, can you think of an algorithm that determines the electric and magnetic fields \vec{E} and \vec{B} using only coincidence data available in \mathcal{E} ? Do you expect the electromagnetic field to be observer-dependent?

7 | Henceforth:

Unless noted otherwise, all frames will be *inertial* (with Cartesian coordinates). \rightarrow We will (almost exclusively) work with *inertial coordinate systems*.

We use the concept of inertial systems because to describe physics by equations, coordinates are a useful tool. As it turns out, Cartesian coordinates allow for particularly simple equations (at least if space is Euclidean). So our concept of inertial systems as defined above is the most useful one.

8 | Physical Models:

Let us fix a bit of terminology:

• **** (*Physical*) *laws* are ontic features of reality (*↑ scientific realism*).

Physical laws can only be *discovered*; they can neither be invented nor modified.

• ** (Physical) models* are algorithms used to describe reality. These algorithms are typically encoded in the language of mathematics.

Physical models are *invented* and can be *modified*; I will use the terms *model* and *theory* interchangeably.

i! These definitions are by no means conventional and you will find many variations in the literature. For the following discussion, it is only important that the terms we use have precise meaning.



i! The validity of models cannot be *proven*; we can only gradually increase our trust in a model by repeated observations (experiments) – or reject it as invalid by demonstrating that its predictions contradict reality ($\uparrow Karl Popper$). Note that models might describe reality only approximately and in specific parameter regimes and still be useful.

You may dismiss this focus on terminology as "philosophical banter." Conceptual clarity, however, is absolutely crucial for science – in particular for RELATIVITY. Whenever there is confusion in physics, it is often rooted in the conceptual fuzziness of our thinking.

1.2. Galilei's principle of relativity

- 9 | Example: <u>Newtonian mechanics</u>
 - i | Definition of the model:
 - \triangleleft Closed system of N massive particles with masses m_i and positions \vec{x}_i .
 - \triangleleft Force exerted by k on i:

$$F_{k \to i}(\vec{x}_k - \vec{x}_i) = (\vec{x}_k - \vec{x}_i) f_{k \leftrightarrow i}(|\vec{x}_k - \vec{x}_i|)$$
(1.15)

It is $f_{k \leftrightarrow i} = f_{i \leftrightarrow k}$ and therefore $F_{k \to i}(\vec{x}_k - \vec{x}_i) = -F_{i \to k}(\vec{x}_i - \vec{x}_k)$.

 \rightarrow Newtonian equations of motion (in some inertial system K):

$$m_{i}\frac{d^{2}\vec{X_{i}}}{dt^{2}} = \sum_{k\neq i}\vec{F_{k\to i}}(\vec{X_{k}} - \vec{X_{i}})$$
(1.16)

We denote with $\vec{X}_i \equiv \vec{X}_i(t)$ coordinate-valued *functions*; i.e., $\vec{x}_i = \vec{X}_i(t)$ determines a spatial point \vec{x}_i for given t.

Remember: This model fully implements "Newton's laws of motion":

1. Lex prima:

A body remains at rest, or in motion at a constant speed in a straight line, unless acted upon by a force.

This is the \checkmark principle of inertia. It is part of the definition of the concept of a Newtonian force used in Eq. (1.16). Note that it is not a consequence of Eq. (1.16) for $F_{k \rightarrow i} \equiv 0$. It rather defines (together with the lex tertia below) the frames and coordinate systems (\leftarrow inertial systems) in which Eq. (1.16) is valid (recall IN).



2. Lex secunda:

When a body is acted upon by a net force, the body's acceleration multiplied by its mass is equal to the net force.

This is just the functional form of Eq. (1.16) in words.

3. Lex tertia:

If two bodies exert forces on each other, these forces have the same magnitude but opposite directions.

This is guaranteed by the property $F_{k \to i} = -F_{i \to k}$ of the forces. Together with the lex secunda this is an expression of momentum conservation. For two particles:

$$m_1 \frac{\mathrm{d}v_1}{\mathrm{d}t} + m_2 \frac{\mathrm{d}v_2}{\mathrm{d}t} = \frac{\mathrm{d}p_1}{\mathrm{d}t} + \frac{\mathrm{d}p_2}{\mathrm{d}t} = F_{2 \to 1} + F_{1 \to 2} = 0$$
(1.17)

This implies in particular that two identical particles $(m_1 = m_2)$ that are both at rest at t = 0 must obey $v_1(t) = -v_2(t)$ for all times (recall IN*).

ii | Application of the model:



As a working hypothesis, let us assume that the model Eq. (1.16) describes the dynamics of massive particles perfectly (from experience we know that there are at least regimes where it is good enough for all practical purposes).

iii | Symmetries of Newtonian mechanics:

To understand the solution space of Eq. (1.16) better, it is instructive to study transformations that map solutions to other solutions.

a | ****** *Galilei transformations*:

We define the following coordinate transformation:

$$G : \mathbb{R}^4 \to \mathbb{R}^4 : \begin{cases} t' = t + s \\ \vec{x}' = R\vec{x} + \vec{v}t + \vec{b} \end{cases}$$
(1.18)

A Galilei transformation G is characterized by 10 real parameters:

- $s \in \mathbb{R}$: Time translation (1 parameter)
- $\vec{b} \in \mathbb{R}^3$: Space translation (3 parameters)
- $\vec{v} \in \mathbb{R}^3$: Boost (3 parameters)
- $R \in SO(3)$: Spatial rotation (3 parameters; rotation axis: 2, rotation angle: 1)



The set of all transformations forms (the matrix representation of) a group:

$$\mathscr{G}_{+}^{\uparrow} = \{ G(R, \vec{v}, s, \vec{b}) \} \quad \text{** Proper orthochronous Galilei group}$$
(1.19)

with group multiplication

$$G_3 = G_1 \cdot G_2 = G(\underbrace{R_1 R_2}_{R_3}, \underbrace{R_1 \vec{v}_2 + \vec{v}_1}_{v_3}, \underbrace{s_1 + s_2}_{s_3}, \underbrace{R_1 \vec{b}_2 + \vec{v}_1 s_2 + \vec{b}_1}_{\vec{b}_3}) \quad (1.20)$$

You derive this multiplication in \bigcirc Problemset 1 and show that the group axioms are indeed satisfied.

As a special case, the multiplication yields the rule for addition of velocities in Newtonian mechanics:

$$G(1, \vec{v}_1, 0, \vec{0}) \cdot G(1, \vec{v}_2, 0, \vec{0}) = G(1, \underbrace{\vec{v}_1 + \vec{v}_2}_{\vec{v}_3}, 0, \vec{0})$$
(1.21)

The full Galilei group is generated by the proper orthochronous transformations together with space and time inversion:

$$\mathcal{G} = \langle \mathcal{G}_{+}^{+} \cup \{P, T\} \rangle \quad \stackrel{\text{\tiny def}}{=} Galilei \, group \tag{1.22a}$$

$$P: (t, \vec{x}) \mapsto (t, -\vec{x})$$
 Space inversion (parity) (1.22b)

$$T: (t, \vec{x}) \mapsto (-t, \vec{x})$$
 Time inversion (1.22c)

b | <u>Galilei covariance & Form-invariance:</u>

Details: September 1

 \triangleleft Coordinate transformation Eq. (1.18)

We express the total differential and the trajectory in the new coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}t'}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}t'} = \frac{\mathrm{d}}{\mathrm{d}t'} \tag{1.23}$$

and

$$\vec{X}_{i}'(t') = R\vec{X}_{i}(t) + \vec{v}t + \vec{b} = R\vec{X}_{i}(t'-s) + \vec{v}(t'-s) + \vec{b}$$
(1.24a)

$$\Leftrightarrow \quad \vec{X}_i(t) = R^{-1} \left[\vec{X}'_i(t') - \vec{v}(t'-s) - \vec{b} \right]$$
(1.24b)

Thus the left-hand side of the Newtonian equation of motion Eq. (1.16) reads in new coordinates:

$$m_{i}\frac{\mathrm{d}^{2}\vec{X}_{i}(t)}{\mathrm{d}t^{2}} = m_{i}\frac{\mathrm{d}^{2}}{\mathrm{d}t'^{2}}R^{-1}\left[\vec{X}_{i}'(t') - \vec{v}(t'-s) - \vec{b}\right] = R^{-1}m_{i}\frac{\mathrm{d}^{2}\vec{X}_{i}'(t')}{\mathrm{d}t'^{2}}$$
(1.25)

Note that the quantity $m_i \frac{d^2}{dt^2} \vec{X}_i(t)$ is not *invariant*; it transforms with an $R^{-1} \in SO(3)$. And the right-hand side:

$$\sum_{k \neq i} \vec{F}_{k \to i} (\vec{X}_k(t) - \vec{X}_i(t)) = R^{-1} \sum_{k \neq i} \vec{F}_{k \to i} (\vec{X}'_k(t') - \vec{X}'_i(t'))$$
(1.26a)



Here we used the form of the force Eq. (1.15), that $\vec{X}_k(t) - \vec{X}_i(t) = R^{-1}[\vec{X}'_k(t') - \vec{X}'_i(t')]$ and $|\vec{X}_k(t) - \vec{X}_i(t)| = |\vec{X}'_k(t') - \vec{X}'_i(t')|$ because of $R \in SO(3)$.

Note that the force on the right-hand side is not *invariant* either; luckily, it transforms with *the same* $R^{-1} \in SO(3)$; it "co-varies" with the left-hand side!

In conclusion, Newton's equation of motion Eq. (1.16) reads in the new coordinates:

$$R^{-1} m_i \frac{\mathrm{d}^2 X_i'(t')}{\mathrm{d}t'^2} = R^{-1} \sum_{k \neq i} \vec{F}_{k \to i} (\vec{X}_k'(t') - \vec{X}_i'(t'))$$
(1.27a)

$$\stackrel{\rightarrow Covariance}{\Leftrightarrow} \underbrace{m_{i} \frac{d^{2}X_{i}'(t')}{dt'^{2}} = \sum_{k \neq i} \vec{F}_{k \rightarrow i}(\vec{X}_{k}'(t') - \vec{X}_{i}'(t'))}_{\rightarrow Form-invariance}}_{\rightarrow Form-invariance}$$
(1.27b)

(You can easily check that this holds for *P* and *T* as well.)

Newton's EOMs Eq. (1.16) are *form-invariant* under Galilei transformations. *Or:* Newton's EOMs Eq. (1.16) are Galilei-*covariant*.

↓ Interlude: Nomenclature

Let X be some group of coordinate transformations (here: $X = \mathcal{G}$ the Galilei group).

• A *quantity* is called *X*-*invariant* if it does not change under the coordinate transformation. Such quantities are called *X*-*scalars*.

An example is the mass m in Eq. (1.16) (which is also constant).

• A *quantity* is called *X*-covariant if it transforms under some given representation of the *X*-group. If this representation is the trivial one (i.e., the quantity does not change at all) this particular *X*-covariant quantity is then also an *X*-scalar.

An example of a Galilei-covariant (but not invariant) quantity is the force $\vec{F}_{k \to i}$ which transforms under a representation of \mathscr{G} .

• An *equation* is called *X*-covariant if the quantity on the left-hand side and on the right-hand side are *X*-covariant (under the same *X*-representation).

An example is Newton's lex secunda Eq. (1.16) where $m_i \frac{d^2}{dt^2} x_i(t)$ transforms in the same (non-trivial) representation as $\vec{F}_{k \to i}$.

• X-covariant *equations* have the feature that a X-transformation leaves them *form-invariant*, i.e., they "look the same" after X-transformations because their left- and right-hand side vary in the same way (they "co-vary"). Note that the quantities in a form-invariant equation do not have to be *invariant*.

An example is again Eq. (1.16) as we just showed. Note that $\vec{x}'_i(t')$ and $\vec{x}_i(t)$ are *different* vectors such that the two sides of the equation as not *in*variant (but *covariant*).



↓Lecture3 [31.10.23]

c | Active symmetries:

There is something additional and particularly useful to be learned from the coordinate transformation above. We showed:

If
$$\vec{X}_i(t)$$
 satisfies $m_i \frac{\mathrm{d}^2 \vec{X}_i(t)}{\mathrm{d}t^2} = \sum_{k \neq i} \vec{F}_{k \to i} (\vec{X}_k(t) - \vec{X}_i(t))$ (1.28a)

then $\vec{X}'_{i}(t')$ satisfies $m_{i} \frac{\mathrm{d}^{2} \vec{X}'_{i}(t')}{\mathrm{d}t'^{2}} = \sum_{k \neq i} \vec{F}_{k \to i} (\vec{X}'_{k}(t') - \vec{X}'_{i}(t'))$ (1.28b)

But *t*′ in the lower statement is just a dummy variable that can be renamed to whatever we want:

If
$$\vec{X}_{i}(t)$$
 satisfies $m_{i} \frac{d^{2} \vec{X}_{i}(t)}{dt^{2}} = \sum_{k \neq i} \vec{F}_{k \to i} (\vec{X}_{k}(t) - \vec{X}_{i}(t))$ (1.29a)

then
$$\vec{X}'_{i}(t)$$
 satisfies $m_{i} \frac{d^{2} \vec{X}'_{i}(t)}{dt^{2}} = \sum_{k \neq i} \vec{F}_{k \to i} (\vec{X}'_{k}(t) - \vec{X}'_{i}(t))$ (1.29b)

Use colors to highlight the changes.

$$\rightarrow \vec{X}'_i(t) = R\vec{X}_i(t-s) + \vec{v}(t-s) + \vec{b}$$
 is a new solution of Eq. (1.16)!

Note that for s = 0 it is $\vec{X}'_i(0) = R\vec{X}_i(0) + \vec{b}$ and $\dot{\vec{X}}_i(0) = R\dot{\vec{X}}_i(0) + \vec{v}$, i.e., the solution $\vec{X}'_i(t)$ satisfies different initial conditions.

 \rightarrow We say:

The Galilei group \mathcal{G} is an * *invariance group* or an *(active) symmetry* of Eq. (1.16).

✤ Interlude: Active and passive transformations

It is important to understand the conceptual difference between the two last points:

- In the previous step we took a specific trajectory (solution of Newton's equation) and expressed it in different coordinates. We then found that the differential equation obeyed by *the same physical trajectory* in these new coordinates "looks the same" as in the old coordinates. We called this peculiar feature of the differential equation "Galilei-covariance" or "form-invariance". This type of a transformation is called *passive* because we keep the physics the same and only change our description of it.
- In the last step, we have shown that there is a dual interpretation to this: If a differential equation is form-invariant under a coordinate transformation, then we can exploit this fact to construct *new solutions* from given solutions (in the same coordinate system!). This type of transformation is called *active* because we keep the coordinate frame fixed and actually *change the physics*. You can therefore think of active transformations/symmetries as "algorithms" to construct new solutions of a differential equation (a quite useful feature since solving differential equations is often tedious).

10 | Galilean relativity:


i | Remember:

The law of inertia holds (by definition) in all inertial systems.

 \rightarrow The "inertial test" **IN** cannot be used to distinguish inertial systems.

This is a tautological statement because we *define* inertial systems in this way!

Empirical fact:

Every mechanical experiment (not just the "inertial test") yields the same result in all inertial systems.

This is not a tautology but an empirically tested feature of reality.

This motivates the following postulate (first given by Galileo Galilei):

§ Postulate 2: Galilei's principle of Relativity GR

No mechanical experiment can distinguish between inertial systems.

i! In this formulation, **GR** encodes a (so far uncontested) empirical fact. In particular, it does neither refer nor rely on (the validity of) any *physical model*, e.g., Newtonian mechanics. As such we should expect that it survives our transition to SPECIAL RELATIVITY.

Here is a more operational formulation of **GR**: You describe a detailed experimental procedure using equipment governed by mechanics (springs, pendula, masses, ...) that can be performed in a closed (but otherwise perfectly equipped) laboratory. Then you copy these instructions without modifications and hand them to scientists with labs in different inertial systems. They all perform your instructions and get some results (e.g. the final velocities of a complicated contraption of pendula). When they report back to you, their results will all be identical. This is the essence of **GR**.

ii | In the language of *models* that describe the mechanical laws faithfully, **GR** can be reformulated:

§ Postulate 3: Galilei's principle of Relativity GR'

The equations that describe mechanical phenomena *faithfully* have the same form in all inertial systems.

If this would not be the case you could distinguish between different inertial systems by checking which formula you have to use to describe your observations. Imagine a rotating (non-inertial) frame where you have to use a modified version of Newton's EOMs (that include additional terms for the Coriolis force) to describe your observations.

Note that "the same form" actually means that the models are *functionally equivalent* (have the same solution space). Functional equivalence is equivalent to the *possibility* to formulate the model (= equation of motion) in the same form.

iii | Under the assumption (!) that Newtonian physics (in particular Eq. (1.16)) describes mechanical phenomena *faithfully*, this implies:

Newton's equations of motion have the same form in all inertial systems.



i! This statement is *not* equivalent to GR or GR' as it relies on an independent empirical claim (namely the validity of Newton's equation as a model of mechanical phenomena).

We can now combine this claim with our (purely mathematical!) finding concerning the invariance group of Newton's equations:



 \rightarrow Preliminary/Historical conclusion:

$$\varphi(K \xrightarrow{R, \vec{v}, s, \vec{b}} K') \stackrel{?}{=} G(R^{-1}, -\vec{v}, -s, -\vec{b}) \in \mathcal{G}$$

Recall that rotating the coordinate *axes* by R makes the coordinates of fixed events rotate in the *opposite* direction R^{-1} ; the same is true for the other transformations.

Is this true?

Since this is a course on RELATIVITY, we should be skeptical (like Einstein) and ask:



11 | Mathematical fact:

The Maxwell equations of electrodynamics are not Galilei-covariant.

Proof: 🔿 Problemset 1

Here for your (and my) convenience the Maxwell equations in vacuum (in cgs units):

Gauss's law (electric): $\nabla \cdot \boldsymbol{E} = 0$ (1.30a)

Gauss's law (magnetic):
$$\nabla \cdot \boldsymbol{B} = 0$$
 (1.30b)

Law of induction:
$$\nabla \times \boldsymbol{E} = -\frac{1}{c} \partial_t \boldsymbol{B}$$
 (1.30c)

Ampère's circuital law:
$$\nabla \times \boldsymbol{B} = \frac{1}{c} \partial_t \boldsymbol{E}$$
 (1.30d)

"Handwavy explanation" for the absence of Galilei symmetry:

The Maxwell equations imply the wave equation for both fields:

$$\left(\nabla^2 - \frac{1}{c^2}\partial_t^2\right) X = \mathbf{0} \quad \text{for } X \in \{E, B\}.$$
(1.31)



Here the speed of light c plays the role of the phase and group velocity of the waves; i.e., all light signals propagate with c. Form-invariance under some coordinate transformation φ implies that the same light signal propagates with *the same velocity* c in all coordinate systems related by φ . This is clearly incompatible with the Galilean law for adding velocities (according to which a signal with velocity u'_x in frame K' propagates with velocity $u_x = u'_x + v_x$ in frame K if $K \xrightarrow{v_x} K'$).

12 | The simplest escape from our predicament:

Maybe there is no relativity principle for electrodynamics?

Reasoning: If we cling to the validity of Newtonian mechanics and Galilean relativity **GR**, we are forced to assume $\varphi = G$ as the transformation between inertial systems. Since the Maxwell equations are *not* form-invariant under these transformations, they look differently in different inertial systems. So there must be a (class of) designated inertial coordinate systems $[K_0]$ in which the Maxwell equations in the specific form Eq. (1.30) you've learned in your electrodynamics course are valid.

- \rightarrow [K₀] = Frame in which the "*luminiferous aether*" is at rest (?)
- 13 | Michelson Morley experiment (plots from [27, 28]):





Michelson's original setup (1881)

Michelson & Morley's improved setup (1887)

- \rightarrow The (two-way) speed of light is the same in all directions.
- \rightarrow There is no "luminiferous aether" [K₀].
- (Or it is pulled along by earth which contradicts the observed \uparrow *aberration of light.*)
- \rightarrow The speed of light *c* cannot be fixed wrt. some designated reference frame [K₀].
- \rightarrow *No experimental evidence* that the Maxwell equations *do not hold* in all inertial systems.
- \rightarrow *Relativity principle* for electrodynamics?!
 - <u>Historical note:</u>

A. Einstein writes in a letter to F. G. Davenport (see Ref. [29]):

[...] In my own development Michelson's result has not had a considerable influence. I even do not remember if I knew of it at all when I wrote my first paper on the subject (1905). The explanation is that I was, for general reasons, firmly convinced how this could be reconciled with our knowledge of electro-dynamics. One can therefore understand why in my personal struggle Michelson's experiment played no role or at least no decisive role.

- \rightarrow The Michelson Morley experiment did *not* kickstart SPECIAL RELATIVITY.
- Modern Michelson-Morley like tests of the isotropy of the speed of light achieve much higher precision than the original experiment. The authors of Refs. [30, 31], for example, report an upper bound of $\Delta c/c \sim 10^{-17}$ on potential anisotropies of the speed of light by rotating optical resonators.
- 14 | Two observations:
 - (1) No evidence that there is no relativity principle for electrodynamics.



(2) Why does Galilean relativity GR treat mechanics differently anyway?

Put differently: Why should mechanics, a branch of physics artificially created by human society, be different from any other branch of physics? This is not impossible, of course, but it certainly lacks simplicity! (To Galilei's defence: At his time "mechanics" was more or less identical to "physics".)

 \rightarrow A. Einstein writes in §2 of Ref. [10] as his first postulate:

1. Die Gesetze, nach denen sich die Zustände der physikalischen Systeme ändern, sind unabhängig davon, auf welches von zwei relativ zueinander in gleichförmiger Translationsbewegung befindlichen Koordinatensystemen diese Zustandsänderungen bezogen werden.

We reformulate this into the following postulate:

§ Postulate 4: (Einstein's principle of) Special Relativity SR

No mechanical experiment can distinguish between inertial systems.

Note the difference to Galilean relativity **GR** according to which no experiment *governed by classical mechanics* can distinguish between inertial systems. Einstein simply extended this idea to all of physics – no special treatment for mechanics!

i! There are various names used in the literature to refer to **SR**. Here we call it the principle of *special* relativity, where the "special" refers to its restriction on *inertial systems* – as compared to the principle of *general* relativity in GENERAL RELATIVITY that refers to *all* frames (\rightarrow *later*). To emphasize its difference to Galilean relativity **GR**, some authors call **SR** the *universal* principle of relativity, where "universal" refers to its applicability on *all* laws of nature (not just the realm of classical mechanics).

15 | But now that there are more contenders (mechanics, electrodynamics, quantum mechanics) all of which must be invariant under the same transformation φ , we have to open the quest for φ again:



The differently colored/shaped trajectories symbolize phenomena of mechanics (red), electrodynamics (blue), and quantum mechanics (green). According to **SR**, *all* of them must be forminvariant under a common coordinate transformation φ .

i! To reiterate: This is *not* a question about symmetry properties of equations or models! It is an experimentally testable fact about reality. There is only *one* correct φ and it is just as real as the three-dimensionality of space.



1.4. Transformations consistent with the relativity principle

Since this is a theory lecture, so we cannot do experiments. Let us therefore weaken the question slightly:

What is most general form of φ consistent with reasonable assumptions about reality?

§ Assumptions 1

- **SR** Special Relativity: There is no distinguished inertial system.
- **IS** *Isotropy:* There is no distinguished direction in space.
- HO Homomgeneity: There is no distinguished place in space or point in time.
- **CO** Continuity: φ is a continuous function (in the origin).

Something is "distinguished" if there exists an experiment that can be used to identify it unambiguously. This derivation follows Straumann [9] with input from Schröder [1] and Pal [32].

Detailed calculations:
Problemset 2

- 1 | Setup:
 - \triangleleft Two inertial systems $K \xrightarrow{R, \vec{v}, s, \vec{b}} K'$.

 \triangleleft Event $E \in \mathcal{E}$ with coordinates $x \equiv (t, \vec{x})_K \in E$ and $x' \equiv (t', \vec{x}')_{K'} \in E$:



We are interested in the transformation $\varphi \equiv \varphi_{R,\vec{v},s,\vec{b}}$ with

$$x' = \varphi(x) \,. \tag{1.32}$$

Note that **SR** forbids us to use the inertial system labels K or K' in the definition of φ ! We can only use the relative parameters (R, \vec{v}, s, \vec{b}) measured in K wrt K'.

2 Affine structure:

Our first goal is to show that φ must be an affine map.

- $i \mid \triangleleft \text{Event } \tilde{E} \in \mathcal{E} \text{ with coordinates } \tilde{x} = x + a \text{ in } K \text{ for some shift } a \in \mathbb{R}^4.$
- ii | Homogeneity $HO \rightarrow$

$$\varphi(x+a) - \varphi(x) \stackrel{!}{=} a'(\varphi, a) \tag{1.33}$$



 $a'(\varphi, a)$: Shift in K' independent of x (this reflects homogeneity in space and time)

Imagine the right-hand side $a'(\varphi, a)$ where *not* independent of x. Then there would be an interval (say, a rod of spatial extend \vec{a}) that has the same length \vec{a} in K no matter where it is located, but *variable* length $\vec{a}(\varphi, \vec{a}, \vec{x})$ in K' as a function of \vec{x} . The observer in K' can then use this "magic rod" to pinpoint absolute positions in space (the same argument works in time, then with a clock instead of a rod).

iii | For x = 0: $a'(\varphi, a) = \varphi(a) - \varphi(0) \rightarrow$

$$\varphi(x+a) = \varphi(x) + \varphi(a) - \varphi(0). \tag{1.34}$$

iv | Let $\Psi(x) := \varphi(x) - \varphi(0) \rightarrow$

$$\Psi(x+a) = \Psi(x) + \Psi(a)$$
 and $\Psi(0) = 0$. (1.35)

This would be satisfied if Ψ were *linear*! But we do not know this yet ...

- $\mathbf{v} \mid \underline{\text{Claim:}} \Psi(x) \text{ continuous at } x = 0 \text{ (follows from } \mathbf{co} \text{)} \Rightarrow \Psi \text{ is linear.}$
 - **a** | Eq. (1.35) $\rightarrow \Psi(nx) = n\Psi(x)$ for $n \in \mathbb{N}$ (show by induction!)
 - **b** | Eq. (1.35) $\rightarrow \Psi(-x) = -\Psi(x)$ (use $\Psi(0) = 0$) $\rightarrow \Psi(nx) = n\Psi(x)$ for $n \in \mathbb{Z}$
 - **c** | \triangleleft Rational number $r = \frac{m}{n}, m, n \in \mathbb{Z} \rightarrow$

$$r\Psi(x) = \frac{m}{n}\Psi(x) = \frac{1}{n}\Psi(mx) = \frac{1}{n}\Psi(nrx) = \frac{n}{n}\Psi(rx) = \Psi(rx). \quad (1.36)$$

- **d** | $\Psi(x)$ continuous at $x = 0 \xrightarrow{\text{Eq. (1.35)}} \Psi(x)$ continuous everywhere. Show this using the definition of continuity, i.e., $\lim_{x\to 0} \Psi(x) = \Psi(0)!$
- e | rΨ(x) = Ψ(rx) for r ∈ Q ^{Ψ continuous}/_→ rΨ(x) = Ψ(rx) for r ∈ ℝ
 Remember that real numbers are defined in terms of (equivalence classes of) limits of rational numbers, i.e., Q is dense in ℝ.
- **f** | In conclusion:

$$\Psi(x+a) = \Psi(x) + \Psi(a) \quad \text{and} \quad \Psi(rx) = r\Psi(x) \tag{1.37}$$

 $\rightarrow \Psi$ is *linear*.

vi | If Ψ is linear, $\varphi(x) = \Psi(x) + \varphi(0)$ is affine:

$$\varphi(x) = \Lambda x + a \tag{1.38}$$

with $\Lambda = \Lambda(R, \vec{v}, s, \vec{b})$ a 4 × 4 matrix and $a = a(R, \vec{v}, s, \vec{b})$ a 4-dimensional vector.

- **3** | The spacetime translation a is simply $a = (-s, -\vec{b})$ [recall Eqs. (1.7) and (1.9)].
 - $\rightarrow \sphericalangle$ Homogeneous transformations (a = 0) in the following:

$$x' = \varphi(x) = \Lambda x \,. \tag{1.39}$$





We already know from our discussion of inertial systems [recall Eq. (1.11)]:
 Rotation group SO(3) must be part of the transformations φ with representation

$$x' = \Lambda_{R^{-1}} x$$
 with $\Lambda_R := \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right)$ where $R \in \mathrm{SO}(3)$. (1.40)

This is just a fancy way to rewrite Eq. (1.11).

- **5** $| \xrightarrow{**} Pure \ boost \ K \xrightarrow{1, \vec{v}, 0, \vec{0}} K':$
 - i $| \langle (t)_K = 0 \rightarrow \vec{x}' = \mathcal{M}\vec{x}$ for an invertible matrix $\mathcal{M} \in \mathbb{R}^{3 \times 3}$: This is the most general transformation for the position labels of the K and K'-clocks at t = 0. Note that we make no statements on the times t' displayed by the K'-clocks at t = 0.

$$\mathcal{M} = R_1 D R_2 = R_1 D R_1^T R = M R \tag{1.41}$$

with $R \in O(3)$ and $M^T = M$.

This follows from the \checkmark singular value decomposition of real matrices with $R_1, R_2 \in O(3)$ and D a diagonal matrix.

ii | With spatial rotations Eq. (1.40) we can always transform the K-coordinates by $\vec{x} \mapsto R^{-1}\vec{x}$ such that $\vec{x}' = \mathcal{M}\vec{x} = \mathcal{M}\vec{x}$ at $t = 0 \rightarrow$

** Pure boost $K \xrightarrow{1,\vec{v},0,\vec{0}} K'$:

$$x' = \Lambda_{\vec{v}} x \quad \Leftrightarrow \quad \begin{cases} t' = a(\vec{v}) t + \vec{b}(\vec{v}) \cdot \vec{x} \\ \vec{x}' = M(\vec{v}) \vec{x} + \vec{e}(\vec{v}) t \end{cases}$$
(1.42)

- $a: \vec{v}$ -dependent scalar
- $\vec{b}, \vec{e}: \vec{v}$ -dependent vectors
- $M^T = M$: \vec{v} -dependent 3 × 3-matrix

Pure boosts are therefore characterized by a symmetric transformation of the spatial coordinates at t = 0 in K. Geometrically, this implies that there are *three* (orthogonal) lines through the origin of K which are mapped onto themselves under the boost (spanned by the eigenvectors of $M(\vec{v})$). The only other possibility is that there is a *single* invariant line, which then coincides with the rotation axis of a spatial rotation mixed into the boost. The *pure* boosts are therefore those boosts *without* any rotation mixed in.

 \rightarrow We focus on pure boosts in the remainder of this derivation:





i! Our characterization of a pure boost does *not* imply that at t = 0 the axes of the two systems K and K' align (as suggested by the sketch and naïvely expected). If this were the case, the eigenbasis of $M(\vec{v})$ would be given by the basis vectors \hat{e}_i in K. Since we do not know the form of $M(\vec{v})$ (yet), we cannot make this assumption! So do not take this sketch literally, it only illustrates *symbolically* the situation of a pure boost in an arbitrary direction.

6 | Isotropy:

Here are two lines of arguments that use isotropy **IS** to restrict the form of Eq. (1.42) further:

- Argument A:
 - i | We claim that isotropy **IS** requires the following multiplicative structure for pure boosts and rotations:

$$\Lambda_R \Lambda_{\vec{v}} \Lambda_{R^{-1}} \stackrel{!}{=} \Lambda_{R\vec{v}} \quad \Leftrightarrow \quad \forall_x : \Lambda_R \Lambda_{\vec{v}} x = \Lambda_R x' \stackrel{!}{=} \Lambda_{R\vec{v}} \Lambda_R x.$$
(1.43a)

$$\Leftrightarrow \quad \forall_x : \Lambda_{\vec{v}} x \stackrel{!}{=} \Lambda_{R^{-1}} \Lambda_{R\vec{v}} (\Lambda_R x) . \tag{1.43b}$$

The reasoning goes as follows:

1. \triangleleft Left-hand side of Eq. (1.43b):

 $x = (t, \vec{x})$ are the coordinates of some event in K and $\Lambda_{\vec{v}} x$ of the same event in K':



2. \triangleleft Right-hand side of Eq. (1.43b):

We consider $y = (t, \vec{y}) := \Lambda_R x = (t, R\vec{x})$ as an *active* transformation, i.e., y denotes a different event that is spatially rotated from x by R. To state our isotropy claim **IS**, we now rotate the coordinate system K'' in which we want to express this event *in the same way*. This implies a rotated boost $\Lambda_{R\vec{v}}$ and a subsequent rotation of the coordinate axes by R via $\Lambda_{R^{-1}}$. (Remember that when rotating the coordinate *axes* by R, the *coordinates* of an event transform by $\Lambda_{R^{-1}}$.):





- 3. Spatial isotropy **IS** is the property that the event x as seen from K' cannot be distinguished from the rotated event y as seen from the rotated system K''; this is Eq. (1.43b).
- ii | Now we can use Eq. (1.42) to rewrite Eq. (1.43a) as

$$t' \stackrel{!}{=} a(R\vec{v})t + \vec{b}(R\vec{v}) \cdot R\vec{x}$$
(1.44a)

$$R\vec{x}' \stackrel{!}{=} M(R\vec{v}) R\vec{x} + \vec{e}(R\vec{v}) t$$
 (1.44b)

- iii | A comparison with Eq. (1.42) (for all t and \vec{x} and arbitrary \vec{v} and R) leads to constraints on the unknown functions:
 - $a(\vec{v}) \stackrel{!}{=} a(R\vec{v}) \rightarrow a(\vec{v}) = a_v$ with $v = |\vec{v}|$

Functions invariant under arbitrary rotations can only depend on the norm $|\vec{v}|$.

 $- \vec{b}(\vec{v}) \stackrel{!}{=} R^T \vec{b}(R\vec{v}) \rightarrow \vec{b}(\vec{v}) = b_v \vec{v}$

Note that $\vec{b}(R\vec{v}) \cdot R\vec{x} = [R^T \vec{b}(R\vec{v})] \cdot \vec{x}$. Let $R_{\hat{v}}$ be some rotation with axis $\hat{v} = \vec{v}/v$ such that $R_{\hat{v}}\vec{v} = \vec{v}$; then $\vec{b}(\vec{v}) \stackrel{!}{=} R_{\hat{v}}^T \vec{b}(\vec{v})$ and therefore $\vec{b}(\vec{v}) \propto \vec{v}$ since rotation matrices have only a single eigenvector.

- $RM(\vec{v}) \stackrel{!}{=} M(R\vec{v})R \rightarrow M(\vec{v}) = c_v \,\mathbb{1} + d_v \,\hat{v}\hat{v}^T$

First recall that $M^T(\vec{v}) = M(\vec{v})$ such that $M(\vec{v})$ can be written as sum of orthogonal projectors (projecting onto its eigenspaces). It is in particular $R_{\hat{v}}M(\vec{v})R_{\hat{v}}^T \stackrel{!}{=} M(\vec{v})$ such that one of the eigenvectors must be $\hat{v} \propto \vec{v}$. The remaining two eigenvectors are orthogonal to \hat{v} and can therefore be mapped onto each other by $R_{\hat{v}}$. Since $R_{\hat{v}}$ commutes with $M(\vec{v})$, their eigenvalues must be degenerate such that the two-dimensional subspace orthogonal to \hat{v} is a degenerate eigenspace. The most general spectral decomposition of $M(\vec{v})$ is then the one given above.

-
$$R\vec{e}(\vec{v}) \stackrel{!}{=} \vec{e}(R\vec{v}) \rightarrow \vec{e}(\vec{v}) = e_v \vec{v}$$

This is the same argument as for $\vec{b}(\vec{v})$.

• Argument B:

A shorter (but less rigorous) line of arguments goes as follows:

- i | To define the unknown functions algebraically, we are only allowed to use the vector \vec{v} and constant scalars. We *cannot* use \vec{x} or *t* due to linearity, and any other constant vector (like $\hat{e}_x = (1, 0, 0)^T$) would pick out some direction and therefore violate isotropy **IS**.
- ii | Since the only *scalar* one can construct from a single vector is its norm, $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$, it must be $a(\vec{v}) = a_v$.
- iii | Similarly, since the only *vector* one can construct from a single vector is a scalar multiplied by the vector itself, it must be $\vec{b}(\vec{v}) = b_v \vec{v}$ and $\vec{e}(\vec{v}) = e_v \vec{v}$.
- iv | Lastly, since $M^T(\vec{v}) = M(\vec{v})$, we can decompose the matrix into orthogonal projectors: $M(\vec{v}) = \sum_i \lambda_i(v) P_i(\vec{v})$. The only projectors that can be defined by a single vector are $P_0 = \hat{v}\hat{v}^T$ and $P_1 = \mathbb{1} - P_0 = \mathbb{1} - \hat{v}\hat{v}^T$ which leads to the most general form $M(\vec{v}) = c_v \mathbb{1} + d_v \hat{v}\hat{v}^T$.

Both arguments lead to the same form for pure boosts $\Lambda_{\vec{v}}$ consistent with isotropy IS:

$$t' = a_v t + b_v \left(\vec{v} \cdot \vec{x} \right) \tag{1.45a}$$

$$\vec{x}' = c_v \, \vec{x} + \frac{d_v}{v^2} \, \vec{v} (\vec{v} \cdot \vec{x}) + e_v \, \vec{v} \, t \tag{1.45b}$$

with $v = |\vec{v}| = |R\vec{v}|$ and $(R\vec{v} \cdot R\vec{x}) = (\vec{v} \cdot \vec{x})$.



7 | \triangleleft Trajectory of origin O' of K':

- In $K': \vec{x}'_{O'} = 0$ (This is the operational *definition* of the origin O'.)
- In $K: \vec{x}_{O'} = \vec{v}t$ (This is the operational *definition* of \vec{v} in $K \xrightarrow{1, \vec{v}, 0, \vec{0}} K'$.)

In Eq. (1.45b):

$$\vec{0} = c_v \, \vec{v}t + \frac{d_v}{v^2} \, \vec{v} (\vec{v} \cdot \vec{v})t + e_v \, \vec{v} \, t \tag{1.46a}$$

$$\vec{v} \neq \vec{0} \& \forall t \quad \Rightarrow \quad 0 = c_v + d_v + e_v \tag{1.46b}$$

8 | Reciprocity:

 $\mathbf{i} \mid \triangleleft \text{Inverse transformation } K' \xrightarrow{\mathbb{1}, \vec{v}', \mathbf{0}, \mathbf{0}} K \text{ from } K' \text{ to } K:$

$$\Lambda_{\vec{v}'}\Lambda_{\vec{v}} = \mathbb{1} \quad \Leftrightarrow \quad \Lambda_{\vec{v}'} = \Lambda_{\vec{v}}^{-1} \,. \tag{1.47}$$

Note that \vec{v}' is the velocity of the origin *O* of *K* as measured in *K'*.

In general: $\vec{v}' = \vec{V}(\vec{v})$ with unknown function \vec{V} .

We assume *reciprocity*: $\vec{v}' = -\vec{v}$ such that

$$\Lambda_{\vec{v}}^{-1} = \Lambda_{-\vec{v}} \,. \tag{1.48}$$

While this is clearly the most reasonable/intuitive assumption, it is not trivial! Recall that \vec{v} is the speed of the origin O' of K' measured with the clocks in K, whereas \vec{v}' is the speed of the origin O of K measured with *different* clocks in K'. So without additional assumptions we cannot conclude that the results of these measurements yield reciprocal results.

However, the assumption of reciprocity can be rigorously derived from relativity SR, isotropy IS and homogeneity HO, see Ref. [33]. Reciprocity is therefore not an independent assumption.

ii | \triangleleft Inverse transformation in Eq. (1.45):

$$t = a_v t' - b_v \left(\vec{v} \cdot \vec{x}' \right) \tag{1.49a}$$

$$\vec{x} = c_v \, \vec{x}' + \frac{d_v}{v^2} \, \vec{v}(\vec{v} \cdot \vec{x}') - e_v \, \vec{v} \, t' \tag{1.49b}$$

iii | Eq. (1.49) in Eq. (1.45) & Eq. (1.46b) $\xrightarrow{\circ}$ (we suppress the v dependence)

$$c^2 = 1$$
, (1.50a)

$$a^2 - ebv^2 = 1, (1.50b)$$

$$e^2 - ebv^2 = 1, (1.50c)$$

$$e(a+e) = 0$$
, (1.50d)

$$b(a+e) = 0$$
. (1.50e)

To show this, use $\vec{v} = (v_x, 0, 0)^T$ with $v_x \neq 0$ and remember that the equations you obtain from plugging Eq. (1.49) into Eq. (1.45) must be valid for all t' and \vec{x}' . Use Eq. (1.46b) to replace $c_v + d_v$ by $-e_v$.

We can conclude:



- $\xrightarrow{\text{Eq. (1.50a)}} c = 1 \ (c = -1 \text{ contradicts } \lim_{v \to 0} \Lambda_{\vec{v}} \stackrel{!}{=} 1)$ $\xrightarrow{\text{Eq. (1.50c)}} e \neq 0 \xrightarrow{\text{Eq. (1.50d)}} a + e = 0$ $\rightarrow \text{Eq. (1.50b)} \equiv \text{Eq. (1.50c) & Eq. (1.50e) \text{ satisfied}}$
- **9** | Collecting results from Eq. (1.50) & Eq. (1.46b):

$$c = 1$$
, $e = -a$, $d = a - 1$, $b = \frac{1 - a^2}{av^2}$. (1.51)

d = a - 1 follows from Eq. (1.46b) and the first two equations. Eq. (1.45) $\xrightarrow{\text{Eq. (1.51)}}$

$$t' = a_v t + \frac{1 - a_v^2}{v a_v} \left(\hat{v} \cdot \vec{x} \right)$$
(1.52a)

$$\vec{x}' = \vec{x} + [a_v - 1] \,\hat{v}(\hat{v} \cdot \vec{x}) - v a_v \,\hat{v} \,t \tag{1.52b}$$

with $\hat{v} := \vec{v}/|\vec{v}|$.

10 $| \triangleleft$ Special boost $\vec{v} = (v_x, 0, 0)^T$ in x-direction:



$$t' = a_v t + \frac{1 - a_v^2}{v_x a_v} x$$
(1.53a)

$$x' = a_v x - v_x a_v t \tag{1.53b}$$

$$y' = y \tag{1.53c}$$

$$z' = z \tag{1.53d}$$

Note that $v = |v_x|$ with $v_x \in \mathbb{R}$.

Matrix form:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \underbrace{ \begin{pmatrix} a_v & \frac{1-a_v^2}{v_x a_v} \\ -v_x a_v & a_v \\ \hline & 1 & 0 \\ 0 & 1 \end{pmatrix}}_{=:\Lambda_{v_x}} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$
(1.54)

In the following, we refer to the upper 2×2 -block as $A(v_x)$.

11 | Group structure:

 $i \mid \text{Relativity principle } SR \rightarrow$

$$\varphi(K' \xrightarrow{R_2, \vec{v}_2, s_2, \vec{b}_2} K'') \circ \varphi(K \xrightarrow{R_1, \vec{v}_1, s_1, \vec{b}_1} K') \stackrel{!}{=} \varphi(K \xrightarrow{R_3, \vec{v}_3, s_3, \vec{b}_3} K'') \quad (1.55)$$

for some parameters $(R_3, \vec{v}_3, s_3, \vec{b}_3)$ that are a function of $(R_i, \vec{v}_i, s_i, \vec{b}_i)_{i=1,2}$.

In words:

The concatenation of a coordinate transformations from K to K' and from K' to K'' must be another coordinate transformation *that is parametrized by data that relates the reference systems* K with K'' directly (without referring to K' in any way).

You may ask why Eq. (1.55) is a constraint on φ in the first place. After all, we could just *define* that

$$\varphi(K \xrightarrow{R_3, \vec{v}_3, s_3, \vec{b}_3} K'') := \varphi(K' \xrightarrow{R_2, \vec{v}_2, s_2, \vec{b}_2} K'') \circ \varphi(K \xrightarrow{R_1, \vec{v}_1, s_1, \vec{b}_1} K').$$
(1.56)

The problem is that the function defined such generically depends on 8 (!) parameters $R_1, \vec{v}_1, s_1, \vec{b}_1, R_2, \vec{v}_2, s_2, \vec{b}_2$ – it is a non-trivial functional constraint on φ that these can be compressed to four parameters $R_3, \vec{v}_3, s_3, \vec{b}_3$. This "compression" is mandated by the relativity principle **SR** according to which all inertial systems must be treated equally. In particular, the transformation between two systems K and K" can only depend on parameters that can be experimentally determined from within these two systems. (The existence of) a third frame K' cannot be of relevance for this transformation as this would make K' special.

Combined with the existence of an inverse transformation (\leftarrow *above*):

 \rightarrow The set of all transformations forms a \downarrow (multiplicative) group.

Note that *associativity* is implicit since we talk about the concatenation of linear/affine maps.

ii | In particular:

$$\Lambda_{v_x}\Lambda_{u_x} \stackrel{!}{=} \Lambda_{w_x} \quad \Leftrightarrow \quad A(v_x)A(u_x) \stackrel{!}{=} A(w_x) \tag{1.57}$$

where $w_x = W(v_x, u_x)$ has to be determined.

• i! Using the restricted form of the boost Eq. (1.54) that followed from previous arguments, it follows indeed that the concatenation of two pure boosts *in the same direction* has again the form of a pure boost (in the same direction). For the arguments that follow, this is sufficient.

However, in general, the multiplicative group structure Eq. (1.55) allows for two boosts to concatenate to a *combination* of boosts and rotations. As we will see \rightarrow *later*, this is indeed what happens: The concatenation of two pure boosts (in different directions) produces a boost with a rotation mixed in (\uparrow *Thomas-Wigner rotation*).

Note that due to Eq. (1.43a) all that follows holds for any pair of *collinear* velocities v
 and u
 i (there is nothing special about the x-direction). Indeed, let R be a rotation that
 maps v
 and u
 to vectors on the x-axis, v
 i = Rv
 and u
 i = Ru
 . Then

$$\Lambda_{\vec{v}}\Lambda_{\vec{u}} \stackrel{1.43a}{=} \Lambda_{R^{-1}}\Lambda_{\vec{v}_{x}}\Lambda_{\vec{u}_{x}}\Lambda_{R} \stackrel{!}{=} \Lambda_{R^{-1}}\Lambda_{\vec{w}_{x}}\Lambda_{R} \stackrel{1.43a}{=} \Lambda_{\vec{w}}$$
(1.58)

where \vec{w} is again collinear with \vec{v} and \vec{u} .

 \rightarrow (use that the diagonal elements of $A(w_x)$ must be equal)

$$\forall_{v_x, u_x} : \quad \frac{1 - a_v^2}{v_x^2 a_v^2} \stackrel{!}{=} \frac{1 - a_u^2}{u_x^2 a_u^2} \tag{1.59}$$

 \rightarrow Universal constant:

$$\kappa := \frac{a_v^2 - 1}{v_x^2 a_v^2} = \text{const}$$
(1.60)



Note: $[\kappa] = \text{Velocity}^{-2}$

$$a_v = \frac{1}{\sqrt{1 - \kappa v_x^2}} \,. \tag{1.61}$$

We use the positive solution for a_v since $\lim_{v\to 0} A(v) \stackrel{!}{=} 1$, i.e., $\lim_{v\to 0} a_v \stackrel{!}{=} 1$.

iii | With this we check: $A(v_x)A(u_x) \stackrel{\circ}{=} A(w_x)$ with

$$w_x = W(v_x, u_x) \stackrel{\circ}{=} \frac{v_x + u_x}{1 + u_x v_x \kappa} \,. \tag{1.62}$$

Eq. (1.62) becomes important later: it tells us how to add velocities in SPECIAL RELATIV-ITY.

12 | Preliminary result:

Eq. (1.52) & Eq. (1.60) \rightarrow Boost $\Lambda_{\vec{v}}$ in direction \hat{v} with velocity $\vec{v} = v\hat{v}$:

$$t' = a_v \left[t - \kappa \, (\vec{v} \cdot \vec{x}) \right]$$
(1.63a)
$$\vec{x}' = \vec{x} + [a_v - 1] \, \hat{v} (\hat{v} \cdot \vec{x}) - a_v \, \vec{v} \, t$$
(1.63b)

with

$$a_v = \frac{1}{\sqrt{1 - \kappa v^2}} \,. \tag{1.64}$$

This is the most general transformation between two inertial coordinate systems that move with relative velocity \vec{v} (with coinciding axes at t = 0) that is consistent with our basic assumptions stated at the beginning of this section: **SR**, **HO**, and **IS**.

The only undetermined parameter left is κ .

1.5. The Lorentz transformation

The purpose of this section is to select the value for κ that describes our reality.

13 | Since $[\kappa] = \text{Velocity}^{-2}$ define formally: $\kappa \equiv 1/v_{\text{max}}^2$.

Why we subscribe the velocity v_{\max} with "max" will become clear below.

- 14 | Three cases:
 - $\kappa = 0 \Leftrightarrow v_{\max} = \infty$:

Eq. (1.63)
$$\Rightarrow$$
 $\begin{pmatrix} t' = t \\ \vec{x}' = \vec{x} - \vec{v} t \end{pmatrix}$ $\stackrel{\text{** Galilei boost}}{}$ (1.65a)

 \rightarrow Maxwell equations are *not* form-invariant under φ .



- \rightarrow Maxwell equations cannot be correct and must be modified.
- \rightarrow Experiment that shows the invalidity of Maxwell equations?

Note that we cannot conclude the *validity* of classical mechanics from this; Newton's equations may still require modifications (without spoiling the Galilean symmetry, of course).

• $\kappa > 0 \Leftrightarrow v_{\max} < \infty$:

Eq. (1.63)
$$\Rightarrow$$
 $t' = \gamma \left(t - \frac{\vec{v} \cdot \vec{x}}{v_{\max}^2} \right)$
 $\vec{x}' = \vec{x} + (\gamma - 1) \hat{v} (\hat{v} \cdot \vec{x}) - \gamma \vec{v} t$ \end{cases} ** Lorentz boost

(1.66a)

with the *** Lorentz factor*

$$\gamma_v \equiv \gamma := \frac{1}{\sqrt{1 - \beta^2}}$$
 and $\beta := v/v_{\text{max}}$. (1.67)

 \rightarrow Newton's equations are *not* form-invariant under φ .

- \rightarrow Classical mechanics cannot be correct and must be modified.
- \rightarrow Experiment that shows the invalidity of Newton's equations?

Similarly, we cannot conclude the *validity* of electrodynamics from this; Maxwell equations may still require modifications (without spoiling the Lorentz symmetry).

 <u>κ < 0</u>: Physically not relevant. (● Problemset 2; we ignore this solution in the following.) This solution is not self-consistent (see e.g. Ref. [32]) and immediately leads to implications that are not observed in nature.

For example, the rule Eq. (1.62) to compute the velocity w_x between K/K'' from the velocities v_x and u_x between K/K' and K'/K'' reads for $\kappa < 0$

$$w_x = \frac{v_x + u_x}{1 - u_x v_x |\kappa|} \,. \tag{1.68}$$

Let $u_x, v_x > 0$ be positive, i.e., K' moves in *positive* x-direction *wrt* K and K'' moves also in *positive* x-direction *wrt* K'. But for large enough velocities $u_x v_x > 1/|\kappa|$ we find $w_x < 0$ such that K'' moves in *negative* x-direction *wrt* K.

No such effect has ever been observed; if you do, let us know!

Note that at no point we used or claimed that v_{max} is the speed of light!

Which transformation describes reality: $v_{\text{max}} < \infty$ or $v_{\text{max}} = \infty$?

15 | Evidence:

• Maximum velocity $v_{\text{max}} \approx c < \infty$ for electrons (plot from Ref. [34]):





 \rightarrow Newton's equations are clearly invalid for high velocities!

See Refs. [34, 35] for more technical details. Note that these results were obtained decades after Einstein published his seminal paper in 1905.

• By contrast:

No evidence for the invalidity of Maxwell equations (on the macroscopic level).

Electrodynamics, as encoded by the Maxwell equations, is of course not a truly fundamental theory as it is the classical limit of a quantum theory: Quantum electrodynamics (QED). For example, the linearity of the Maxwell equations (= EM waves cannot scatter off each other) is an *approximation*; in QED photons *can* (weakly) scatter off each other! This is why I emphasize that Maxwell theory is experimentally valid only on the *macroscopic level*. Note, however, that QED has the same spacetime symmetry group as electrodynamics, namely Lorentz transformations.

16 | Hence it is reasonable stipulate $v_{\text{max}} < \infty$ and postulate:

The transformations φ between inertial systems are given by *Lorentz transformations*. These transformations must be (part of) the spacetime symmetries of *all* physical theories.

The last statement is often rephrased as follows:

All (fundamental) theories must be *form-invariant* (covariant) under Lorentz transformations.

This is just **SR** all over again: The equations of models that describe reality must "look the same" (more precisely: be functionally equivalent) in all inertial systems. Since the transformations between inertial systems are given by Lorentz transformations (and not Galilean transformations, as historically anticipated), this requires their form-invariance under Lorentz transformations.

\rightarrow SPECIAL RELATIVITY restricts the structure of all fundamental theories of physics!

This is what is meant by the statement that SPECIAL RELATIVITY is a *theoretical framework* (German: *Rahmentheorie*) or "meta theory": It provides a "recipe" (*ordering principle*) of how to construct consistent theories of physics. The Standard Model of particle physics, for example, is form-invariant under Lorentz transformations, and if you propose an extension thereof (for example to give neutrinos a mass) you better make sure that the terms you write down are also form-invariant under Lorentz transformations (otherwise you will not be taken seriously!). Note,



however, that this perspective prevents an important insight: What we really study is an entity called *spacetime*, and this entity has a property: Lorentz symmetry. Since all our (fundamental) physical theories are formulated *on* spacetime, it should not come as a surprise that the Lorentz symmetry of spacetime shows up all over the place.

- **17** | Interpretion of v_{max} :
 - $\mathbf{i} \mid \triangleleft \text{Systems } K \xrightarrow{v_x} K' \text{ and signal with velocity } \frac{\mathrm{d}x'}{\mathrm{d}t'} = u'_x$:



Question: What is the velocity $u_x = \frac{dx}{dt}$ of this signal in *K*?

ii | Remember (Group structure!):

$$\varphi(K' \xrightarrow{v_2} K'') \circ \varphi(K \xrightarrow{v_1} K') = \varphi(K \xrightarrow{v_3} K'') \quad \text{with} \quad v_3 = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{v_{\text{max}}^2}} \,. \tag{1.69}$$

Let $v_1 = v_x$ and $v_2 = u'_x$ so that $v_3 = u_x$ (i.e., the signal is at rest in the origin of K''). You can also derive this by computing the time derivative of the position of the signal in K using a Lorentz transformation; you will do this properly when you derive a more general addition of velocities (\bigcirc Problemset 2).

iii | Addition formula for collinear velocities:

$$u_{x} = \frac{v_{x} + u'_{x}}{1 + \frac{v_{x}u'_{x}}{v_{\max}^{2}}}$$
(1.70)

Because of isotropy **IS** this formula must be true in all directions (not just in x-direction) as long as the two velocities to be added are parallel. We still keep the index x to signify that these are not absolute values of velocities.

- Note that for $v_{\max} \to \infty$ we get back the "conventional" (= Galilean) additivity of velocities:

$$u_{x} = (v_{x} + u'_{x}) \left[1 - \frac{v_{x}u'_{x}}{v_{\max}^{2}} + \dots \right] \stackrel{v_{\max} \to \infty}{=} v_{x} + u'_{x}$$
(1.71)

From this expansion and the validity of classical mechanics for small velocities (in particular its law for adding velocities), we can also conclude that v_{max} must be *large compared to everyday experience*.

• A historically influential experiment that (in hindsight) can be explained by the relativistic addition of velocities Eq. (1.70) is the ↑ *Fizeau experiment* [36,37] (see also ↑ *Fresnel drag coefficient*). The Fizeau experiment was one of the crucial hints that led Einstein to SPECIAL RELATIVITY.



 $\mathsf{iv} \mid \ \sphericalangle \ 0 \le v_x, u_x' \le v_{\max} \text{:} \ (\tilde{v}_x := v_x/v_{\max} \text{ so that } 0 \le \tilde{v}_x, \tilde{u}_x \le 1)$

$$u_x = v_{\max} \frac{\tilde{v}_x + \tilde{u}'_x}{1 + \tilde{v}_x \tilde{u}'_x} \le v_{\max}$$
(1.72)

Here we used that $a + b \le 1 + ab$ for numbers $0 \le a, b \le 1$.

- \rightarrow "Addition" of velocities Eq. (1.70) never exceeds v_{max} .
- $\rightarrow v_{\text{max}}$ plays the role of a *maximum velocity*.
- **v** | \triangleleft Signal with maximum velocity in K': $u'_x = v_{\text{max}}$:

$$u_{x} = \frac{v_{\max} + v_{x}}{1 + \frac{v_{\max}v_{x}}{v_{\max}^{2}}} = v_{\max}\frac{v_{\max} + v_{x}}{v_{\max} + v_{x}} = v_{\max}$$
(1.73)

Note that the result is completely independent of the velocity v_x of K'!

 \rightarrow Whatever moves with the maximum velocity v_{max} does so *in all inertial systems*!

Please appreciate how counterintuitive this effect is *from the perspective of everyday experience*! But also notice that we didn't have to postulate it: The relativity principle **SR** together with the *existence* of a (finite) maximum velocity is sufficient.

If you think about it: Assuming a maximum velocity (in the absence of a preferred reference frame) automatically invalidates the simple Galilean law of additive velocities. So it is actually not surprising at all that the maximum velocity must be independent of the reference system.



↓Lecture4 [07.11.23]

18 | Experiments (in particular: the validity of Maxwell equations) show:

 $v_{\rm max} = c = 299\,792\,458\,{\rm m\,s^{-1}}$

(1.74)

Note that since 1983 the value of c in the international system of units (SI) is *exact* by definition.

A. Einstein incorporated this insight in §2 of Ref. [10] as his second postulate:

2. Jeder Lichtstrahl bewegt sich im "ruhenden" Koordinatensystem mit der bestimmten Geschwindigkeit V, unabhängig davon, ob dieser Lichstrahl von einem ruhenden oder bewegten Körper emittiert ist.

Note that at the time it was conventional to denote the speed of light with a capital V. The convention switched to our now standard lower-case c just a few years later. For more historical background:

https://math.ucr.edu/home/baez/physics/Relativity/SpeedOfLight/c.html

We can condense this into:

§ Postulate 5: Constancy of the speed of light SL

The speed of light is independent of the inertial system in which it is measured.

Comments:

• If you take the validity of the Maxwell equations for granted, then $v_{max} = c < \infty$ (and thereby SL) follows immediately from the relativity principle SR because then the Maxwell equations must be valid in all inertial systems. But you've learned in your course on electro-dynamics that the wavelike solutions of these equations always propagate with group velocity c in vacuum. This is only possible if the speed of light plays the role of the limiting velocity: $v_{max} = c$.

Einstein acknowledges as much at the beginning of Ref. [11]. However, **SL** is empirically weaker than claiming the validity of Maxwell's equations (after all, there could be alternative equations that also predict the velocity c of wavelike solutions). At the time when Einstein formulated **SL** in [10], he also worked on the photoelectric effect (another of his *annus mirabilis* papers [38]). The postulation of "quanta of light" is the foundation of quantum mechanics, but cannot be explained by Maxwell's equations. It is therefore reasonable to assume that Einstein didn't want to rely on the validity of this specific theory when formulating his SPECIAL RELATIVITY. He therefore opted for the empirically weaker (but still sufficient) assumption **SL**.

• If you derive the transformation φ using *both* postulates **SR** and **SL** the derivation is shorter (see e.g. [1] or [5]); one then of course doesn't find the Galilei transformations as an option. Note, however, that the relativity principle **SR** is a reasonable and intuitive starting point that doesn't need much convincing (after all, we witness the relativity of Newtonian mechanics in our everyday life). By contrast, the speed of light postulate **SL** clashes directly with our everyday experience (how velocities add up, that is). Through our elaborate derivation we learned how much is already implied by the simple, reasonable assumption of relativity. We



only had to check whether there is any evidence of a finite maximum velocity v_{max} . The counterintuitive feature that this velocity is the same viewed from all inertial systems was then a necessary conclusion from our derivation.

† Note: Finite speed of causality (Locality)

Another insight from our **SR**-based derivation of the Lorentz transformation is that the formulation of the speed-of-light postulate **SL** is conceptually misleading:

• The constant v_{max} and its role as a maximum velocity followed *without* referring to light (or electrodynamics) in any way!

Put bluntly: SPECIAL RELATIVITY is *not* about the "strange behavior" of light!

• The relevant speed for SPECIAL RELATIVITY is the *speed of causality*: How fast can information travel, i.e., one event affect another. v_{max} is the maximum speed of *causal interactions*, irrespective of the mediator of these interactions.

In our world, the fastest and most salient information carrier just happens to be the electromagnetic field ("light"). For example, to synchronize our clocks with light signals, it wasn't the light *per se* we were interested in; we just used it as carrier of information to correlate the clocks.

- Given the relativity principle **SR** and our derivation in Section 1.4, we showed that there are only two possibilities: (1) There is *no* upper bound on velocities (Galilean symmetry) or (2) there *is* such an upper bound v_{\max} (Lorentz symmetry). In the latter case, every signal that propagates with v_{\max} in some frame automatically does so in all inertial systems. (Which immediately leads to the counterintuitive conclusion, akin to **SL**, that there are signals the velocity of which does not depend on the velocity of the observer.)
- We could replace **SL** therefore by the (empirically weaker) postulate that there are *no instantaneous actions* at a distance (this is essentially a statement about *locality*). This modified postulate implies the existence of a maximal velocity $v_{\text{max}} < \infty$ which, in turn, selects the Lorentz transformation as the correct symmetry. That $v_{\text{max}} = c$ is then a fact to be discovered by experiments.
- It turns out that *everything with vanishing rest mass* travels at this maximum speed $v_{\text{max}} = c$. Since photons are the only elementary particles that are massless and can be easily detected, we just happen to refer to this maximum velocity as "speed of light."

For example: Without Higgs symmetry breaking, the W^{\pm} and Z bosons of the weak interaction are massless and would propagated with light velocity, just as the photon (the weak interactions would then be no longer "weak"). For a long time it was believed that neutrinos are massless as well, and therefore would also propagate with the speed of light (today we know that they have a very tiny mass).

19 | Special Lorentz transformations = Lorentz boosts:

Now that everything is settled, let us write down our final result in their conventional form.

i! These are not the most general (homogeneous) Lorentz transformations since we omit rotations, parity and time inversion. We will discuss the structure of the full homogeneous Lorentz group and its inhomogeneous generalization (\rightarrow *Poincaré group*) later. To discuss the "fancy" phenomena of SPECIAL RELATIVITY, the transformations below are sufficient.



$$\Lambda(K \xrightarrow{\vec{v}} K'): \begin{cases} ct' = \gamma \left(ct - \beta \vec{x} \cdot \hat{v}\right) \\ \vec{x}' = \vec{x} + (\gamma - 1)(\vec{x} \cdot \hat{v}) \cdot \hat{v} - \gamma \vec{v}t \end{cases}$$
(1.75)

(Since we now settled on Lorentz transformation for φ , we write $\varphi = \Lambda$ henceforth.) with $\beta \equiv v/c$ and the *Lorentz factor*

$$\gamma_v \equiv \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}.$$
 (1.76)

ii | Special case: Boost in x-direction $(\vec{v} = v_x \hat{x})$:

$$\Lambda(K \xrightarrow{v_x} K'): \begin{cases} ct' = \gamma \left(ct - \frac{v_x}{c}x\right) \\ x' = \gamma(x - v_x t) \\ y' = y \\ z' = z \end{cases}$$
(1.77)

20 | <u>State of affairs:</u>

Now that we know the spacetime symmetry φ of reality, we have quite a to-do list:

• We will have to *modify* Newton's equations to replace their Galilean by a Lorentz symmetry, without changing their predictions for small velocities $v \ll c$ (\checkmark correspondence principle).

 \rightarrow Relativistic mechanics

• We can keep the Maxwell equations in their current form ©.

Note that we still have to check that they are really Lorentz covariant (Problemset ?)!

In the end we will come up with a neat notation that allows us to rewrite (not modify!) the Maxwell equations in a compact form to make their Lorentz symmetry apparent.

- Similar to classical mechanics, we will have to replace the *Schrödinger equation* in quantum mechanics by a modified version with Lorentz symmetry.
 - → Relativistic quantum mechanics (Klein-Gordon and Dirac equation)

But before we do all the heavy work:

Simple implications of this transformation? (> *below* and next lectures)

With "simple" we refer to implications that follow without imposing a model-specific dynamics (= equation of motion). We will refer to these implications as *kinematic* because they follow from fundamental constraints on the degrees of freedom of all relativistic theories.



1 \triangleleft Trajectory of a light signal in *x*-direction in *K*:

$$x(t) = ct, y = 0, z = 0$$
 (1.78)

Trajectory of the same signal in K' with $K \xrightarrow{v_x} K'$:

$$x'(t') = ct', y' = 0, z' = 0$$
 (1.79)



This follows from our previous discussion: signals propagating with $c = v_{\text{max}}$ do so in all inertial systems!

You can also simply calculate this using the Lorentz boost Eq. (1.77):

$$ct' = \gamma \left(ct - \frac{v_x}{c} ct \right) \tag{1.80a}$$

and
$$x' = \gamma(ct - v_x t) = ct'$$
. (1.80b)

 \rightarrow

$$(ct)^2 - x^2 = 0 = (ct')^2 - (x')^2$$
 is a *frame-independent* quantity. (1.81)

Note that the separate summands $[(ct)^2$ etc.] are *not* frame-independent! This finding motivates the definition of the ...

2 *** Spacetime interval*:

Details:
Problemset 2

 \triangleleft Two events $E_1 \ni (t_1, \vec{x}_1)_K$ and $E_2 \ni (t_2, \vec{x}_2)_K$ with temporal and spatial separation

$$(\Delta t)_K := t_1 - t_2 \text{ and } (\Delta \vec{x})_K := \vec{x}_1 - \vec{x}_2.$$
 (1.82)

Then the *spacetime interval* between E_1 and E_2 is denoted $(\Delta s)^2 \equiv \Delta s^2$ and defined as

$$(\Delta s)^2 := (c \,\Delta t)_K^2 - (\Delta \vec{x})_K^2 \,. \tag{1.83}$$

We omit the subscript K from Δs because it is frame-independent (\rightarrow *next*).

In our example above it was $\Delta t = t - 0$ and $\Delta \vec{x} = (x - 0, 0 - 0, 0 - 0)$, i.e., we considered the interval between the event in the origin $x_0 = (0, \vec{0})$ and the events along the trajectory (ct, x(t), 0, 0) of the light signal.



3 | The importance of Δs^2 stems from the following fact:

The spacetime interval Δs^2 is independent of the frame in which it is calculated.

This means that given two events, all observers agree on the numerical value of the interval Δs^2 between these two events.

Proof: Use Eq. (1.75) to calculate (Details:) Problemset 2)

$$(ct')^{2} = \left[\gamma \left(ct - \beta \vec{x} \cdot \hat{v}\right)\right]^{2}$$
(1.84a)

$$(\vec{x}')^2 = \left[\vec{x} + (\gamma - 1)(\vec{x} \cdot \hat{v}) \cdot \hat{v} - \gamma \vec{v}t\right]^2$$
(1.84b)

$$\Rightarrow (ct')^2 - (\vec{x}')^2 \stackrel{\circ}{=} (ct)^2 - (\vec{x})^2 + \underbrace{\dots}_{=0}$$
(1.84c)

Note that we do not have to do the computation for two events and an interval Δt and $\Delta \vec{x}$ since the special Lorentz transformations are *linear*.

This proves the invariance under *special* Lorentz transformations (= Lorentz boosts). It is easy to see that the invariance is also valid for inhomogneous shifts in time and space (these drop out in the intervals Δt etc.) and spatial rotations Λ_R [since $(\Delta \vec{x})^2$ is clearly invariant under rotations]. We will come back to this when we discuss the structure of the Lorentz group in more detail (\rightarrow *later*).

4 | Two events E_1 and E_2 are in one of three possible (frame-independent) relations:

$$\Delta s^{2} \begin{cases} > 0 & E_{1} \text{ and } E_{2} \text{ are } \overset{*}{*} \textit{time-like separated} \\ = 0 & E_{1} \text{ and } E_{2} \text{ are } \overset{*}{*} \textit{light-like separated} \\ < 0 & E_{1} \text{ and } E_{2} \text{ are } \overset{*}{*} \textit{space-like separated} \end{cases}$$

Note that Δs^2 can be *negative* so that Δs^2 should be read as a symbol rather than defining an imaginary number Δs . For the special case of *time-like* intervals, however, Δs^2 indeed defines a real number $\Delta s = \sqrt{\Delta s^2}$ which we will later relate to the time measured by moving clocks (the so called *proper time*).

All events that are light-like separated from an event E (*wlog* in the origin) satisfy

$$\Delta s^2 = 0 \quad \Leftrightarrow \quad (ct)^2 = (\vec{x})^2 \quad \Leftrightarrow \quad |ct| = |\vec{x}| \tag{1.86}$$

which determines the *** light cone* of *E*:

(1.85)





Here we show the light cone of an event E in a space time with *two* spatial dimensions x and y. The light cone in our 3 + 1 dimensional space time is a higher-dimensional generalization which obeys the same equations.

- Time-like events satisfy $\Delta s^2 > 0 \Leftrightarrow |ct| > |\vec{x}|$ which characterizes the (disconnected) interior of the light cone. The manifold with $ct > |\vec{x}| \ge 0$ is called * *future light cone* (of *E*) whereas the events with $-ct > |\vec{x}| \ge 0$ make up the * *past light cone* (of *E*).
- Space-like events satisfy $\Delta s^2 < 0 \Leftrightarrow |ct| < |\vec{x}|$ which characterizes the (connected) spacetime volume outside the light cone.

5 | Causality:

The importance of the threefold classification of spacetime intervals stems from the following observations.

 $i \mid \triangleleft$ Actions of (homogeneous) Lorentz transformations:

Since Δs^2 is invariant under Lorentz transformations, the manifold of events characterized by a specific value $\Delta s^2 = \pm C$ ($C \ge 0$) must be mapped onto itself under these transformations: Events on these hyperbolic manifolds cannot leave their manifolds under Lorentz transformations.

Invariant hyperbolae:

time-like:
$$\Delta s^2 = C > 0 \Rightarrow ct = \pm \sqrt{C + |\vec{x}|^2}$$
 (1.87a)

light-like:
$$\Delta s^2 = C = 0 \implies ct = \pm |\vec{x}|$$
 (1.87b)

space-like:
$$\Delta s^2 = -C < 0 \implies ct = \pm \sqrt{|\vec{x}|^2 - C}$$
 (1.87c)



This picture leads immediately to two conclusions:

- ii | \triangleleft Two *distinct* events $E_1 \ni (t_1, \vec{x}_1)_K$ and $E_2 \ni (t_2, \vec{x}_2)_K$ with coordinates in K:
 - If $\Delta s^2 \ge 0$ (= time-like or light-like), then

either
$$\forall_K : (t_1)_K > (t_2)_K$$
 or $\forall_K : (t_1)_K < (t_2)_K$. (1.88)

This means that for time-like or light-like separated events all observers agree on their temporal ordering! Note that they do not necessarily agree on the time $(t_1)_K - (t_2)_K$ elapsed between the two events.

Proof: Assume $(t_1)_A < (t_2)_A$ and $(t_1)_B > (t_2)_B$ for two inertial systems A and B. Because of the continuity of Lorentz transformations there must exist a frame C with $(t_1)_C = (t_2)_C$. But in this frame $(\Delta s)_C^2 = -(\Delta \vec{x})_C^2 \ge 0$ such that $(\vec{x}_1)_C = (\vec{x}_2)_C$ and therefore $E_1 = E_2$ (which contradicts our assumption that the two events are distinct).

Proof by picture!

• If $\Delta s^2 < 0$ (= space-like), then

$$\exists_{A,B}: (t_1)_A > (t_2)_A \text{ and } (t_1)_B < (t_2)_B.$$
 (1.89)

This means that for space-like separated events there are always observers who see E_1 happening before E_2 while other observers see E_1 happening after E_2 . The temporal order of space-like separated events is therefore observer-dependent!

Proof: Problemset ?

Proof by picture!

iii | Conventional relation of time order and causality:

$$E_1$$
 can causally affect $E_2 \implies E_1$ happens before E_2 (1.90)

Since causality should be an objective, observer-independent fact, and we just showed that only time- and light-like separated events have an observer-independent temporal order, it is reasonable to define the following ...

... \checkmark *(strict) partial order* \prec on the set \mathcal{E} of events:

$$E_1 \prec E_2 \quad :\Leftrightarrow \quad \Delta s^2 \ge 0 \quad \text{and} \quad t_1 < t_2 : \quad ``E_1 \text{ can affect } E_2 " \tag{1.91}$$
$$E_1 \succ E_2 \quad :\Leftrightarrow \quad \Delta s^2 \ge 0 \quad \text{and} \quad t_1 > t_2 : \quad ``E_2 \text{ can affect } E_1 " \tag{1.92}$$



This is a *partial* order because for $\Delta s^2 < 0$ there is *no* relation between E_1 and E_2 (we denote this by $E_1 \searrow E_2$).

To be a partial order, one has to show *irreflexivity* (which is trivial since t < t is not true) and *transitivity*. To show transitivity, show that $\Delta s_{1,2}^2 \ge 0$ and $\Delta s_{2,3}^2 \ge 0$ together with $t_2 > t_1$ and $t_3 > t_2$ implies $\Delta s_{1,3}^2 \ge 0$ and $t_3 > t_1$ (use the triangle inequality).

iv | This definition of causality is consistent with our previous findings that no signal can travel faster than the speed of light c:



- $E \prec E_1$: There exists a signal trajectory $\vec{x}(t)$ with $\left|\frac{d\vec{x}(t)}{dt}\right| \leq c$ connecting the two events (blue in the sketch).
- $E \searrow E_3$: Any trajectory $\vec{x}(t)$ connecting the two events (red in the sketch) has some segment with $\left|\frac{d\vec{x}(t)}{dt}\right| > c$ (yellow in the sketch). Since this is physically impossible, there is no signal of any kind that can mediate causal influence from E to E_3 (and vice versa).

This follows from an application of (a generalization of) the \downarrow mean value theorem.

6 | Since the causal structure (\mathcal{E}, \prec) is observer independent:

There is no relativity of causality in SPECIAL RELATIVITY!

If one observer states that E_1 can causally affect E_2 , then *all* observers will agree on this statement.

7 | *Fun fact:*

If one starts from the causal structure (\mathcal{E}, \prec) and derives the group of \uparrow *causality-preserving automorphisms* Φ ,

$$E_1 \prec E_2 \Leftrightarrow \Phi(E_1) \prec \Phi(E_2),$$
 (1.93)

one again finds the homogeneous Lorentz transformations (boosts & rotations) that we constructed above (plus space-inversion, spacetime dilations and translations), see Ref. [39] for more details. Most interestingly, for the proof neither a continuity assumption on Φ nor a topology on \mathcal{E} is required; all this follows (at least in 2 + 1 spacetime dimensions and more) from the partial order \prec .



1.7. **‡** Relativity, compressibility, and the anthropic principle

The statements in this section are not specific to Einstein's relativity principle SR.

- **1** | Relativity principles ...
 - ... are statements about (the existence of) symmetries of spacetime.
 - ... imply the versatility of models to predict events from many viewpoints.
 - ... are statements about an *a priori* unnecessary *simplicity* of nature.
- **2** | Imagine a world *without any* relativity principle:

The equations (models) that capture physical laws faithfully are different from frame to frame.

 \rightarrow Your brain must learn arbitrary many *different* models adapted to all possible reference frames to anticipate the future in all situations.

- → Biologically impossible (your brain capacity is finite, building models is expensive)
- **3** | Example: Catching balls:



Notice that most reference frames that we naturally encounter are (approximately) inertial only in x and y direction (the axes that are locally parallel to earth's surface) and constantly accelerated in z direction (the axis perpendicular to earth's surface; the acceleration is $g \approx 9.81 \text{ m/s}^2$). The non-relativistic symmetries that relate these frames are a *subgroup* of the full Gallilei group (excluding rotations around the x and y axes as well as "large" translations). Our brain contains only models for these frames (equipped with Cartesian coordinates). Have you ever tried throwing or catching a ball in frames with acceleration in x or y directions (like a centrifuge)?

SouTube Video: The artificial gravity lab (Tom Scott)

Note that it is not *impossible* to train specific models for other frames to which the relativity principle of our everyday experience does not apply (after some practice you *can* throw and catch balls in a centrifuge of constant angular velocity). But this is just *one* additional model and even this is not implemented in our brains by default!

- 4 | Relativity principle
 - \rightarrow Descriptions of natural phenomena are highly *compressible*.
 - \rightarrow Only *few* models (equations) are necessary to anticipate the future.
- 5 | Anthropic principle:

Question: Why are there spacetime symmetries / relativity principles in the first place?

Answer: Because if there were none, evolution would most likely be impossible, hence we would be unable to ask the question.



Note that evolution relies on the somewhat reliable proliferation of information over time. This seems only possible if the individuals carrying this information survive. Surviving in environments with life-threatening phenomena (thunderstorms, predators, ...) relies on its (approximate) *predictability* by (approximate) *models* that are learned evolutionary and/or by experience.

For this argument to work some form of "ensemble interpretation" of reality is required (e.g. \uparrow *multiverses*) [40].



2. Kinematic Consequences

In this chapter we study implications of the special Lorentz transformations Eq. (1.75) and Eq. (1.77) that follow without imposing a model-specific dynamics (= equations of motion). We refer to these implications as *kinematic* because they follow from fundamental constraints on the degrees of freedom of all relativistic theories. The phenomena we will encounter are therefore *features of spacetime itself* – and not of some entities that live on/in (or couple to) spacetime.

i! The phenomena we will encounter are *not* "illusions" (in the sense that we "see" things differently than they "really are"). Remember that we precisely defined what we mean by observers/reference frames; in particular, we emphasized that we do not "look" at anything, we *measure* events in a systematic way, using a well-defined structure called \leftarrow *inertial system*. All phenomena we will encounter are derived from and to be understood in this operational, physically meaningful context.

2.1. Length contraction and the Relativity of Simultaneity

1 $| \triangleleft$ Inertial systems $A \xrightarrow{v_x} A'$ with rod on x'-axis and at rest in A':

Remember that $A \xrightarrow{v_x} A'$ denotes a boost in x-direction with v_x (as measured in A) where the spatial axes of both A and A' coincide at t = 0:



In such situations, we refer to A' as the ** rest frame of the rod and A as the ** lab frame (some call A the ** stationary frame). In the following, coordinates of events in the inertial system A' are marked by primes.

2 | First, we have to define what we mean by the "lenght" of an object:

"Length" is an intrinsically non-local concept. It is not something you can measure or define at a single point in space. Consequently, there are no "length-events" in \mathcal{E} . Thus we need an algorithm (= operational definition) of what we mean by "length".

 \triangleleft Two event *types*:

$$\{e_L\} = \{ \langle \text{Left end of rod detected} \rangle \}$$
(2.1a)

$$\{e_R\} = \{\langle \text{Right end of rod detected} \rangle\}$$
(2.1b)

Think of an event *type* as a set (equivalence class) of all elementary events that you deem \uparrow *type-identical* (but not \uparrow *token-identical*). In the example given here, there will be many events e_L in



spacetime that signify "Left end of rod detected" (if there is one rod, there will be one such event for each time t); these are *different* events of the *same type* $\{e_L\}$.

One could even declare that the event *type* $\{e_L\}$ *is* what we refer to as "the left end of the rod."

 \rightarrow Algorithm LENGTH to compute "Length of Rod" in system K at time t:

LENGTH:

→ Input: Coincidences \mathcal{E} , Inertial system label K, Time t

 \leftarrow **Output:** Length l_K of rod at time t as measured in K

- 1. Find (unique) event $L \in \mathcal{E}$ with $\{e_L\} \in L$ and $(t, \vec{l})_K \in L$.
- 2. Find (unique) event $R \in \mathcal{E}$ with $\{e_R\} \in R$ and $(t, \vec{r})_K \in R$.
- 3. Return $l_K := |\vec{l} \vec{r}|$.

Here, $\{e_L\} \in L$ is shorthand for $\{e_L\} \cap L \neq \emptyset$. In words: the coincidence class L contains an event of the *type* "Left end of rod detected".

Note that we define "length" as the spatial distance between the two ends of the rod *at the same time t* (as measured by the clocks in K). I hope you agree that this is what one typically means by "length."



↓Lecture 5 [14.11.23]

- **3** | We now apply this algorithm twice, in the lab frame A and the rest frame A':
 - $\mathbf{i} \mid \underline{\text{Rest frame } A':}$

** Proper length \equiv ** Rest length := Length of rod in A':

$$l_0 := \text{LENGTH}(\mathcal{E}, t'_0; A') = |l'_0 - \vec{r}'_0| = |l'_0 - r'_0|$$
(2.2)

with simultaneous clock events $(t'_0, \vec{l}'_0)_{A'} \in L_0$ and $(t'_0, \vec{r}'_0)_{A'} \in R_0$.

The time t'_0 that we choose is irrelevant since the rod is (by definition) at rest in A'. Since the rod lies on the x'-axis, it is $\vec{l}'_0 = (l'_0, 0, 0)$ and $\vec{r}'_0 = (r'_0, 0, 0)$.

The subscript "0" in L_0 indicates that this is a specific event (coincidence class) we selected in A' to compute the length of the rod. It does *not* mean "as seen from the rest frame A'" or anything like that. Remember that coincidence classes in \mathcal{E} are objective information!

ii | Lab frame A:

Length of moving rod in *A*:

$$l := \text{LENGTH}(\mathcal{E}, t; A) = |l - \vec{r}|$$
(2.3)

with simultaneous clock events $(t_l, \vec{l})_A \in L$ and $(t_r, \vec{r})_A \in R$ with $t_l = t_r = t$.

The time *t* that we choose might be irrelevant as well, but we do not know this yet.

i! There is no reason to assume that the events L_0/R_0 chosen in A' to measure the length of the rod are identical to the events L/R used in $A: L_0 \neq L$ and $R_0 \neq R$ in general.

4 | How does l_0 relate to l?

i | In Section 1.5 we did a lot of hard work to compute the transformation φ which transforms the coordinates of an event in one inertial system into the coordinates of *the same event* in another inertial system. We identified the transformation as the Lorentz transformation:

$$\Lambda(A \xrightarrow{v_x} A') : [E]_A = (t, \vec{x}) = x \mapsto \Lambda_{v_x} x = x' = (t', \vec{x}') = [E]_{A'}$$
(2.4)

ii | So let us use this tool [namely Eq. (1.77)] to obtain the coordinates of the events L and R (used for the length measurement in A) in the rest frame A' of the rod:

$$[L]_{A'} = \begin{cases} ct'_{l} = \gamma \left(ct_{l} - \frac{v_{x}}{c} l_{x} \right) \\ l'_{x} = \gamma (l_{x} - v_{x}t_{l}) \\ l'_{y} = l_{y} \\ l'_{z} = l_{z} \end{cases} \text{ and } [R]_{A'} = \begin{cases} ct'_{r} = \gamma \left(ct_{r} - \frac{v_{x}}{c} r_{x} \right) \\ r'_{x} = \gamma (r_{x} - v_{x}t_{r}) \\ r'_{y} = r_{y} \\ r'_{z} = r_{z} \end{cases}$$
(2.5)

Here we use $\vec{l} = (l_x, l_y, l_z)$ and $\vec{r} = (r_x, r_y, r_z)$. Since we declared that the rod is fixed on the x'-axis of A', and $\{e_L\} \in L$ and $\{e_R\} \in R$, it must be $l'_y = l'_z = r'_y = r'_z = 0$, and therefore $\vec{l} = (l_x, 0, 0)$ and $\vec{r} = (r_x, 0, 0)$. That is, the rod is not rotated by the boost and always lies on the x-axis of A as well. In particular: $l = |\vec{l} - \vec{r}| = |l_x - r_x|$.

 \rightarrow Two immediate conclusions:



a | In A' the two events L and R are no longer simultaneous:

$$t_l = t_r \text{ in } A \quad \text{but} \quad t'_l \neq t'_r \text{ in } A' \text{ (since } l_x \neq r_x \text{)}.$$
 (2.6)

 \rightarrow The simultaneity of events is *observer-dependent*.

This ambiguity of simultaneity can be graphically illustrated in a spacetime diagram (for details on how to draw the (t', x')-axes in $A: \bigoplus$ Problemset 2):



- As a side note, this calculation implies that not only is it generally *not* true that $L_0 = L$ and $R_0 = R$, it is actually *impossible* (at least for both pairs).
- In the sketch above, the "interior of rod"-events are painted gray. One is tempted to ask: Which "line" of these events *is* the rod? The counterintuitive answer is that this depends on the observer: For *A*-observers, *horizontal* lines of gray events make up "the rod", whereas for the *A'*-observer *tilted* lines are "the rod". It is actually more reasonable to think of the complete area of gray events as "the rod", just as the event type $\{e_L\}$ is "the left edge" of the rod. This suggests that our intuitive concept of the *instantaneous existence of extended objects* which feels so natural to us is, to some extend, misleading.
- **b** | In A' the coordinate distance is different:

$$|l'_{x} - r'_{x}| \stackrel{t_{l} = t_{r}}{=} \gamma |l_{x} - r_{x}| \stackrel{v_{x} \neq 0}{\neq} |l_{x} - r_{x}| = l$$
(2.7)

i! The time-dependence cancels so that the expressions are time-independent.

At this point, it is a bit premature to identify the left-hand side as the *rest length* l_0 of the rod because these are spatial coordinates of events that are *not simultaneous*! (Remember that the length of any object in any frame is defined as the coordinate distance of *simultaneous* events.)

However, since A' is (by definition) the *rest frame* of the rod, the position labels of the A'-clocks adjacent to the ends of the rod are the same for all events:

$$\left. \begin{array}{c} l'_{x} \stackrel{\{e_{L}\} \in L}{=} l'_{0} \\ r'_{x} \stackrel{\{e_{R}\} \in R}{=} r'_{0} \end{array} \right\} \Rightarrow |l'_{x} - r'_{x}| = |l'_{0} - r'_{0}| = l_{0}$$

$$(2.8)$$



 \rightarrow ** Length contraction \equiv ** Lorentz contraction:

A rod of rest length l_0 is *shorter* if measured from an inertial system in relative motion:

$$l = l_0 \sqrt{1 - \frac{v^2}{c^2}} \quad \stackrel{v \neq 0}{<} \quad l_0 \tag{2.9}$$

- i! Due to isotropy, this result is true for any length of extended objects *in the direction of the boost*. A rod along the *y*'-axis, for example, is contracted according to Eq. (2.9) for a boost in *y*-direction, but not for a boost in *x*-direction.
- The rod is just a proxy for *any* physical object; the Lorentz contraction therefore affects all physical objects in the same way. The contraction is not a dynamical feature of the object itself (like a force that compresses the atomic lattice) but an intrinsic property of space(time).
- Note that we say above "if measured from …" and not "as viewed from …" This distinction is important: If you ask how you would *visually perceive* extended objects flying by (or how they look on a picture taken by a camera) you have to factor in that the photons bouncing of the object at different points take different times to reach your eye (our the camera sensor). If you do the math (● Problemset 3), this additional optical effect leads to the surprising result that 3D objects actually do *not look* "squeezed" but *rotated*. This implies in particular that a moving sphere still *looks* like a sphere and not like an ellipse (↑ *Penrose-Terrell effect* [41, 42], see also Ref. [43]).

You can experience this effect (among others) in the educational game "A Slower Speed of Light," which has been developed by the MIT Game Lab for educational purposes, and can be downloaded here for Windows, Mac, and Linux (€ Problemset 3):

Download "A Slower Speed of Light"

You should always keep in mind, however, that this "looking" is *not* what we refer to as *observing* in RELATIVITY; the latter has been defined operationally as a measurement procedure at the beginning of this course.

2.2. Time dilation

1 \triangleleft Inertial systems $A \xrightarrow{v_x} A'$ and a clock \vec{x}' at rest in A':



- **2** $| \triangleleft$ Two events:
 - $A'-\text{Clock } \vec{x}' \text{ meets } A\text{-clock } \vec{x}_0: \quad (t'_0, \vec{x}')_{A'} \sim (t_0, \vec{x}_0)_A \in E_0$ (2.10a) $A'-\text{Clock } \vec{x}' \text{ meets } A\text{-clock } \vec{x}_1: \quad (t'_1, \vec{x}')_{A'} \sim (t_1, \vec{x}_1)_A \in E_1$ (2.10b)



i! The two events E_0 and E_1 relate *three* different clocks: The single A'-clock \vec{x}' and two *different* A-clocks \vec{x}_0 and \vec{x}_1 .

3 As for length, the concept of "duration" cannot be defined locally in spacetime. We therefore need an operational definition (algorithm) of "duration":

DURATION:

- → **Input:** Two events E_0 and E_1 , Inertial system label K
- \leftarrow **Output:** Time interval Δt_K between events as measured in K
 - 1. Find (unique) clock event $(t_0, \vec{x}_0)_K \in E_0$.
 - 2. Find (unique) clock event $(t_1, \vec{x}_1)_K \in E_1$.
 - 3. Return $\Delta t_K := t_1 t_0$.

Hopefully you agree that this is a reasonable definition of the duration (or time interval) between two events.

4 We can now apply this algorithm to determine the time elapsed between E_0 and E_1 :

In
$$A'$$
: $\Delta t' = \text{DURATION}(E_0, E_1; A') = t'_1 - t'_0$ Measured by a single clock! (2.11a)
In A : $\Delta t = \text{DURATION}(E_0, E_1; A) = t_1 - t_0$ Measured by two clocks! (2.11b)

- **5** | <u>How does Δt relate to $\Delta t'$?</u>
 - i | Since $(t'_0, \vec{x}')_{A'} \sim (t_0, \vec{x}_0)_A$ and $(t'_1, \vec{x}')_{A'} \sim (t_1, \vec{x}_1)_A$, we can use the Lorentz transformation to translate between the coordinates: Inverse of Eq. (1.77)

Remember that $\Lambda_{\vec{v}}^{-1} = \Lambda_{-\vec{v}}$ because of reciprocity; the inverse Lorentz transformation can then be obtained by substituting $v_x \mapsto -v_x$:

$$[E_0]_A = \begin{cases} ct_0 = \gamma \left(ct'_0 + \frac{v_x}{c} x' \right) \\ x_0 = \gamma \left(x' + v_x t'_0 \right) \end{cases} \text{ and } [E_1]_A = \begin{cases} ct_1 = \gamma \left(ct'_1 + \frac{v_x}{c} x' \right) \\ x_1 = \gamma \left(x' + v_x t'_1 \right) \end{cases}$$
(2.12)

We omit the other two coordinates since they are invariant anyway; the transformation of the spatial coordinate is also not necessary for the following derivation.

ii | Subtracting the equations for the time coordinate of both events yields:

$$c(t_1 - t_0) = \gamma c(t_1' - t_0')$$
(2.13)

Note that in the inverse Lorentz transformation Eq. (2.12) the position coordinate in A' is x' for *both* events because the *same* A'-clock takes part in both coincidences.

- iii | <u>** Time dilation:</u>
 - \rightarrow The moving clocks in A' run slower than the stationary clocks in A:

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \stackrel{v \neq 0}{>} \Delta t_0 \tag{2.14}$$

We renamed $\Delta t' \equiv \Delta t_0$ to emphasize the analogy to the *proper length* l_0 :

 Δt_0 : ** *Proper time* elapsed in A' between E_0 and E_1 Δt : Time elapsed in A between E_0 and E_1



- The characteristic feature of the *proper time* Δt₀ between two (time-like separated) events E₀ and E₁ is that it can be measured by a *single* inertial clock that takes part in both events. All other time intervals must be measured by subtracting the reading of *two different* clocks. Eq. (2.14) tells you that these time intervals are always longer than the proper time Δt₀.
- i! Due to isotropy, our result above is true for boosts in any direction.

Note that in the derivation above, we did *not* impose any special constraints on the positions of the clocks (except that they coincide pairwise at E_0 and E_1). In particular, we did not assume (despite the sketch suggesting this) that the clocks are located on the x/x'-axis. All clocks in A' are slowed down in the same way, irrespective of their location!

• This result does *not* contradict our assumption that all clocks are type-identical (= run with the same rate if put next to each other at rest) because the two events needed to compare the tick rate of moving clocks necessarily describe coincidences between *different* pairs of clocks.

6 | Relativity principle:

Because of the relativity principle **SR** time dilation must be completely *symmetrical*: The A'-clocks run slower compared to the A-clocks, and the A-clocks run slower compared to the A' clocks. That this is indeed that case (without being a clock "paradox") is best illustrated in a symmetric spacetime diagram:



The existence of the "median frame" A'' between $A \xrightarrow{v_x} A'$ can be easily shown with the addition for collinear velocities Eq. (1.70). This symmetric form of a spacetime diagram is sometimes called \uparrow Loedel diagram [44] and makes the symmetry between inertial frames manifest; in particular, the units on the axes of A and A' are identical (they are not identical to the units of A'', tough). In this symmetric form, the t'-axis is orthogonal to the x-axis and the t-axis to the x'-axis. Note that because of the relativistic addition of velocities, it is $A'' \xrightarrow{\tilde{v}_x} A'$ and $A'' \xrightarrow{-\tilde{v}_x} A$ with $\tilde{v}_x = v_x \frac{\gamma}{1+\gamma}$ and $\tan(\varphi) = \frac{\tilde{v}_x}{c}$ (\bigcirc Problemset 3). Only in the non-relativistic limit $v_x/c \to 0$ one finds $\tilde{v}_x = \frac{v_x}{2}$ as naïvely expected.

Note that due to the relativity of simultaneity, the two observers use *different* pairs of clock-events to decide which of the two origin clocks runs slower:

• For A the two clock events \tilde{D} and C are simultaneous such that one has to conclude that the (blue) A'-clock runs slower than the (red) A-clock.

• By contrast, for the observer A' the two events D and \tilde{C} are simultaneous such that one has to conclude that the (red) A-clock runs slower than the (blue) A'-clock.

It is evident from the diagram that there is no disagreement about coincidences of events (or readings of clocks). It is just the observer-dependent concept of simultaneity that leads to the seemingly "paradoxical" reciprocity of time dilation.

7 | Experiments:

• Muon decay [45]:

Muons quickly decay into electrons (and neutrinos):

$$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e \,.$$
 (2.15)

This decay can be readily observed in storage rings of particle colliders like CERN. The lifetime of muons *at rest* (measured by clocks in an inertial laboratory frame) is $\tau_{\mu}^{0} \approx 2.1948(10) \,\mu$ s. However, the lifetime of muons in flight (close to the speed of light) is measured to be $\tau_{\mu} \approx 64.368(29) \,\mu$ s, i.e., much longer! If one carefully takes into account the speed of the muons and additional experimental imperfections, this result fits Eq. (2.14) with deviations of only $\sim 0.1 \%$ [45].

Notes:

- In the rest frame of the flying *muons* one would measure the usual lifetime $\tau_{\mu}^{0} \approx 2.1948(10) \,\mu\text{s}$. However, in this frame, the *laboratory* is *Lorentz contracted* such that the muon reaches exactly the same point in space where it decays in this "shorter" lifetime. Note how time-dilation and Lorentz contraction provide different explanations for the same experimental obervation.
- One can also use different particle species to study time dilation, for example *pions* (a sort of meson, i.e., a hadron with one quark and one antiquark) [46].
- Hafele-Keating experiment [47, 48]:

In 1971, J.C. Hafele and R. E. Keating took four Cesium atomic clocks along commerical jet flights around the globe twice: once eastward and once westward. Compared to a reference clock on the ground, the clocks on the eastward flight lost on average $\sim 59 \text{ ns}$ (= they ran slower) and the clocks on the westward flight gained $\sim 273 \text{ ns}$ (= they ran faster). To understand this qualitatively, note that the reference clock on the ground is *rotating* (together with earth) and therefore is *not* an inertial clock. Therefore imagine an (approximately) inertial reference system flying along earth around the sun, and from this system look down on the north pole; earth is now slowly rotating beneath you. From this inertial system, the eastward flight has higher velocity than the reference clock, which, in turn, has higher velocity than the westward clock runs slower than the reference clock which runs slower than the westward clock (this is also true if the clocks are accelerated, \rightarrow *below*). These theoretical considerations are explained in [47].

2.3. Addition of velocities

Details: Problemset 2

 $1 \mid \triangleleft$ Particle moving with $\vec{u}' = \frac{d\vec{x}'}{dt'}$ in system K' and inertial system K with $K \xrightarrow{\vec{v}} K'$:





2 | Velocity \vec{u} in K:

$$\vec{u} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \equiv \vec{v} \oplus \vec{u}' \stackrel{\circ}{=} \frac{1}{1 + \frac{\vec{v} \cdot \vec{u}'}{c^2}} \left[\vec{v} + \frac{\vec{u}'}{\gamma_v} + \frac{\gamma_v}{c^2(1 + \gamma_v)} (\vec{u}' \cdot \vec{v}) \vec{v} \right]$$
(2.16)

Proof: Use Eq. (1.75) (● Problemset 2).

i! The relativistic addition of velocities \oplus is in general not commutative $(\vec{v} \oplus \vec{u} \neq \vec{u} \oplus \vec{v})$ nor associative $[\vec{v} \oplus (\vec{u} \oplus \vec{w}) \neq (\vec{v} \oplus \vec{u}) \oplus \vec{w}]$. As you can easily see from Eq. (2.16), it is also not linear: $(\lambda \vec{v}) \oplus (\lambda \vec{u}) \neq \lambda (\vec{v} \oplus \vec{u})$. Be careful: There are different notations (in particular: orderings) used in the literature.

3 | \triangleleft Non-relativistic limit ($c \rightarrow \infty \Rightarrow \gamma_v \rightarrow 1$):

$$\lim_{c \to \infty} \vec{v} \oplus \vec{u}' = \lim_{c \to \infty} \vec{u}' \oplus \vec{v} = \vec{v} + \vec{u}'$$
(2.17)

 \rightarrow Galilean addition of velocities

4 | Special case: $\vec{v} = (v_x, 0, 0)$:

$$u_{x} \stackrel{\circ}{=} \frac{v_{x} + u'_{x}}{1 + \frac{v_{x}u'_{x}}{c^{2}}}, \quad u_{y} \stackrel{\circ}{=} \frac{u'_{y}/\gamma_{v}}{1 + \frac{v_{x}u'_{x}}{c^{2}}}, \quad u_{z} \stackrel{\circ}{=} \frac{u'_{z}/\gamma_{v}}{1 + \frac{v_{x}u'_{x}}{c^{2}}}.$$
 (2.18)

i! Note that also the transverse components of \vec{u}' are modified, but in a different way than the collinear component u'_x . For $\vec{u}' = (u'_x, 0, 0)$ we get our previous result for collinear velocities Eq. (1.70) back.

5 | Thomas-Wigner rotation [49, 50]:

Remember that for *collinear* addition of velocities the concatenation of two boosts yields another boost: $\Lambda_{v_x} \Lambda_{u_x} = \Lambda_{w_x}$ [recall Eq. (1.57)].

As a straightforward (but tedious) calculation using two general boosts Eq. (1.75) shows, this is *not* true in general: $\Lambda_{\vec{v}}\Lambda_{\vec{u}} \neq \Lambda_{\vec{w}}$ with $\vec{w} = \vec{u} \oplus \vec{v}$. Rather one finds

$$\Lambda_{\vec{v}}\Lambda_{\vec{u}} = \Lambda_{\vec{u}\oplus\vec{v}}\Lambda_{R(\vec{u},\vec{v})} \tag{2.19}$$

with the * Thomas-Wigner rotation $R(\vec{u}, \vec{v}) \in SO(3)$ (we omit the explicit form of $R(\vec{u}, \vec{v})$ here).

This is not in contradiction with our general addition for velocities above because there we were only interested in the velocity of a moving particle (which you can identify with the origin of its rest frame K''); we completely ignored the *axes* of K''. The Thomas-Wigner rotation tells you that the concatenation of two *pure* boosts is *not* a pure boost in general.


2.4. Proper time and the twin "paradox"

1 | ⊲ Time-like trajectory $\mathcal{P} \subseteq \mathcal{E}$ of a spaceship with departure $D \in \mathcal{P}$ and arrival $A \in \mathcal{P}$. ⊲ Coordinate parametrization $\vec{x}(t)$ of \mathcal{P} in system K with

departure
$$[D]_K = (t_D, \vec{x}_D)$$
 and arrival $[A]_K = (t_A, \vec{x}_A)$: (2.20)

Formally, \mathcal{P} is a set of coincidence classes parametrized in K by the clock events $(t, \vec{x}(t))_K$:

$$\mathcal{P} = \{ [(t, \vec{x}(t))_K] \mid t \in [t_D, t_A] \} \subseteq \mathcal{E} .$$

$$(2.21)$$

This suggests the formal notation $[\mathcal{P}]_K = (t, \vec{x}(t)).$



2 | Thought experiment:

The spaceship takes a clock along and resets it to $\tau_D = \tau(t_D)$ at departure D.

What is the reading $\tau_A = \tau(t_A)$ of the clock at arrival *A*?

We assume that the clock in the spaceship is type-identical to the clocks used for inertial observers.

3 | Idea:

Approximate the trajectory by a *polygon* of N segments i = 1, ..., N separated by time steps t_i (with $t_0 := t_D$ and $t_N := t_A$):



i | Let $\Delta t_i := t_{i-1} - t_i$ and $\Delta \vec{x}_i := \vec{x}(t_{i-1}) - \vec{x}(t_i)$

For each segment, there is an inertial frame K' with a t'-axis that follows the spacetime segment (because all segments are time-like!). This is the instantaneous rest frame of the spaceship where the clock in the spaceship and the origin clock of K' are at the same place and at rest relative to each other. Since the clocks are type-identical, the time $\Delta \tau_i$ accumulated by the spaceship clock on this segment is identical to the time $\Delta t'_i$ elapsed for the origin



clock of K' on this segment: $\Delta \tau_i = \Delta t'_i$. This time is equal to the spacetime interval $(\Delta s'_i)^2 = (c \Delta t'_i)^2 - 0$ because the origin clock is at rest in K' (so that $\Delta \vec{x}'_i = \vec{0}$). But remember that the spacetime interval $(\Delta s'_i)^2$ is Lorentz invariant so that we can calculate *the same number* in any inertial system: $(\Delta s'_i)^2 = (\Delta s_i)^2 = (c \Delta t_i)^2 - (\Delta \vec{x}_i)^2$.

In summary, on the *i* th interval, the spaceship clock accumulates the time

$$\Delta \tau_i = \frac{\Delta s_i}{c} := \frac{\sqrt{\Delta s_i^2}}{c} = \frac{\sqrt{(c\Delta t_i)^2 - (\Delta \vec{x}_i)^2}}{c} = \Delta t_i \sqrt{1 - \frac{(\Delta \vec{x}_i / \Delta t_i)^2}{c^2}} \quad (2.22)$$

The above chain of arguments provided us with a *physical interpretation* for the Lorentz invariant spacetime interval $(\Delta s)^2 > 0$ of time-like separated events: It measures (up to a factor of c) the time accumulated by an inertial (= unaccelerated) clock that takes part in both events.

ii | Continuum limit $N \to \infty$ $(v(t) := |\vec{v}(t)| = |\vec{x}(t)|)$:

$$d\tau = \frac{ds}{c} = dt \sqrt{1 - \frac{\dot{x}(t)^2}{c^2}} \quad \Leftrightarrow \quad \frac{dt}{d\tau} = \gamma_{v(t)}$$
(2.23)

Note that this is just an infinitesimal version of the time-dilation formula Eq. (2.14) with $\Delta t \rightarrow dt$ and $\Delta t_0 \rightarrow d\tau$.

Since $(\Delta s)^2 = (\Delta s')^2$ is Lorentz invariant:

$$K \xrightarrow{\Lambda} K': dt \sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} = \frac{ds}{c} = \frac{ds'}{c} = dt' \sqrt{1 - \frac{\dot{\vec{x}}'(t')^2}{c^2}}$$
 (2.24)

You can check this also explicitly using the Lorentz transformation Eq. (1.75).



↓Lecture6 [21.11.23]

iii \rightarrow ** *Proper time* accumulated by the spaceship clock along the trajectory \mathcal{P} :

$$\Delta \tau[\mathcal{P}] = \lim_{N \to \infty} \sum_{\substack{\text{Segment}\\i=1}}^{N} \Delta \tau_i = \int_{\mathcal{P}} d\tau = \int_{\mathcal{P}} \frac{ds}{c} := \int_{t_D}^{t_A} dt \sqrt{1 - \frac{\dot{\dot{x}}(t)^2}{c^2}}$$
(2.25)

- As constructed, the proper time $\Delta \tau[\mathcal{P}]$ of a time-like trajectory \mathcal{P} , parametrized by $\vec{x}(t)$ for $t \in [t_0, t_1]$, is the time elapsed by a clock that follows this trajectory in spacetime.
- i! This result is valid for *accelerated* clocks.

In general, SPECIAL RELATIVITY *can* described the physics of accelerated objects as long as the descpription of the process is given in an inertial coordinate system (as is the case here).

- i! The right-most expression in Eq. (2.25) yields the same result *in all inertial systems* K [recall Eq. (2.24)]. This is why $\tau[\mathcal{P}]$ is a function of the *event trajectory* \mathcal{P} and not its coordinate parametrization $\vec{x}(t)$. This is important: It tells us that all inertial observers will agree on the reading of the spaceship clock τ_A at arrival A (although their *parametrization* $\vec{x}(t)$ may look different).
- Note that since $\vec{x}(t)$ is assumed to be time-like, it is $\forall_t : |\vec{x}(t)| < c$ such that the radicand is always non-negative.
- τ [•] is a *functional* of the trajectory \mathcal{P} ; this is why we use square-brackets.
- **4** Which trajectory \mathcal{P}^* between the two events D and A maximizes the proper time $\Delta \tau$?



i $\mid D$ and A are *time-like* separated $\rightarrow \exists$ Inertial system K' = K(D, A) with

$$[D]_{K'} = (t'_D = 0, \vec{x}'_D = \vec{0}) \text{ and } [A]_{K'} = (t'_A, \vec{x}'_A = \vec{0})$$
 (2.26)

That is, without loss of generality, we can Lorentz transform into an inertial system where the two events happen at the *same* location (and by translations we can assume that this location is the origin $\vec{0}$ and that the coordinate time is $t'_D = 0$ at D). We label the time and space coordinate in K' by t' and \vec{x}' . Because of the relativity principle **SR**, K' is as good as any system to describe events.

ii | Time of an arbitrary path $\mathcal{P} \ni D$, A with $[\mathcal{P}]_{K'} = (t', \vec{x}'(t'))$:

$$\Delta \tau[\mathcal{P}] = \int_{t'_D}^{t'_A} \mathrm{d}t' \sqrt{1 - \frac{\dot{\vec{x}}'(t')^2}{c^2}} \stackrel{\vec{x}'(t') = \vec{0}}{\leq} \int_{t'_D}^{t'_A} \mathrm{d}t' = t'_A - t'_D = \Delta \tau[\mathcal{P}^*] \quad (2.27)$$



Here \mathcal{P}^* is the trajectory between D and A that is parametrized by the constant function $\vec{x}'(t') \equiv \vec{0}$ in K'. In other inertial systems, this trajectory will not be constant; however, it is inertial, i.e., \mathcal{P}^* is described by a trajectory between D and A with uniform velocity.

Check this by applying a Lorentz transformation to the coordinates $(t', \vec{0})_{K'}$!

 \rightarrow Clocks that travel along the *inertial trajectory* \mathcal{P}^* between D and A collect the largest proper time $\tau^* = \Delta \tau [\mathcal{P}^*]$.

Collecting the "largest time" means that the these clocks run the *fastest*.

5 | It is important to let this result sink in:

Let K' be the rest frame of earth (which is located in the origin $\vec{0}$) and consider two twins of age τ_D :

- Twin S departs with a Spaceship at D, flies away from earth, turns around and returns to earth at A. Twin S therefore follows a trajectory similar to \mathcal{P}_2 in the sketches above.
- Twin E stays on Earth. He follows the inertial trajectory \mathcal{P}^* in the sketches above.

We just proved above:

(Age of **Twin S** at A) = $\Delta \tau[\mathcal{P}_2] + \tau_D < \Delta \tau[\mathcal{P}^*] + \tau_D =$ (Age of **Twin E** at A)

This is the famous ** Twin "paradox": Twin S aged less than Twin E.

- **6** Why there is *no* paradox:
 - If you don't see why the above result should be paradoxical:

Good! Move along. Nothing to see here! ©

- Why one *could* conclude that the above result is paradoxical (= logically inconsistent):
 - From the view of **Twin E**, **Twin S** speeds around quickly, thus time-dilation tells him that **Twin S** should age slower. And indeed, when **Twin S** returns, he actually didn't age as much.
 - Now, you conclude, due to the relativity principle **SR**, we could also take the perspective of **Twin S** (i.e., our system of reference is now attached to the spaceship). Then **Twin S** would conclude that time-dilation makes **Twin E** (who now, together with earth, speeds around quickly) age more slowly. But this does not match up with the above result that, when both twins meet again at *A*, **Twin S** is the younger one! *Paradox*!

The resultion is quite straightforwad:

The invocation of the relativity principle **SR** in the last point is not admissible! Remember that **SR** only makes claims about the equivalence of *inertial systems*. Now have a look at the trajectory \mathcal{P}_2 of the spaceship again: it is clearly accelerated and cannot be inertial. And that there *is* at least a period where the spaceship (and **Twin S**) is accelerating is a *neccessity* for **Twin S** to *return* to **Twin E** (at least in flat spacetimes, but not so in curved ones [51])! This implies that the reunion of both twins at A requires at least one of them to *not* stay in an inertial system. This breaks the symmetry between the two twins and explains why the result can be (and is) asymmetric.

ⁱ! For historical (and anthropocentric) reasons, the "twin paradox" is called a "paradox." We stick to this term because we have to – and not because it is appropriate name. The term "paradox" suggests an intrinsic inconsistency of RELATIVITY. As we explained above: *This is not the case*. All "paradoxes" in RELATIVITY are a consequence of unjustified, seemingly "intuitive" reasoning. The root cause is almost always an inappropriate, vague notion of "absolute simultaneity" that cannot be operationalized.



• An overview on different geometric approaches to rationalize the phenomenon can be found in Ref. [52].

Below are two widely used spacetime diagrams of an idealized version where **Twin S** changes inertial systems only once from S_D to S_A halfway through the journey at R. You can think of this as an instantenous acceleration at the kink. Note, however, that the acceleration itself is dynamically irrelevant for the arguments; it is only important that the inertial frames in which **Twin S** departs and returns are not the same:



- In the left diagram the slices of simultaneity in the two systems S_D and S_A are drawn. As predicted by time-dilation (and mandated by **SR**), **Twin S** observes the clocks of **Twin E** to run slower during his "inertial periods", i.e., while he stays in a single inertial system. However, the moment **Twin S** "jumps" from S_D to S_A at R, his notion of simultaneity changes instantaneously: In S_D , R and R_D are simultaneous; in S_A , however, R and R_A are simultaneous. Due to this jump, the record of **Twin S** contains now a temporal gap for events on earth (highlighted interval). It is this "missing" time interval that overcompensates the slower running clocks on earth (as observed from S_D and S_A) and makes **Twin S** conclude that **Twin E** ages faster (in agreement with the actual outcome of the experiment).

If you wonder what happened to the (missing) observations of events in the triangle $R_A R R_D$: there is a nice explanation in Schutz [5]. (The bottom line is that **Twin S** constructs a bad coordinate system by stopping the recording of events in system S_D when he reaches R.)

- In the right diagram, we draw *light signals* ("pings") of an earth-bound clock next to **Twin E** sent to **Twin S**. **Twin S** receives these signals and measures their period. This idealizes how **Twin S** sees (not observes!) the clocks ticking on earth (and, by proxy, how fast **Twin E** ages). It is important to understand the difference between this "seeing" and our operational definition of observing (using the contraption called an \leftarrow inertial system, as used in the left diagram). As demonstrated by the diagram, **Twin S** first sees the clock on earth ticking *slower*; but when he turns around at *R*, the clocks on earth (apparantly) speed up significantly. In the end, this speedup overcompensates for the slowdown during the first part of the journey so that **Twin S** again arrives at the (correct) conclusion that **Twin E** ages faster. Note that the speedup of the earth-bound clock seen by **Twin S** during the second half of his journey does not contradict time-dilation



because *seeing* is not *observing*. This is similar to the \uparrow *Penrose-Terrell effect* in that a genuine relativistic effect (here: time-dilation) is distorted by an additional "imaging effect" due to the finite speed of light.

- In our careful derivation above, we not only showed that **Twin S** ages less than **Twin E**; we also showed that this conclusion is *independent* of the inertial observer! Thus we know that there will be *no dispute* about the different ages between different inertial observers.
- The Hafele-Keating experiment [47, 48] and the muon decay experiments [45], mentioned previously in the context of time-dilation, are experimental confirmations of the twin "paradox." So our theoretical prediction above (that **Twin S** ages less than **Twin E**) is experimentally confirmed. End of discussion.
- Our derivation of the accumulated proper time along trajectories in spacetime is both mathematically sound and experimentally confirmed. This qualifies SPECIAL RELATIVITY as a successfull theory of physics. *Operationally* there is nothing to complain about: the theory does its job to produce quantiative predictions of real phenomena. So why do so many people (physicists included) – despite the various efforts to visualize the phenomenon – have this nagging feeling of dissatisfaction that they cannot get rid of? The reason, so I would argue, is the human brain and its proclivity to inject concepts of absolute simulateneity into its model building. This qualifies the historical overemphasis of the twin "paradox" as a *meta problem*: The question to study is not how to "solve" the twin "paradox" (as we showed above, there is nothing to solve); the question to study is why so many peoply thought (and still think) that there is a problem in the first place. This *meta problem* is an actual problem to study; but it falls into the domain of cognitive science, and not physics!
- **7** | Two lessons to be learned from this:

You can outlive your inertial-system-dwelling peers by changing inertial systems (= accelerating) at least once.

- i! You "live longer" when speeding around than your twin on earth, i.e., when you return, your twin might be 80 and have reached the end of his lifespan while you are still in your fourties. This is a real, observable effect, not an illusion of sorts. However, "living longer" does *not* mean that you somehow have "more time to spend" than your twin because all physical phenomena in your spaceship experience the same effect. It is not your metaboslism that slows down wrt. other physical phenomena around you, it is *time* itself. Put differently: If you and your twin both try to read as many books as possible during your lifetimes (say one per month), both of you will have read roughly the same amount of books when either of you dies (say at the age of 80).
- The mere fact that our universe *really* allows for this (at least in theory) makes it much more interesting than its boring alternative: a Galilean universe.

and

Phenomena like length contraction and the twin "paradox" are physically *real*. Their "paradoxical" flavor is a phenomenon of human cognition, not physics.

This is why we put "paradox" always in quotes in the context of RELATIVITY.



3. Mathematical Tools I: Tensor Calculus

In this chapter we introduce tensor calculus (\uparrow *Ricci calculus*) for general coordinate transformations φ (which will be useful both in SPECIAL RELATIVITY and GENERAL RELATIVITY). The coordinate transformations φ relevant for SPECIAL RELATIVITY are Lorentz transformations (and therefore linear) which simplifies expressions often significantly (\rightarrow *Chapter 4*). However, this special feature of coordinate transformations in SPECIAL RELATIVITY is not crucial for the discussions in this chapter.

Goal: Construct Lorentz covariant (form invariant) equations

(for mechanics, electrodynamics, quantum mechanics)

Question: How to do this systematically?

Note that (we suspect that) Maxwell equations *are* Lorentz covariant. Clearly this is not obvious and requires some work to prove; we say that the Lorentz covariance is *not manifest*: it is there, but it is hard to see. Conversely, without additional tools that make Lorentz covariance more obvious, it is borderline impossible to *construct* Lorentz covariant equations from scratch (which we must do for mechanics and quantum mechanics!).

We are therefore looking for a "toolkit" that provides us with elementary "building blocks" and a set of rules that can be used to construct Lorentz covariant equations. This toolbox is known as *tensor calculus* or \uparrow *Ricci calculus*; the "building blocks" are tensor fields and the rules for their combination are given by index contractions, covariant derivatives, etc. The rules are such that the expressions (equations) you can build with tensor fields are *guaranteed* to be Lorentz covariant. This implies in particular that if you can rewrite any given set of equations (like the Maxwell equations) in terms of these rules, you automatically show that the equations were Lorentz covariant all along. We then say that the Lorentz covariance is *manifest*: one glance at the equation is enough to check it.

Later, in GENERAL RELATIVITY, our goal will be to construct equations that are invariant under *arbitrary* (differentiable) coordinate transformations (not just global Lorentz transformations). Luckily, the formalism we introduce in this chapter is powerful enough to allow for the construction of such \rightarrow *general covariant* equations as well. This is why we keep the formalism in this chapter as general as possible, and specialize it to SPECIAL RELATIVITY in the next Chapter 4. The discussion below is therefore already a preparation for GENERAL RELATIVITY; it is based on Schröder [1] and complemented by Carroll [53].

3.1. Manifolds, charts and coordinate transformations

1 | D-dimensional Manifold

= Topological space that *locally* "looks like" *D*-dimensional Euclidean space \mathbb{R}^{D} :





- i! In RELATIVITY, the manifold of interest is the set of coincidence classes \mathcal{E} ; it makes up the D = 4-dimensional manifold we call *spacetime*.
- A space that "locally looks like ℝ^D" is formalized as a ↑ *topological space* that is locally ↑ *homeomorphic* to Euclidean space ℝ^D. The structure defined in this way is then called a ↑ *topological manifold*.

2 Differentiable Manifolds:

We want to formalize this idea and introduce additional structure to the manifold so that we can differentiate functions on it:



 $i \mid ** Coordinate system / Chart (U, u):$

$$u: \quad U \subseteq M \to u(U) \subseteq \mathbb{R}^D$$
 (3.1a)

$$u^{-1}: u(U) \subseteq \mathbb{R}^D \to U \subseteq M$$
 (3.1b)

 $U \subseteq M$: open subset of M; u and u^{-1} are continuous and $u \circ u^{-1} = 1$.

U = M is allowed. This is the situation we assumed so far in SPECIAL RELATIVITY: Our inertial coordinate systems cover all of spacetime $M = \mathcal{E}$.

ii | \triangleleft Two charts (U, u) and (V, v) and let $U \cap V \neq 0$:

$$\varphi := v \circ u^{-1} : u(U \cap V) \to v(U \cap V) \tag{3.2a}$$

$$\varphi^{-1} := u \circ v^{-1} : v(U \cap V) \to u(U \cap V)$$
(3.2b)

φ : ** Coordinate transformation / Transition map

U = M = V and $U \cap V = M$ is allowed. This is the situation we assume so far in SPECIAL RELATIVITY where $(U = \mathcal{E}, u)$ and $(V = \mathcal{E}, v)$ correspond to the coordinate systems of two different inertial systems. The coordinate transformation φ would then be a Lorentz transformation (defined on $U \cap V = \mathcal{E}$).

iii | Atlas := Family of charts $(U_i, u_i)_{i \in I}$ such that $M = \bigcup_{i \in I} U_i$

This definition of an atlas formalizes the notion of an atlas in real life (of the book variety): It contains many charts that, taken together, cover the complete manifold (typically earth). The different charts (on different pages of the book) all overlap on their edges such that you can draw any route on earth without gaps.

All φ, φ^{-1} differentiable $\rightarrow M$: ** *Differentiable Manifold*



- φ and φ^{-1} are maps from \mathbb{R}^D to itself. It is therefore clear what "differentiable" means.
- In mathematics one is of course more precise about the degree of differentiability of the transition functions, and subsequently assigns this degree to the manifold. For example, if all coordinate transformations are infinitely often differentiable (= smooth), the manifold is called a ↑ *smooth manifold*. We are sloppy in this regard: For us all functions are differentiable as often as we need them to be.

In RELATIVITY we will only be concerned with *differentiable* manifolds.

3 | Example:



 \rightarrow In general, a manifold *cannot* be covered by a single chart (Earth, mathematically S^2 , needs at least two charts). In SPECIAL RELATIVITY this is not a problem: There we assume that spacetime is a flat (pseudo-)Euclidean space $\mathcal{E} \simeq \mathbb{R}^4$ and the coordinates given by our inertial systems cover all of spacetime. Later, in GENERAL RELATIVITY, this will not necessarily be the case.

3.2. Scalars

- **4** | * Scalar (field) := Function $\phi : M \to \mathbb{R}/\mathbb{C}$
 - If ϕ maps to $\mathbb{R}(\mathbb{C})$, we call ϕ a real (complex) scalar field.
 - i! φ is a geometric object because it only depends on the manifold itself. It does not rely on charts/coordinates and does not depend on a particular set of charts you might choose to parametrize the manifold. The notion of a mathematical object to be "geometric in nature" or "independent of the choice of coordinates" is *absolutely crucial* for the understanding of GENERAL RELATIVITY. The reason why these "geometric objects" are so important for physics is the following insight that took physicists (including Einstein) a long time to fully comprehend and implement mathematically:

Coordinates (charts) do *not* represent physical entities. They are (useful) "mathematical auxiliary structures."

• One reason why it is so hard for us to grasp and implement the "physical irrelevance" of coordinates is, so I believe, that the first (and often only) coordinates we encounter in school are *Cartesian coordinates*. They are particularly intuitive because they are simply the *distances* of a point to some coordinate axes. Distances are a geometric property and physically relevant



(you can measure them with rods); they are not the invention of mathematicians. This makes students draw the (wrong) conclusion that coordinates in general have intrinsic physical meaning. The problem is that coordinates *are* inventions of mathematicians; they do not share the ontological status of physical quantities like lengths etc. To undo this misconception is key to understand GENERAL RELATIVITY (\rightarrow much later).

- Since both M and ℝ/ℂ are ↑ topological spaces, it makes sense to ask whether (or require that) φ is continuous. It does not make sense to ask whether φ is differentiable (and what is derivative is) because, in general, M does neither come with a notion of "distance" between two points in M nor can you add or subtract points (M does not have to be a ↓ metric space and/or a ↓ linear space). So an expression like ∂_pφ(p) does not make sense (→ below)!
- We just declared that coordinates are "not physical." The problem is that *without* coordinates it is really hard (at least for physicists) to do actual calculations with the geometric objects we are interested in (for example: compute derivatives). In addition, comparing theoretical predictions with experimental observations typically requires some sort of coordinate representation. Our *< inertial systems*, for example, are elaborate measurement devices that produce a specific coordinate representation of the observed events.

This is why we always assume in the following that we have one (or more) charts that allow us to parametrize a (part of the) manifold, and then express the geometric quantities as functions of these coordinates. This means for the scalar field:

 \triangleleft Two overlapping charts *u* and *v*:

$$\Phi(x) := \phi(u^{-1}(x)) \quad x \in u(U \cap V)$$
(3.3a)

$$\bar{\Phi}(\bar{x}) := \phi(v^{-1}(\bar{x})) \quad \bar{x} \in v(U \cap V)$$
(3.3b)

 Φ and $\overline{\Phi}$ are functions on (subsets of) \mathbb{R}^D ; in contrast to ϕ which is a function on the manifold M. In an abuse of notation, some authors do not make this distinction and write ϕ and $\overline{\phi}$ instead. $\stackrel{\circ}{\longrightarrow}$

$$\overline{\Phi}(\overline{x}) = \Phi(x)$$
 for $\overline{x} = \varphi(x)$ with $\varphi = v \circ u^{-1}$. (3.4)

Note that $\overline{\Phi}(\overline{x}) \stackrel{\text{def}}{=} \phi(p) \stackrel{\text{def}}{=} \Phi(x)$ with $u^{-1}(x) = p = v^{-1}(\overline{x})$.

- IN RELATIVITY we typically work in a particular chart (coordinate system). Thus we write our fields as functions of coordinates (and not points on the manifold); e.g., when working with scalars, we typically work with Φ (and not ϕ).
- i! The special transformation of a field Eq. (3.4) (given as function of coordinates) tells us that it actually encodes a geometric, chart-independent function ϕ (given as function of points on the manifold). This idea will be prevalent throughout this chapter and is the basis of our modern formulation of RELATIVITY: We work with functions that depend on specific coordinates (and therefore change when we transition to another chart); however, these functions satisfy certain transformation laws [like Eq. (3.4)] that guarantee that they actually encode geometric, chart-independent objects (which is what physics is about).
- As a function of coordinates, scalar fields are those fields the values of which do not change under coordinate transformations. A typical example would be the temperature as a function of position: When you move your coordinate system, the temperature of a particular point in space still is the same (only your coordinates of this particular point have changed!). This is exactly what Eq. (3.4) demands.



Note that being a scalar (field) does not simply mean "being a number." The z-component of the electric field strength $E_z(x)$, for example, assigns a number to every point x; however, it does *not* transform like Eq. (3.4) under coordinate transformations. (Do you see why? What happens to E_z if you rotate your coordinate system?)

- In the literature, you will find the notation Φ
 = Φ to characterize scalars. This does not mean Φ
 (x) = Φ(x) for all x ∈ ℝ^D (which characterizes form-invariance or functional equivalence), but rather Φ(x
) = Φ(x) (which characterizes scalar fields). Note that with x = φ⁻¹(x
) it follows Φ(x
) = Φ(φ⁻¹(x
)) such that the function Φ
 is typically not functionally equivalent to Φ. This ambiguity is the price we have to pay if we want to express geometric objects in terms of coordinates.
- Since Φ: ℝ^D → ℝ, it is well-defined what "differentiability" of Φ means. So expressions like ^{∂Φ(x)}/_{∂x^k} make sense now (if Φ is differentiable). One then defines that φ is differentiable on M iff Φ is differentiable for all charts of an atlas of M.

3.3. Covariant and contravariant vector fields

Are scalar fields the only geometric objects that can be defined on a manifold? The answer is *no*, there are many more! And these objects are not just toys for mathematicians: they are necessary to represent physical quantities like the electromagnetic field. Unfortunately, the definition of these quantities is not so straightforward as for scalars. We will not be mathematically precise in our discussion; however, it is important to understand the conceptual ideas:

6 | ** Tangent space $T_p M$ at $p \in M$

= Vector space of directional derivative operators with evaluation at $p \in M$ (=derivations)

These operators can be applied to differentiable functions on the manifold (i.e., scalar fields).



• The tangent space T_pM is the mathematical formalization of the intuitive concept of the plane ℝ² that you can attach tangentially at any point p of a two-dimensional manifold. The problem with this picture is that it only works if you *embed* the manifold M into a higher-dimensional Euclidean space. Mathematically, such an approach is not satisfying because it presupposes additional structure to characterize the manifold (which, as it turns out, is not needed). Physically, the approach is also problematic: The manifold we are interested in is all of spacetime *E*. But *E* is all there is, it is (to the best of our knowledge) not embedded into anything. It is therefore crucial that we can work with manifolds "stand alone", without assuming any embedding into a higher-dimensional space. The price we have to pay is that tangent vectors must be defined, rather abstractly, as directional derivative operators.



- There is a different tangent space $T_p M$ at every point $p \in M$; these vector spaces all have the same dimension D (like the manifold) and are therefore all isomorphic. However, without additional structure, there is no natural connection (isomorphism) between these different vector spaces at different points. The disjoint union of all tangent spaces is called \uparrow *tangent bundle TM*.
- Mathematically, the vectors in the tangent space can be defined as equivalence classes of smooth curves through p with the same derivative (with respect to their parametrization) at p. This equivalence class corresponds to a particular directional derivative that one can apply to smooth functions on the manifold at p. We do not need this abstract "bootstrapping procedure" for T_pM in the following.

 \triangleleft Chart (U, u) with coordinates $x = (x^0, x^1, \dots, x^D)$

 $\rightarrow *$ Coordinate basis $\{\partial_i \equiv \frac{\partial}{\partial x^i}\}$ for $T_p M$

Recall that partial derivatives are special kinds of directional derivatives (namely in the direction where you keep all but one coordinate fixed). You can therefore think of ∂_i as the tangent vector at $p \in M$ that points into the x^i -direction mapped by u^{-1} onto the manifold.



↓Lecture7 [28.11.23]

7 | Since $T_p M$ is a vector space for each point p of the manifold M, we can define *fields* on M that assign to each point p a tangent vector:

** Vector field: $A(p) = \sum_{i=1}^{D} A^{i}(x)\partial_{i}$ with x = u(p)

At every point $p \in M$ the vector field yields a tangent vector $A(p) = \sum_i A^i(u(p))\partial_i \in T_p M$.

- **8** $| \triangleleft \text{Coordinate transformation } \bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x})$
 - \rightarrow Chain rule:

$$\underbrace{\frac{\partial}{\partial \bar{x}^{i}}}_{\bar{\partial}_{i}} = \sum_{k=1}^{D} \frac{\partial x^{k}}{\partial \bar{x}^{i}} \underbrace{\frac{\partial}{\partial x^{k}}}_{\partial_{k}}$$
(3.5)

 \rightarrow For x = u(p) and $\bar{x} = v(p)$ this is a *basis change* on the tangent space $T_p M$ from one coordinate basis $\{\partial_i\}$ to another coordinate basis $\{\bar{\partial}_i\}$ via the (invertible) matrix $\frac{\partial x^k}{\partial \bar{x}^i}$:



9 $| \triangleleft$ Vector field A and expand it in different coordinate bases:

$$\sum_{i} A^{i}(x)\partial_{i} = A(p) = \sum_{i} \bar{A}^{i}(\bar{x})\bar{\partial}_{i}$$
(3.6)

with x = u(p) and $\bar{x} = v(p)$.

- i! The vector field A is a geometric object, just as the scalar field ϕ was. That it does *not* depend on the chosen chart is the statement of this equation.
- You learned this (with different notation and without the x/p-dependency) in your first course on linear algebra: Given a vector space V, a vector v ∈ V, and a basis {ei} with V = span {ei}, you can encode the vector in a basis-dependent set of numbers vi called components via linear combination: v = ∑i viei. The same vector can be encoded by different components v'_i in a different basis {ei'_i}: v = ∑i vie'_i. In our terminology, the vector v is a "geometric object" that does not depend on your choice of basis; only its components do. In this context, the gist of the story is that v represents something physical (like the velocity of a particle). The components v_i do so only indirectly because they depend on your choice of the basis {ei'_i} and this choice does not bear any physical meaning.

Eq.
$$(3.6) \rightarrow$$

$$A = \sum_{i} A^{i}(x)\partial_{i} \stackrel{!}{=} \sum_{i} \bar{A}^{i}(\bar{x})\bar{\partial}_{i} \stackrel{\text{Eq. (3.5)}}{=} \sum_{k} \underbrace{\left[\sum_{i} \frac{\partial x^{k}}{\partial \bar{x}^{i}} \bar{A}^{i}(\bar{x})\right]}_{\stackrel{!}{=} A^{k}(x)} \partial_{k}$$
(3.7)



This motivates the following definition (we replace $x \leftrightarrow \bar{x}$ and the indices $i \leftrightarrow k$):

10 $| \langle D$ -tuple $\{A^i(x)\}$ of fields (in some chart with coordinates x):

** Contravariant vector field
$$\{A^{i}(x)\}$$
 : \Leftrightarrow $\bar{A}^{i}(\bar{x}) = \sum_{k=1}^{D} \frac{\partial \bar{x}^{i}}{\partial x^{k}} A^{k}(x)$ (3.8)

Contravariant vector (field) \rightarrow Superscript indices!

This is a *convention* which relates syntax and semantics and is at the heart of *tensor calculus*. The idea is that whenever you are given a collection of fields $A^i(x)$, you immediately know that they transform like Eq. (3.8) under coordinate transformations. (Unfortunately, there are exceptions to this rule, e.g., the \rightarrow *Christoffel symbols*.)

- i! Not every *D*-tuple of fields transforms as Eq. (3.8). To deserve the name "contraviarant vector (field)," (and superscript indices) one has to check this transformation law explicitly!
- The rationale of Eq. (3.8) is the same as that of Eq. (3.4): Whenever we find a family of fields that transform under coordinate transformations as Eq. (3.8), we immediately know that together they encode a geometric, chart-independent object on the manifold that can be used to describe a physical quantity.
- **11** | (Counter)Examples:
 - \triangleleft Only *linear* coordinate transformations: $\bar{x} = \varphi(x) = \Lambda x$
 - \triangleleft Coordinate functions $X^i(x) := x^i$ as fields:

$$\underbrace{\bar{X}^{i}(\bar{x})}_{\bar{x}^{i}} = \sum_{k=1}^{D} \Lambda^{i}_{k} \underbrace{X^{k}(x)}_{x^{k}} = \sum_{k=1}^{D} \underbrace{\frac{\partial \bar{x}^{i}}{\partial x^{k}}}_{\Lambda^{i}_{k}} X^{k}(x)$$
(3.9)

 \rightarrow Coordinate functions are contravariant vectors for linear transition maps.

This is useful in SPECIAL RELATIVITY because there we only consider global Lorentz transformations (which are linear).

• $\triangleleft D$ scalar fields $\Phi^i(x)$ (i = 1, ..., D):

For general
$$\bar{x} = \varphi(x)$$
: $\bar{\Phi}^i(\bar{x}) = \Phi^i(x) \neq \sum_{k=1}^D \underbrace{\frac{\partial \bar{x}^i}{\partial x^k}}_{\neq \delta^i_k} \Phi^k(x)$ (3.10)

- $\rightarrow \{\Phi^{i}(x)\}$ are *not* components of a contravariant vector field.
 - You see: not every collection of D fields is a vector!
 - $i! \delta_k^i$ is the Kronecker symbol: $\delta_k^i = 1$ for i = k and $\delta_k^i = 0$ for $i \neq k$. The notation δ_{ik} is *not* used in tensor calculus (\rightarrow *later*).
- **12** | Reminder: \checkmark *Dual spaces*



i | Remember: Linear algebra

Consider the vector space $V = \mathbb{R}^D$ and a column vector $\vec{v} = (v_1, \ldots, v_D)^T \in V$ (a $1 \times D$ -matrix). Let $\vec{w}^T = (w_1, \ldots, w_D)$ be a row vector (a $D \times 1$ -matrix). We can then perform a matrix multiplication between the vectors and interpret it as a linear map \vec{w}^T acting on the vector \vec{v} and producing a number:

$$\vec{w}^T : \vec{v} \in V \mapsto \vec{w}^T \cdot \vec{v} = (w_1 \quad \dots \quad w_D) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_D \end{pmatrix} = \sum_i w_i v_i \in \mathbb{R} .$$
 (3.11)

In mathematical parlance \vec{w}^T is a *linear functional* on the vector space V. All linear functionals of this form make up another vector space V^* called the \downarrow *dual space* of V. You can think of V^* as the vector space of all D-dimensional *row* vectors and V as the vector space of all D-dimensional *column* vectors. The elements of the dual space are referred to as a \downarrow *covectors*.

ii | Remember: Quantum mechanics

In quantum mechanics, the state of a physical system is described by \checkmark *state vectors* in some Hilbert space \mathcal{H} (which is a special kind of vector space). Vectors in this space are written as \checkmark *kets*: $|\Psi\rangle \in \mathcal{H}$. You can produce a \checkmark *bra* $\langle \Psi | = |\Psi \rangle^{\dagger}$ by applying the complex transpose operator. As in the example above, the bra $\langle \Psi |$ is a covector from the dual space \mathcal{H}^* ; indeed, it acts as a linear functional on state vectors via the inner product of the Hilbert space:

$$\langle \Psi || \Phi \rangle := \langle \Psi | \Phi \rangle \in \mathbb{C}$$
 (3.12)

This is the gist of the famous ↓ *Dirac bra-ket notation*.

- iii | Hopefully these examples convinced you that the dual space is just as important and useful as the vector space itself.
 - \rightarrow Dual space of the tangent space $T_p M$?

Given a coordinate basis $\{\partial_i\} \in T_p M$ of a vector space, there is a standard way to define a basis of of the dual space $T_p^* M$:

 \downarrow Dual basis {dxⁱ} with

$$dx^{i}(\partial_{j}) := \delta^{i}_{j} = \frac{\partial x^{i}}{\partial x^{j}}$$
(3.13)

- $\rightarrow \{dx^i\}$ is a basis of the ** Cotangent space T_p^*M
- T_p^*M is the dual space of T_pM ; it is common to write T_p^*M and not $(T_pM)^*$.
- **13** | Since T_p^*M is just another vector space for each point p of the manifold M, we can again define *fields* on M that map into this space:

** Covector field:
$$B(p) = \sum_{i=1}^{D} B_i(x) dx^i$$
 with $x = u(p)$

14 | Just like the coordinate basis, the dual coordinate basis depends on the chart and changes under coordinate transformations:

 \triangleleft Coordinate transformation $\bar{x} = \varphi(x)$:

$$\mathrm{d}\bar{x}^{i} = \sum_{k=1}^{D} \frac{\partial \bar{x}^{i}}{\partial x^{k}} \,\mathrm{d}x^{k} \tag{3.14}$$



• Check that this is the correct transformation for the dual coordinate basis:

$$d\bar{x}^{i}(\bar{\partial}_{j}) = \left[\sum_{k} \frac{\partial \bar{x}^{i}}{\partial x^{k}} dx^{k}\right] \left(\sum_{l} \frac{\partial x^{l}}{\partial \bar{x}^{j}} \partial_{l}\right)$$
$$= \sum_{k,l} \frac{\partial \bar{x}^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \bar{x}^{j}} \underbrace{dx^{k}(\partial_{l})}_{\delta_{l}^{k}} = \sum_{k} \frac{\partial \bar{x}^{i}}{\partial x^{k}} \frac{\partial x^{k}}{\partial \bar{x}^{j}} = \delta_{j}^{i} \quad \textcircled{o} \qquad (3.15)$$

• You might recognize Eq. (3.14): This is simply the rule to compute the \checkmark total differential of the function $\bar{x} = \varphi(x)$. This is no coincidence and explains why we use the differential notation dx^i for the dual vectors: The objects dx^i that we physicists like to illustrate as "infinitesimal shifts" in x^i are actually *linear functionals* (\uparrow 1-forms).

15 | Now we can play the same game on T_p^*M as before on T_pM :

 \triangleleft Covector field *B* and expand it in different dual coordinate bases:

$$\sum_{i} B_i(x) \mathrm{d}x^i = B(p) = \sum_{i} \bar{B}_i(\bar{x}) \mathrm{d}\bar{x}^i$$
(3.16)

with x = u(p) and $\bar{x} = v(p)$.

i! The covector field B is another geometric object, just as the vector field A was. That it does *not* depend on the chosen chart is the statement of this equation. For $(3.16) \rightarrow$

Eq.
$$(3.16) \rightarrow$$

$$B = \sum_{i} B_{i}(x) dx^{i} \stackrel{!}{=} \sum_{i} \bar{B}_{i}(\bar{x}) d\bar{x}^{i} \stackrel{\text{Eq.}(3.14)}{=} \sum_{k} \underbrace{\left[\sum_{i} \frac{\partial \bar{x}^{i}}{\partial x^{k}} \bar{B}_{i}(\bar{x})\right]}_{\stackrel{!}{=} B_{k}(x)} dx^{k}$$
(3.17)

This motivates the following definition (we replace $x \leftrightarrow \bar{x}$ and the indices $i \leftrightarrow k$):

16 $| \triangleleft D$ -tuple $\{B_i(x)\}$ of fields (in some chart with coordinates x):

** Covariant vector field
$$\{B_i(x)\}$$
 : \Leftrightarrow $\bar{B}_i(\bar{x}) = \sum_{k=1}^{D} \frac{\partial x^k}{\partial \bar{x}^i} B_k(x)$ (3.18)

Covariant vector (field) \rightarrow Subscript indices!

The rationale of Eq. (3.18) is the same as that of Eq. (3.8): Whenever we find a family of fields that transform under coordinate transformations as Eq. (3.18), we immediately know that together they encode a geometric, chart-independent object on the manifold that can be used to describe a physical quantity. To indicate that this object is a *covariant* vector field, we use *subscript* indices.

17 | Example:

First, let us introduce an even shorter notation for partial derivatives: $\Phi_{,i} \equiv \partial_i \Phi$

Following our index convention, the lower index in these expressions is only warranted if the field transforms as a covariant vector field according to Eq. (3.18). Let us check this:

$$\bar{\Phi}_{,i}(\bar{x}) = \bar{\partial}_i \bar{\Phi}(\bar{x}) \stackrel{\text{Eq. (3.4)}}{=} \sum_{k=1}^{D} \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial \Phi(x)}{\partial x^k} = \sum_{k=1}^{D} \frac{\partial x^k}{\partial \bar{x}^i} \Phi_{,k}(x)$$
(3.19)



 \rightarrow The gradient of a scalar is a covariant vector field.

18 What happens if we apply a covector field on a vector field at each point $p \in M$?

$$\phi(p) := B(p)A(p) = \sum_{i,j} B_i(x)A^j(x) \underbrace{dx^i(\partial_j)}_{\delta_i^i} = \sum_i A^i(x)B_i(x) =: \Phi(x) \quad (3.20)$$

 $\rightarrow \Phi(x)$ must be a scalar!

This is a good point to introduce a new (and very convenient) notation:

*** Einstein sum convention:*

$$\sum_{i=1}^{D} A^{i}(x)B_{i}(x) \equiv \underbrace{A^{i}(x)B_{i}(x)}_{\substack{** \text{ Einstein summation} \\ \text{ # Contraction}}} = A^{l}(x)B_{l}(x)$$
(3.21)

The *Einstein sum convention* or *Einstein summation* is a syntactic convention according to which a sum is automatically implied (but not written) whenever two indices show up twice in an expression and one is up (contravariant) and one down (covariant). Note that such indices are "dummy indices" in the sense that you can rename them to whatever you want (as long as you do not use the same letter for other indices already!). The sum over one co- and one contravariant index is called a *contraction*.

With this new notation it is straightforward to check that Φ transforms according to Eq. (3.4) by using the transformations Eq. (3.8) and Eq. (3.18):

$$\bar{\Phi}(\bar{x}) = \bar{A}^{i}(\bar{x})\bar{B}_{i}(\bar{x}) = \left[\frac{\partial\bar{x}^{i}}{\partial x^{k}}A^{k}(x)\right] \left[\frac{\partial x^{l}}{\partial\bar{x}^{i}}B_{l}(x)\right]$$
(3.22a)

$$= \underbrace{\frac{\partial \bar{x}^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \bar{x}^{i}}}_{\text{Chain rule} \to \delta_{k}^{l}} A^{k}(x)B_{l}(x) = A^{l}(x)B_{l}(x) = \Phi(x)$$
(3.22b)

The intermediate expression contains *three* sums over the colored indices (which we don't write)!

\rightarrow The contraction of a contra- and a covariant vector field yields a scalar field.

- **19** | <u>Note on nomenclature:</u>
 - If you compare Eq. (3.18) with Eq. (3.5) you find that the *components* B_i of a covector field transform like the *basis vectors* ∂_i of the tangent space. We say the components *covary* ("vary together") with the basis. This is why they are called *covariant*.
 - A comparison of Eq. (3.8) and Eq. (3.14) shows that the components Aⁱ of a vector field transform like the basis dxⁱ of the cotangent space which is the *inverse* ("opposite") transformation as for the basis of the tangent space ∂_i. Thus we say the components Aⁱ contravary ("vary opposite to") the basis ∂_i. This is why they are called contravariant.

3.4. Higher-rank tensors

You learned in your linear algebra course that two vector spaces V and W can be used to construct a new vector space $V \otimes W$ called the \checkmark *tensor product*. This allows us to generalize the notion of contra- and covariant *vector* fields to *tensor* fields, all of which are geometric, chart-independent objects defined on the manifold that are needed to describe physical quantities:



20 | An ** (absolute) (p, q)-tensor (field) T of rank r = p + q

$$T^{i_{1}i_{2}...i_{p}}_{j_{1}j_{2}...j_{q}} \equiv T^{i_{1}i_{2}...i_{p}}_{j_{1}j_{2}...j_{q}}(x) \text{ or } T^{I}_{J} \equiv T^{I}_{J}(x), \qquad (3.23)$$

with \checkmark multi-indices $I = (i_1 \dots i_p)$ and $J = (j_1 \dots j_q)$,

transforms like the tensor product of p contravariant and q covariant vector fields:

$$\underbrace{\bar{T}^{i_{1}\dots i_{p}}_{j_{1}\dots j_{q}}(\bar{x})}_{=:\frac{\partial \bar{x}^{i_{1}}}{\partial x^{m_{1}}}\cdots\frac{\partial \bar{x}^{i_{p}}}{\partial x^{m_{p}}}\right]}_{=:\frac{\partial \bar{x}^{I}}{\partial x^{M}}}\underbrace{\left[\frac{\partial x^{n_{1}}}{\partial \bar{x}^{j_{1}}}\cdots\frac{\partial x^{n_{q}}}{\partial \bar{x}^{j_{q}}}\right]}_{=:\frac{\partial x^{N}}{\partial \bar{x}^{J}}}\underbrace{T^{m_{1}\dots m_{p}}_{n_{1}\dots n_{q}}(x)}_{=T^{M}_{N}(x)}$$
(3.24)

There are r = p + q sums in this transformation rule (Einstein summation!).

- i! It is important that we do *not* write contra- and covariant indices above each other like so: T_j^i (at least not with additional knowledge about the tensor). This will become important below.
- Henceforth we always encode tensor fields by their chart-dependent *components*. The actual tensor field is of course chart-independent and maps each point $p \in M$ to an element of the tensor product

$$\underbrace{T_p M \otimes \cdots \otimes T_p M}_{p \text{ factors}} \otimes \underbrace{T_p^* M \otimes \cdots \otimes T_p^* M}_{q \text{ factors}} .$$
(3.25)

like so

$$T(p) = \sum_{I,J} T^{i_1 \dots i_p}_{j_1 \dots j_q} (x) \,\partial_{i_1} \otimes \dots \otimes \partial_{i_p} \otimes \mathrm{d} x^{j_1} \otimes \dots \otimes \mathrm{d} x^{j_q} \,. \tag{3.26}$$

 Note that while tensors (more precisely: tensor components) are indicated by upper and lower indices (corresponding to their rank), not every object that is conventionally written with upper and lower indices does encode a tensor. For example, the transformation matrices ^{3xi}/_{3xm}, which describe a basis change on T^{*}_pM, do not encode a tensor field.

21 | Examples:

Scalar
$$\Phi(x) \rightarrow (0, 0)$$
-tensor
Contravariant vector $A^i(x) \rightarrow (1, 0)$ -tensor
Covariant vector $B_i(x) \rightarrow (0, 1)$ -tensor
Tensor product $T^i{}_j(x) := A^i(x)B_j(x) \rightarrow (1, 1)$ -tensor (Check this!)

22 | Properties:

• Equality:

$$A = B \quad :\Leftrightarrow \quad \forall_{i_1 \dots i_p} \forall_{j_1 \dots j_q} : A^{i_1 \dots i_p}_{j_1 \dots j_q} = B^{i_1 \dots i_p}_{j_1 \dots j_q} \qquad (3.27)$$

• Symmetry:

$$T \text{ (anti-)symmetric in } k \text{ and } l \quad :\Leftrightarrow \quad T^{\dots k \dots l \dots} = (-) T^{\dots l \dots k \dots}$$
(3.28)



Every contra- or covariant rank-2 tensor can be decomposed into a sum of symmetric and antisymmetric tensors:

$$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{=:T_{(ij)}} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{=:T_{(ij)}} = T_{(ij)} + T_{[ij]}.$$
(3.29)

23 | Constructing tensors:

New tensors can be constructed from known tensors as follows (Proofs:) Problemset 4):

• Sum of (p,q)-tensors A and B yields (p,q)-tensor C:

$$C^{i_1...i_p}_{j_1...j_q} := A^{i_1...i_p}_{j_1...j_q} + B^{i_1...i_p}_{j_1...j_q}$$
(3.30a)

or
$$C_J^I := A_J^I + B_J^I$$
 (3.30b)

• Product of (p,q)-tensor A and scalar Φ yields (p,q)-tensor C:

$$C^{I}{}_{J} := \Phi A^{I}{}_{J} \tag{3.31}$$

• Tensor product of (p,q)-tensor A and (r,s)-tensor B yields (p+r,q+s)-tensor C:

$$C^{IK}{}_{JL} := A^I{}_J \cdot B^K{}_L \tag{3.32}$$

• Contractions:

Summing over a pair of contra- and covariant indices yields a tensor of rank (p - 1, q - 1):

$$\tilde{A}^{i_1\dots\bullet\dots i_p}_{j_1\dots\bullet\dots j_q} := A^{i_1\dots k\dots i_p}_{j_1\dots k\dots j_q}$$
(3.33)

The • indicates that the index summed over on the right side is missing in the list.

Proof: OProblemset 4

A special case of a contraction (in combination with a tensor product) is the scalar obtained from a contra- and a covariant vector field above:

$$\Phi = C^i_{\ i} = A^i B_i . \tag{3.34}$$

• Quotient theorem:

$$AB = C$$
 tensor for all tensors $B \implies A$ is tensor (3.35)

Here, $\stackrel{A}{AB}$ denotes (potentially multiple) contractions between indices of A and B (but not within A and B).

As an example, rewrite an arbitrary contravariant vector Aⁱ as Aⁱ = δⁱ_j A^j with Kronecker symbol δⁱ_j. The above theorem then implies that δⁱ_j transforms as a (1, 1)-tensor (verify this using the definition!). Hence we actually should write δⁱ_j instead of δⁱ_j. However, because the Kronecker symbol is symmetric in its indices, this simplified notation is allowed (*→ later*).



- Special case:

$$A_{ik}B^k = C_i$$
 covector for all vectors $B^k \Rightarrow A_{ik}$ is (0, 2)-tensor (3.36)

Proof:
Problemset 4

24 | <u>Relative tensors:</u>

i | Relative tensor are a generalization of the (absolute) tensors defined above. This generalization is useful because most of the rules for computing with tensors discussed so far carry over to relative tensors.

A * relative tensor of weight $w \in \mathbb{Z}$ picks up an additional power w of the \checkmark Jacobian determinant under coordinate transformations:

$$\bar{R}^{I}{}_{J}(\bar{x}) = \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{w} \frac{\partial \bar{x}^{I}}{\partial x^{M}} \frac{\partial x^{N}}{\partial \bar{x}^{J}} R^{M}{}_{N}(x) \quad \text{with weight } w \in \mathbb{Z}$$
(3.37)

and Jacobian determinant

$$\det\left(\frac{\partial x}{\partial \bar{x}}\right) := \sum_{\sigma \in S_D} (-1)^{\sigma} \prod_{i=1}^D \frac{\partial x^i}{\partial \bar{x}^{\sigma_j}} \,. \tag{3.38}$$

Here S_D is the group of permutations σ on D elements.

Since $\bar{x} = \varphi(x)$ is invertible, $x = \varphi^{-1}(\bar{x})$, it is $\frac{\partial \bar{x}}{\partial x} = \left(\frac{\partial x}{\partial \bar{x}}\right)^{-1}$ and therefore det $\left(\frac{\partial \bar{x}}{\partial x}\right) = \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{-1}$.

- ii | Examples:
 - (Absolute) tensors \equiv Relative tensors of weight w = 0
 - Volume form: Relative tensor of weight w = -1:

$$d^{D}\bar{x} = d^{D}x \det\left(\frac{\partial \bar{x}}{\partial x}\right) = d^{D}x \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{-1}$$
(3.39)

Remember the rule for integration by substitution with multiple variables!

• *** Tensor density* $\mathcal{L}(x) :=$ Relative tensor of weight $w = +1 \rightarrow$

$$S = \int \underbrace{\mathrm{d}^{D} x \mathcal{L}(x)}_{\text{Absolute tensor}} = \int \mathrm{d}^{D} \bar{x} \bar{\mathcal{L}}(\bar{x})$$
(3.40)

In this example, we assume that $\mathcal{L}(x)$ is a *scalar* tensor density such that its integral is a (absolute) *scalar* quantity.

In \uparrow relativistic field theories (like electrodynamics), the Lagrangian density $\mathcal{L}(x)$ is a scalar tensor density such that the \checkmark action S becomes a scalar.

• Let $i_1, i_2, \ldots, i_D \in \{1, 2, \ldots, D\}$ and define the * *Levi-Civita symbol* as

 $\varepsilon^{I} \equiv \varepsilon^{i_{1}i_{2}...i_{D}} := \begin{cases} +1 & I \text{ even permutation of } 1, 2, ..., D \\ -1 & I \text{ odd permutation of } 1, 2, ..., D \\ 0 & (\text{at least}) \text{ two indices equal} \end{cases}$ (3.41)



An even (odd) permutation of 1, 2, ..., D is constructed by an even (odd) number of *transpositions* (= exchanges of only two indices).

 $\stackrel{\circ}{\rightarrow}$

$$\bar{\varepsilon}^{I} = \varepsilon^{I} \stackrel{\circ}{=} \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{+1} \frac{\partial \bar{x}^{I}}{\partial x^{J}} \varepsilon^{J}$$
(3.42)

- $\rightarrow \varepsilon^{I} = \varepsilon^{i_{1}i_{2}...i_{D}}$ is a (D, 0)-tensor density
 - i! ε^I = ε^I is true by definition: ε is a symbol defined by Eq. (3.41); this definition is independent of the coordinate system. In Eq. (3.42) we compare this trivial transformation with that of a (relative) tensor and conclude that it is equivalent to the statement that ε^I transforms as a (D, 0)-tensor density with weight w = +1. This knowledge is helpful in tensor calculus to construct covariant expressions that contain Levi-Civita symbols (→ below).
 - To show this, note that the Levi-Civita symbol can be used to compute determinants:

$$\det\left(\frac{\partial \bar{x}}{\partial x}\right) = \sum_{\sigma \in S_D} (-1)^{\sigma} \prod_{i=1}^D \frac{\partial \bar{x}^i}{\partial x^{\sigma_j}} = \frac{\partial \bar{x}^1}{\partial x^{j_1}} \cdots \frac{\partial \bar{x}^D}{\partial x^{j_D}} \varepsilon^{j_1 \dots j_D} .$$
(3.43)

Details: SProblemset 4



↓Lecture8 [05.12.23]

3.5. The metric tensor

A differentiable manifold M does not automatically allow us to measure the length of curves, the angles of intersecting lines, or the area/volume of subsets of the manifold; to do so, we need a *metric* on M (which is an additional piece of information). While the continuity structure (an atlas) that comes with M determines its *topology*, the metric determines its *geometry* (= shape). The same manifold M can be equipped with *different* metrics; this corresponds to different geometries of the same topology (a potato and an egg both have the topology of a sphere, nonetheless they are geometrically distinct).

A differentiable manifold together with a (pseudo-)metric is called \uparrow (pseudo-)Riemannian manifold. In SPE-CIAL RELATIVITY and GENERAL RELATIVITY, spacetime is modeled by such (pseudo-)Riemannian manifolds where the metric is used to represent spatial and temporal distances between events.

25 | <u>Motivation:</u>

On linear spaces V, it is convenient to define an \checkmark *inner product* (like in quantum mechanics where you consider Hilbert spaces and use their inner product to compute probabilities and transition amplitudes).

Recall the definition of a (real) inner product:

$$\langle \bullet | \bullet \rangle : V \times V \to \mathbb{R}$$
 with ... (3.44a)

Symmetry:
$$\langle x|y \rangle = \langle y|x \rangle$$
 (3.44b)

(Bi)linearity: $\langle ax + by | z \rangle = a \langle x | z \rangle + b \langle y | z \rangle$ (3.44c)

Positive-definiteness:
$$x \neq 0 \Rightarrow \langle x | x \rangle > 0$$
 (3.44d)

Once you have an inner product, you get a norm, and subsequently a metric for free:

$$\underbrace{\langle x|y \rangle}_{\text{Inner product}} \Rightarrow \underbrace{\|x\| := \sqrt{\langle x|x \rangle}}_{\text{Norm}} \Rightarrow \underbrace{d(x, y) := \|x - y\|}_{\text{Metric}}$$
(3.45)

Thus an inner product is a rather versatile structure and nice to have!

Problem: We cannot define a inner product on the manifold directly because M is not a linear space.

However: We can introduce an inner product on each of its tangent spaces $T_p M! \rightarrow$

26 | A *Riemannian (Pseudo-)Metric* ds² := Symmetric, non-degenerate (0, 2)-tensor field:

$$ds^{2}: M \ni p \mapsto \underbrace{\left(ds_{p}^{2}: T_{p}M \times T_{p}M \to \mathbb{R}\right)}_{\text{Bilinear & symmetric & non-degenerate}}$$
(3.46a)

$$ds_{p}^{2} \text{ bilinear } \Rightarrow ds_{p}^{2} \in T_{p}^{*}M \otimes T_{p}^{*}M$$

$$\Rightarrow ds_{p}^{2} = \sum_{i,j=1}^{D} g_{ij}(x) dx^{i} \otimes dx^{j} \equiv g_{ij}(x) dx^{i} dx^{j}$$
(3.46b)
with $g_{ij} = g_{ji}$ (symmetry) and $g = \det(g_{ij}) \neq 0$ (non-degeneracy).



• The tensor product is *non-commutative*: $dx^i \otimes dx^j \neq dx^j \otimes dx^i$. However, you can always decompose a tensor product as

$$dx^{i} \otimes dx^{j} = \underbrace{\frac{1}{2}(dx^{i} \otimes dx^{j} + dx^{j} \otimes dx^{i})}_{=:dx^{i} \vee dx^{j}} + \underbrace{\frac{1}{2}(dx^{i} \otimes dx^{j} - dx^{j} \otimes dx^{i})}_{=:dx^{i} \wedge dx^{j}}$$
(3.47)

with the symmetrized tensor product $dx^i \vee dx^j$ and the anti-symmetrized tensor product $dx^i \wedge dx^j$ (\uparrow medge product).

Since g_{ij} is assumed to be symmetric, only the symmetric component survives:

$$g_{ii}(x)dx^{i} \otimes dx^{j} = g_{ii}(x)dx^{i} \vee dx^{j} \equiv g_{ii}(x)dx^{i}dx^{j}$$
(3.48)

This means that when writing $dx^i dx^j$ in the above formula, you can be sloppy and either mean $dx^i \otimes dx^j$ or, equivalently, $dx^i \vee dx^j$. You will find both conventions in the literature. I will use $dx^i dx^j \equiv dx^i \vee dx^j$ so that $dx^i dx^j = dx^j dx^i$.

- It would be more appropriate to write $g = g_{ij} dx^i dx^j$ for the metric (0, 2)-tensor; it is conventional, however, to reserve g for the determinant det (g_{ij}) so that we are stuck with ds^2 for the metric. Note that the d in ds^2 does *not* refer to an \uparrow *exterior derivative*, it is purely symbolical.
- To define a proper ↓ *inner product* on T_pM, we should demand ↓ *positive-definiteness* instead of *non-degeneracy*. This, however, is often (for example in RELATIVITY) too restrictive; as it turns out, non-degeneracy is all we need for an isomorphism between T_pM and T^{*}_pM ("pulling indices up and down", → *below*). This is why *negative* eigenvalues of g_{ij} are fine for many purposes, and motivates the concept of a → *signature*:

27 | Signature:

Since $g_{ij}(x) = g_{ji}(x)$ and $det(g_{ij}(x)) \neq 0$

 $\rightarrow g_{ii}(x)$ has r positive and s negative real eigenvalues for all $p \in M$

Since $det(g_{ij}(x)) \neq 0$, these numbers must be the *same* for all $p \in M$.

 \rightarrow (r, s): ** Signature of the metric ds²

This classification does not depend on the coordinate basis (*Sylvester's law of inertia*).

• (r > 0, s = 0)

 \rightarrow ds²: Riemannian metric \rightarrow (M, ds²): ** Riemannian manifold

I.e., g_{ij} has only positive eigenvalues for all $p \in M$ and is therefore \checkmark *positive-definite*. This produces a true, positive-definite inner product on $T_p M$.

• (r > 0, s > 0)

 \rightarrow ds²: pseudo-Riemannian metric \rightarrow (M, ds²): ** pseudo-Riemannian manifold

I.e., g_{ij} has both positive and negative eigenvalues and is therefore \downarrow *indefinite*.

- (r > 0, s = 1) or (r = 1, s > 0):

 \rightarrow ds²: Lorentzian metric \rightarrow (M, ds²): ** Lorentzian manifold

In RELATIVITY we are only interested in metric tensors with one positive and three negative eigenvalues (equivalently: three positive and one negative eigenvalue). Mathematically speaking, spacetime is then a four-dimensional Lorentzian manifold and a special case of a pseudo-Riemannian manifold.

28 | Example: (Details:) Problemset 4)

i $| \triangleleft D = 2$ Euclidean space $E_2 \equiv (\mathbb{R}^2, ds_E^2)$ The Euclidean metric in Cartesian coordinates $x^1 = x$ and $x^2 = y$ reads:

$$ds_E^2 := dx^2 + dy^2 = g_{ij}(x) dx^i dx^j \quad \text{with} \quad (g_{ij}) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Signature}}.$$
 (3.49)

This is consistent with the notion of dx and dy as infinitesimal shifts in coordinates and ds^2 as the infinitesimal distance (squared) that corresponds to this shift:



ii | We can now transition to a new chart, namely polar coordinates $\bar{x}^1 = r$ and $\bar{x}^2 = \theta$. The induced basis change on the cotangent space is given by the total differential of the coordinate functions Eq. (3.14):

$$\varphi^{-1}:\begin{cases} x = r\cos(\theta) & \text{Eq. (3.14)} & dx = \cos(\theta) \, dr - r\sin(\theta) \, d\theta \\ y = r\sin(\theta) & \Rightarrow & dy = \sin(\theta) \, dr + r\cos(\theta) \, d\theta \end{cases}$$
(3.50)

iii | We find the components of the metric tensor field in the new basis $\{d\bar{x}^1 = dr, d\bar{x}^2 = d\theta\}$:

$$\mathrm{d}s^{2} \stackrel{\circ}{=} \mathrm{d}r^{2} + r^{2}\mathrm{d}\theta^{2} = \bar{g}_{ij}(\bar{x})\,\mathrm{d}\bar{x}^{i}\,\mathrm{d}\bar{x}^{j} \quad \text{with} \quad (\bar{g}_{ij}) = \underbrace{\begin{pmatrix} 1 & 0\\ 0 & r^{2} \end{pmatrix}}_{\substack{\mathrm{Signature}\\(2,0)}}.$$
 (3.51)

This expression is again compatible with infinitesimal shifts in the (new) coordinates r and θ :



• The Euclidean plane E_2 is therefore an example for a Riemannian manifold with metric signature (2, 0); its distinctive feature is that it is *flat*.



 Note that here we compute *the same* infinitesimal length in different coordinates (with the same result)! We did not change the *metric*, only the *coordinates* and thereby the coordinate basis in which we express the metric tensor. This is *flat* Euclidean space in *curvilinear coordinates*. By contrast, later in GENERAL RELATIVITY we will study curved (non-flat, non-Euclidean) metric tensors, i.e., we will modify the geometry of space(time) itself.

29 | Since the metric ds^2 is a (0, 2)-tensor field:

$$\bar{g}_{ij}(\bar{x}) \mathrm{d}\bar{x}^i \mathrm{d}\bar{x}^j = \mathrm{d}s^2 = g_{ij}(x) \mathrm{d}x^i \mathrm{d}x^j$$
 (3.52)

Eq. (3.14) $\xrightarrow{\circ}$

$$\bar{g}_{ij}(\bar{x}) = \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial x^m}{\partial \bar{x}^j} g_{lm}(x)$$
(3.53)

The metric (components) transforms as any other (0, 2) tensor. Nothing special! Side note:

Let
$$g := \det(g_{ii})$$
 and $\bar{g} := \det(\bar{g}_{ii})$

$$\sqrt{|\bar{g}|} = \left| \det \left(\frac{\partial x}{\partial \bar{x}} \right) \right| \sqrt{|g|}$$
(3.54)

 $\rightarrow \sqrt{|g|}$ is a *pseudo* scalar tensor density of weight w = +1. The "pseudo" indicates that the absolute value of the Jacobian determinant shows up, cf. Eq. (3.37).

 $\triangleleft g < 0 \xrightarrow{\text{Eq. (3.39)}} d^D x \sqrt{-g} \text{ is a scalar } (\rightarrow later)!$

30 | Length of curves on M:

One immediate benefit of having a Riemannian manifold is that we can now compute the length of curves $\gamma(t)$ on M (parametrized by $t \in [a, b]$ and given in some chart):

$$L[\gamma] \equiv \int_{\gamma} \mathrm{d}s := \int_{a}^{b} \sqrt{g_{ij}(\gamma(t)) \frac{\mathrm{d}\gamma^{i}(t)}{\mathrm{d}t} \frac{\mathrm{d}\gamma^{j}(t)}{\mathrm{d}t}} \,\mathrm{d}t$$
(3.55)

$$\equiv \int_{a}^{b} \|\dot{\gamma}(t)\|_{\gamma(t)} \,\mathrm{d}t \tag{3.56}$$

i! If ds^2 is a true *pseudo* metric (i.e., g_{ij} has at least one negative eigenvalue), one must make sure that the chosen curve γ does not produce negative values under the square root. In RELATIVITY these will be \uparrow *time-like* curves.

Example:

Let γ be the circle with radius R in the Euclidean plane E_2 . A possible parametrization in Cartesian coordinates (with origin in the center of the circle) is $\vec{\gamma}_{xy}(t) = (x_t, y_t) = (R \cos(t), R \sin(t))$ with $0 \le t < 2\pi$ so that one finds for the circumference:

$$L = \int_{\gamma} \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_0^{2\pi} \sqrt{\dot{x}_t^2 + \dot{y}_t^2} \,\mathrm{d}t \,\stackrel{\circ}{=} 2\pi R \tag{3.57}$$



The same length can of course be calculated with the parametrization $\vec{\gamma}_{r\theta}(t) = (r_t, \theta_t) = (R, t)$ and $0 \le t < 2\pi$ in polar coordinates:

$$L = \int_{\gamma} \sqrt{\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2} = \int_0^{2\pi} \sqrt{\dot{r}_t^2 + r_t^2 \dot{\theta}_t^2} \,\mathrm{d}t \,\stackrel{\circ}{=} 2\pi R \tag{3.58}$$

Details: September 4

31 | Besides computing lengths of curves (and other geometric quantities, → *later*), there is another benefit of having a metric tensor:

Pulling indices down:

$$\tilde{T}^{i_1} \dots \square \dots i_p \square \dots \square := g_{ik} T^{i_1} \dots k \dots i_p \square \dots \square := g_{ik} (3.59)$$

- $\rightarrow \tilde{T}$ is a tensor of type (p-1, q+1)
 - In Eq. (3.59) we indicate "empty" slots for indices by □ to emphasize that in each index "column" an index can either be *up* (contravariant) or *down* (covariant). It is conventional to omit the □-markers. Note that this explains why you never should write two indices directly above each other (except for special cases, → *below*).

Furthermore, since g is fixed, it makes sense to label \tilde{T} again by T (note that the difference between the original tensor and the new one is manifest in the different index patterns!):

$$\tilde{T}^{i_1\dots\square\dots i_p\square\dots\square}_{\square\dots i\dots\square j_1\dots j_q} \mapsto T^{i_1\dots\dots i_p}_{i_1\dots j_1\dots j_q}$$
(3.60)

Example:

$$A^{i}{}_{j}{}^{k}{}_{l} := g_{jm}A^{imk}{}_{l} \tag{3.61}$$

• This convention matches perfectly with the computation of an inner product (which is determined by the metric tensor g) of two contravariant vectors:

$$\langle A, B \rangle \stackrel{\text{def}}{=} g_{ij} A^i B^j \stackrel{\text{def}}{=} \underbrace{A^i B_i}_{\text{Scalar}}$$
(3.62)

32 | Pulling indices *up*:

We would like to have a (2, 0)-tensor g^{ij} with the property

$$\delta_j^k T^j = T^k \stackrel{!}{=} g^{ki} T_i \stackrel{\text{def}}{=} g^{ki} g_{ij} T^j .$$
(3.63)

 g^{ij} allows us to revert the pulling-down of indices defined by the metric g_{ij} . Note that g^{ij} is a *different* tensor than g_{ij} , we could call it \tilde{g}^{ij} ; however, it is conventional to denote it with the same label due to the following close relationship with g:

$$g^{ki}g_{ij} \stackrel{!}{=} \delta^k_j \tag{3.64}$$

This is an implicit equation for g^{ki} !



 $\rightarrow g^{ij}$ is the *inverse matrix* of g_{ij} (Which always exists because ds^2 is non-degenerate: $det(g_{ij}) \neq 0$.)

 \rightarrow In general:

$$\tilde{T}^{i_1\dots i_p} \square \dots j_1 \dots \square \square \dots j_q := g^{jk} T^{i_1\dots i_p} \square \dots \square \square \dots \square \square \square \square \dots \square j_q$$
(3.65)

 $\rightarrow \tilde{T}$ is a tensor of type (p+1, q-1)

• Again we relabel \tilde{T} to T and omit the \Box -markers:

$$\tilde{T}^{i_1\dots i_p} \square \dots \square j_1\dots \square \dots j_q \quad \mapsto \quad T^{i_1\dots i_p} \qquad j_1\dots \dots j_q \tag{3.66}$$

• Example:

$$A^{ijkl} := g^{lm} A^{ijk}{}_m \tag{3.67}$$

• With these new definitions, we can now raise and lower contractions:

$$A^{i}B_{i} = A^{i}\delta_{i}^{j}B_{j} = A^{i}g_{ik}g^{kj}B_{j} = A^{i}g_{ik}B^{k} = A_{k}B^{k} = A_{i}B^{i}$$
(3.68)

• What happens if you pull the indices of the Kronecker symbol up or down?

$$\delta^{ij} := g^{jk} \delta^i_{\ k} = g^{ij} \quad \text{and} \quad \delta_{ij} := g_{ik} \delta^k_{\ j} = g_{ij} \tag{3.69}$$

 $i! \delta^{ij} \equiv g^{ij}$ and $\delta_{ij} \equiv g_{ij}$ denote the metric and its inverse!

 \rightarrow We never use the notation δ^{ij} and δ_{ij} to prevent confusion!

• Note that in general

$$g^{jk}T^{i}_{\ k} = T^{ij} \neq T^{ji} = g^{jk}T^{\ i}_{\ k} \ . \tag{3.70}$$

This means that the "column" in which the index is located is *important*, and notations like T_k^i are ill defined (if you pull k up by g^{jk} , do you get T^{ij} or T^{ji} ?). However, if the tensor is *symmetric*, $T^{ij} = T^{ji}$, this does not matter and you can get away with the sloppy notation T_k^i . This explains why writing δ_k^i for the Kronecker symbol is fine: $g^{ji} = g^{jk} \delta_k^i$ is symmetric.

33 | <u>Mathematical side note:</u>

"Pulling indices up and down" is mathematically the application of an \downarrow isomorphism between $T_p M$ and $T_p^* M$:

$$g(\bullet, \bullet): T_p M \ni A \mapsto g(A, \bullet) \in T_p^* M$$

$$(3.71)$$

This has nothing to do with differential geometry or manifolds in particular; it is a general feature of non-degenerate bilinear forms on vector spaces. In differential geometry, this canonical isomorphism between the tangent bundle TM and the cotangent bundle T^*M is know as \uparrow musical isomorphism.

For example, you are using the same kind of isomorphism all the time in quantum mechanics, namely whenever you "dagger" a *ket* $|\Psi\rangle$ to obtain a *bra* $\langle\Psi|$:

$$(\bullet)^{\dagger}: \mathcal{H} \ni |\Psi\rangle \mapsto \langle \Psi| \equiv |\Psi\rangle^{\dagger} \in \mathcal{H}^* \quad \text{with} \quad \langle \Psi||\Phi\rangle \stackrel{!}{=} \langle \Psi|\Phi\rangle_{\mathcal{H}} \quad \text{for all } |\Phi\rangle \in \mathcal{H}.$$
(3.72)



Note how the bra *bra* $\langle \Psi |$ associated to the *ket* $|\Psi \rangle$ is *defined* via the inner product $\langle \bullet | \bullet \rangle_{\mathcal{H}}$ (and therefore metric) of the Hilbert space (\uparrow *Riesz representation theorem*)!

This leads to a nice dictionary between concepts in tensor calculus (and therefore RELATIVITY) and the bra-ket formalism of quantum mechanics:

	Relativity (fixed $p \in M$)	Quantum mechanics
Inner product space	$T_p M$	${\cal H}$
Basis	$\{\partial_i\}$	$\{ i\rangle\}$
Vector	$A = A^i \partial_i$	$ \Psi angle=\Psi_i i angle$
Dual space	T_p^*M	\mathcal{H}^*
Dual basis	$\{dx^i\}$	$\{\langle i \}$
	$\mathrm{d}x^i(\partial_j) = \delta^i_j$	$\langle i j \rangle = \delta_{ij}$
Covector	$B = B_i \mathrm{d} x^{i}$	$\langle \Psi = \Psi_i^* \langle i $
Inner product	$g(A_1, A_2) = g_{ij} A_1^i A_2^j$	$\langle \Psi \Phi angle$
Tensor	$A = A^{ij} \partial_i \otimes \partial_j$	$ \Psi angle\otimes \Phi angle\equiv \Psi angle \Phi angle$
	$B = B_{ij} \mathrm{d} x^i \otimes \mathrm{d} x^j$	$\langle \Psi \otimes \langle \Phi \equiv \langle \Psi \langle \Phi $
Operator	$T = T^i_{\ i} \ \partial_i \otimes \mathrm{d} x^j$	$ \Phi angle\otimes\langle\Psi \equiv \Phi angle\langle\Psi $
Trace	T^i_i	$\mathrm{Tr}[\Phi angle\langle\Psi]$
Scalar	$BA = B_i A^i = g_{ij} B^i A^j$	$\langle \Psi \Phi angle = \langle \Psi \Phi angle$
Pulling indices down	$A_i = g_{ij} A^j$	$\langle\Psi = \Psi angle^{\dagger}$
Pulling indices up	$A^i = g^{ij} A_j$	$ \Psi angle=\langle\Psi ^{\dagger}$

3.6. Differentiation of tensor fields

34 | Remember: $\partial_i \Phi$ is covariant vector if Φ is scalar. However:

 \triangleleft Contravariant vector A^i :

$$\bar{A^{i}}_{,k} \equiv \frac{\partial \bar{A^{i}}}{\partial \bar{x}^{k}} = \frac{\partial x^{m}}{\partial \bar{x}^{k}} \frac{\partial}{\partial x^{m}} \left[\frac{\partial \bar{x}^{i}}{\partial x^{l}} A^{l} \right] = \underbrace{\frac{\partial^{2} \bar{x}^{i}}{\partial x^{m} \partial x^{l}} \frac{\partial x^{m}}{\partial \bar{x}^{k}} A^{l}}_{\neq 0 \text{ (in general) } \odot} + \underbrace{\frac{\partial x^{m}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{i}}{\partial x^{l}} \frac{\partial A^{l}}{\partial x^{m}}}_{(1, 1) \text{ tensor } \odot}$$
(3.73)

Here we used the transformation of \bar{A}^i [Eq. (3.8)] and $\bar{\partial}_k$ [Eq. (3.5)] and the product rule.

- \rightarrow In general: $\frac{\partial \bar{A}^i}{\partial \bar{x}^k}$ is not a tensor!
- 35 | How to define a derivative of tensor fields that again transforms as a tensor?To solve this problem, we first need a new field:

to solve this problem, we first need a new field:

 \rightarrow ** Christoffel symbols (of the second kind):

$$\Gamma^{i}_{\ kl} := \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$$
(3.74)

• The Christoffel symbols are symmetric in the lower two indices: $\Gamma^{i}_{kl} = \Gamma^{i}_{lk}$

• i! Despite the index notation, the Christoffel symbols are *not* tensors:

$$\bar{\Gamma}^{i}_{kl} \stackrel{\circ}{=} \frac{\partial \bar{x}^{i}}{\partial x^{m}} \frac{\partial x^{n}}{\partial \bar{x}^{k}} \frac{\partial x^{p}}{\partial \bar{x}^{l}} \Gamma^{m}_{np} - \underbrace{\frac{\partial x^{n}}{\partial \bar{x}^{k}} \frac{\partial x^{p}}{\partial \bar{x}^{l}} \frac{\partial^{2} \bar{x}^{i}}{\partial x^{n} \partial x^{p}}}_{\text{No tensor!}}$$
(3.75)



This is why they are called "symbols" and not "tensors"!

• There are also Christoffel symbols of the *first* kind:

$$\Gamma_{ikl} := g_{ij} \Gamma^{j}_{\ kl} = \frac{1}{2} \left(g_{ik,l} + g_{il,k} - g_{kl,i} \right)$$
(3.76)

- Mathematically, the Christoffel symbols are the coefficients (in some basis) of the ↑ Levi-Civita connection which is determined by the metric tensor g^{ij} (→ later).
- **36** | \triangleleft Contravariant vector \bar{A}^i and contract it with $\bar{\Gamma}^i_{kl}$:

$$\bar{\Gamma}^{i}_{kl}\bar{A}^{l} = \underbrace{\frac{\partial \bar{x}^{i}}{\partial x^{m}} \frac{\partial x^{n}}{\partial \bar{x}^{k}} \Gamma^{m}_{np}}_{(1, 1)\text{-tensor}} \underbrace{\left[\frac{\partial x^{p}}{\partial \bar{x}^{l}} \bar{A}^{l}\right]}_{A^{p}} - \underbrace{\frac{\partial x^{n}}{\partial \bar{x}^{k}} \frac{\partial^{2} \bar{x}^{i}}{\partial x^{n} \partial x^{p}}}_{\text{Problematic term in Eq. (3.73)}} \underbrace{\left[\frac{\partial x^{p}}{\partial \bar{x}^{l}} \bar{A}^{l}\right]}_{(3.77)}$$

Idea: Add Eq. (3.73) and Eq. (3.77) to cancel the problematic term:

$$\bar{A}^{i}_{,k} + \bar{\Gamma}^{i}_{kp} \bar{A}^{p} = \underbrace{\frac{\partial x^{m}}{\partial \bar{x}^{k}} \frac{\partial \bar{x}^{i}}{\partial x^{l}} \left[A^{l}_{,m} + \Gamma^{l}_{mp} A^{p} \right]}_{(1,1) \text{-tensor } \odot \odot}$$
(3.78)

37 | This motivates the definition of the **** *Covariant derivative:*

Contrav

Scalar:
$$\Phi_{:k} := \Phi_{:k}$$
 (3.79a)

ariant vector:
$$A^{i}_{;k} := A^{i}_{,k} + \Gamma^{i}_{kl} A^{l}$$
 (3.79b)

Covariant vector:
$$B_{i;k} := B_{i,k} - \Gamma^l_{ik} B_l$$
 (3.79c)

- With this definition, $A^{i}_{:k}$ is a (1, 1)-tensor and $B_{i:k}$ is a (0, 2)-tensor!
- With this definition, the product rule is valid for the covariant derivative:

$$(A^{i}B_{i})_{;k} = (A^{i}B_{i})_{,k} \stackrel{\circ}{=} A^{i}_{;k}B_{i} + A^{i}B_{i;k}$$
(3.80)

• The construction of higher-rank tensors by tensoring contra- and covariant vectors Eq. (3.32) and the definitions of the covariant derivative above Eq. (3.79) can be used to construct covariant derivatives of arbitrary tensor fields. For example:

$$T^{i}_{k;l} := T^{i}_{k,l} + \Gamma^{i}_{ml} T^{m}_{\ k} - \Gamma^{m}_{\ kl} T^{i}_{\ m}$$
(3.81)

• With this generalization, we can apply the covariant derivative multiple times. For example:

$$A^{i}_{;k;l} \equiv \left(A^{i}_{;k}\right)_{;l} \tag{3.82}$$

• The covariant derivative is *not commutative* in general:

$$A^{i}_{;k;l} - A^{i}_{;l;k} \neq 0 \tag{3.83}$$

 \rightarrow Riemann curvature tensor \rightarrow general relativity (\rightarrow *later*)

(This is not the case for the "normal" derivative: $A^{i}_{,k,l} = A^{i}_{,l,k}$.)



38 | <u>Conclusion</u>:

If you can formulate an equation that describes a physical theory in terms of tensors, it can always be brought into the form

$$T^{I}_{\ J}(x) = 0.$$
 (3.84)

(This equation is meant to hold for all values of indices I and J and all coordinate values x.) Here is an example:

The (inhomogeneous) Maxwell equations on an arbitrary (potentially curved) spacetime read:

$$\underbrace{F^{\mu\nu}_{;\nu} + \frac{4\pi}{c} J^{\mu}}_{=:T^{\mu}(x)} = 0$$
(3.85)

with current density J^{μ} and field strength tensor $F^{\mu\nu} = g^{\mu\rho}g^{\nu\pi}(A_{\pi;\rho} - A_{\rho;\pi})$.

How does Eq. (3.84) look like in any other coordinate system $\bar{x} = \varphi(x)$? Easy:

$$\bar{T}^{I}_{J}(\bar{x}) = \frac{\partial \bar{x}^{I}}{\partial x^{M}} \frac{\partial x^{N}}{\partial \bar{x}^{J}} \underbrace{T^{M}_{N}(x)}_{=0} = 0 \quad \Leftrightarrow \quad \bar{T}^{I}_{J}(\bar{x}) = 0.$$
(3.86)

This means:

Tensor equations are automatically form-invariant under *arbitrary* coordinate transformations; we say they exhibit *** (manifest) general covariance.*

The "manifest" means that checking general covariance is just a matter of checking whether the equation "looks right", i.e., whether it is built from tensors following the rules discussed in this chapter. If a property of an equation is manifest, you don't have to do calculations to verify it!

In the next chapter, we take a step back and specialize the allowed coordinate transformations to the Lorentz transformations of SPECIAL RELATIVITY. We can then use the form-invariance of equations built from "Lorentz tensors" to construct Lorentz covariant equations from scratch – which was our original goal!

↓Lecture9 [12.12.23]



4. Formulation on Minkowski Space

In this section we briefly reformulate what we already learned about SPECIAL RELATIVITY in terms of tensor calculus. We use this notation in subsequent chapters to make classical and quantum mechanics relativistic, and reformulate electrodynamics in a form where its Lorentz covariance is manifest. It also allows a smooth transition into GENERAL RELATIVITY.

The formulation of SPECIAL RELATIVITY on a unified, four-dimensional spacetime manifold goes back to Hermann Minkowski, Albert Einstein's former professors of mathematics at ETH. Minkowski writes in the notes of his lecture "Raum und Zeit" delivered 1908 in Cologne [54]:

Die Anschauungen über Raum und Zeit, die ich Ihnen entwickeln möchte, sind auf experimentellphysikalischem Boden erwachsen. Darin liegt ihre Stärke. Ihre Tendenz ist eine radikale. Von Stund' an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und nur noch eine Art Union der beiden soll Selbständigkeit bewahren.

Einstein, a physicist all through, didn't appreciate this mathematical reformulation of his theory at first. According to Sommerfeld, he (Einstein) commented:

Seit die Mathematiker über die Relativitätstheorie hergefallen sind, verstehe ich sie selbst nicht mehr.

Einstein later changed his views and acknowledged that without Minkowski's introduction of spacetime as a four-dimensional manifold, the development of GENERAL RELATIVITY would have been impossible.

For a historical account on the role of Minkowski, and his relationship (or absence thereof) to Einstein, see Ref. [55].

4.1. Minkowski space

1 | <u>Manifold:</u>



(4.1)

(4.2)

It is a well founded, but nonetheless empirical assumption that the spacetime manifold of events has the *topology* of \mathbb{R}^4 . Note that at this point we do not impose restrictions on the *geometry* of spacetime, e.g., whether it is flat or curved; this follows below when we settle on a metric.

2 | <u>Charts:</u>

In SPECIAL RELATIVITY, we restrict the coordinate systems to the ones that correspond to inertial observers / inertial coordinate systems:

$$(\mathcal{E}, K) \quad \leftrightarrow \quad \text{Inertial (coordinate) systems } K \in \mathcal{J}$$



The coordinates are the ones obtained by an \uparrow *inertial observer*:

$$K: \ \mathcal{E} \ni E \ \mapsto \ K(E) := [E]_K = x$$
with $x^{\mu} = (x^0, x^1, x^2, x^3)^T = (ct, x, y, z)^T = (ct, \vec{x})^T$
(4.3)
(4.4)

- i! Henceforth, *Greek* indices μ, ν,... run over 0, 1, 2, 3 where μ = 0 denotes the time component and μ = 1, 2, 3 denote the spatial components. *Roman* indices *i*, *j*,... run only over the spatial components 1, 2, 3.
- i! We multiply the time *t* with the speed of light to measure times and distances in the same units.
- Since we assumed that our inertial systems cover all of spacetime, the domains on which the coordinate functions are defined are the complete manifold.
- The notation above is very suggestive: You can think of our inertial systems, namely the calibrated latticework of clocks and rods, as physical manifestations of the coordinate map of the corresponding chart. That is, an inertial system is a measurement device, or function, which assigns to every event $E \in \mathcal{E}$ the coordinate tuple $x = K(E) = (ct, \vec{x})_K \in E$.

3 | Transition maps:

i | We worked hard in Section 1.4 to derive and select the correct coordinate transformations between different inertial systems. The most general ones have the form of ...

 $\begin{array}{l}
 Inhomogenous \text{ Lorentz transformations} \\
 Poincaré transformations
\end{array} \left\{ : \quad \bar{x} = \varphi(x) = \Lambda x + a \quad (4.5)
\end{array}$

with $a \in \mathbb{R}^4$ arbitrary and $\Lambda \in \mathbb{R}^{4 \times 4}$ a \uparrow Lorentz transformation.

For the special case $a = 0 \in \mathbb{R}^4$ we found:

Homogeneous Lorentz transformations:
$$\bar{x} = \varphi(x) = \Lambda x$$
 (4.6)

ii | Since these transformations are affine, we find immediately:

$$\frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}} = \Lambda^{\mu}{}_{\nu} \quad \text{and} \quad \frac{\partial x^{\mu}}{\partial \bar{x}^{\nu}} = (\Lambda^{-1})^{\mu}{}_{\nu} \equiv \Lambda^{\mu}{}_{\nu}$$
(4.7)

Recall that the derivative of a linear (affine) map is simply the matrix which defines the map.

;! We use the tensor-inspired notation $\Lambda^{\mu}{}_{\nu}$ for the matrix elements of Λ to allow for welldefined contractions with the metric (\Rightarrow *later*). In $\Lambda^{\mu}{}_{\nu}$, the upper index μ denotes the *rows*, the lower index ν the *columns* of the matrix. The notation $\Lambda^{\nu}{}_{\nu}{}^{\mu}$ for the components of the inverse transformation matrix Λ^{-1} is purely conventional at this point; it will turn out to be consistent with pulling indices up and down with the Minkowski metric (\Rightarrow *below*).

This allows us to rewrite the coordinate transformation Eq. (4.5) in tensor notation:

$$\bar{x}^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} + a^{\mu} \tag{4.8}$$



;! The matrix-vector product Λx is now given by the Einstein summation (index contraction) highlighted blue. We will stick to this notation whenever possible. Since we are now in the world of tensor calculus, it is strongly discouraged to think of and write rank-2 tensors as "matrices" and contractions as matrix-vector products Λx (even though Λ does not represent the components of a tensor). It is less error-prone (and simpler) to perform computations using the index notation introduced in Chapter 3.

- iii | Writing down the most general homogeneous Lorentz transformation is very complicated (and unnecessary). Here we provide the two special Lorentz transformations (boosts) discussed earlier in the new matrix notation, and an example for a spatial rotation about the *z*-axis:
 - Lorentz boost in *x*-direction $K \xrightarrow{v_x} \overline{K} \ (\beta_x = v_x/c)$:

Eq. (1.77)
$$\rightarrow \quad \Lambda^{\mu}{}_{\nu} = [\Lambda_{\nu_{x}}]^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta_{x}\gamma & 0 & 0\\ -\beta_{x}\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}$$
(4.9)

• Lorentz boost in \hat{v} -direction $K \xrightarrow{\vec{v}} \bar{K}$ $(v = |\vec{v}| \text{ and } \tilde{\gamma} := \gamma - 1)$:

$$\begin{aligned} & \operatorname{Eq.} (1.75) \rightarrow \quad \Lambda^{\mu}{}_{\nu} = \left[\Lambda_{\vec{v}} \right]^{\mu}{}_{\nu} = \\ & \begin{pmatrix} \gamma & -\beta_{x}\gamma & -\beta_{y}\gamma & -\beta_{z}\gamma \\ -\beta_{x}\gamma & 1 + \tilde{\gamma}v_{x}^{2}/v^{2} & \tilde{\gamma}v_{x}v_{y}/v^{2} & \tilde{\gamma}v_{x}v_{z}/v^{2} \\ -\beta_{y}\gamma & \tilde{\gamma}v_{x}v_{y}/v^{2} & 1 + \tilde{\gamma}v_{y}^{2}/v^{2} & \tilde{\gamma}v_{y}v_{z}/v^{2} \\ -\beta_{z}\gamma & \tilde{\gamma}v_{x}v_{z}/v^{2} & \tilde{\gamma}v_{y}v_{z}/v^{2} & 1 + \tilde{\gamma}v_{z}^{2}/v^{2} \end{pmatrix}_{\mu\nu} \end{aligned}$$

$$\end{aligned}$$

$$(4.10)$$

• Spatial rotation $K \xrightarrow{R_z(\theta), \vec{0}} \bar{K}$ by θ in *xy*-plane:

$$\Lambda^{\mu}{}_{\nu} = [R_{z}(\theta)]^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}$$
(4.11)

- 4 Metric tensor:
 - $i \mid$ We elevate the spacetime manifold M to a pseudo-Riemannian (and Lorentzian) manifold by



introducing the following pseudo-Riemannian metric tensor (given in inertial coordinates):

$$\text{** Minkowski metric } ds^{2} \begin{cases} := (c dt)^{2} - (d\vec{x})^{2} \\ = (dx^{0})^{2} - (dx^{1})^{2} - (dx^{2})^{2} - (dx^{3})^{2} \\ = \eta_{\mu\nu} dx^{\mu} dx^{\nu} \end{cases}$$
with metric components $\eta_{\mu\nu} = \eta^{\mu\nu} = \underbrace{\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}}_{\text{Signature } (1,3) \equiv (+,-,-,-)}$

$$(4.12a)$$

- The components η_{µν} of this metric tensor in Eq. (4.12b) are the same for all inertial coordinate systems [→ Eq. (4.21) below].
- Recall that $\eta^{\mu\nu}$ is the matrix inverse of $\eta_{\mu\nu}$.
- \rightarrow We call the spacetime manifold equipped with this metric ...

** Minkowski space:
$$\mathbb{R}^{1,3} \equiv (\mathcal{E} \simeq \mathbb{R}^4, \mathrm{d}s^2)$$
 (4.13)

- We will always use $\eta_{\mu\nu}$ to denote the components of the Minkowski metric (in an inertial coordinate chart) to distinguish it from a generic metric g_{ij} .
- Note that, informally speaking, ds² this is the infinitesimal form of the *← invariant* spacetime interval Eq. (1.83) we introduced earlier (*→ below*).
- Minkowski space is therefore an example of a ← Lorentzian manifold. By fixing a metric, we fixed the geometry of spacetime. As we will see in our discussion of GENERAL RELATIVITY, the distinctive feature of Minkowski space is that it is *flat* (it has no curvature). It will turn out that, in reality, this assumption is only valid to some degree: The tenet of GENERAL RELATIVITY is that the *deviations* of spacetime from flat Minkowski space are what we expercience as gravity!
- ii | With the metric we can measure "lengths" of trajectories on spacetime:

 \triangleleft Time-like trajectory $\gamma : s \mapsto x^{\mu}(s)$ for $s \in [s_a, s_b]$ in $\mathbb{R}^{1,3} \rightarrow$

$$L[\gamma] \stackrel{3.55}{=} \int_{s_a}^{s_b} \sqrt{\eta_{\mu\nu} \frac{\mathrm{d}x^{\mu}(s)}{\mathrm{d}s} \frac{\mathrm{d}x^{\nu}(s)}{\mathrm{d}s}} \,\mathrm{d}s \tag{4.14a}$$

$$\stackrel{4.12b}{=} \int_{s_a}^{s_b} \sqrt{[\dot{x}^0(s)]^2 - [\dot{x}^1(s)]^2 - [\dot{x}^2(s)]^2 - [\dot{x}^3(s)]^2} \,\mathrm{d}s \tag{4.14b}$$

Choose parametrization $s := x^0/c \equiv t$ (4.14c)

$$= \int_{t_a}^{t_b} \underbrace{\sqrt{c^2 - \vec{v}^2(t)}}_{> 0 \text{ (time-like)}} dt$$
(4.14d)

$$\stackrel{2.25}{=} c\,\Delta\tau[\gamma] \tag{4.14e}$$

Thus the "length" $L[\gamma]$ of time-like curves in $\mathbb{R}^{1,3}$ is the \leftarrow proper time $\Delta \tau[\gamma]$ along the curve defined in Eq. (2.25) (multiplied by *c*); this explains why the Minkowski metric ds^2 is the right choice for SPECIAL RELATIVITY.



4.2. Four vectors and tensors

5 | Tensors are defined as in Chapter 3, with the restriction to D = 4 and that only homogeneous Lorentz transformations Eq. (4.7) are considered as transition maps. To emphasize this, we introduce a new nomenclature:

Tensor calculus	SPECIAL RELATIVITY
Contravariant vector A^i	Contraviarant $*$ <i>Lorentz vector</i> / 4-vector A^{μ}
Covariant vector B_i	Covariant $*$ Lorentz vector / 4-vector B_{μ}
(Mixed) tensor T^{i}_{j}	(Mixed) $*$ Lorentz tensor / 4-tensor T^{μ}_{ν}
Scalar Φ	$*$ Lorentz scalar Φ

Then a generic (p, q) tensor transforms under the coordinate transformation Eq. (4.7) as:

$$\bar{T}^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q}(\bar{x}) = \left[\Lambda^{\mu_1}_{\ \rho_1}\dots\Lambda^{\mu_p}_{\ \rho_p}\right] \left[\Lambda^{\pi_1}_{\nu_1}\dots\Lambda^{\pi_q}_{\nu_q}\right] T^{\rho_1\dots\rho_p}_{\ \pi_1\dots\pi_q}(x)$$
(4.15)

6 | With the Minkowski metric, we can reformulate our classification for 4-vectors [recall Eq. (1.85)]:

A *light-like* 4-vector is also called *** null*.

i! We use this classification scheme also for generic Lorentz vectors that are not coordinate differences between a pair of events (\rightarrow *below*). Since the pseudo-norm $X^{\mu}X_{\mu} = X^2$ is a Lorentz scalar, this classification is independent of the inertial system.

7 | Coordinate functions:

It is a particular feature of *linear* coordinate transformations (here: homogeneous Lorentz transformations) that the coordinate functions themselves transform as contravariant vector fields:

 \triangleleft Coordinate field $X^{\mu}(x) := x^{\mu} \rightarrow$

$$\underbrace{\bar{X}^{\mu}(\bar{x})}_{\bar{x}^{\mu}} = \Lambda^{\mu}{}_{\nu}\underbrace{X^{\nu}(x)}_{x^{\nu}} = \underbrace{\frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}}}_{\Lambda^{\mu}{}_{\nu}}X^{\nu}(x)$$
(4.17)

We make the identification $X^{\mu}(x) \equiv x^{\mu}$ and don't write $X^{\mu}(x)$ henceforth.

Consequently, we can construct *** covariant coordinates* (a covariant vector field) via the metric by pulling the index down:

$$x_{\mu} := \eta_{\mu\nu} x^{\nu} = (x^0, -x^1, -x^2, -x^3) = (ct, -\vec{x})$$
(4.18)

i! To pull the index of a contravariant vector down, you multiply the spatial components by -1.



$$\Delta x^2 \equiv \Delta x^{\mu} \Delta x_{\mu} \tag{4.19a}$$

$$\stackrel{\text{def}}{=} \eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu} \tag{4.19b}$$

$$= (\Delta x^{0})^{2} - (\Delta x^{1})^{2} - (\Delta x^{2})^{2} - (\Delta x^{3})^{2}$$
(4.19c)

$$\stackrel{\text{def}}{=} \Delta s^2 \tag{4.19d}$$

Remember [Eq. (1.84)]: $\Delta s^2 = \Delta \bar{s}^2$ for arbitrary Lorentz transformations \rightarrow

$$\underbrace{\eta_{\mu\nu}\,\Delta x^{\mu}\Delta x^{\nu}}_{\Delta s^{2}} = \underbrace{\eta_{\rho\pi}\,\Delta \bar{x}^{\rho}\Delta \bar{x}^{\pi}}_{\Delta \bar{s}^{2}} = \left[\eta_{\rho\pi}\,\Lambda^{\rho}{}_{\mu}\Lambda^{\pi}{}_{\nu}\right]\,\Delta x^{\mu}\Delta x^{\nu} \tag{4.20}$$

Since this is true for all events $\Delta x^{\mu} \xrightarrow{\circ}$

$$\Lambda^{\rho}{}_{\mu}\Lambda^{\pi}{}_{\nu}\eta_{\rho\pi} = \eta_{\mu\nu} \tag{4.21}$$

Concluding Eq. (4.21) from Eq. (4.20) is non-trivial because we consider "norms" $\eta_{\mu\nu} \Delta x^{\mu} \Delta x^{\nu}$ and not "inner products" $\eta_{\mu\nu} \Delta x^{\mu} \Delta y^{\nu}$. However, for *symmetric*, real matrices A and B, it is true that if $\vec{x}^T A \vec{x} = \vec{x}^T B \vec{x}$ for all real vectors \vec{x} , then A = B. This is so because A - B is a symmetric matrix that can be diagonalized by an orthogonal matrix and $\vec{x}^T (A - B) \vec{x} = 0$. The last condition implies that all eigenvalues of A - B are zero and therefore A - B = 0. Alternatively you can use the \checkmark *polarization identity* to show that the invariance of the Minkowski (pseudo) norm implies the invariance of the Minkowski (pseudo) inner product.

We say:

Lorentz transformations are
$$\checkmark$$
 isometries of Minkowski space. (4.22)

With det $(\eta_{\mu\nu}) \neq 0$, a corollary of Eq. (4.21) is:

$$\det\left(\Lambda^{\mu}_{\nu}\right) = \pm 1 \tag{4.23}$$

If you want to write Eq. (4.21) in the old matrix notation, make the identifications $\Lambda^{\mu}{}_{\nu} = \Lambda_{\mu\nu}$ and $\eta_{\mu\nu} = \eta_{\mu\nu}$. Here, subscripts of bold symbols denote the entries of *matrices* as usual (first index: row; second index: column). Equations that contain matrices (bold symbols) do *not* comply with the syntax of tensor calculus (which is why you should avoid them!).

Eq. (4.21) then reads in matrix notation:

$$\Lambda^{T}_{\mu\rho}\eta_{\rho\pi}\Lambda_{\pi\nu} = \eta_{\mu\nu} \quad \Leftrightarrow \quad \Lambda^{T}\eta\Lambda = \eta$$
(4.24)

Here we defined the *transposed matrix* as $\Lambda_{\mu\rho}^T := \Lambda_{\rho\mu}$, i.e., the matrix where rows and columns are swapped. Eq. (4.24) immediately implies det(Λ^T) det(η) det(Λ) = det(η); using det(η) $\neq 0$ and det(Λ^T) = det(Λ), we find det(Λ) = ± 1 .


$$\Lambda^{\rho}_{\ \mu} \left[\Lambda^{\pi}_{\ \nu} \eta_{\rho \pi} \eta^{\nu \sigma} \right] = \delta^{\sigma}_{\mu} \tag{4.25}$$

We can therefore conclude that:

$$\Lambda_{\rho}^{\ \sigma} := \eta_{\rho\pi} \eta^{\nu\sigma} \Lambda_{\ \nu}^{\pi} = (\Lambda^{-1})^{\sigma}{}_{\rho} \tag{4.26}$$

Note that this is consistent with our definition in Eq. (4.7).

In the literature (e.g. Schröder [1]) the concept of a "transposed" transformation is introduced. We refer to it as "*pseudo-adjoint*" transformation instead and label it by *. It is defined analogous to proper adjoints on proper inner product spaces:

$$\underbrace{\eta_{\mu\nu}\Lambda^{\nu}{}_{\rho}}_{=:\Lambda_{\mu\rho}} x^{\rho}y^{\mu} \stackrel{\text{def}}{=} \langle y,\Lambda x \rangle \stackrel{!}{=} \langle \Lambda^*y,x \rangle \stackrel{\text{def}}{=} \underbrace{\eta_{\mu\nu}(\Lambda^*)^{\mu}{}_{\rho}}_{=:(\Lambda^*)_{\nu\rho}} x^{\nu}y^{\rho}.$$
(4.27)

This yields as reasonable definition for the pseudo-adjoint:

$$(\Lambda^*)_{\mu\nu} := \Lambda_{\nu\mu} \quad \Rightarrow \quad (\Lambda^*)^{\mu}_{\ \nu} = \Lambda_{\nu}^{\ \mu} \stackrel{\text{Eq. (4.26)}}{=} (\Lambda^{-1})^{\mu}_{\ \nu} \,. \tag{4.28}$$

One can then define a corresponding matrix Λ^* such that $(\Lambda^*)^{\mu}{}_{\nu} = \Lambda^*_{\mu\nu}$ and use $(\Lambda^{-1})^{\mu}{}_{\nu} = \Lambda^{-1}_{\mu\nu}$ to rewrite the above equation as

$$\mathbf{\Lambda}^* = \mathbf{\Lambda}^{-1} \,. \tag{4.29}$$

Recall that the pseudo-adjoint is implicitly defined via the inner product. At no point did we claim that the pseudo-adjoint matrix is given by the *transposed* matrix Λ^T (which is defined by swapping rows and columns)! To find a relation to the latter, we can rewrite Eq. (4.26) in matrix language:

$$\boldsymbol{\Lambda}_{\sigma\rho}^{-1} = \boldsymbol{\eta}_{\rho\pi} \boldsymbol{\Lambda}_{\pi\nu} \boldsymbol{\eta}_{\nu\sigma}^{-1} = (\boldsymbol{\eta} \boldsymbol{\Lambda} \boldsymbol{\eta})_{\rho\sigma} = (\boldsymbol{\eta} \boldsymbol{\Lambda}^T \boldsymbol{\eta})_{\sigma\rho} \,. \tag{4.30}$$

Here we used that $\eta^{-1} = \eta = \eta^T$ and that $M_{ab}^T := M_{ba}$ for any matrix M. So finally:

$$\Lambda^* = \Lambda^{-1} = \eta \Lambda^T \eta \,. \tag{4.31}$$

The take home message is that the *transpose* of a Lorentz transformation (given by swapping columns and rows) is *not* its inverse (there are additional minuses sprinkled in by the metric)! By contrast, the *pseudo-adjoint* (defined via the pseudo-inner product) *is* identical to the inverse.

Warning: In the literature you will find the notation T instead of * (e.g. Schröder [1]). Then one finds the (correct) relation $(\Lambda^T)^{\mu}{}_{\nu} = \Lambda_{\nu}{}^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}$. The problem is that this notation *suggests* that $(\Lambda^T)^{\mu}{}_{\nu} \stackrel{\epsilon}{=} \Lambda^T_{\mu\nu}$ and therefore $\Lambda^{-1} \stackrel{\epsilon}{=} \Lambda^T$. As shown above, *both equations are wrong!*

10 Covariant derivative:

i | Since in inertial coordinate systems the Minkowski metric is given by $\eta_{\mu\nu}$, it follows immediately for the Christoffel symbols Eq. (3.74):

$$\Gamma^{i}_{\ kl} = \frac{1}{2} \eta^{im} (\underbrace{\eta_{mk,l}}_{0} + \underbrace{\eta_{ml,k}}_{0} - \underbrace{\eta_{kl,m}}_{0}) = 0$$
(4.32)

i! If you would transform into *curvilinear (non-inertial) coordinates*, the Christoffel symbols would *not* vanish – even on flat Minkowski space (S Problemset 5). That simple partial



derivatives produce Lorentz tensors is therefore a special feature of Minkowski space in inertial coordinates.

Eq. (3.79)

Lorentz Scalar:
$$\Phi_{;\mu} := \Phi_{,\mu} = \partial_{\mu} \Phi$$
 (4.33a)

Contravariant Lorentz vector:
$$A^{\mu}_{;\nu} := A^{\mu}_{,\nu} = \partial_{\nu}A^{\mu}$$
 (4.33b)

Covariant Lorentz vector:
$$B_{\mu;\nu} := B_{\mu,\nu} = \partial_{\nu} B_{\mu}$$
 (4.33c)

ii | <u>***</u> 4-Gradient:

This allows us to think of the differential operator ∂_{μ} itself as a *covariant* Lorentz vector and motivates the introduction of its *contravariant* components:

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}} = (\frac{1}{c}\partial_t, +\vec{\nabla})^T$$
 (4.34a)

$$\partial^{\mu} := \eta^{\mu\nu} \partial_{\nu} = \frac{\partial}{\partial x_{\mu}} = \left(\frac{1}{c} \partial_{t}, -\vec{\nabla}\right)$$
(4.34b)

Using Eq. (3.5), the transformation laws match that of co- and contravariant Lorentz vectors, respectively:

$$\bar{\partial}_{\mu} = \frac{\partial}{\partial \bar{x}^{\mu}} = \Lambda_{\mu}^{\nu} \frac{\partial}{\partial x^{\nu}} = \Lambda_{\mu}^{\nu} \partial_{\nu}$$
(4.35a)

$$\bar{\partial}^{\mu} = \frac{\partial}{\partial \bar{x}_{\mu}} = \Lambda^{\mu}{}_{\nu} \frac{\partial}{\partial x_{\nu}} = \Lambda^{\mu}{}_{\nu} \partial^{\nu} .$$
(4.35b)

i! The *covariant* 4-gradient (index *down*) is the partial derivative wrt. the *contravariant* coordinates (index *up*) and vice versa.

iii | These transformation properties immediately suggest two Lorentz scalars that can be constructed from 4-gradients $(A^{\mu} = (A^0, \vec{A}))$:

** 4-divergence:
$$\partial A := \partial_{\mu} A^{\mu} = \partial^{\mu} A_{\mu} = \frac{1}{c} \partial_{t} A^{0} + \vec{\nabla} \cdot \vec{A}$$
 (4.36a)
** 4-Laplacian: $\Box \equiv \partial^{2} := \partial_{\mu} \partial^{\mu} = \left(\frac{1}{c} \partial_{t}\right)^{2} - \vec{\nabla}^{2}$ (4.36b)

The 4-Laplacian \Box is also known as \checkmark *d'Alembert operator*.

Examples:

In electrodynamics (→ *later*) the gauge potential transforms as a contravariant Lorentz vector A^μ = (¹/_cφ, A).

The \downarrow Lorenz gauge is defined as $\partial_{\mu}A^{\mu} = 0$; it is Lorentz invariant since the 4divergence is a Lorentz scalar: $\bar{\partial}_{\mu}\bar{A}^{\mu}(\bar{x}) = \partial_{\mu}A^{\mu}(x)$.

Note: The Lorenz gauge is named after \uparrow *Ludvig Lorenz*; by contrast, the Lorentz transformation is named after \uparrow *Hendrik Lorentz*. Thus: *The Lorenz gauge* (no "t") *is Lorentz invariant*.



• In vacuum (and in Lorenz gauge), the gauge field of electrodynamics satisfies the wave equation

$$\partial^2 A^{\mu} = \left[\left(\frac{1}{c} \partial_t \right)^2 - \vec{\nabla}^2 \right] A^{\mu} = 0.$$
(4.37)

Since ∂^2 is a Lorentz scalar and A^{μ} a Lorentz vector, $\partial^2 A^{\mu}$ transforms as a contraviarant Lorentz vector and the equation is *manifestly Lorentz covariant*:

$$\partial^2 A^{\mu}(x) = 0 \quad \Leftrightarrow \quad \partial^2 \bar{A}^{\mu}(\bar{x}) = 0.$$
 (4.38)

• If we have a scalar field Φ , we can construct a manifestly Lorentz covariant wave equation:

$$(\partial^2 + m^2)\Phi(x) = 0 \quad \Leftrightarrow \quad (\bar{\partial}^2 + m^2)\bar{\Phi}(\bar{x}) = 0.$$
(4.39)

The parameter *m* is arbitrary and plays the role of a mass (spectral gap) of the excitations. This equation is known as \uparrow *Klein-Gordon equation* and describes, for example, the classical equation of motion of the Higgs field (without interactions).

11 | Relative tensors $\rightarrow *$ *Lorentz pseudo tensor*:

Since $det(\Lambda) = \pm 1$, the classification of tensors simplifies:

Tensor:
$$\bar{T}^{M}_{N}(\bar{x}) = \Lambda^{M}_{R}\Lambda^{P}_{N}T^{R}_{P}(x)$$
 (4.40a)
Pseudo tensor: $\bar{T}^{M}_{N}(\bar{x}) = \det(\Lambda) \Lambda^{M}_{R}\Lambda^{P}_{N}T^{R}_{P}(x)$ (4.40b)

Here we use again a multi-index notation: $M = \mu_1, \ldots, \mu_p$ etc. Recall that $det(\Lambda) = \pm 1$; pseudo tensors therefore pick up an additional minus sign under parity or time inversion (\rightarrow *later*).

 \rightarrow Relative tensors of *odd* weight w are pseudo tensors under Lorentz transformations.

Example:

The Levi-Civita symbol is a Lorentz pseudo tensor [recall Eq. (3.42)]:

$$\bar{\varepsilon}^{\mu\nu\rho\pi} = \varepsilon^{\mu\nu\rho\pi} = \det(\Lambda) \Lambda^{\mu}_{\ \mu'} \Lambda^{\nu}_{\ \nu'} \Lambda^{\rho}_{\ \rho'} \Lambda^{\pi}_{\ \pi'} \varepsilon^{\mu'\nu'\rho'\pi'} \,. \tag{4.41}$$

This means that if you contract a Levi-Civita symbol with an actual (0, 4) Lorentz tensor like $F_{\mu\nu}F_{\rho\pi}$ (the tensor product of two electromagnetic field trength tensors), you obtain a *pseudo* (*Lorentz*) scalar:

$$\bar{\Phi}(\bar{x}) := \bar{\varepsilon}^{\mu\nu\rho\pi} \bar{F}_{\mu\nu} \bar{F}_{\rho\pi} \stackrel{\circ}{=} \det(\Lambda) \, \varepsilon^{\mu\nu\rho\pi} F_{\mu\nu} F_{\rho\pi} = \det(\Lambda) \, \Phi(x) \,. \tag{4.42}$$

Since this is a quadratic (pseudo) scalar quantity, you might try to add it to the Lagrangian of Maxwell theory ($\theta \in \mathbb{R}$):

$$\tilde{\mathcal{L}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \theta \varepsilon^{\mu\nu\rho\pi} F_{\mu\nu} F_{\rho\pi} .$$
(4.43)

(This Lagrangian is now only invariant under Lorentz transformations with $det(\Lambda) = +1$.)

The new term is called $\uparrow \theta$ -term. One can show that it is a total derivative and therefore does not affect the classical equations of motion (Maxwell's equations). However, for non-abelian generalizations of electrodynamics like \uparrow quantum chromodynamics (\uparrow Yang-Mills theories), it does affect the theory (\uparrow Strong CP-problem [56]).

Note that we did not use the metric tensor $\eta_{\mu\nu}$ to construct the term $\varepsilon^{\mu\nu\rho\pi} F_{\mu\nu} F_{\rho\pi}$ (as compared to $F^{\mu\nu} F_{\mu\nu}$ where we need it to pull two indices up); this makes the θ -term an example of a so called \uparrow *topological* term (\uparrow *topological field theory*): the term doesn't "see" the *geometry* of spacetime! In condensed matter physics, the term plays a role in the description of \uparrow *topological insulators* [57].

- 12 | In the next chapter we want to construct a relativistic version of classical mechanics (using the framework of tensors calculus to make the equations Lorentz covariant). As a preparation, we can already define two 4-vectors with physical interpretation:
 - i | 4-velocity:

Question: What is a reasonable definition for a relativistic (= Lorentz covariant) velocity? \triangleleft Particle trajectory $x^{\mu}(\lambda)$ parametrized by λ :

$$x^{\mu}(\lambda) = \begin{pmatrix} ct(\lambda) \\ \vec{x}(\lambda) \end{pmatrix} \quad \Rightarrow \quad \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda} = \begin{pmatrix} \frac{\mathrm{d}ct}{\mathrm{d}\lambda} \\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}\lambda} \end{pmatrix}$$
(4.44)

First try: $\lambda = t$ (coordinate time) \rightarrow

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}t} = \begin{pmatrix} c\\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} c\\ \vec{v}(t) \end{pmatrix} \tag{4.45}$$

with coordinate velocity $\vec{v}(t)$.

Problem:

 $\frac{dx^{\mu}}{dt}$ is *not* a contravariant Lorentz vector because $dt \neq d\bar{t}$ is not a Lorentz scalar. That is:

$$\frac{\mathrm{d}\bar{x}^{\mu}}{\mathrm{d}\bar{t}} \neq \Lambda^{\mu}{}_{\nu}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}t} \tag{4.46}$$

 \rightarrow Eq. (4.45) is useless to construct Lorentz covariant equations!

Idea: The \leftarrow *Proper time* τ is a Lorentz scalar [Eq. (2.24)]: $d\tau = d\overline{\tau} \rightarrow \text{Set } \lambda = \tau$:

** 4-velocity:
$$u^{\mu} := \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \begin{pmatrix} c \frac{\mathrm{d}t}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\vec{x}}{\mathrm{d}\tau} \end{pmatrix} = \gamma_{v} \begin{pmatrix} c \\ \vec{v} \end{pmatrix}$$
 (4.47)

Here we used $\frac{dt}{d\tau} = \gamma_{v(t)}$ [recall Eq. (2.23)].

By construction, the 4-velocity is a contravariant Lorentz vector: $\bar{u}^{\mu} = \Lambda^{\mu}{}_{\nu}u^{\nu}$. \triangleleft Pseudo-norm:

$$u^{2} = \eta_{\mu\nu} u^{\mu} u^{\nu} = (u^{0})^{2} - (\vec{u})^{2} \stackrel{\circ}{=} c^{2} > 0$$
(4.48)

\rightarrow *Time-like* 4-vector

In Minkowski space, u^{μ} is the tangent at x^{μ} of the world line $x^{\mu}(\tau)$.

ii <u>4-acceleration:</u>

Following the same line of arguments above, the 4-acceleration is then defined as the derivative of the 4-velocity wrt. the proper time:

$$\overset{*}{*} 4\text{-acceleration:}$$

$$b^{\mu} := \frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} = \begin{pmatrix} c \frac{\mathrm{d}\gamma_{v(t)}}{\mathrm{d}\tau} \\ \frac{\mathrm{d}[\gamma_{v(t)}\vec{v}(t)]}{\mathrm{d}\tau} \end{pmatrix} \stackrel{\circ}{=} \begin{pmatrix} \gamma_{v}^{4} \frac{\vec{v} \cdot \vec{a}}{c} \\ \gamma_{v}^{2} \vec{a} + \gamma_{v}^{4} \frac{\vec{v} \cdot \vec{a}}{c^{2}} \vec{v} \end{pmatrix}$$

$$(4.49)$$



Here $\vec{a} := \frac{d\vec{v}(t)}{dt}$ is the coordinate acceleration or 3-acceleration.

It is now easy to show that $b^2 = b_{\mu}b^{\mu} < 0$ is a *space-like* Lorentz vector and that

$$\frac{\mathrm{d}(u^{\mu}u_{\mu})}{\mathrm{d}\tau} = \frac{\mathrm{d}(c^2)}{\mathrm{d}\tau} = 0 \quad \Rightarrow \quad u^{\mu}b_{\mu} = 0, \qquad (4.50)$$

i.e., the 4-acceleration is always "orthogonal" (in terms of the Minkowski metric) to the 4-velocity.

4.3. The complete Lorentz group

Details: Problemset 5

 The Lorentz group is a matrix group defined as the homogenous isometry group of the Minkowski metric η:

** Lorentz group:
$$O(1,3) := \left\{ \Lambda \in \mathbb{R}^{4 \times 4} \, \middle| \, \Lambda^T \eta \Lambda = \eta \right\}$$
 (4.51)

with identification $\Lambda^{\mu}{}_{\nu} = \Lambda_{\mu\nu}$ and $\eta_{\mu\nu} = \eta_{\mu\nu}$.

• As shown previously [Eq. (4.21) and Eq. (4.24)], the matrix constraint in Eq. (4.51) is equivalent to the property

$$\eta_{\mu\nu} x^{\mu} y^{\nu} \stackrel{\text{def}}{=} \eta(x, y) \stackrel{!}{=} \eta(\Lambda x, \Lambda y) \stackrel{\text{def}}{=} \left[\eta_{\rho\pi} \Lambda^{\rho}{}_{\mu} \Lambda^{\pi}{}_{\nu}\right] x^{\mu} y^{\nu}$$
(4.52)

for all 4-vectors x, y. Namely, the transformations Λ do not change the inner product (and thereby length) of arbitrary vectors; maps with this feature are called \uparrow *isometries*.

If you replace the Minkowski metric η_{μν} = diag (+1, -1, -1, -1) by the Euclidean metric δ_{μν} = diag (+1, +1, +1, +1), the homogeneous isometry constraint becomes Λ^T Λ = 1 since δ = 1 is the identity matrix; this constraint characterizes *orthogonal matrices*. The homogenous isometry group of a D = 4 Euclidean space is therefore O(4): the group of four-dimensional *rotations* and *reflections*.

2 Continuous Lorentz transformations:

i | Mathematical fact: O(1, 3) is a \uparrow Lie group (= a group that is also a differentiable manifold)

To be precise: O(1, 3) is a 6-dimensional (\rightarrow below) \uparrow non-compact \downarrow non-abelian disconnected (\rightarrow below) real matrix Lie group with components that are not \uparrow simply connected.

 \rightarrow In a neighborhood of 1, elements of Lie groups can be written as exponentials:

$$\Lambda = \exp(X) \quad \text{with} \quad X \in \mathfrak{o}(1,3) \tag{4.53}$$

where o(1, 3) denotes the \uparrow *Lie algebra* (= vector space with a Lie bracket):

$$\mathfrak{o}(1,3) = \left\{ X \in \mathbb{R}^{4 \times 4} \mid \exp(t X) \in \mathcal{O}(1,3) \text{ for all } t \in \mathbb{R} \right\}.$$
(4.54)



ii | The isometry constraint on the group elements can be translated into the Lie algebra:

$$\Lambda^{T} \eta \Lambda = \eta \quad \stackrel{\text{Eq. (4.53)}}{\longleftrightarrow} \quad X^{T} = -\eta X \eta \tag{4.55}$$

 \rightarrow Most general form of *X*:

$$X = \begin{pmatrix} 0 & a & b & c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{pmatrix} \quad \text{with} \quad a, \dots, f \in \mathbb{R}$$
(4.56)

Proof: 🔿 Problemset 5

 \rightarrow

• $\dim(\mathfrak{o}(1,3)) = 6$

This is why O(1, 3) is a 6-dimensional Lie group.

• $\operatorname{Tr}[X] = 0 \Rightarrow \det \Lambda = \det[\exp(X)] = \exp(\operatorname{Tr}[X]) = 1$

 \rightarrow All Lorentz transformations connected to the identity have positive determinant. Recall that we found previously det $\Lambda = \pm 1$, so we should not expect to find *all* elements of O(1, 3) in this way.

iii | <u>Generators</u> = Basis of o(1, 3) [58]:

We use the shorthand +(-) for +1(-1).

Interpretation:

 \rightarrow

$$\exp\left(\varphi \boldsymbol{L}_{x}\right) \stackrel{\circ}{=} \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cos\varphi & -\sin\varphi\\ 0 & 0 & \sin\varphi & \cos\varphi \end{pmatrix} = \boldsymbol{\Lambda}_{R_{x}(\varphi)} \quad \rightarrow \textit{Rotation around x-axis} \quad (4.58a)$$

$$\exp\left(-\theta \mathbf{K}_{x}\right) \stackrel{\circ}{=} \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0\\ -\sinh\theta & \cosh\theta & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{\Lambda}_{v_{x}} \quad \rightarrow Boost \text{ in } x \text{-direction} \qquad (4.58b)$$

with \leftarrow rapidity tanh $\theta = \frac{v_x}{c} \in (-1, 1)$ (\bigcirc Problemset 3) and rotation angle $\varphi \in [0, 2\pi)$.

L_x, L_y, L_z :	Generators of <i>rotations</i>	(4.59a)
K_x, K_y, K_z :	Generators of boosts	(4.59b)



An arbitrary element of O(1, 3) that is connected to the identity can then be written as

$$\mathbf{\Lambda} = \exp\left(\sum_{i} \varphi_{i} \mathbf{L}_{i} - \sum_{i} \theta_{i} \mathbf{K}_{i}\right) \quad \text{with} \quad i \in \{x, y, z\}.$$
(4.60)

In particular [58]:

Pure boost:
$$\Lambda_{\vec{v}} \equiv \Lambda_{\vec{\theta}} = \exp\left(-\vec{\theta} \cdot \vec{K}\right)$$
 (4.61a)

Pure rotation:
$$\Lambda_{R_{\vec{\varphi}}} = \exp\left(\vec{\varphi} \cdot \vec{L}\right)$$
 (4.61b)

with rotation angle $\varphi = |\vec{\varphi}|$, rotation axis $\hat{\varphi} = \vec{\varphi}/\varphi$, and rapidity vector

$$\vec{\theta} \equiv \vec{\theta}(\vec{v}) := \hat{v} \tanh^{-1}\left(\frac{v}{c}\right) \,. \tag{4.62}$$

i! The rapidity vector $\vec{\theta}$ is *not* given by the rapidities $\tanh^{-1} \frac{v_i}{c}$ of the components v_i of \vec{v} .

iv | Lie algebra:

The Lie bracket (= commutator) on the Lie *algebra* determines the multiplicative structure of the Lie *group* via the \checkmark *Baker-Campbell-Hausdorff formula*:

$$\exp(X) \cdot \exp(Y) = \exp\left(X + Y + \frac{1}{2}\left[X, Y\right] + \dots\right). \tag{4.63}$$

 \rightarrow The Lie algebra o(1, 3) determines the (local) group structure of O(1, 3):

Eq. (4.57) $\xrightarrow{\circ}$

$$\begin{bmatrix} L_i, L_j \end{bmatrix} = \varepsilon^{ijk} L_k \tag{4.64a}$$

$$\begin{bmatrix} L_i, K_j \end{bmatrix} = \varepsilon^{ijk} K_k \tag{4.64b}$$
$$\begin{bmatrix} K \cdot K \cdot \end{bmatrix} = -\varepsilon^{ijk} I_k \tag{4.64c}$$

$$[K_i, K_j] = -\varepsilon^{ij\kappa} L_k \tag{4.64c}$$

Some comments and implications:

• ¡! Because of Eq. (4.64) [and Eq. (4.63)], you *cannot* simply combine exponentials:

$$\exp\left(-\vec{\theta}\cdot\vec{K}\right)\cdot\exp\left(\vec{\varphi}\cdot\vec{L}\right)\neq\exp\left(\vec{\varphi}\cdot\vec{L}-\vec{\theta}\cdot\vec{K}\right),\qquad(4.65a)$$

$$\exp\left(-\vec{\theta}\cdot\vec{K}\right)\cdot\exp\left(-\vec{\theta}'\cdot\vec{K}\right)\neq\exp\left(-(\vec{\theta}+\vec{\theta}')\cdot\vec{K}\right)\,,\qquad(4.65b)$$

$$\exp\left(\vec{\varphi}\cdot\vec{L}\right)\cdot\exp\left(\vec{\varphi}'\cdot\vec{L}\right)\neq\exp\left(\left(\vec{\varphi}+\vec{\varphi}'\right)\cdot\vec{L}\right)\,.$$
(4.65c)

This is why the concatenation of Lorentz transformations is quite complicated in general.

- Eq. (4.64a) is written in physics often as $[L_i, L_j] = i\hbar\varepsilon^{ijk}L_k$ with angular momentum operators L_k . In this notation, they generate rotations $U_{\vec{\omega}} = \exp(\frac{i}{\hbar}\vec{\omega}\vec{L})$. The additional phase *i* in the commutation relation matches a corresponding factor in an alternative definition of the generators \vec{L} . (Recall that the L_i in Eq. (4.57) are *anti-Hermitian* whereas in physics we often prefer *Hermitian* operators.)
- Eq. (4.64a) shows that $o(3) := \text{span} \{L_x, L_y, L_z\}$ forms a *subalgebra* of o(1, 3). On the group level, this means that the group of spatial rotations SO(3) is a subgroup of the full Lorentz group O(1, 3).



By contrast, Eq. (4.64c) shows that the boost generators $\{K_x, K_y, K_z\}$ do *not* form a subalgebra, but mix with rotations. This implies that there is no "subgroup of pure boosts" in O(1, 3). In particular:

$$\Lambda_{\vec{v}}\Lambda_{\vec{u}} = \Lambda_{\vec{u}\oplus\vec{v}}\Lambda_{R(\vec{u},\vec{v})} \tag{4.66}$$

with the \leftarrow *Thomas-Wigner rotation* $R(\vec{u}, \vec{v}) \in SO(3)$ [recall Section 2.3].

• There is a more compact, 4-vector-inspired notation for the 6 generators in Eq. (4.57), namely [59]:

$$\left(J^{\alpha\beta}\right)^{\mu}_{\nu} \equiv \left(J^{\alpha\beta}\right)_{\mu\nu} := \eta^{\alpha\mu}\delta^{\beta}_{\nu} - \eta^{\beta\mu}\delta^{\alpha}_{\nu} \,. \tag{4.67}$$

Inspection shows that (O Problemset 5)

1

$$L_x = J^{23} = -J^{32}, \qquad K_x = J^{01} = -J^{10},$$
 (4.68a)

$$L_y = J^{31} = -J^{13}, \qquad K_y = J^{02} = -J^{20},$$
 (4.68b)

$$L_z = J^{12} = -J^{21}, \qquad K_z = J^{03} = -J^{30}.$$
 (4.68c)

The three equations of the Lie algebra Eq. (4.64) can then be condensed into a single equation [59]:

$$[\boldsymbol{J}^{\mu\nu}, \boldsymbol{J}^{\rho\sigma}] = \eta^{\nu\rho} \boldsymbol{J}^{\mu\sigma} - \eta^{\mu\rho} \boldsymbol{J}^{\nu\sigma} - \eta^{\nu\sigma} \boldsymbol{J}^{\mu\rho} + \eta^{\mu\sigma} \boldsymbol{J}^{\nu\rho} .$$
(4.69)

This form is useful to construct other representations of the Lorentz group, especially in relativistic quantum mechanics (\rightarrow *Dirac equation*).

 It is a useful mathematical fact that every continuous Lorentz transformation of the form Eq. (4.60) can be decomposed *uniquely* as follows:

$$\mathbf{\Lambda} = \mathbf{\Lambda}_{\vec{v}} \mathbf{\Lambda}_{R} = \mathbf{\Lambda}_{R} \mathbf{\Lambda}_{\vec{w}}$$
(4.70a)
with parameters:
$$\frac{v_{i}}{c} = -\frac{\mathbf{\Lambda}_{i0}}{\mathbf{\Lambda}_{00}}, \ \frac{w_{i}}{c} = -\frac{\mathbf{\Lambda}_{0i}}{\mathbf{\Lambda}_{00}} \text{ and } R_{ij} = \mathbf{\Lambda}_{ij} - \frac{\mathbf{\Lambda}_{i0} \mathbf{\Lambda}_{0j}}{1 + \mathbf{\Lambda}_{00}}$$
(4.70b)

 $\Lambda_{\vec{v}}$ and Λ_R are defined in Eq. (4.61a) [or Eq. (1.75)] and Eq. (4.61b) [or Eq. (1.40)].

The proof can be found in Ref. [60]. This decomposition, sometimes referred to as * rotation-boost decomposition, relates to the mathematical concept of \uparrow *Cartan decompositions* [61].

If we use the multiplicative law $\Lambda_R \Lambda_{\vec{v}} \Lambda_{R^{-1}} = \Lambda_{R\vec{v}}$ [recall Eq. (1.43a)] and choose R such that $R\vec{v} = (v_x, 0, 0)^T$, we can also find a decomposition of the form

$$\mathbf{\Lambda} = \mathbf{\Lambda}_{R_1} \mathbf{\Lambda}_{v_x} \mathbf{\Lambda}_{R_2} \tag{4.71}$$

with appropriately chosen rotations $R_1, R_2 \in SO(3)$ and a boost in x-direction by v_x .

3 | Discrete generators:



It is easy to verify that the following two matrices also belong to O(1, 3):

$$\overset{*}{*} \overset{Parity:}{P: (t, \vec{x}) \mapsto (t, -\vec{x})} \Rightarrow P^{\mu}_{\nu} \equiv P_{\mu\nu} := \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}$$

$$\overset{*}{*} \overset{Time reversal:}{T: (t, \vec{x}) \mapsto (-t, \vec{x})} \Rightarrow T^{\mu}_{\nu} \equiv T_{\mu\nu} := \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}_{\mu\nu}$$

$$(4.72)$$

In contrast to the continuous group elements above: det(P) = det(T) = -1

- $\rightarrow P$ and T are not generated by boosts or rotations!
- **4** | Structure of the Lorentz group:

Combining the discrete transformation P and T with the continuous transformations $\Lambda = \exp(X)$ yields the complete group O(1, 3). Let us study its structure:

 $\mathbf{i} \mid \det(\Lambda) = \pm 1 \rightarrow$

$$O(1,3) = \underbrace{L_+}_{\det(\Lambda)=+1} \cup \underbrace{L_-}_{\det(\Lambda)=-1}$$
(4.74)

All Lorentz transformations that are continuously connected to 1 are in L_+ . One can transition between L_+ and L_- by applying either T or P.

ii | In addition, we find:

$$1 = \eta_{00} \stackrel{4.21}{=} \left(\Lambda_{0}^{0}\right)^{2} - \sum_{k=1}^{3} \left(\Lambda_{0}^{k}\right)^{2} \le \left(\Lambda_{0}^{0}\right)^{2}$$
(4.75)

Thus $\Lambda_0^0 \neq 0$ and $\operatorname{sign}(\Lambda_0^0) = \pm 1$ can be used to characterize Lorentz transformations. Note that $\operatorname{sign}(P_0^0) = +1$ but $\operatorname{sign}(T_0^0) = -1$ and $\operatorname{sign}((PT)_0^0) = -1$.

iii | Neither det(Λ) = ±1 nor sign(Λ_0^0) = ±1 can be changed by continuously deforming a Lorentz transformation.

 \rightarrow Four disconnected components of O(1, 3):

$$L^{\uparrow}_{+}: \operatorname{det}(\Lambda) = +1 \quad \operatorname{and} \quad \operatorname{sign}(\Lambda^{0}_{0}) = +1 \quad (\mathbb{1} \in L^{\uparrow}_{+})$$
 (4.76a)

$$L_{-}^{\uparrow}: \quad \det(\Lambda) = -1 \quad \text{and} \quad \operatorname{sign}(\Lambda_{0}^{0}) = +1 \quad (P \in L_{-}^{\uparrow})$$
(4.76b)

$$L_{+}^{*}$$
: det $(\Lambda) = +1$ and sign $(\Lambda_{0}^{0}) = -1$ $(PT \in L_{+}^{\downarrow})$ (4.76c)

$$L^{\downarrow}_{\perp}$$
: det $(\Lambda) = -1$ and sign $(\Lambda^0_{\ 0}) = -1$ $(T \in L^{\downarrow}_{\perp})$ (4.76d)



Graphically:



iv | Subgroups: We can define the following four subgroups of O(1, 3):

** Proper LG:	$\mathrm{SO}(1,3) \equiv L_+ := L_+^{\uparrow} \cup L_+^{\downarrow}$	(4.77a)
** Orthochronous LG:	$\mathcal{O}^+(1,3) \equiv L^{\uparrow} := L^{\uparrow}_+ \cup L^{\uparrow}$	(4.77b)
** Proper orthochronous LG:	$\mathrm{SO}^+(1,3) \qquad := L_+^{\uparrow}$	(4.77c)
** Orthchorous LG:	$L_0 := L_+^{\uparrow} \cup L^{\downarrow}$	(4.77d)

Note that subgroups must contain the identity 1!

In Greek, "chrónos" ($\chi\rho\delta\nu\sigma\varsigma$) means "time" and "chóros" ($\chi\delta\rho\sigma\varsigma$) means "space". According to modern physics, Einstein's principle of relativity **SR** reads formally:

> All fundamental theories of nature must be invariant under the *proper orthochronous Lorentz group* $SO^+(1,3)$.

• This does not prevent specific theories to have *additional* symmetries. \uparrow *Quantum electrodynamics* (QED), for example, is invariant under the *full* Lorentz group O(1, 3). This means that phenomena of electromagnetism – and its interaction with charged particles – are also symmetric under time inversion T and parity P.

So far, observations suggest that, besides the electromagnetic force, also gravity and the strong force are symmetric under P and T. (Interestingly, there is no formal reason *why* the strong force should *not* break P and T; the fact that it does *not* violate these symmetries is called the \uparrow *strong CP problem*).

• However, today we know that there *are* terms in the standard model of particle physics that *violate* both P and T. For example, the weak interaction (responsible for radioactive β -decay) violates parity P strongly (\uparrow Wu experiment). This means that you can use



experiments that depend on the weak interaction to tell the difference between our world and its mirror image (or a right-handed and a left-handed coordinate system). There are also weak terms (concerning quarks) that violate time reversal T ($\uparrow CP$ *violation*). As a consequence, the standard model as a whole is only invariant under the proper orthochronous Lorentz group SO⁺(1, 3).

This explains why we can only require symmetry under $SO^+(1, 3)$, and not the full Lorentz group O(1, 3): We already *know* by experiments that the latter is *not* a fundamental symmetry of nature!

• The fact that there are processes that violate parity symmetry P contradicts our everyday experience: If you run an experiment using equipment found in a school physics lab and put a mirror next to it, there is no way to decide whether you are watching the experiment directly or via the mirror (i.e., parity inverted). The reason is that the physics we experience in everyday life is goverened by electrodynamics and gravity, both of which are invariant under P. To unveil that nature secretly violates P, you must perform an experiment that involves the weak interaction (that is: a particle physics experiment). This is what Chien-Shiung Wu did in her now famous $\uparrow Wu$ experiment. At the time, the result (that P is not a symmetry of nature) was unexpected and groundbreaking.

So if you are surprised that P is not a symmetry of nature, you are not alone. Here is how Wolfang Pauli reacted to the result of the Wu experiment [62]:

At one point, Temmer found himself in the presence of eminence grise Wolfgang Pauli, who asked for the latest news from the United States. Temmer told him that parity was no longer to be assumed conserved. "That's total nonsense" averred the great man. Temmer: "I assure you the experiment says it is not." Pauli (curtly): "Then it must be repeated!"

4.4. ‡ Why is spacetime 3+1 dimensional?

Given the discussions in Chapter 3 and Chapter 4 it is clear that the mathematical formalism allows for straightforward generalizations to higher- (or lower-) dimensional spacetime manifolds with arbitrary signatures; these suggest spacetimes with various numbers of spatial and temporal dimensions.

It is therefore natural to ask:

Is there anything special about our 3 + 1-dimensional world?

What follows is not a *proof* that spacetime must be 3 + 1 dimensional. Our goal is to argue that all spacetimes, except ours with *three* space and *one* time dimension, face severe problems that, most likely, would not allow for complex life.

The following discussion is based on Tegmark [40, 63].

1 \triangleleft Pseudo-Riemannian manifold of signature (t, s) with metric

$$g_{ij} = \operatorname{diag}(\underbrace{+1, \dots, +1}_{t}, \underbrace{-1, \dots, -1}_{s})$$
(4.78)

- This is the generalization of Minkowski space to a (flat) spacetime manifold with, naïvely, *t* time and *s* space-dimensions.
- Most of our discussions in this chapter can be transferred to this more general setting.



$$\left(\partial^{2} + m^{2}\right)\Phi = \underbrace{\sum_{i=1}^{t} \frac{\partial^{2}\Phi}{\partial x^{i^{2}}}}_{t \times \text{Time}\,(?)} - \underbrace{\sum_{i=t+1}^{s+t} \frac{\partial^{2}\Phi}{\partial x^{i^{2}}}}_{s \times \text{Space}\,(?)} + m^{2}\Phi = 0 \tag{4.79}$$

- Recall that $\partial^2 = g^{ij} \partial_i \partial_j$ where g^{ij} is given by (the inverse of) Eq. (4.78).
- The Klein-Gordon equation (KGE) is the simplest covariant field equation. It describes the time evolution of a scalar field of mass m. It is ubiquitous in relativistic physics (especially in \uparrow quantum field theory).
- For example, the components of the electromagnetic field in vacuum are described by the KGE for m = 0 and (t, s) = (1, 3) (which is then referred to as \checkmark *wave equation*):

$$\partial^2 E_i = \frac{1}{c^2} \partial_t^2 E_i - \nabla^2 E_i = 0, \qquad (4.80a)$$

$$\partial^2 B_i = \frac{1}{c^2} \partial_t^2 B_i - \nabla^2 B_i = 0.$$
(4.80b)

This motivates in Eq. (4.79) the (tentative) identification of the coordinates with positive sign as "time coordinates", and the ones with a negative sign as "space coordinates":

The difference between time and space is just a sign!

In the following, we use the KGE as a proxy for more general relativistic field equations.

 \rightarrow Possible combinations of t time and s space dimensions:



3 Partial differential equations (PDE):

The general KGE in Eq. (4.79) is an example of a partial differential equation (PDE). The theory of PDEs has been thoroughly developed by mathematicians and a lot is known about their solvability. The problem of solving a PDE, given some boundary/initial conditions, is known as \uparrow *Cauchy* problem:

- * Well-posed (Cauchy) problem: Given some boundary/initial data, there exists a unique solution to the PDE that satisfies these conditions, and this solution is *robust*. Here "robust" means that if you slightly modify the boundary/initial conditions, the solution also changes only slightly. Put differently: The solutions are not *chaotic* and you can use them to extrapolate reliably from boundary/initial states with finite errorbars. This is a crucial feature to use PDEs for *predictions* in the real world.
- * Ill-posed (Cauchy) problem: Given some boundary/initial data, there either exist multiple • solutions to the PDE that satisfy these conditions, or the unique solution is not robust. In both cases, the PDE cannot be used for predictions in the real world.

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 $\mathbf{i} \mid \mathbf{i} \in (t = 0, s) \text{ or } (t, s = 0) \rightarrow \text{Eq. } (4.79) = \uparrow \text{ Elliptic PDE}$

This corresponds to spacetimes that are \leftarrow *Riemannian manifolds*.

Elliptic PDEs have well-posed boundary problems:



- \rightarrow One *cannot* use Eq. (4.79) to make predictions \odot
- \rightarrow No coordinate in Eq. (4.79) qualifies as a "time coordinate".



ii $| \langle (t \ge 2, s \ge 2) \text{ or } (t \ge 2, s \ge 2) \rightarrow \text{Eq. } (4.79) = \uparrow Ultrahyperbolic PDE$

This corresponds to spacetimes that are generic ← *pseudo-Riemannian manifolds*.

A similar but more involved chain of arguments holds also for ultrahyperbolic PDEs [40,63].

 \rightarrow One *cannot* use Eq. (4.79) to make predictions \odot



iii | $(t = 1, s \ge 1)$ or $(t \ge 1, s = 1) \rightarrow \text{Eq.}(4.79) = \uparrow$ Hyperbolic PDE This corresponds to spacetimes that are \leftarrow Lorentzian manifolds.

Hyperbolic PDEs have well-posed *initial value problems*:





 \rightarrow We *can* use Eq. (4.79) to make predictions \bigcirc

- 4 | Stability:
 - \triangleleft Newtonian Gravity in $s \ge 4$ spatial dimensions:
 - $\rightarrow \downarrow$ *Two-body problem* has no stable orbits (only scattering and attraction solutions).
 - \rightarrow No stable planetary systems possible \circledast
 - \triangleleft Hydrogen atom in $s \ge 4$ spatial dimensions:
 - \rightarrow Schrödinger equation has no bound states.
 - \rightarrow No stable atoms possible \odot

The opposite cases with $t \ge 4$ and s = 1 are equivalent if one interprets space as time and vice versa (which is necessary to use hyperbolic PDEs to predict "the future", \rightarrow *below*).





5 | Simplicity:

GENERAL RELATIVITY in $s \leq 2$ spatial dimensions \rightarrow No gravity (\rightarrow *later*)!

 \rightarrow No stars, no planets, no orbits \circledast

The opposite cases with $t \le 2$ and s = 1 are equivalent if one interprets space as time and vice versa (which is necessary to use hyperbolic PDEs to predict "the future", \Rightarrow *below*).

 \rightarrow





6 | Tachyon world:

i! In the literature both Lorentzian signatures (1, 3) and (3, 1) are used to formulate SPECIAL REL-ATIVITY. Formulations in signature (3, 1) have nothing to do with the Tachyon sector discussed here since they compensate for the global minus in their equations. For example, the KGE in signature (-, +, +, +,) reads $(-\partial^2 + m^2)\Phi = 0$ which is equivalent to the KGE $(\partial^2 + m^2)\Phi = 0$ in signature (+, -, -, -). The point here is that we do *not* add this additional minus:

Eq. (4.79)
$$\xrightarrow[\text{Time} \Leftrightarrow \text{Space}]{} (-\partial^2 + m^2)\Phi = 0 \Leftrightarrow (\partial^2 - m^2)\Phi = 0 \quad (4.81)$$

In more detail:

For t = 3 and s = 1 the KGE reads

$$\underbrace{\frac{\partial^2 \Phi}{\partial (x^1)^2} + \frac{\partial^2 \Phi}{\partial (x^2)^2} + \frac{\partial^2 \Phi}{\partial (x^3)^2}}_{3 \times \text{Time (?)}} - \underbrace{\frac{\partial^2 \Phi}{\partial (x^4)^2}}_{1 \times \text{Space (?)}} + m^2 \Phi = 0.$$
(4.82)

But because this an hyperbolic PDE, the Cauchy problem is only well-posed with initial conditions on a hypersurface spanned by $\{x^1, x^2, x^3\}$. Put differently: The PDE allows predictions only in x^4 -direction! Thus we should interpret x^4 as *time* and $\{x^1, x^2, x^3\}$ as *space*:

$$\underbrace{\frac{\partial^2 \Phi}{\partial (x^1)^2} + \frac{\partial^2 \Phi}{\partial (x^2)^2} + \frac{\partial^2 \Phi}{\partial (x^3)^2}}_{3 \times \text{Space } (!)} - \frac{1}{c^2} \underbrace{\frac{\partial^2 \Phi}{\partial t^2}}_{1 \times \text{Time } (!)} + m^2 \Phi = 0$$
(4.83)

with $ct \equiv x^4$. But this KGE is equivalent to

$$\left(\frac{1}{c^2}\partial_t^2 - \nabla^2 - m^2\right)\Phi = (\partial^2 - m^2)\Phi = 0.$$
(4.84)

Thus the "transposed" situation ($t \ge 1$, s = 1) is equivalent to the situation (t = 1, $s \ge 1$) with *negative* square-masses in the equations. Fields with negative square-mass (equivalently: imaginary mass) are called \uparrow *tachyonic fields* or \uparrow *tachyons* for short.

 \rightarrow All massive particles are \uparrow *tachyons* [64]

i! Tachyonic fields are not science fiction; they do exist $(\rightarrow below)$ but, contrary to the features assigned to them in science fiction, do *not* allow for faster-than-light propagation of information.

\rightarrow Tachyon fields herald vacuum instabilities [65] \otimes

The spontaneous symmetry breaking of the \uparrow *Higgs mechanism* is an example of this phenomenon: The Higgs field has a negative square-mass which is responsible for the "Mexican hat potential."

 \rightarrow



The consequence is spontaneous symmetry breaking, which, in this context, can be reframed as "tachyon condensation." On the new, symmetry broken vacuum, excitations are *not* tachyons with negative square-mass but Higgs bosons with positive square-mass.



7 | These arguments support the following *hypothesis*:

Only a spacetime with 1 time and 3 space dimensions supports observers like us.

What does this line of arguments explain? Well, if you would randomly construct universes by dicing the number of space and time dimensions, only the ones with *one time* and *three space* dimensions have the chance to develop complex observers like us (who then wonder why their universe is 3 + 1-dimensional). Thus the arguments above are important for "ensemble interpretations" of reality, like certain \uparrow *multiverse hypotheses* or superstring theories (which can predict a plethora of different spacetime dimensions) [40, 66].

↓Lecture 10 [19.12.23]



5. Relativistic Mechanics

Equipped with the machinery of Chapter 4, we can finally construct a relativistic (Lorentz covariant) version of classical mechanics.

5.1. The relativistic point particle

1 | *⊲* Point particle in $\mathbb{R}^{1,3}$ with trajectory $x^{\mu}(\tau)$:



2 | It is reasonable to define the relativistic momentum of a massive particle as follows:

** 4-momentum :
$$p^{\mu} := mu^{\mu} = m \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \begin{pmatrix} m\gamma_{v}c\\ m\gamma_{v}\vec{v} \end{pmatrix} \equiv \begin{pmatrix} p^{0}\\ \vec{p} \end{pmatrix}$$
 (5.1)
with (rest) mass *m* and ** *3-momentum* \vec{p} . (5.2)

• i! The mass *m* is the good old (inertial) mass we would assign to the particle in classical mechanics; it is a measure of the particles resistance to changes in its state of motion. You can determine it by applying a (weak) force to the particle *at rest* and observing its initial acceleration:
$$m = F/a$$
. This mass is an intrinsic property of the particle and does not depend on velocity. It is sometimes called *rest mass*, but we will simply call it *mass*.

- Since the 4-velocity u^μ is a Lorentz vector, the 4-momentum is also a Lorentz vector; i.e., under a Lorentz transformation Λ the 4-momentum transforms as p^μ = Λ^μ_ν p^ν.
- We will later rederive the expression for the 4-momentum as the conserved \downarrow *Noether charge* for translations in spacetime.
- **3** | The spatial part of the momentum (the 3-momentum \vec{p}) is related to the velocity as follows:

$$\vec{p} = m\gamma_{v}\vec{v} = \frac{m\vec{v}}{\underbrace{\sqrt{1 - \frac{v^{2}}{c^{2}}}}_{\text{RELATIVITY}}} \xrightarrow{\beta \ll 1} \underbrace{m\vec{v}}_{\text{Newtonian}} \underbrace{m\vec{v}}_{\text{mechanics}}$$
(5.3)



- i! In SPECIAL RELATIVITY the kinetic momentum is no longer proportional to the velocity. In particular for $v \rightarrow c$ the momentum of a massive particle *diverges*.
- The non-relativistic limit ($v \ll c \Rightarrow \beta \ll 1 \Rightarrow \gamma_v \approx 1$) is consistent with the Newtonian (non-relativistic!) relation $\vec{p} = m\vec{v}$ for the kinetic momentum; the 3-momentum \vec{p} is therefore the proper relativistic version of the momentum in Newtonian mechanics.
- This explains why the above definition for the 4-momentum is reasonable and why the mass *m* must be identified with the mass used in Newtonian mechanics.
- At this point it is unclear how to interpret the time-component $p^0 = m\gamma_v c$ of $p^{\mu} (\Rightarrow below)$. Eq. (4.48)

$$p^{2} = p^{\mu} p_{\mu} = (p^{0})^{2} - \vec{p}^{2} \stackrel{\text{def}}{=} m^{2} u^{2} \stackrel{4.48}{=} m^{2} c^{2} > 0$$
(5.4)

- \rightarrow The mass *m* is a *Lorentz scalar*: $m^2 = p^2/c^2$
 - The 4-momentum is a *time-like* 4-vector for massive particles.
 - This means that the mass *m* can be measured/computed in every inertial system by measuring/computing the 4-momentum p^{μ} and its pseudo-norm p^2 . The numerical result will always be the same, namely m^2c^2 .

5 | Equation of motion (EOM):

- i We want an EOM that ...
 - ... is manifestly Lorentz covariant \rightarrow Lorentz tensor equation
 - ...reduces to Newton's equation of motion

$$m\vec{a} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F} \quad \text{with} \quad \vec{p} = m\vec{v}$$
 (5.5)

in the non-relativistic limit (correspondence principle).

ii | Suggestion:

$$mb^{\mu} = \frac{\mathrm{d}p^{\mu}}{\mathrm{d}\tau} = K^{\mu} \equiv \begin{pmatrix} K^{0} \\ \vec{K} \end{pmatrix} \quad \text{with} \quad \overset{*}{*} \text{ 4-force } K^{\mu} \,. \tag{5.6}$$

Because this is a equation built from Lorentz vectors, it is form-invariant (Lorentz covariant) by construction:

$$mb^{\mu} = K^{\mu} \Leftrightarrow m\Lambda^{\nu}{}_{\mu}b^{\mu} = \Lambda^{\nu}{}_{\mu}K^{\mu} \Leftrightarrow m\bar{b}^{\nu} = \bar{K}^{\nu}$$
(5.7)

This is of course only so if the 4-force transforms like a Lorentz vector.

iii | \triangleleft Instantaneous rest frame (IRF) K_0 :

a | At any time there is an inertial coordinate system K_0 in which the (potentially accelerated) particle is at rest *at this very moment* (if the particle is accelerating, it is also accelerating in this frame).

$$mb_0^{\mu} \stackrel{4.49}{=} \begin{pmatrix} 0\\ m\vec{a}_0 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0\\ \vec{F}_0 \end{pmatrix} = K_0^{\mu}$$
 (5.8)



This follows from the correspondence principle: In the IRF the particle is in the non-relativistic, Newtonian limit. Thus its coordiante acceleration \vec{a}_0 must be given by Newton's equation of motion: $m\vec{a}_0 = \vec{F}_0$.

with

• ** Proper acceleration \vec{a}_0

The proper acceleration is the coordinate accelaration (3-acceleration) that you can measure (e.g., with an accelerometer) in the IRF K_0 of the particle.

It follows immediately that the norm of the proper acceleration is a Lorentz scalar:

$$b^2 = b^{\mu} b_{\mu} = -|\vec{a}_0|^2 < 0 \tag{5.9}$$

• ** Proper force \vec{F}_0

The proper force is the Newtonian force (3-force) you can measure (e.g., with a spring balance) in the IRF K_0 of the particle.

b | We *demand* that this equation is Lorentz covariant, i.e., that b_0^{μ} and K_0^{μ} transform as contravariant Lorentz 4-vectors. We can then use a Lorentz boost to transform back into the lab frame in which the particle has coordinate velocity \vec{v} :

Eq. (1.75)

4-acceleration:
$$b^{\mu} = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu} b^{\nu}_{0} \stackrel{1.75}{=} \begin{pmatrix} \gamma_{v} \frac{\vec{a}_{0} \cdot \vec{v}}{c} \\ \vec{a}_{0} + \frac{\gamma_{v} - 1}{v^{2}} (\vec{a}_{0} \cdot \vec{v}) \vec{v} \end{pmatrix}$$
 (5.10a)

4-force:
$$K^{\mu} = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu} K^{\nu}_{0} \stackrel{1.75}{=} \begin{pmatrix} \gamma_{v} \frac{\vec{F}_{0} \cdot \vec{v}}{c} \\ \vec{F}_{0} + \frac{\gamma_{v} - 1}{v^{2}} (\vec{F}_{0} \cdot \vec{v}) \vec{v} \end{pmatrix}$$
 (5.10b)

We will use these expressions later!

- iv | On the other hand, we can return to Eq. (5.6) and study the 4-force K^{μ} in more detail:
 - **a** | \triangleleft Spatial components of Eq. (5.6):

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}\tau} = \gamma_{v(t)}\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{K} \quad \Leftrightarrow \quad \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\vec{K}}{\gamma_{v}} =: \vec{F} \quad \Leftrightarrow \quad \vec{K} = \gamma_{v}\vec{F} \quad (5.11)$$

with ** 3-force \vec{F} .

Here $\frac{d\vec{p}}{dt}$ denotes the change in momentum measured in coordinate time; it makes sense identify this quantity with the relativistic analog of the Newtonian force.

b | What is the time component K^0 of the 4-force? \triangleleft

$$0 \stackrel{4.50}{=} m b^{\mu} u_{\mu} \stackrel{5.6}{=} K^{\mu} u_{\mu} = K^{0} u^{0} - \vec{K} \cdot \vec{u} \stackrel{4.47}{=} \gamma_{v} (K^{0} c - \vec{K} \cdot \vec{v})$$
(5.12)

 \rightarrow

$$K^{0} = \frac{\vec{K} \cdot v}{c} \stackrel{5.11}{=} \frac{\gamma_{v}}{c} \vec{F} \cdot \vec{v}$$
(5.13)

c | In summary, the 4-force in terms of the 3-force and the 3-velocity reads

4-force:
$$K^{\mu} = \begin{pmatrix} \gamma_v \frac{\vec{F} \cdot \vec{v}}{c} \\ \gamma_v \vec{F} \end{pmatrix}$$
 (5.14)

Example:

In our discussion of electrodynamics (\rightarrow *Chapter* **6**) we will find the following expression for the 3-force acting on a charged particle in an electromagnetic field:

$$\vec{F} = q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}$$
(5.15)

This is the conventional \checkmark *Lorentz force*.

This example demonstrates that the 3-force \vec{F} is indeed the proper relativistic analog of Newtonian forces. Note, however, that it is only the *component* of the 4-force and thus does not transform nicely under Lorentz transformations.

d | Spatial part of Eq. (5.6)
$$\xrightarrow{\text{Eq.}(5.14)}$$

$$\vec{F} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(m\gamma_v \vec{v} \right) = \langle \text{Change in 3-momentum} \rangle$$
 (5.16)

The Newtonian equation $\vec{F} = \frac{d\vec{p}}{dt}$ therefore remains valid in SPECIAL RELATIVITY for the 3-force \vec{F} and the 3-momentum \vec{p} . By contrast, $\vec{p} = m\gamma_v \vec{v}$ is different from the Newtonian relation $\vec{p} = m\vec{v}$ between momentum and velocity.

e | Temporal part of Eq.
$$(5.6)$$
 $\xrightarrow{Eq. (5.14)}$

$$\frac{\mathrm{d}p^{0}}{\mathrm{d}\tau} = \gamma_{v} \frac{\mathrm{d}p^{0}}{\mathrm{d}t} = \gamma_{v} \frac{\vec{F} \cdot \vec{v}}{c} \quad \Rightarrow \quad \frac{\mathrm{d}(cp^{0})}{\mathrm{d}t} = \vec{F} \cdot \vec{v} \tag{5.17}$$

 $\rightarrow \vec{F} \cdot \vec{v}$: Work performed by \vec{F} on particle

 $\rightarrow E = c p^0$: Total energy of particle

Note that we can actually only conclude $E = cp^0 + \text{const}$ from the differential equation above. We will later see that the constant must be set to zero because p^0 is the conserved Noether charge that derives from time translations.

The time component of the EOM Eq. (5.6) can therefore be written as:

$$\vec{F} \cdot \vec{v} = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left(m\gamma_v c^2 \right) = \langle \text{Change in energy} \rangle$$
 (5.18)

We will discuss the expression for the energy in Section 5.2 below.

6 | Above we expressed the 4-force in terms of the proper force \vec{F}_0 and in terms of the 3-force \vec{F} . Equating the two expressions yields a relation between the 3-force \vec{F}_0 measured in the IRF and the 3-force \vec{F} measured in the lab frame:



(5.19)

Eq. (5.10b) & Eq. (5.14) \rightarrow

3-force
$$\vec{F}$$
 as function of proper force \vec{F}_0 and velocity \vec{v} :
$$\vec{F} = \frac{\vec{F}_0}{\gamma_v} + \left(1 - \frac{1}{\gamma_v}\right) \frac{\vec{F}_0 \cdot \vec{v}}{v^2} \vec{v}$$

Recall that the proper force is the Newtonian force you would measure with a spring scale in the IRS of the particle. In contrast to Newtonian mechanics, the force \vec{F} measured from a frame in relative motion is *different* from $\vec{F_0}$. In the non-relativistic limit $\gamma_v \approx 1$ we find $\vec{F} \approx \vec{F_0}$ and this distinction becomes irrelevant (as assumed by Newtonian mechanics).

7 | A similar comparison yields a relation between the 3-acceleration in the IRF (the proper acceleration) and the 3-acceleration in the rest frame:

Eq. (5.10a) & Eq. (4.49) \rightarrow

3-acceleration
$$\vec{a}$$
 as function of proper acceleration \vec{a}_0 and velocity \vec{v} :
 $\vec{a} \stackrel{\circ}{=} \frac{1}{\gamma_v^2} \left[\vec{a}_0 - \left(1 - \frac{1}{\gamma_v} \right) \frac{\vec{v} \cdot \vec{a}_0}{v^2} \vec{v} \right]$
(5.20)

This is again in sharp contrast to Newtonian mechanics where, as a consequence of absolute time, acceleration does not depend on the velocity of the reference frame. In the non-relativistic limit for $\gamma_v \approx 1$ we find $\vec{a} \approx \vec{a}_0$, consistent with Newtonian mechanics.

8 | Sanity check:

If we integrate the equation of motion Eq. (5.16), we find:

$$\int_{0}^{T} \vec{F} dt = \frac{m\vec{v}_{T}}{\sqrt{1 - \frac{v_{T}^{2}}{c^{2}}}} - \text{const}.$$
(5.21)

For a finite 3-force $|\vec{F}| < \infty$ and finite time $T < \infty$, and non-zero mass $m \neq 0$, it follows for the final velocity \vec{v}_T :

$$\frac{m|\vec{v}_T|}{\sqrt{1 - \frac{v_T^2}{c^2}}} < \infty \quad \Rightarrow \quad |\vec{v}_T| < c \,. \tag{5.22}$$

Thus the dynamics does not allow massive particles to reach the speed of light, no matter how strong the force or how long the acceleration! This is in direct contradiction to Newtonian mechanics and by now experimentally well-confirmed (\rightarrow *below*).

5.2. Momentum, Energy, and Mass

9 | To summarize, the 4-momentum of a *massive particle* can be written as:

$$p^{\mu} = m u^{\mu} = \begin{pmatrix} p^{0} \\ \vec{p} \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix} = \begin{pmatrix} \gamma_{v} m c \\ \gamma_{v} m \vec{v} \end{pmatrix}$$
(5.23)

10 | The relativistic energy of a *massive particle* is then (as a function of 3-velocity):

** Relativistic energy:
$$E = cp^0 = \gamma_v mc^2 = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$
 (5.24)

With

$$m^2 c^2 \stackrel{5.4}{=} p^2 = (p^0)^2 - (\vec{p})^2 = E^2/c^2 - \vec{p}^2$$
 (5.25)

we find the alternative expression as a function of 3-momentum:

** Energy-momentum relation:
$$E = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$$
 (5.26)

- This expression is also valid in the massless case m = 0 (\rightarrow below).
- Eq. (5.25) has actually two solutions: E = ±√p²c² + m²c⁴. In relativistic mechanics (and relativistic single-particle quantum mechanics), we can ignore the negative energy solution and consider only time-like 4-momenta p^μ that point into the future light-cone. In quantum field theory, where interacting particles can be destroyed and produced, these negative energy solutions necessitate the introduction of ↑ *antiparticles* (like the positron).
- For fixed mass m, Eq. (5.25) determines a 3-dimensional hypersurface in the 4-dimensional "energy-momentum space" spanned by 4-momenta p^µ = (p⁰, p) ∈ ℝ⁴. For m ≠ 0 this hypersurface is a hyperboloid of two sheets E = ±√p²c² + m²c⁴ (for m = 0 it is a cone: E = ±c|p|). This hypersurface is called *** mass shell*. If a 4-momentum satisfies the energy-momentum relation (with either sign) we say that it is "on-shell"; if not, it is "off-shell". In quantum field theory, real particles that can be measured are always on-shell; intermediate "virtual particles" in scattering processes can be off-shell.

11 | Rest energy:

i | \triangleleft Rest frame K_0 of the particle where $\vec{p} = 0$:

$$p_0^{\mu} = \begin{pmatrix} p_0^0 \\ \vec{0} \end{pmatrix} = \begin{pmatrix} E_0/c \\ \vec{0} \end{pmatrix}$$
(5.27)

For these considerations, it does not matter whether the particle is accelerating and this is an IRF, or whether the particle is in inertial motion and has a fixed rest frame. Formally, since $p^2 = m^2 c^2 > 0$ is a time-like Lorentz vector, there is always an inertial frame in which $p^0 \neq 0$ and $\vec{p} = 0$.

 \rightarrow

** Rest energy :
$$E_0 = mc^2$$
 (5.28)

This is Einstein's famous principle of equivalence of (inertial) mass and (rest) energy.

• i! The total energy E is the time-component of a 4-vector: $p^{\mu} = (E/c, \vec{p})^T$; thus it makes sense to refer to the rest energy E_0 – which is the component of this 4-vector in the rest frame K_0 , i.e., the particular frame where $\vec{p} = \vec{0}$.



- i! By contrast, the *mass* is a *Lorentz scalar*, namely $p^2 = m^2 c^2$; hence it is the same in all inertial systems and it does not make sense to refer to the *rest mass* m_0 as this term suggests that there is a "non-rest mass" (which there isn't).
- Einstein first derived the mass-energy equivalence in his Annus Mirabilis paper *Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig?* [11]. In the paper, the equation is not given verbatim but encoded in the following statement:

Gibt ein Körper die Energie L in Form von Strahlung ab, so verkleinert sich seine Masse um L/V^2 .

Einstein concludes:

Die Masse eines Körpers ist ein Maß für dessen Energieinhalt; [...]. Es ist nicht ausgeschlossen, daß bei Körpern, deren Energie in hohem Maße veränderlich ist (z.B. bei den Radiumsalzen), eine Prüfung der Theorie gelingen wird.

Einstein further elaborates on the relativistic energy relation and its implications in [67]. He provides self-contained step-by-step derivation in Ref. [68]. Additional insight was provided over the years with alterantive derivations by various authors [69–71].

The derivation by Feigenbaum and Mermin in [71] is particularly insightful as it follows Einsteins original derivation in [11] closely without invoking electrodynamics. They also demonstrate that the heart of relativistic mechanics is actually Eq. (5.24) (where mc^2 appears as a *coefficient*), and not Eq. (5.28) (which is conventional).

• Note 1: Some comments on $E_0 = mc^2$

Eq. (5.28) is arguably the most famous equation in physics. The popularization of scientific concepts is often accompanied by simplifications and distortions. This is also the case for $E_0 = mc^2$:

- $E_0 = mc^2$ is often written as $E = mc^2$. This is either wrong or misleading (depending on the interpretation of the symbols); in any case, it is not consistent with modern conventions in RELATIVITY (\Rightarrow below).
- $E_0 = mc^2$ is by no means Einstein's most important equation. This is why it is not refered to as "Einstein equation;" this honor goes to

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}$$
(5.29)

which are also known as the \rightarrow *Einstein field equations*; these form the basis of GENERAL RELATIVITY and are empirically of much greater value than Eq. (5.28). Luckily, the Einstein field equations look daunting and are not nearly as accessible as $E_0 = mc^2$; hence they weren't seized (and multilated) by pop culture like $E_0 = mc^2$ was.

• How statements are phrased determines our conceptualization of the world. The often heard phrase

" $E_0 = mc^2$ says that mass can be converted into energy"

makes me think of "mass" as a sort of coal that can be lighted and then produces energy (maybe in form of light and heat or an atomic explosion). I am quite convinced that there are many who got "conceptually derailed" by statements like this, and hence think of Einstein's revelation as modern-day equivalent of an early human realizing, perhaps by witnessing a lightning strike, that wood can be kindled to produce heat. This is completely off the mark. $E_0 = mc^2$ says that rest energy and inertial mass are *equivalent*; not that they can be "converted" into each other. It means that the Lorentz symmetry of spacetime necessitates that our concepts of "energy" (as a quantity that can make things change in time) and "inertial mass" (as a quantity that measures how hard it is to make the state of motion of an object change in time) are like two sides of the same coin. Note that we did not arrive at the equation by studying the microscopic dynamics and interactions of matter (like we do in quantum mechanics, and especially quantum field theory); the equivalence of rest energy and mass is a consequence of the symmetries of spacetime alone. One can take $E_0 = mc^2$ thus as a hint at the unanswered questions "What is time?" and "What is inertia?" because energy is the generator of time translations (think of the time-evolution operator in quantum mechanics) and mass quantifies the phenomenon of inertia.

To drive the point home, here a few examples:

- An atom in an excited electronic state is heavier than the same atom in the ground state.
- A battery gets lighter when being discharged.
- A chunk of metal is heavier when it is hot.
- If you put an atomic bomb into an opaque, completely sealed "super box" that survives the explosion, the weight of the box does not change when the bomb goes off. This makes it clear that mass is not "converted" into energy.
- If the box is made out of "super glass" that lets only photons escape, the box gets lighter by E_{phot}/c^2 if the photons carry away the energy E_{phot} .
- For these reasons, $E_0 = mc^2$ is *not* a magical blueprint to build atomic bombs. The equation is only relevant in this context because it provides a nice "shortcut" to compute the energies that the fission (splitting) of isotopes can yield (or cost, depending on the isotopes). Because one could measure the rest masses of isotopes rather easily (using mass spectrometry [72]) – but had almost no clue how to describe the inner workings (and therefore binding energies) of said nulei - the equation allowed for a straightforward survey of the periodic table to identify suitable isotopes that would yield energy under fission. $E_0 = mc^2$ is not the reason why atomic weapons work, and these weapons are not so powerful "because they convert mass into energy." This is pure nonsense. If you discharge the battery of your phone, it also looses mass - because rest energy and mass are equivalent: $E_0 = mc^2$! And yes, this mass difference is much smaller than the mass difference accompanied by a nuclear explosion. But this is not the *reason*; the reason is that the strength of electromagnetic interactions – which govern chemical processes (like discharging your battery) - is dwarfed by the strength of the strong interaction (and its residual, the nuclear force) - which governs nuclear reactions.

In a nutshell:

When studying reaction processes (of any sort), the change of restmass predicted by $E_0 = mc^2$ is an \uparrow *epiphenomenon*. The mass change is not causal; it cannot be, because it is a consequence of the symmetries of spacetime, and not of the inner workings of matter.

ii | Unfortunately, the notation and interpretation of SPECIAL RELATIVITY has changed since its inception. In former times it was conventional to introduce the concept of a

** Relativistic mass:
$$m_r := \gamma_v m = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$
 (5.30)



which depends on velocity. With this definition, the relativistic relation between 3-velocity and 3-momentum reads $\vec{p} = m_r \vec{v}$ and parallels the Newtonian relation $\vec{p} = m\vec{v}$. The relativistic energy relation then reads $E = m_r c^2$.

The concept of a velocity-dependent, relativistic mass is avoided in most modern treatments of RELATIVITY (and in this script). While this is mostly a matter of concepts and semantics, there are good reasons why the concept of a velocity dependent mass is less useful than it might seem (\rightarrow below).

Here a few comments on various notations that you might encounter:

$$E_0 = mc^2$$
Correct (a) $E = mc^2$ Only makes sense if $m = m_r$ (which we don't use). $E_0 = m_0 c^2$ Why m_0 ? There is only $m!$ $E = m_0 c^2$ Energy is frame-dependent. Do you mean E_0 ? Otherwise: Wrong!

For more details and explanations see Refs. [73-75].

iii $| \rightarrow$ Take home message:

There is only one mass: the rest mass m (which we call *mass*). Thus mass does *not* depend on velocity.

This convention is used by almost all modern textbooks on RELATIVITY.

Unfortunately the old conventions (using relativistic, velocity-dependent masses) are still used by school books and popular science books.

iv Aside: Why introducing velocity depended masses leads nowhere.

If you are still inclined to think in terms of a velocity-dependent, relativistic mass m_r , here is a compelling argument why this is a useless and artificial concept that needs to die:

The 3-component of the relativistic equation of motion Eq. (5.16) reads

$$\vec{F} = \frac{\mathrm{d}}{\mathrm{d}t} \left(m\gamma_{v}\vec{v} \right) = m\gamma_{v}\vec{a} + m\gamma_{v}^{3}\frac{\vec{v}\cdot\vec{a}}{c^{2}}\vec{v}$$
(5.31)

with two extreme cases:

$$\vec{v} \parallel \vec{a} \implies \vec{F} \stackrel{\circ}{=} m \gamma_v^3 \vec{a}$$
 (5.32a)

$$\vec{v} \perp \vec{a} \quad \Rightarrow \quad \vec{F} = m\gamma_v \vec{a}$$
 (5.32b)

If you insist on introducing a "mass" as the proportionality factor between 3-force and 3acceleration to quantify the inertial response of an object at finite velocity, you are not only forced ([©]) to make this mass velocity dependent, you also need *two* masses:

"Longitudinal mass":
$$m_{\parallel} := m\gamma_{\nu}^{3}$$
 (5.33)

"Transverse mass":
$$m_{\perp} := m \gamma_v$$
 (5.34)

The above result demonstrates that the concept of a mass as a measure for inertia is not very useful in SPECIAL RELATIVITY. More precisely, the result shows that the quantitites m_{\parallel} and m_{\perp} are *relational properties* between an object and an observer (they depend on the state of motion of the observer); they are not *intrinisic* properties of the object itself. Only the restmass m qualifies as such an intrinisc property. The velocity dependence of m_{\parallel} and m_{\perp} is not an intrinisc feature of matter, it is a feature of spacetime.



This is why in modern textbooks there is only one mass m (the rest mass) which does *not* depend on v, and one has to accept that the Newtonian relation $\vec{p} = m\vec{v}$ is no longer valid. The concepts of "longitudinal mass" and "transverse mass" (and velocity dependent mass, for that matter) are therefore no longer used in modern literature.



↓Lecture11 [09.01.24]

12 $| \triangleleft$ Non-relativistic limit:

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \approx \underbrace{mc^2}_{\substack{\text{Rest} \\ \text{energy}}} + \underbrace{\frac{1}{2}mv^2}_{\substack{\text{Newtonian} \\ \text{kinetic} \\ \text{energy}}} + \mathcal{O}\left(\beta^4\right)$$
(5.35)

This shows again that the correspondence principle is satisfied: For small velocities compared to c, the kinetic energy of Newtonian mechanics is (up to a constant shift given by the rest energy) a good proxy for the true energy of the particle.

13 | The kinetic energy is: $E_{kin} = E - E_0 = E - mc^2$

 \rightarrow The velocity of a relativistic particle as a function of its *kinetic energy* is:

$$\beta^2 = \left(\frac{v}{c}\right)^2 = 1 - \left[\frac{mc^2}{E_{\rm kin} + mc^2}\right]^2 \quad \xrightarrow{\beta \ll 1} \quad \frac{2E_{\rm kin}}{mc^2} \tag{5.36}$$

Note that in the non-relativistic limit it is $E_{\rm kin} \ll mc^2$.

This velocity dependence has been confirmed experimentally to high precision; for example with accelerated electrons [34] (see Refs. [34, 35] for more technical details):

$$1.5 - (\frac{\sqrt{c}}{2})^{2} = 2E_{k}/m_{e}c^{2}$$

$$(\frac{\sqrt{c}}{2})^{2} = 1 - [m_{e}c^{2}/(m_{e}c^{2} + E_{k})]^{2}$$

$$0.5 - (\frac{E_{k}}{2})^{2} - (\frac{\sqrt{c}}{2})^{2} + (\frac{\sqrt{c}}{2$$

 \rightarrow The relativistic energy relation Eq. (5.24) is correct \odot

14 | Massless particles:

So far we considered only particles with non-vanishing mass $m \neq 0$. The definition of the momentum Eq. (5.1) and the relativistic energy Eq. (5.24) cannot be directly applied to particles without mass. However:

 $\mathbf{i} \mid \triangleleft \mathbf{Eq.} (5.26) \text{ with } m \rightarrow 0$:

$$E = |\vec{p}|c$$
 (linear dispersion) (5.37)

 \triangleleft Eq. (5.4) with $m \rightarrow 0$:

$$p^2 = 0$$
 (light-like) $\Rightarrow p^{\mu} = \begin{pmatrix} |\vec{p}| \\ \vec{p} \end{pmatrix}$ (5.38)



i! We take this as the *definition* of the 4-momentum for massless particles (it is the only definition that is consistent with $p^{\mu} = mu^{\mu}$ in the limit of vanishing mass). Note that there is no finite 4-velocity u^{μ} associated to massless particles.

ii | The fact that p^{μ} becomes light-like for massless particles already suggests that they move with the speed of light. We can verify this:

$$\left. \begin{array}{c} E = \gamma_v m c^2 \\ \vec{p} = \gamma_v m \vec{v} \end{array} \right\} \quad \Rightarrow \quad E = \left| \vec{p} \right| \frac{c^2}{v} \quad \frac{m \to 0}{\text{Eq. (5.37)}} \quad \left| \vec{p} \right| c \tag{5.39}$$

This limit is only consistent if $v \to c$ for $m \to 0$:

All particles with vanishing mass move with the speed of light.

(5.40)

- Examples: Photons, Gravitons (if they exist)
- Massless particles do not have a rest frame.

You would need a boost with v = c to reach such a frame; but such boosts are not defined (because the Lorentz factor diverges in this limit).

- i! The relativistic energy $E = \gamma_v mc^2$ holds only for massive particles. For massless particles it does *not* follow E = 0 but rather $E = |\vec{p}|c \neq 0$. So photons do have energy and momentum, but no mass (neither rest- nor any other type of mass). You are also not allowed to use the "forbidden" equation $E = m_r c^2$ and declare $m_r = E/c^2 = |\vec{p}|/c$ as the "dynamic mass" of the photon because (1) we argued above that this concept is not as useful as it sounds, and (2) you only renamed momentum, so what's the point. And if you are afraid that later in GENERAL RELATIVITY– our photons will not be deflected by stars or sucked into black holes because they "have no mass": I assure you, they will; they have energy and momentum, that's enough.
- This demonstrates why the "speed of light" is sort of a misnomer in this context, and we should have stuck to our v_{max} (but then all our equations would look different from the literature). Then it would be conceptually clear that *every* particle with vanishing rest mass "runs into" the universal speed limit v_{max} .

5.3. Action principle and conserved quantities

In this section we study a more formal (and more versatile) approach to describe the dynamics of relativistic systems, namely in terms of the Lagrangian and the action. We do this for the free particle (no force!) and consider electromagnetic forces in the next Chapter 6.

- 1 | Action of free massive particle:
 - i | < Trajectory γ parametrized by $x^{\mu} = x^{\mu}(\lambda)$ with $\lambda \in [\lambda_a, \lambda_b]$ and $x^{\mu}(\lambda_a) = a^{\mu}, x^{\mu}(\lambda_b) = b^{\mu}$

Remember the characteristic property of the trajectory of a free particle (Section 2.4): The proper time (= Minkowski distance) is *maximized* along the trajectory!

$$\rightarrow$$
 Action: $S[\gamma] := \alpha \int_{\gamma} ds = \alpha \int_{\lambda_a}^{\lambda_b} \sqrt{\eta_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}} d\lambda$ (5.41)



with $\dot{x}^{\mu} \equiv \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda}$.

The prefactor α is undetermined so far (\rightarrow *next step*).

i! The parameter λ has no physical interpretation in this formulation as this action is reparametrization invariant (\Rightarrow Section 5.4).

ii | Correspondence principle $\rightarrow \alpha = -mc$

To determine the parameter α , consider the non-relativistic limit of the Lagrangian in coordinate time parametrization $\lambda = t$:

$$\tilde{L} = \alpha \sqrt{c^2 - \dot{\vec{x}}^2} = \alpha c \sqrt{1 - \frac{v^2}{c^2}} \quad \xrightarrow{\beta \ll 1} \quad L \approx \alpha c - \frac{\alpha v^2}{2c} \qquad (5.42)$$

The non-relativistic limit yields – up to a constant that doesn't change the equations of motion – the Lagrangian with Newtonian kinetic energy if we set $\alpha = -mc$.

iii | Lagrangian:

$$L(x^{\mu}, \dot{x}^{\mu}) = -mc\sqrt{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}} = -mc\sqrt{\dot{x}_{\mu}\dot{x}^{\mu}}$$
(5.43)

- i! This Lagrangian is only valid for *massive* particles.
- The Lagrangian Eq. (5.43) is fully specified as is; there is no need to fix a specific parametrization. In this form, the Lagrangian [more precisely: the action Eq. (5.41)] has a gauge symmetry: the parametrization λ is arbitrary (→ Section 5.4).
- On the contrary, if you fix a parametrization (= fix a gauge), e.g., by identifying λ with the coordinate time λ = t ≡ x⁰/c ("static gauge") or the proper time λ = τ ("proper time gauge"), you obtain different (but physically equivalent) Lagrangians which have no longer a gauge symmetry:

$$\lambda \stackrel{!}{=} t \quad \Leftrightarrow \quad c\lambda \stackrel{!}{=} x^0 \quad \Rightarrow \qquad \tilde{L}_t(\vec{x}, \dot{\vec{x}}) = -mc^2 \sqrt{1 - \dot{\vec{x}}^2/c^2} \,, \tag{5.44a}$$

$$\lambda \stackrel{!}{=} \tau \; \Leftrightarrow \dot{x}^{\mu} \dot{x}_{\mu} \stackrel{!}{=} c^2 \; \Rightarrow \; \tilde{L}_{\tau}(x^{\mu}, \dot{x}^{\mu}) = -mc^2 \,. \tag{5.44b}$$

We denote gauge-fixed Lagrangians by \tilde{L} and the gauge-invariant Lagrangian Eq. (5.43) by L. In the following we often work with the latter and choose specific parametrizations at the end of our calculations to express results in known quantities.

2 | Euler-Lagrange equations:

$$\delta S \stackrel{!}{=} 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^{\sigma}} - \underbrace{\frac{\partial L}{\partial x^{\sigma}}}_{=0} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{-mc\dot{x}_{\sigma}}{\sqrt{\dot{x}_{\mu}\dot{x}^{\mu}}} = 0 \tag{5.45}$$

These are 4 differential equations ($\sigma = 0, 1, 2, 3$)!

 \rightarrow Equations of motion in the "proper time gauge" $\lambda = \tau$ [where $\dot{x}_{\mu}\dot{x}^{\mu} = u^2 = c^2$]:

$$m\frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} = \frac{\mathrm{d}p^{\mu}}{\mathrm{d}\tau} = 0 \tag{5.46}$$

This is Eq. (5.6) for vanishing 4-force ©

3 | \triangleleft Action in "static gauge" $\lambda = t = x^0/c$:

$$S[\gamma] \stackrel{\lambda = \frac{x^0}{c}}{\equiv} \tilde{S}_t[\vec{x}(t)] = \int_{t_a}^{t_b} \tilde{L}_t(\vec{x}, \dot{\vec{x}}) \, \mathrm{d}t = -mc^2 \int_{t_a}^{t_b} \sqrt{1 - \frac{\dot{\vec{x}}^2}{c^2}} \, \mathrm{d}t \tag{5.47}$$

i | Canonical momenta ($\vec{v} = \dot{\vec{x}}$):

$$\vec{p} = \frac{\partial \tilde{L}_t}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$
(5.48)

This is the expression for the relativistic 3-momentum Eq. (5.3) we found before, now derived as the canonical momentum of a Lagrangian.

ii | Hamiltonian:

$$\tilde{H}_t = \vec{p} \cdot \vec{v} - \tilde{L}_t \stackrel{\circ}{=} \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = cp^0 \stackrel{5.26}{=} c\sqrt{\vec{p}^2 + m^2 c^2}$$
(5.49)

This is just the relativistic energy Eq. (5.24) we found before, now derived from a Lagrangian.

• ⊲ Non-relativistic limit:

$$\tilde{H}_{t} = mc^{2}\sqrt{1 + \frac{\vec{p}^{2}}{m^{2}c^{2}}} \stackrel{\frac{\vec{p}^{2}}{2m} \ll mc^{2}}{\approx} \underbrace{mc^{2}}_{\substack{\text{Rest} \\ \text{energy}}} + \underbrace{\frac{\vec{p}^{2}}{2m}}_{\substack{\text{Newtonian} \\ \text{kinetic} \\ \text{energy}}}$$
(5.50)

• i! Contrary to the action Eq. (5.47), this Hamiltonian also makes sense for *massless* particles:

$$\tilde{H}_t \stackrel{m=0}{=} |\vec{p}|c \tag{5.51}$$

4 | Noether's (first) theorem:

Details:
Problemset 6

 x^{μ} cyclic \rightarrow Spacetime translations $x^{\mu} + \delta \varepsilon^{\mu}$ are *continuous symmetries* of S

These transformations correspond to the inhomogeneous part of Poincaré transformations: $\bar{x}^{\mu} = x^{\mu} + a^{\mu}$. Every relativistic theory must have this symmetry; for field theories one obtains then four conserved currents: \rightarrow *Energy momentum tensor*.

↓ Noether's theorem → ↓ Conserved Noether charges Q_{μ} : (set $\lambda = t$ as the coordinate time)

$$Q_{\mu} \equiv \begin{cases} \text{Time translation} \Rightarrow \text{Energy } E/c \\ \text{Space translations} \Rightarrow \text{Momentum } \vec{p} \end{cases}$$
(5.52)
$$= -\frac{\partial L}{\partial \dot{x}^{\mu}} = \frac{mc \dot{x}_{\mu}}{\sqrt{c^2 - \vec{v}^2}} = \begin{pmatrix} \frac{1}{c} \frac{mc^2}{\sqrt{1 - \beta^2}} \\ -\frac{m\vec{v}}{\sqrt{1 - \beta^2}} \end{pmatrix} = p_{\mu}$$
(5.53)



- Because x^{μ} are cyclic coordinates, we can obtain the Noether charges directly from the Lagrangian as $\frac{\partial L}{\partial \dot{x}^{\mu}}$; the additional minus is conventional to connect to our definition of the 4-momentum.
- i! This shows that our definition of the 4-momentum is consistent, and the identification of its time-component p^0 as the total energy was correct: By definition, *energy* is the Noether charge that corresponds to translation invariance in *time*. Similarly, *momentum* is the charge for translation invariance in *space*.
- 5 | Noether charges for *homogeneous* Lorentz transformations?

Any relativistic theory is also invariant under (proper orthochronous) Lorentz transformations, $\bar{x}^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\mu}$; for these there must exist additional conserved Noether charges:

Infinitesimal Lorentz transformations $x^{\mu} + \delta \varepsilon^{\mu}{}_{\nu} x^{\nu}$ are *continuous symmetries* of S

The infinitesimal transformation is antisymmetric: $\delta \varepsilon^{\mu}{}_{\nu} = -\delta \varepsilon_{\nu}{}^{\mu}$, \bigcirc Problemset 5.

 $\stackrel{\circ}{\rightarrow}$ Conserved Noether charges:

** Angular momentum (tensor):
$$L^{\mu\nu} = x^{\mu} p^{\nu} - x^{\nu} p^{\mu}$$
 (5.54)

This is an example of an antisymmetric (2,0) Lorentz tensor.

Proof:
Problemset 6

 $i \mid \triangleleft$ Spatial components:

$$L^{23} = x^{2}p^{3} - x^{3}p^{2} = l_{1} L^{31} = x^{3}p^{1} - x^{1}p^{3} = l_{2} L^{12} = x^{1}p^{2} - x^{2}p^{1} = l_{3}$$
 with 3-angular momentum $\vec{l} = \vec{x} \times \vec{p}$. (5.55)

 \rightarrow 3-angular momentum \vec{l} is not (part of a) Lorentz vector but of a (2,0) tensor!

It is not surprising that invariance under spatial rotations $SO(3) \subset O(1, 3)$ implies angular momentum conservation.

ii $| \triangleleft Mixed$ components:

$$L^{10} = x^{1} \gamma_{v} mc - ctp^{1} = cn_{1}$$

$$L^{20} = x^{2} \gamma_{v} mc - ctp^{2} = cn_{2}$$

$$L^{30} = x^{3} \gamma_{v} mc - ctp^{3} = cn_{3}$$
(5.56)

with ** dynamic mass moment

$$\vec{n} := m\gamma_v \left(\vec{x} - t\vec{v}\right) = \frac{E}{c^2}\vec{x} - t\vec{p} = \text{const}.$$
(5.57)

This is the relativistic version of the \checkmark *center-of-mass theorem*.

The center of mass (COM) is now the center of energy (COE). Since \vec{n} (and E) is conserved, we can set t = 0 to find $\vec{n} = E/c^2 \vec{x}_0$, which is the initial center of energy of the system (times E/c^2).

For many particles this is slightly less trivial: One finds analogously the conserved quantity

$$\vec{N} = \sum_{i} \vec{n}_{i} = \sum_{i} \left(\frac{E_{i}}{c^{2}} \vec{x}_{i} - t \vec{p}_{i} \right) = \text{const}.$$
(5.58)



Division by the total (also conserved) energy $E = \sum_{i} E_{i}$ yields

$$\vec{X}_{\text{COE}}(t) := \frac{\sum_{i} E_{i} \vec{x}_{i}}{\sum_{i} E_{i}} = t \frac{c^{2} \vec{P}}{E} + \text{const} \equiv t \vec{V}_{\text{COE}} + \text{const}$$
(5.59)

with the total 3-momentum $\vec{P} = \sum_i \vec{p}_i$. Thus the * *center of energy* \vec{X}_{COE} moves in a straight line with constant velocity \vec{V}_{COE} . Note that the center of energy becomes the Newtonian center of mass in the non-relativistic limit where $E_i \approx E_{i,0} = m_i c^2$.

6 $| \triangleleft$ Multiple particles (covariantly coupled by fields):

The above arguments can be directly generalized to many (non-interacting) particles. This immediately yields the sum of the 4-momenta of these particles as conserved quantity. Interactions between the particles must be covariantly mediated by fields – which also carry 4-momentum (\rightarrow *Chapter 6*):

Conserved Noether charge:
** Total 4-momentum:
$$P^{\mu} := \sum_{i} p_{i}^{\mu} + p_{\text{Fields}}^{\mu}$$
(5.60)

with

• p_i^{μ} the 4-momentum of particle *i*, and

- p_{Fields}^{μ} the total 4-momentum of the fields mediating the interations.
- **7** | \triangleleft Scattering process:



Long before and after the interactions play a role we can approximate the system by non-interacting particles and set $p_{\text{Fields}}^{\mu} = 0 \rightarrow$

$$\sum_{i} p_{\mathrm{in},i}^{\mu} = \sum_{j} p_{\mathrm{out},j}^{\mu}$$
(5.61)

 \rightarrow Conservation of *energy* ($\mu = 0$) and *momentum* ($\mu = 1, 2, 3$)

- In RELATIVITY, conservation of total energy and total momentum is combined into the conservation of 4-momentum.
- We will denote the 4-momenta of *massive* particles (solid lines) with p^μ and the 4-momenta of *massless* particles with q^μ (wiggly lines).



Examples:

- i | Particle decay: \triangleleft Radioactive Nucleus \rightarrow Nucleus 1 & Nucleus 2
 - \rightarrow Energy-momentum conservation:



 \triangleleft Center-of-mass frame where $\vec{p} = \vec{p}_1 + \vec{p}_2 = 0 \xrightarrow{\circ}$

$$mc^2 = m_1c^2 + E_{\rm kin,1} + m_2c^2 + E_{\rm kin,2}$$
 (5.63)

 \rightarrow Decay only possible if

$$m \ge m_1 + m_2 \tag{5.64}$$

If $E_{\text{kin},1} \neq 0$ or $E_{\text{kin},2} \neq 0$, it is $m \neq m_1 + m_2$.

 \rightarrow The rest mass of composite objects is *not additive*.

Composite objects also contain binding energy (potential energy) which contributes to the rest mass of the object.

 $\stackrel{\circ}{\rightarrow}$

$$E_{\rm kin,1} = \frac{(m-m_1)^2 c^2 - m_2^2 c^2}{2m}$$
(5.65)

In the COM frame, the kinetic energy of the two decay products is constant and depends only on the masses of the particles. So if you find a non-trivial energy distribution for the products of a decay process, there must at least three particles be produced (of which you might not be able to detect all). This is how the neutrino was predicted by Pauli from the decay of the neutron: $n \rightarrow p + e^- + \bar{\nu}_e$.

ii | Particle creation:

(

Note that a single massless (light-like) particle (like a photon) cannot decay into two massive (time-like) particles because $(p_1 + p_2)^2 = q^2 = 0$ cannot be solved if $p_i^2 = m_i^2 c^2 > 0$. Indeed (we set c = 1): With the \downarrow *Cauchy-Schwarz inequality* we find

$$m_1 m_2 + \vec{p}_1 \cdot \vec{p}_2 \le \sqrt{m_1^2 + \vec{p}_1^2} \sqrt{m_2^2 + \vec{p}_2^2} = p_1^0 p_2^0$$
(5.66a)

$$\Rightarrow \quad 0 < m_1 m_2 \le p_1 \cdot p_2 \tag{5.66b}$$

so that for arbitrary m_1 and m_2 (particle creation: $q^{\mu} = p_1^{\mu} + p_2^{\mu}$)

$$(p_1 + p_2)^2 = m_1^2 + m_2^2 + 2p_1 \cdot p_2 > 0 \implies \text{Time-like}$$
 (5.67)

Furthermore, for $m_1 = m_2$ (scattering: $p_1^{\mu} - p_2^{\mu} = q^{\mu}$):

$$(p_1 - p_2)^2 = m_1^2 + m_2^2 - 2p_1 \cdot p_2$$
(5.68a)

$$\leq m_1^2 + m_2^2 - 2m_1m_2 \stackrel{m_1 = m_2}{=} 0 \stackrel{p_1 \neq p_2}{\Longrightarrow} \text{Space-like}$$
(5.68b)

(For the Cauchy-Schwarz inequality, *equality* holds iff the two vectors are linearly dependent; for $m_1 = m_2$ this is only possible if $p_1 = p_2$, i.e., in the trivial case of no scattering.)

Eq. (5.67) shows that two particles (of arbitrary masses) can never annihilate into a single photon, and, vice versa, a single photon can never create a pair of massive particles. This is reason why we need an additional (heavy) nucleus for the creation of a particle & antiparticle pair from a photon.

By contrast, Eq. (5.68) tells us that a single massive particle cannot emit or absorb a single photon *if it cannot change its mass* (i.e., has no different energy states). This is true for free elementary particles like electrons (an electron cannot be excited, it always has the same mass). Thus a free electron cannot emit a single photon. If the massive particle in question *has* different internal energy states (and therefore the two masses m_1 and m_2 can be different), this argument does not hold. This is why atoms can spontaneously emit or absorb single photons.

 \triangleleft Photon (+Nucleus) \rightarrow Electron & Positron (+Nucleus)

 \rightarrow Energy-momentum conservation:



With the mass M of the nucleus and the momentum/energy $|\vec{q}| = E_{\gamma}/c$ of the incoming photon, we find

$$\underbrace{\left(\frac{E_{\gamma} + Mc^2}{c}\right)^2 - \left(\frac{E_{\gamma}}{c}\right)^2}_{\text{Rest frame of nucleus}} = P_{\text{in}}^2 \stackrel{!}{=} P_{\text{out}}^2 = \underbrace{\left(\frac{E_{\text{Nuc}} + E_{e^-} + E_{e^+}}{c}\right)^2}_{\text{COM frame of system}}$$
(5.70)

where the right hand side was evaluated in the COM frame with $\vec{P}_{out} = \vec{0}$ and the left hand side in the rest frame of the nucleus (which is allowed since $P^2 = P^{\mu}P_{\mu}$ is a Lorentz scalar).

Please appreciate the subtlety of this evaluation: The 4-momentum conservation Eq. (5.69) is Lorentz *covariant*. Therefore you *cannot* evaluate the left hand side P_{in}^{μ} in one inertial system and the right hand side P_{out}^{μ} in another. However, in any inertial system Eq. (5.69) implies $P_{in}^2 = P_{out}^2$ where left and right hand side are now Lorentz *invariant*; hence you can evaluate the two sides in *different* inertial systems.

$\stackrel{\circ}{\rightarrow}$ Threshold for particle creation:

$$E_{\gamma,\min} = 2m_e^2 c^2 \left(1 + \frac{m_e}{M}\right) > 2m_e c^2$$
(5.71)

The threshold follows for vanishing kinetic energy of the products in the COM frame.

The threshold energy is larger than twice the rest energy of the electron $2m_ec^2$ (the positron has the same mass as the electron) because the scattering products necessarily aquire kinetic energy in the initial rest frame of the nucleus (to carry the momentum of the photon).

iii | <u>Annihilation</u>: \triangleleft Electron & Positron \rightarrow Photon & Photon



 \rightarrow Energy-momentum conservation:

$$P_{\rm in}^{\mu} \equiv \underbrace{p_1^{\mu} + p_2^{\mu}}_{\rm in} = \underbrace{q_1^{\mu} + q_2^{\mu}}_{\rm out} \equiv P_{\rm out}^{\mu} \quad (5.72)$$

⊲ COM frame:

$$P_{\rm in}^{\mu} = \begin{pmatrix} E_{e^-/c} \\ \vec{p} \end{pmatrix} + \begin{pmatrix} E_{e^+/c} \\ -\vec{p} \end{pmatrix} = \begin{pmatrix} |\vec{q}| \\ \vec{q} \end{pmatrix} + \begin{pmatrix} |\vec{q}| \\ -\vec{q} \end{pmatrix} = P_{\rm out}^{\mu}$$
(5.73)

Using that electron and positron have the same mass m_e , we find for the energy of the emitted photons:

$$E_{\gamma} = c \sqrt{\vec{p}^2 + m_e^2 c^2}$$
(5.74)

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Note that the individual rest massess of particles in scattering processess are not conserved: $p_1^2 = p_2^2 = m_e^2 c^2 > 0$ for the incoming electron and the positron, but $q_1^2 = q_2^2 = 0$ for the outgoing photons. The rest mass of the *composite system* remains the same, though. In particular, the two photons together have the same rest mass as the electron-positron system before: $P_{\text{out}}^2 = P_{\text{in}}^2 = 4(\vec{p}^2 + m_e^2 c^2) > 0.$

 \rightarrow The rest masses of individual particles are *not conserved*.

iv | Compton scattering: \triangleleft Electon & Photon \rightarrow Electron & Photon

Details: Problemset 6

Compton scattering is an example of \downarrow *elastic scattering* where the total kinetic energy is conserved and the rest energies of in- and outgoing particles remains the same.

(5.75)

 \rightarrow Energy-momentum conservation:

$$\underbrace{q_{1}^{\mu} + p_{1}^{\mu}}_{\text{in}} = \underbrace{q_{2}^{\mu} + p_{2}^{\mu}}_{\text{out}}$$

.75) P2 5 42 With $q_1^2 = q_2^2 = 0$ and $p_1^2 = p_2^2 = m_e^2 c^2$ one finds:

$$\underbrace{E_1 E_2/c^2(1-\cos\theta)}_{\text{Rest frame of }e^-} \stackrel{\circ}{=} \underbrace{q_1 \cdot q_2 = p \cdot (q_1 - q_2)}_{\text{Lorentz invariant}} \stackrel{\circ}{=} \underbrace{m_e c(E_1/c - E_2/c)}_{\text{Rest frame of }e^-}$$
(5.76a)

$$\Rightarrow \quad \frac{1}{E_2} - \frac{1}{E_1} = \frac{1}{m_e c^2} (1 - \cos \theta) \tag{5.76b}$$

Here the left and right hand sides are evaluated in the rest frame of the electron: $p_1^{\mu} =$ $(m_e c, \vec{0})^T$; θ is the angle between incoming and outgoing photon (scattering angle):





With the photon energy $E_i = hc/\lambda_i$ we find the change in wavelength due to scattering:

$$\Delta \lambda = \lambda_2 - \lambda_1 = \underbrace{\frac{h}{m_e c}}_{\lambda_e} (1 - \cos \theta)$$
(5.77)
with $\stackrel{*}{\to} Compton mayelength \lambda_e \text{ of the electron}$

- With Compton scattering one can measure the Compton wavelength of the electron and thereby determine the Planck constant *h*.
- Because the Compton wavelength is the natural length scale associated to a massive quantum particle, it appears in many field equations of relativistic quantum mechanics (Klein-Gordon equation, Dirac equation, ...).

5.4. **‡** Reparametrization invariance

The action of the free relativistic particle Eq. (5.41) has the peculiar property of "reparametrization invariance", a feature that plays an important role in GENERAL RELATIVITY, and is also crucial for the quantization of the relativistic string in string theory (\uparrow *Nambu-Goto action*).

1 | ⊲ Trajectory γ parametrized by $x^{\mu}(\lambda)$ for $\lambda \in [\lambda_a, \lambda_b]$.

 \triangleleft Diffeomorphism $\varphi : [\lambda_a, \lambda_b] \rightarrow [\lambda_a, \lambda_b]$ with $\lambda_{a/b} = \varphi(\lambda_{a/b})$ and write $\tilde{\lambda} = \varphi(\lambda)$.

Diffeomorphism = Bijective map where both the map and its inverse are continuously differentiable.

 \rightarrow Define new trajectory $\tilde{\gamma}$ via $\tilde{x}^{\mu}(\tilde{\lambda}) := x^{\mu}(\varphi^{-1}(\tilde{\lambda})) = x^{\mu}(\lambda)$ with $\tilde{\lambda} \in [\lambda_a, \lambda_b]$.

 $\tilde{x}^{\mu}(\tilde{\lambda})$ is a *reparametrization* of $x^{\mu}(\lambda)$: \tilde{x}^{μ} and x^{μ} are *different* functions on $[\lambda_a, \lambda_b]$ that parametrize the *same* trajectory in spacetime $\mathbb{R}^{1,3}$.


 \rightarrow Action of new trajectory:

$$S[\tilde{\gamma}] \stackrel{\text{def}}{=} -mc \int_{\lambda_a}^{\lambda_b} \sqrt{\dot{\tilde{x}}_{\mu}(\lambda)} \dot{\tilde{x}}^{\mu}(\lambda)} \, \mathrm{d}\lambda$$
(5.78a)

Rename the dummy variable: $\lambda \rightarrow \tilde{\lambda}$

$$= -mc \int_{\lambda_a}^{\lambda_b} \sqrt{\dot{\tilde{x}}_{\mu}(\tilde{\lambda})} \dot{\tilde{x}}^{\mu}(\tilde{\lambda})} \, \mathrm{d}\tilde{\lambda} \tag{5.78b}$$

Use $\tilde{x}^{\mu}(\tilde{\lambda}) = x^{\mu}(\lambda)$ and the chain rule

$$= -mc \int_{\lambda_a}^{\lambda_b} \sqrt{\dot{x}_{\mu}(\lambda)} \frac{d\lambda}{d\tilde{\lambda}} \dot{x}^{\mu}(\lambda) \frac{d\lambda}{d\tilde{\lambda}} d\tilde{\lambda}$$
(5.78c)

Substitution in the integral: $\tilde{\lambda} = \varphi(\lambda)$

$$= -mc \int_{\lambda_a}^{\lambda_b} \sqrt{\dot{x}_{\mu}(\lambda)} \dot{x}^{\mu}(\lambda) \, \mathrm{d}\lambda \tag{5.78d}$$

$$\stackrel{\text{def}}{=} S[\gamma] \tag{5.78e}$$

 \rightarrow S is invariant under diffeomorphisms on parameter space.

 \rightarrow ** Reparametrization invariance (RI)

2 | Infinitesimal generators:

i | Consider infinitesimal deformations $\varepsilon(\lambda)$ of the parametrization (i.e., $|\varepsilon(\lambda)| \ll 1$ for all λ):

$$\hat{\lambda} = \varphi(\lambda) \equiv \lambda + \varepsilon(\lambda) \tag{5.79}$$

With this we find:

$$x^{\mu}(\lambda) \stackrel{\text{def}}{=} \tilde{x}^{\mu}(\tilde{\lambda}) = \tilde{x}^{\mu}(\lambda + \varepsilon(\lambda)) = \tilde{x}^{\mu}(\lambda) + \varepsilon(\lambda)\partial_{\lambda}\tilde{x}^{\mu}(\lambda) + \mathcal{O}(\varepsilon^{2})$$
(5.80)

ii | The infinitesimal variation of the trajectory is:

$$\delta_{\varepsilon} x^{\mu} := \tilde{x}^{\mu}(\lambda) - x^{\mu}(\lambda) \tag{5.81a}$$

$$= -\varepsilon(\lambda)\partial_{\lambda}x^{\mu}(\lambda) + \mathcal{O}(\varepsilon^{2})$$
(5.81b)

$$\equiv G_{\varepsilon} x^{\mu} + \mathcal{O}(\varepsilon^2) \tag{5.81c}$$

Note that we can replace \tilde{x}^{μ} by x^{μ} in linear order of ε .

 $\rightarrow **$ *Generators* of one-dimensional diffeomorphisms:

$$G_{\varepsilon} = -\varepsilon(\lambda)\partial_{\lambda}$$
 for arbitrary (infinitesimal) $\varepsilon(\lambda)$. (5.82)

iii | We can expand $\varepsilon(\lambda)$ into a Taylor series $\varepsilon(\lambda) = \sum_n \frac{\varepsilon_n}{n!} \lambda^n$ to write

$$G_{\varepsilon} = \sum_{n} \frac{\varepsilon_{n}}{n!} \left(-\lambda^{n} \partial_{\lambda} \right) \equiv \sum_{n} \frac{\varepsilon_{n}}{n!} G_{n} \,. \tag{5.83}$$

 \rightarrow <u>Basis</u> of generators that generate infinitesimal reparametrizations is given by

$$G_n = -\lambda^n \partial_\lambda \quad \text{for } n \in \mathbb{N}_0.$$
 (5.84)



\rightarrow RI = <u>Infinite-dimensional</u> continuous symmetry group

Note that in particular $\varepsilon(\lambda)$ can be chosen such that it is non-zero only for a compact subinterval of $[\lambda_a, \lambda_b]$, i.e., reparametrization invariance is a *local* symmetry (local in parameter space).

- \rightarrow RI is a gauge symmetry
- **3** | Conserved quantities:

You know from your course on classical mechanics that Noether's theorem assigns a conserved quantity to each continuous symmetry of an action. What are these quantities for the infinitely many symmetry transformations G_{ε} associated to RI?

i | \triangleleft Variation of the Lagrangian $L = -mc\sqrt{\dot{x}_{\mu}\dot{x}^{\mu}}$ under G_{ε} :

$$\delta_{\varepsilon}L = \frac{\partial L}{\partial \dot{x}^{\mu}} \delta_{\varepsilon} \dot{x}^{\mu} \tag{5.85a}$$

Use
$$\delta_{\varepsilon} \dot{x}^{\mu} := \dot{\tilde{x}}^{\mu} - \dot{x}^{\mu} = \partial_{\lambda} (\delta_{\varepsilon} x^{\mu})$$
:

$$\stackrel{5.81}{=} -\frac{mc\dot{x}_{\mu}}{\sqrt{\dot{x}_{\sigma}\dot{x}^{\sigma}}} \,\partial_{\lambda} \left[-\varepsilon(\lambda)\dot{x}^{\mu} \right] \tag{5.85b}$$

$$= \frac{mc}{\sqrt{\dot{x}_{\sigma}\dot{x}^{\sigma}}} \left[\dot{x}_{\mu}\dot{\varepsilon}(\lambda)\dot{x}^{\mu} + \dot{x}_{\mu}\varepsilon(\lambda)\ddot{x}^{\mu} \right]$$
(5.85c)

$$= mc\sqrt{\dot{x}_{\mu}\dot{x}^{\mu}}\dot{\varepsilon}(\lambda) + mc\varepsilon(\lambda)\partial_{\lambda}\sqrt{\dot{x}_{\mu}\dot{x}^{\mu}}$$
(5.85d)

$$= \frac{\mathrm{d}}{\mathrm{d}\lambda} \underbrace{\left[m c \varepsilon(\lambda) \sqrt{\dot{x}_{\mu} \dot{x}^{\mu}} \right]}_{=:K_{\varepsilon}(\lambda, \dot{x}^{\mu})} = \frac{\mathrm{d}K_{\varepsilon}}{\mathrm{d}\lambda}$$
(5.85e)

 $\rightarrow \delta_{\varepsilon} L$ is a total derivative $\rightarrow G_{\varepsilon}$ is a continuous symmetry of S

Note that in Eq. (5.78) we assumed $\lambda_{a/b} = \varphi(\lambda_{a/b})$ which corresponds to $\varepsilon(\lambda_{a/b}) = 0 = K_{\varepsilon}(\lambda_{a/b}, \dot{x}^{\mu})$ such the boundary terms vanish and the action is completely invariant.

ii | \checkmark Noether's (first) theorem \rightarrow

For each continuous symmetry $\delta_{\varepsilon} x^{\mu} = G_{\varepsilon} x^{\mu}$ there is a conserved Noether charge:

$$Q_{\varepsilon} \stackrel{*}{=} \delta_{\varepsilon} x^{\mu} \frac{\partial L}{\partial \dot{x}^{\mu}} - K_{\varepsilon} \stackrel{5.85e}{=} \varepsilon(\lambda) mc \frac{\dot{x}^{\mu} \dot{x}_{\mu}}{\sqrt{\dot{x}_{\sigma} \dot{x}^{\sigma}}} - \varepsilon(\lambda) mc \sqrt{\dot{x}_{\mu} \dot{x}^{\mu}} = 0 \qquad (5.86)$$

\rightarrow The Noether charge corresponding to G_{ε} vanishes identically!

"Vanishing identically" means that $Q_{\varepsilon}(\lambda, x^{\mu}, \dot{x}^{\mu}) \equiv 0$ for *all* functions $x^{\mu}(\lambda)$, and not just those that satisfy the equations of motion.

- Naïvely, we expected infinitely many conserved quantities from the infinitely many symmetry generators G_n . We found them, but quite surprisingly, they turned out to be trivially zero. This is a general feature of *local* or *gauge* symmetries; here we use the reparametrization invariance of the relativistic free particle only as an example.
- So while the conserved charges of local symmetries are trivial, such symmetries have other non-trivial implications: they enforce *constraints* on the equations of motion, so that they are no longer independent. Mathematically, this is described by *toether's* second *theorem*.
- 4 We can illustrate the implications of Noether's second theorem for the relativistic free particle:

i | The Lagrangian

$$L = -mc\sqrt{\dot{x}_{\mu}\dot{x}^{\mu}} \tag{5.87}$$

leads to the conjugate momenta

$$p_{\sigma} = \frac{\partial L}{\partial \dot{x}^{\sigma}} = -\frac{mc\dot{x}_{\sigma}}{\sqrt{\dot{x}_{\mu}\dot{x}^{\mu}}}$$
(5.88)

which satisfy the identity

$$p^2 = p^{\mu} p_{\mu} = m^2 c^2 \tag{5.89}$$

- Eq. (5.89) is an *identity*, i.e., it holds for arbitrary trajectories x^μ(λ). In particular, x^μ(λ) does not need to satisfy the equations of motion for Eq. (5.89) to be valid. In Hamiltonian mechanics, such constraints are called *** primary constraints*. So our four canonical momenta p^μ are not independent!
- Eq. (5.89) is equivalent to:

$$\frac{\mathrm{d}p^2}{\mathrm{d}\lambda} = 0 \quad \Leftrightarrow \quad \left(\frac{\mathrm{d}p^{\mu}}{\mathrm{d}\lambda}\right)p_{\mu} = 0 \tag{5.90}$$

ii | \triangleleft Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^{\sigma}} - \frac{\partial L}{\partial x^{\sigma}} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial \dot{x}^{\sigma}} = \frac{\mathrm{d}p_{\sigma}}{\mathrm{d}\lambda} = 0$$
(5.91)

 \rightarrow *Four* differential equations ($\sigma = 0, 1, 2, 3$) for *four* undetermined functions $x^{\mu}(\lambda)$.

However: Eq. (5.91) not independent:

$$p^{\mu} \frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial \dot{x}^{\mu}} = p_{\mu} \frac{\mathrm{d}p^{\mu}}{\mathrm{d}\lambda} \stackrel{5.90}{=} 0$$
(5.92)

- Eq. (5.92) is again an *identity*, i.e., valid for *all* functions x^μ, and not only those that satisfy the equations of motion.
- As a consequence, the system of equations of motion Eq. (5.91) effectively looses one of the four equations, and is therefore *underdetermined*.

Put differently, if you specify a spacetime position $x^{\mu}(\lambda = 0)$ and its first derivative $\dot{x}^{\mu}(\lambda = 0)$ (note that the Euler-Lagrange equations are second-order differential equations), the equations of motion do *not* determine a unique solution $x^{\mu}(\lambda)$. Mathematically speaking, the initial value problem is ill-posed. This is the characteristic property of a *gauge theory*.

- This makes sense in the light of reparametrization invariance: If x^μ(λ) solves the equations of motion, you can construct a new solution x̃^μ(λ) = x^μ(φ(λ)) where φ is some diffeomorphism that is the identity except for a compact subinterval somewhere in the interior of [λ_a, λ_b]. In particular, x̃^μ(λ) = x^μ(λ) in the neighborhood of λ_a, such that the two solutions cannot be distinguished by their initial value and derivative. Note how important the *locality* of the symmetry is for this argument to hold!
- This is a special case of \uparrow Noether's second theorem [76,77].

oreti



iii | The fact that our theory is a gauge theory has another, at first glance surprising, consequence:

$$H = p_{\mu} \dot{x}^{\mu} - L = -\frac{mc \dot{x}_{\mu} \dot{x}^{\mu}}{\sqrt{\dot{x}_{\mu} \dot{x}^{\mu}}} + mc \sqrt{\dot{x}_{\mu} \dot{x}^{\mu}} = 0$$
(5.93)

 \rightarrow The (canonical) Hamiltonian vanishes identically

i! This does *not* mean that there is no time-evolution in our system. The Hamiltonian Eq. (5.93) describes the "parameter evolution" in λ – which, as we have seen, can be modified arbitrarily by gauge transformations; λ has therefore *no physical significance*.

This phenomenon will become important for the interpretation of the Einstein field equations in GENERAL RELATIVITY.

If one fixes a gauge, the Hamiltonian that describes evolution in this parameter is non-zero in general. E.g., for the "static gauge" λ = t = x⁰/c one finds the Hamiltonian Eq. (5.49) which coincides with the relativistic energy of the particle.

↓Lecture 12 [16.01.24]

6. Relativistic Field Theories I: Electrodynamics

6.1. A primer on classical field theories

We start with a general discussion of classical field theories on Minkowski space; Maxwell's electrodynamics is the prime example for such theories (\rightarrow *next section*).

Details: Chapter 1 of my QFT script [20]

6.1.1. Remember: Classical mechanics of "points"

With "points" we mean a discrete set of degrees of freedom.

- **1** $| \triangleleft$ Degrees of freedom q_i labeled by $i = 1, \ldots, N$
- 2 | Lagrangian $L(\{q_i\}, \{\dot{q}_i\}, t) = T V$ We write q for $\{q_i\} = \{q_1, \dots, q_N\}$. T is the kinetic, V the potential energy.
- **3** | Action $S[q] = \int dt L(q(t), \dot{q}(t), t) \in \mathbb{R}$ This is a *functional* of trajectories q = q(t).
- 4 | Hamilton's principle of least action:

$$\frac{\delta S[q]}{\delta q} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \delta S = \int \mathrm{d}t \,\delta L \stackrel{!}{=} 0 \tag{6.1}$$

 δ denotes functional derivatives/variations.

5 | Euler-Lagrange equations (i = 1, ..., N):

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{6.2}$$

6.1.2. Analogous: Lagrangian Field Theory

Now we consider a *continuous* set of degrees of freedom:

- **6** $| < \text{One or more fields } \phi(x) \text{ on spacetime } x \in \mathbb{R}^{1,3} \text{ with derivatives } \partial_{\mu}\phi(x)$ If there are multiple fields we label them by indices: $\phi_k(x)$. In the following we suppress these indices for the sake of simplicity.
- 7 | $\overset{*}{*}$ Lagrangian density $\mathcal{L}(\phi, \partial\phi, x)$ Most general form: $\mathcal{L}(\{\phi_k\}, \{\partial_\mu \phi_k\}, \{x^\mu\})$. (No explicit x^μ -dependence in the following!)

 $\rightarrow \text{Lagrangian } L = \int_{\text{Space}} d^3x \ \mathcal{L}(\phi, \partial \phi)$ (We omit the "density" in the following.)



8 | Action:

$$S[\phi] = \int dt L = \int dt \, d^3x \, \mathcal{L}(\phi, \partial\phi) = \frac{1}{c} \int_{\text{Spacetime}} d^4x \, \mathcal{L}(\phi, \partial\phi) \tag{6.3}$$

 $S[\phi]$ is a functional of "field trajectories" in $\mathbb{R}^{1,3}$.

9 | Action principle:

The classical field evolutions of the system extremize the action:

$$\delta S[\phi] \stackrel{!}{=} 0 \tag{6.4}$$

This variation can be evaluated along the same lines as for the classical mechanics of points:

$$0 \stackrel{!}{=} \delta S[\phi] = \int \mathrm{d}^4 x \, \delta \mathcal{L} \tag{6.5a}$$

$$= \int \mathrm{d}^{4}x \,\left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \delta (\partial_{\mu} \phi) \right\}$$
(6.5b)

Add zero and use $\delta(\partial_{\mu}\phi) = \partial_{\mu}(\delta\phi)$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \right) \right\}$$
(6.5c)

Gauss theorem

$$= \int_{\text{Boundary}} \mathrm{d}\sigma_{\mu} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \underbrace{\delta\phi}_{=0} + \int \mathrm{d}^{4}x \underbrace{\left\{\frac{\partial \mathcal{L}}{\partial\phi} - \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\right)\right\}}_{=0} \delta\phi \qquad (6.5d)$$

- Note that ϕ is fixed on the boundary and therefore $\delta \phi = 0$.
- The second term vanishes because the integral must vanish for arbitrary variations $\delta\phi$.
- **10** | Euler-Lagrange equations (one for each field):

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\mu} \phi)} \right) = 0$$
(6.6)

- Note the *Einstein summation* over repeated indices.
- These equations are manifestly Lorentz covariant if \mathcal{L} is a Lorentz scalar; such field theories are called ** relativistic field theories*.
- If there are multiple fields ϕ_k , there is one Euler-Lagrange equation per field (it is straightforward to generalize the derivation above).

11 | <u>Hamiltonian formalism:</u>

Just like for the mechanics of points, we can define:

$$\pi := \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \qquad \text{** Momentum density conjugate to } \phi \tag{6.7}$$

Like $\phi(x)$, the momentum is a *field*: $\pi(x)$. Here it is $\dot{\phi}(x) \equiv \partial_0 \phi(x)$.

 \rightarrow

$$\mathcal{H}(\pi,\phi,\nabla\phi) := \pi \dot{\phi} - \mathcal{L}(\phi,\partial\phi)$$
 ** Hamiltonian density

(6.8)



- Here $\dot{\phi}$ is to be expressed as a function of the conjugate momentum via Eq. (6.7).
- The argument $\partial \phi$ of \mathcal{L} is short for $\{\partial_{\mu}\phi\}$ or $\{\nabla \phi, \dot{\phi}\}$.

$$\rightarrow$$

$$H := \int d^3x \, \mathcal{H} \qquad \overset{*}{**} Hamiltonian \tag{6.9}$$

For given fields $\pi(x)$ and $\phi(x)$, *H* is a (potentially constant) function of time. By contrast, the Hamiltonian *density* \mathcal{H} is a function of space \vec{x} and time *t*.

6.2. Electrodynamics: Covariant formulation and Lagrange function

We now want to reformulate Maxwell's electrodyamics in this formalism, i.e., we want to find a Lagrangian density (and an associated action) such that the Euler-Lagrange equations are the Maxwell equations.

1 <u>Remember:</u>

 $i \mid \checkmark Maxwell equations (in cgs units):$

Magnetic Gauss's law:	$\nabla \cdot \vec{B} = 0$	(6.10a)
Maxwell-Faraday law:	$\nabla \times \vec{E} + \frac{1}{c} \partial_t \vec{B} = 0$	(6.10b)
Electric Gauss's law:	$ abla \cdot \vec{E} = 4\pi ho$	(6.10c)
Ampère's law:	$\nabla \times \vec{B} - \frac{1}{c} \partial_t \vec{E} = \frac{4\pi}{c} \vec{j}$	(6.10d)

with charge density $\rho(x)$ and current density $\vec{j}(x)$ that satisfy the * *continuity* equation

$$\partial_t \rho + \nabla \cdot \vec{j} = 0.$$
 (6.11)

This follows from the two inhomogeneous Maxwell equations Eqs. (6.10c) and (6.10d). Note that here ρ and \vec{j} are external fields and not dynamic degrees of freedom. The statement is therefore that only for external fields that satisfy Eq. (6.11) the Maxwell equations yield solutions for \vec{E} and \vec{B} .

ii | Homogeneous Maxwell equations (HME) Eq. (6.10a) & Eq. (6.10b)

 $\stackrel{\circ}{\rightarrow} \exists$ "Scalar" potential φ and "Vector" potential \vec{A} :

$$\vec{E} = -\nabla \varphi - \frac{1}{c} \partial_t \vec{A} \quad \text{and} \quad \vec{B} = \nabla \times \vec{A}$$
 (6.12)

- Constraining the fields \vec{E} and \vec{B} to this form satisfies the *homogeneous Maxwell equations* Eqs. (6.10a) and (6.10b) automatically.
- Because of the two homogeneous Maxwell equations, the *six* fields $\{E_x, E_y, E_z, B_x, B_y, B_z\}$ are not independent so that all degrees of freedom can be encoded in the *four* fields $\{\varphi, A_x, A_y, A_z\}$. This suggests a reformulation of Maxwell's theory in terms of these "potentials".



iii | Gauge transformation:

 \triangleleft Arbitrary function λ : $\mathbb{R}^{1,3} \rightarrow \mathbb{R}$ and define

$$\vec{A'} := \vec{A} + \nabla \lambda$$
 and $\varphi' := \varphi - \frac{1}{c} \partial_t \lambda$ (6.13)

This transformation of fields is called a $\stackrel{*}{*}$ gauge transformation (\rightarrow below).

 $\stackrel{\circ}{\rightarrow} \vec{E} = \vec{E}'$ and $\vec{B} = \vec{B}'$

 \rightarrow The potentials φ and \vec{A} are not unique.

iv | Inhomogeneous Maxwell equations (IME) Eqs. (6.10c) and (6.10d) in terms of the potentials:

Eq. (6.10c)
$$\Leftrightarrow \nabla^2 \varphi + \frac{1}{c} \partial_t (\nabla \cdot \vec{A}) = -4\pi\rho$$
 (6.14a)

Eq. (6.10d)
$$\Leftrightarrow \nabla^2 \vec{A} - \frac{1}{c^2} \partial_t^2 \vec{A} = -\frac{4\pi}{c} \vec{j} + \nabla \left(\nabla \cdot \vec{A} + \frac{1}{c} \partial_t \varphi \right)$$
 (6.14b)

In this form, electrodynamics is a *gauge theory* because it has a *local* symmetry, namely the transformation Eq. (6.13). Indeed, it is straightforward to show that if (φ, \vec{A}) is a solution of Eq. (6.14), then $(\varphi', \vec{A'})$ given by Eq. (6.13) is another solution. Since $\lambda(x)$ is arbitrary, one can choose continuously differentiable $\lambda(x)$ that vanish everywhere except for a compact region of spacetime. This makes Eq. (6.13) a *local* symmetry transformation of the PDE system Eq. (6.14); such local symmetries are called $\frac{3}{4}$ gauge transformations, and models that feature such symmetries are referred to as $\frac{3}{4}$ gauge theories. The locality of the symmetry has profound implications:



Thus, if we want a deterministic theory (meaning: a theory with predictive power), we *cannot* interpret the gauge fields (φ, \vec{A}) as physical (= observable) degrees of freedom. Our only choice (to save predictability) is to identify the *equivalence classes* $[(\varphi, \vec{A})]$ of field configurations that are related by (local) gauge transformations as physical states; this is the defining property of a gauge theory. In a nutshell: local symmetries must be interpreted as gauge symmetries and fields related by such transformations are mathematically redundant descriptions of the *same* physical state.

 $\mathbf{v} \mid$ Eq. (6.14) Gauge theory \rightarrow Fix a gauge:

$$\nabla \cdot \vec{A} + \frac{1}{c} \partial_t \varphi \stackrel{!}{=} 0 \quad \text{** Lorenz gauge (LG)}$$
(6.15)

It is straightforward to show that for any given (φ, \vec{A}) there is a gauge transformation λ such that $(\varphi', \vec{A'})$ satisfies Eq. (6.15).



Eq. (6.10c)
$$\Leftrightarrow \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\varphi = \frac{4\pi}{c}c\rho$$
 (6.16a)
Eq. (6.10d) $\Leftrightarrow \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\vec{A} = \frac{4\pi}{c}\vec{j}$ (6.16b)

- Expressed in potentials in the Lorenz gauge, the inhomogeneous Maxwell equations become a set of four decoupled wave equations.
- We do not have to consider the *homogeneous* Maxwell equations in the gauge field representation because Eq. (6.12) ensures that Eqs. (6.10a) and (6.10b) are automatically satisfied.
- **2** | <u>Observation</u>: Charge $dq = \rho d^3x$ in volume $dV = d^3x$ independent of inertial system:



$$\rho d^{3}x = \bar{\rho} d^{3}\bar{x} \implies \underbrace{\rho d^{3}x}_{4-\text{vector}} \underbrace{dx^{\mu}}_{4-\text{vector}} = \rho d^{3}x dt \frac{dx^{\mu}}{dt} = \underbrace{\frac{1}{c} d^{4}x}_{\text{Eq. (4.23)}} \underbrace{\rho \frac{dx^{\mu}}{dt}}_{4-\text{vector}} \qquad (6.17)$$

This suggests that charge and current density are actually components of a Lorentz 4-vector:

$$j^{\mu} := \rho \frac{\mathrm{d}x^{\mu}}{\mathrm{d}t} = \begin{pmatrix} c\rho\\ \rho \vec{v} \end{pmatrix} = \begin{pmatrix} c\rho\\ \vec{j} \end{pmatrix} \quad \text{** 4-current (density)}$$
(6.18)

with \ast charge density $\rho = \rho(x)$ and \ast current density $\vec{j} = \vec{j}(x) = \rho(x)\vec{v}(x)$.

- In the argument above, the trajectory x
 i(t) in x^μ = (ct, x
 i(t)) parametrizes the movement of the infinitesimal volume dV = d³x with charge dq = ρdV; the coordinate velocity v
 i(t) = dx
 i is therefore the velocity of the charge distribution at position x
 i(t) at time t: v
 i(x). Thus, in general, the current density j
 j(x) = ρ(x)v
 i(x) depends on position and time via the charge density ρ(x) and the velocity field v
 i(x).
- That the charge density ρ is not a Lorentz scalar is intuitively clear as it is defined as charge per volume. Volumes, however, are clearly not Lorentz invariant because they are Lorentz contracted. Since the charge (not the charge density!) is Lorentz invariant (this is an observational fact), the ratio of charge by volume must change under boosts.



 $\mathbf{3} \mid$ Eq. (6.18) and Eq. (6.16) suggest the compact notation

$$Eq. (6.16a) Eq. (6.16b) $\partial^2 A^{\mu} = \frac{4\pi}{c} j^{\mu}$ (IME in LG) (6.19)$$

Remember that $\partial^2 = \Box = \frac{1}{c^2} \partial_t^2 - \nabla^2$.

with

$$A^{\mu} := \begin{pmatrix} \varphi \\ \vec{A} \end{pmatrix} \quad \overset{*}{*} \text{ 4-potential} \tag{6.20}$$

The covariant components of the gauge field are $A_{\mu} = (\varphi, -\vec{A})$.

The transformation of the 4-potential must be that of a Lorentz 4-vector:

$$\bar{\partial}^2 = \partial^2 : \text{Scalar} [\text{Eq. (4.36b)}]$$

$$\bar{j}^{\mu} = \Lambda^{\mu}{}_{\nu} j^{\nu} : 4\text{-vector} [\text{Eq. (6.18)}]$$

$$\rightarrow \quad \bar{A}^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} : 4\text{-vector} \quad (6.21)$$

With this transformation, the Maxwell equations in their simple formulation Eq. (6.19) are *manifestly* Lorentz covariant:

$$\partial^2 A^{\mu} = \frac{4\pi}{c} j^{\mu} \qquad \xrightarrow{K \xrightarrow{R, \vec{v}, s, b}} \bar{K} \qquad \bar{\partial}^2 \bar{A}^{\mu} = \frac{4\pi}{c} \bar{j}^{\mu} \tag{6.22}$$

- 4 We can now rewrite our previous equations in tensor notation:
 - i | The Lorenz gauge condition can be compactly written as:

$$\partial A \equiv \partial_{\mu} A^{\mu} = 0$$
 (Lorenz gauge) (6.23)

\rightarrow The Lorenz gauge is Lorentz invariant

Note: The Lorenz gauge is named after \uparrow *Ludvig Lorenz*; by contrast, the Lorentz transformation is named after \uparrow *Hendrik Lorentz*. Thus: *The Lorenz gauge* (no "t") *is Lorentz invariant*.

ii | The continuity equation also becomes very simple (and Lorentz covariant):

$$\partial j \equiv \partial_{\mu} j^{\mu} = 0$$
 (Continuity equation) (6.24)

iii | The gauge transformation can be written as follows:

$$A^{\prime\mu} = A^{\mu} - \partial^{\mu}\lambda$$
 (Gauge transformation) (6.25)

Recall that $\partial^{\mu} = (\frac{1}{c}\partial_t, -\nabla).$



5 | Let us summarize our findings so far:

$$\begin{array}{ll} \text{Maxwell equations} : \quad \partial^2 A^{\mu} = \frac{4\pi}{c} j^{\mu} \\ \text{Lorenz gauge} : \quad \partial A = 0 \\ \text{Continuity equation} : \quad \partial j = 0 \end{array} \right\} \quad \xrightarrow{K \longrightarrow \bar{K}} \quad \begin{cases} \bar{\partial}^2 \bar{A}^{\mu} = \frac{4\pi}{c} \bar{j}^{\mu} \\ \bar{\partial} \bar{A} = 0 \\ \bar{\partial} \bar{j} = 0 \end{cases} \quad (6.26)$$

 \rightarrow Electrodynamics satisfies Einstein's principle of Special Relativity SR

- In contrast to Newtonian mechanics, electrodynamics was a relativistic theory all along and there was no need to modify it. It's Lorentz covariance was simply not manifest and required a bit of work to unveil.
- The treatment above relies on (1) expressing the Maxwell equations in terms of the gauge fields and (2) choosing a particular gauge (the Lorenz gauge). While this is mathematically legit (and not restrictive), it would be nice to have manifestly Lorentz covariant expressions (1) withou fixing a gauge and (2) in terms of the physically observable fields *E* and *B*.

To achieve both goals, we first need a new tensorial quantity:

- **6** | Field strength tensor:
 - i | Motivation: We are looking for the simplest field that ...
 - ... is gauge-invariant (i.e., has a physical interpretation).
 - ... is Lorentz covariant (i.e., can be used to construct Lorentz covariant equations).
 - ii | \triangleleft Discretized spacetime on a (hypercubic) lattice (here we consider the *xy*-plane):
 - The gauge field A^{μ} lives on *edges* in μ -direction.
 - The gauge transformation λ lives on *vertices* of the lattice.



 \rightarrow Discretized gauge transformation:

$$A'_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}\,\mu} = A_{\boldsymbol{x},\boldsymbol{x}+\boldsymbol{e}\,\mu} + \underbrace{\frac{1}{\varepsilon} \left(\lambda_{\boldsymbol{x}+\boldsymbol{e}\,\mu} - \lambda_{\boldsymbol{x}} \right)}_{\sim \partial_{\mu}\lambda} \tag{6.27}$$

 \rightarrow Sums along paths *P* transform non-trivially only at their "start site" *s* and "end site" *e*:

$$\sum_{e \in P} A'_e = \sum_{e \in P} A_e + \frac{1}{\varepsilon} \left(\lambda_e - \lambda_s \right)$$
(6.28)

Edges *e* are pairs of adjacent lattice sites, e.g., $e = (x, x + e_x)$ with lattice vector $|e_x| = \varepsilon$. \rightarrow Sums $\sum_{e \in L} A_e$ along *closed loops L* are *gauge-invariant* (because s = e)!



 \rightarrow Smallest gauge-invariant loop (= loop around a single face f = yx):

$$F_{yx} := A_{x,x+e_x} + A_{x+e_x,x+e_x+e_y} - A_{x+e_y,x+e_x+e_y} - A_{x,x+e_y}$$
(6.29a)
$$= (A_{x,x+e_x} - A_{x+e_y,x+e_x+e_y}) - (A_{x,x+e_y} - A_{x+e_x,x+e_y+e_x})$$

$$\xrightarrow{\varepsilon \to 0} \partial_y A_x - \partial_x A_y$$
(6.29b)

iii | This motivates the definition:

$$F_{\mu\nu} := \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \quad \stackrel{*}{*} Field \ strength \ tensor \ (FST) \tag{6.30a}$$

$$\stackrel{6.12}{=} \begin{pmatrix} 0 & E_{x} & E_{y} & E_{z} \\ -E_{x} & 0 & -B_{z} & B_{y} \\ -E_{y} & B_{z} & 0 & -B_{x} \\ -E_{z} & -B_{y} & B_{x} & 0 \end{pmatrix}_{\mu\nu} \tag{6.30b}$$

Details:
Problemset 7

- $\rightarrow F_{\mu\nu}$ is a (0, 2) Lorentz tensor
 - The FST is gauge-invariant by construction. You can also check this by applying the gauge transformation Eq. (6.25).
 - It is easy to see that the FST has the following properties:

Antisymmetry:
$$F^{\mu\nu} = -F^{\nu\mu}$$
 (6.31a)

Tracelessness:
$$F^{\mu}_{\ \mu} = g_{\mu\nu}F^{\mu\nu} = 0$$
 (6.31b)

- i! When we write " E_x ", we refer to the x-component of the original electric field \vec{E} as it occurs in the Maxwell equations Eq. (6.10). In this context, an expression like E^x does not make sense since \vec{E} is not a 4-vector but the component of a rank-2 tensor.
- iv | Using that $\varepsilon^{\mu\nu\alpha\beta}$ is a Lorentz pseudo-tensor [recall Eq. (4.41)], we can define:

$$\tilde{F}^{\mu\nu} := \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \quad \stackrel{*}{**} Dual field strength tensor (DFST)$$

$$\stackrel{3.41}{\stackrel{6.30}{=}} \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}_{\mu\nu}$$
(6.32a)
$$(6.32b)$$

Details: September 7

- $\rightarrow \tilde{F}^{\mu\nu}$ is a (2,0) *pseudo* Lorentz tensor
 - The dual field-strength tensor will be useful below.
 - $\tilde{F}^{\mu\nu}$ is obtained from $F^{\mu\nu}$ (*contravariant*!) by the substitution $\vec{E} \mapsto \vec{B}$ and $\vec{B} \mapsto -\vec{E}$. [Just like in vacuum the homogeneous Maxwell equations Eqs. (6.10a) and (6.10b) transform into the "inhomogeneous" ones Eqs. (6.10c) and (6.10d)].



7 | Transformation of the electromagnetic field:

The field strength tensor Eq. (6.30) has the useful properties that (1) we know how it transforms under Lorentz transformations, and (2) we know how it relates to the observable fields \vec{E} and \vec{B} . Hence we can use it to derive the transformation of the electromagnetic field when transitioning from one inertial system to another.

i | The (contravariant) FST transforms under a Lorentz transformation Λ as follows:

$$\underbrace{\bar{F}^{\mu\nu}(\bar{x})}_{\{\bar{E}_i(\bar{x}), \bar{B}_i(\bar{x})\}} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} \underbrace{F^{\alpha\beta}(x)}_{\{E_i(x), B_i(x)\}}$$
(6.33)

Here it is $F^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} F_{\alpha\beta}$ as usual.

ii | \triangleleft Boost $\Lambda_{\vec{v}}$ [Eq. (4.10)]:

$$\vec{\bar{E}}(\bar{x}) \stackrel{\circ}{=} \gamma \left[\vec{E}(x) + \frac{1}{c}\vec{v} \times \vec{B}(x)\right] - (\gamma - 1)\frac{\vec{v} \cdot \vec{E}(x)}{v^2}\vec{v}$$
(6.34a)

$$\vec{B}(\vec{x}) \stackrel{\circ}{=} \gamma \left[\vec{B}(x) - \frac{1}{c} \vec{v} \times \vec{E}(x) \right] - (\gamma - 1) \frac{\vec{v} \cdot \vec{B}(x)}{v^2} \vec{v}$$
(6.34b)

with $x^{\mu} = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu} \bar{x}^{\nu}$.

i! Note that on the left-hand side the arguments are \bar{x} and on the right-hand side x!

 \rightarrow Electric and magnetic fields "mix" under boosts!

• Please appreciate what we showed: If you start from Maxwell Eq. (6.10) and perform an arbitrary Lorentz boost $\bar{x}^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$, transforming the derivatives as $\bar{\partial}_{\mu} = \Lambda_{\mu}{}^{\nu}\partial_{\nu}$, you obtain a set of horribly looking PDEs. But if you recombine the equations appropriately, group the terms according to Eq. (6.34) and define the new fields $\vec{E}(\bar{x})$, $\vec{B}(\bar{x})$, the equations look again like Eq. (6.10), only with bars over coordinates and fields.

You *could* show this directly, without ever introducing the gauge field A^{μ} and without using the machinery of tensor calculus (this is what Einstein did for a boost in z-direction in his 1905 paper "Zur Elektrodynamik bewegter Körper" [10]); but hopefully you agree that our more advanced route (using the gauge field and tensor calculus) is a more elegant approach.

• Because of our motivation from Einstein's principle of Special Relativity **SR**, we frame our discussion in the terminology of *passive* transformations (= coordinate transformation): The same electromagnetic field that looks like $\vec{E}(x)$, $\vec{B}(x)$ in an inertial system K looks like $\vec{E}(\bar{x})$, $\vec{B}(\bar{x})$ in another system \bar{K} .

Because we showed that the Maxwell equations satisfy **SR**, they have exactly the same form in \bar{K} as in K. This, however, allows you to interpret the transformation *actively*: If you are given a solution of Maxwell equations $\vec{E}(x)$, $\vec{B}(x)$, then, for any \vec{v} , the *new* functions $\vec{E}(\bar{x})$, $\vec{B}(\bar{x})$ defined by Eq. (6.34) and $x^{\mu} = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu} \bar{x}^{\nu}$ are again solutions (in the same coordinates). This shows that the Lorentz group is (part of) the invariance group of the PDE system Eq. (6.10) we call Maxwell equations (just like the Galiei group was an invariance group of Newton's equation, recall Section 1.2).



↓Lecture13 [23.01.24]

iii | \triangleleft Non-relativistic limit:

Eq. (6.34)
$$\xrightarrow{\gamma \approx 1} \begin{cases} \vec{E}(\vec{x}) \approx \vec{E}(x) + \frac{1}{c}\vec{v} \times \vec{B}(x) \\ \vec{B}(\vec{x}) \approx \vec{B}(x) - \frac{1}{c}\vec{v} \times \vec{E}(x) \end{cases}$$
 (6.35)

- The interconversion between magnetic and electric fields happens already in linear order of v/c.
- The separation of the electromagnetic field into "electric" and "magnetic" components is observer dependent!
- Example: A charge at rest has a non-zero electric field, but a vanishing magnetic field. The *same* charge as seen from an inertial system in relative motion gives rise to a current that is accompanied by a non-vanishing magnetic field perpendicular to the direction of motion and the electric field. This is a direct consequence of Eq. (6.35): $\vec{B}(\vec{x}) \approx -\frac{1}{c}\vec{v} \times \vec{E}(x) \neq \vec{0}.$
- iv | \triangleleft Special case: Boost Λ_{v_x} in x-direction: Eq. (6.34) $\xrightarrow{\vec{v} = (v_x, 0, 0)}$

$$\bar{E}_x = E_x$$
, $\bar{E}_y = \gamma \left[E_y - \frac{v}{c} B_z \right]$, $\bar{E}_z = \gamma \left[E_z + \frac{v}{c} B_y \right]$, (6.36a)

$$\bar{B}_x = B_x$$
, $\bar{B}_y = \gamma \left[B_y + \frac{v}{c} E_z \right]$, $\bar{B}_z = \gamma \left[B_z - \frac{v}{c} E_y \right]$, (6.36b)

(Here the fields in \overline{K} on the left-hand side are functions of \overline{x} whereas the fields in K on the right-hand side are functions of x.)

 \rightarrow

- The field components *parallel* to the boost remain unchanged.
- The *perpendicular* components mix and get enhanced by a Lorentz factor $\gamma > 1$.
- Einstein derived this transformation directly (without using gauge fields and tensor notation) in his 1905 paper "Zur Elektrodynamik bewegter Körper" [10]; you follow this path in Problemset 7.
- v | Lorentz scalars:

The electric and magnetic field components transform in a complicated way under Lorentz transformations. Is it possible to combine them into scalar quantities? Thanks to our knowledge of tensor calculus and the field strength tensor, this question is easy to answer:

a We can construct a scalar by contracting the FST with itself:

$$F^{\mu\nu}F_{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} \stackrel{\circ}{=} 2(\vec{B}^2 - \vec{E}^2)$$
(6.37)

 \rightarrow If $|\vec{E}| \ge |\vec{B}|$ is true in one IS, it is true in all IS.

b | We can construct a *pseudo* scalar by contracting the FST with the DFST:

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}F_{\mu\nu} \stackrel{\circ}{=} -4(\vec{E}\cdot\vec{B})$$
(6.38)

 \rightarrow If $\vec{E} \perp \vec{B}$ is true in one IS, it is true in all IS.



Some comments:

- Note that *F̃^{μν} F̃_{μν}* [≙] − *F^{μν} F_{μν}* (use contraction identities for Levi-Civita symbols to show this, *⇒* Problemset 7); i.e., the two quantities above exhaust all elementary gauge-invariant scalar fields that we can construct (*A^μA_μ* is of course also a scalar, but not a gauge-invariant one).
- The combination of Eq. (6.37) and Eq. (6.38) can be used to infer whether inertial systems exist in which either the electric or magnetic field vanishes. For example: If $\tilde{F}^{\mu\nu}F_{\mu\nu} = 0$ and $F^{\mu\nu}F_{\mu\nu} > 0$, it is possible to find an inertial system where $\vec{E} = 0$ and $\vec{B} \neq 0$ (but not the other way around). If $\tilde{F}^{\mu\nu}F_{\mu\nu} \neq 0$ there is no inertial system in which one of the fields vanishes.

8 | Manifest covariant form of the Maxwell equations:

Using the FST and the DFST, we can write the Maxwell equations manifestly covariant without using the gauge field and/or fixing a gauge (cf. Eq. (6.19)):

- i | The equations we look for must be ...
 - ...manifestly covariant (\rightarrow tensor equations).
 - ...linear in the FST or the DFST (the ME are linear in \vec{E} and \vec{B}).
 - ... use one 4-divergence ∂_{μ} (the ME are first-order PDEs).
 - $\rightarrow \triangleleft$

$$\partial_{\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_{\nu}(\partial_{\rho}A_{\sigma} - \partial_{\sigma}A_{\rho}) = \varepsilon^{\mu\nu\rho\sigma}\partial_{\nu}\partial_{\rho}A_{\sigma} = 0 \qquad (6.39a)$$

$$\partial_{\nu}F^{\mu\nu} = \partial_{\nu}(\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}) = \partial^{\mu}(\partial A) - \partial^{2}A^{\mu}$$
(6.39b)

ii | The homogeneous ME Eqs. (6.10a) and (6.10b) must be *identically* true if the fields are given in terms of gauge fields. Eq. (6.39a) then suggests that the homogeneous ME are:

$$\partial_{\nu}\tilde{F}^{\mu\nu} = 0$$
 Homogeneous ME (?) (6.40)

To check this evaluate:

$$\partial_{\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_{\nu}F_{\rho\sigma}$$
(6.41a)

$$= \frac{1}{6} \varepsilon^{\mu\nu\rho\sigma} \left(\partial_{\nu} F_{\rho\sigma} + \partial_{\sigma} F_{\nu\rho} + \partial_{\rho} F_{\sigma\nu} \right)$$
(6.41b)

$$= \frac{1}{2} \sum_{\nu < \rho < \sigma} \varepsilon^{\mu\nu\rho\sigma} \left(\partial_{\nu} F_{\rho\sigma} + \partial_{\rho} F_{\sigma\nu} + \partial_{\sigma} F_{\nu\rho} \right)$$
(6.41c)

Here we used that the Levi-Civita symbol is invariant under cyclic permutations of (subsets) of indices and that the FST (and the Levi-Civita symbol) is antisymmetric in its indices. Note that for every fixed μ there are 3! = 6 non-vanishing assignments of indices $\nu\rho\sigma$. However, pairs of terms like $\varepsilon^{\mu\nu\rho\sigma}\partial_{\nu}F_{\rho\sigma} = \varepsilon^{\mu\nu\sigma\rho}\partial_{\nu}F_{\sigma\rho}$ are identical, so that only 3 distinct terms remain. These can be w.l.o.g. written like cyclic permutations as in Eq. (6.41c). Note that for a fixed index μ , the sum contains only one non-vanishing summand.

$$\rightarrow$$

$$\underbrace{\forall_{\nu < \rho < \sigma} : \partial_{\nu} F_{\rho\sigma} + \partial_{\rho} F_{\sigma\nu} + \partial_{\sigma} F_{\nu\rho} = 0}_{\uparrow Bianchi \, identity \, (4 \text{ equations})} \Leftrightarrow \underbrace{\forall_{\mu} : \partial_{\nu} \tilde{F}^{\mu\nu} = 0}_{(4 \text{ equations})} \tag{6.42}$$



It is straightforward to check by hand, using Eq. (6.30), that the four Bianchi identities correspond to the four homogeneous Maxwell Eqs. (6.10a) and (6.10b). For example:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} \stackrel{\circ}{=} -\nabla \cdot B = 0 \quad \Leftrightarrow \quad \text{Eq. (6.10a)}$$
(6.43)

Details: SProblemset 7

- As shown in Eq. (6.39a), the homogeneous ME are *identities* if the FST is expressed in terms of gauge fields.
- By contrast, if the FST is expressed in terms of physical fields *E* and *B* [as given in Eq. (6.30)], the equation ∂_ν *F̃^{μν}* = 0 becomes a non-trivial constraint on the field configurations.
- iii | \triangleleft Lorenz gauge Eq. (6.23) \rightarrow

Eq. (6.39b)
$$\Rightarrow \partial_{\nu} F^{\mu\nu} = -\partial^2 A^{\mu}$$
 (6.44)

Compare Eq. (6.19) (inhomogeneous ME in Lorenz gauge):

$$-\partial^2 A^\mu = -\frac{4\pi}{c} j^\mu \tag{6.45}$$

This suggests that the inhomogeneous ME are:

$$\partial_{\nu}F^{\mu\nu} = -\frac{4\pi}{c}j^{\mu}$$
 Inhomogeneous ME (?) (6.46)

It is straightforward to check by hand that these four equations are equivalent to the four inhomogeneous ME Eqs. (6.10c) and (6.10d) using Eq. (6.30). For example for $\mu = 0$:

$$\partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} \stackrel{\circ}{=} -\nabla \cdot \vec{E} = -\frac{4\pi}{c} j^0 = -4\pi\rho \quad \Leftrightarrow \quad \text{Eq. (6.10c)}$$

$$(6.47)$$

Details:
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• In this form, the continuity equation Eq. (6.24) follows trivially from the antisymmetry of the FST:

$$\partial_{\mu}j^{\mu} = -\frac{c}{4\pi}\partial_{\nu}\partial_{\mu}F^{\mu\nu} = 0 \tag{6.48}$$

• If you express the FST in terms of the gauge field, the inhomogeneous ME read (without fixing a gauge!):

$$\partial^2 A^{\mu} - \partial^{\mu} (\partial A) = \frac{4\pi}{c} j^{\mu}$$
(6.49)

This equation becomes Eq. (6.19) in the Lorenz gauge Eq. (6.23). It is easy to check that this equation is still gauge symmetric under the transformation Eq. (6.25).

iv | In summary, the 8 (=1+3+1+3=4+4) Maxwell equations can be written in the covariant form:

Homogeneous ME: $\partial_{\nu} \tilde{F}^{\mu\nu} = 0$ (6.50a) Inhomogeneous ME: $\partial_{\nu} F^{\mu\nu} = -\frac{4\pi}{c} j^{\mu}$ (6.50b)



- i! Using Eqs. (6.30) and (6.32), these equations make sense without introducing the gauge field.
- In particular, this means that the Maxwell equations written in their conventional form Eq. (6.10) (i.e., as two scalar and two vector equations) remain *not* invariant under Lorentz transformations for each equation separately, rather the magnetic Gauss law mixes with the Maxwell-Faraday law, and the electric Gauss law mixes with Ampère's law. This explains why showing the Lorentz covariance of the PDE system Eq. (6.10) is quite messy and complicated without using the tensor formalism. This is why we say that its Lorentz covariance *is not manifest*. By contrast, the Lorentz covariance of the formulation Eq. (6.50) *is manifest* as these are tensor equations.

9 | Lagrangian formulation:

Our final goal is to make a connection to the formalism introduced in Section 6.1 and obtain the Lorentz covariant Maxwell equations as the Euler-Lagrange equations of some action/Lagrangian:

i | It is convenient to construct the Lagrangian as a function of the gauge fields A^{μ} because in this formulation the HME are identically satisfied:

$$\partial_{\nu}\tilde{F}^{\mu\nu} \equiv 0 \quad \Rightarrow \quad \mathcal{L} = \mathcal{L}(A, \partial A)$$
(6.51)

 \rightarrow Only the inhomogeneous ME must follow as Euler-Lagrange equations

Note that the counting matches: We have four fields A^{μ} and thus four Euler-Lagrange equations – just as we have four IME: $\partial_{\nu} F^{\mu\nu} = -\frac{4\pi}{c} j^{\mu}$.

- ii We have the following hints to construct a reasonable Lagrangian density:
 - The IME are Lorentz covariant. This can be ensured by a Lagrangian density that is a Lorentz (pseudo) scalar.
 - The Maxwell equations are linear (superposition principle!); thus the Lagrangian must be quadratic in the fields.
 - The IME are gauge invariant. This can be ensured by a Lagrangian density that is gauge invariant up to a total derivative (here: surface term) which does not affect the equations of motion.
 - \rightarrow Most general form:

$$\mathcal{L}(A,\partial A) = a_1 F^{\mu\nu} F_{\mu\nu} + a_2 \underbrace{\tilde{F}^{\mu\nu}}_{\text{Surface}} F_{\mu\nu} + a_3 \underbrace{\tilde{F}^{\mu\nu}}_{\text{Kerm}} \underbrace{\tilde{F}^{\mu\nu}}_{\text{Kerm}} F_{\mu\nu} + a_4 \underbrace{A_{\mu} j^{\mu}}_{\text{Gauge inv.}}$$
(6.52)

$$\underbrace{\tilde{F}^{\mu\nu}}_{\text{term}} F_{\mu\nu} + a_3 \underbrace{\tilde{F}^{\mu\nu}}_{\text{Kerm}} F_{\mu\nu} + a_4 \underbrace{A_{\mu} j^{\mu}}_{\text{term}}$$
(6.52)

Details:
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• It is straightforward to check that

$$\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} \stackrel{\circ}{=} -F^{\mu\nu}F_{\mu\nu} \tag{6.53}$$

so that we can drop the a_3 -term without loss of generality.



• One can also check that

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})(\partial_{\rho}A_{\sigma} - \partial_{\sigma}A_{\rho})$$
(6.54a)

$$=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_{\mu}A_{\nu}\partial_{\rho}A_{\sigma}+\partial_{\nu}A_{\mu}\partial_{\sigma}A_{\rho}-\partial_{\nu}A_{\mu}\partial_{\rho}A_{\sigma}-\partial_{\mu}A_{\nu}\partial_{\sigma}A_{\rho}) \quad (6.54b)$$

$$= 2\varepsilon^{\mu\nu\rho\sigma}(\partial_{\mu}A_{\nu})(\partial_{\rho}A_{\sigma}) \tag{6.54c}$$

$$= 2 \underbrace{\partial_{\mu} \varepsilon^{\mu\nu\rho\sigma} (A_{\nu}\partial_{\rho}A_{\sigma})}_{\text{Surface term}}$$
(6.54d)

so that the a_2 -term has no effect on the equations of motion and we can drop it as well.

Note: The a_2 -term is known as the $\uparrow \theta$ -term and is special because it is topological (it does not "feel" the geometry of spacetime). This is easy to see: One does not need a metric tensor to construct it because the contravariant indices of the DFST stem from the Levi-Civita symbol! Despite being a surface term, such terms are important when one quantizes the theory and/or when the gauge theory is non-Abelian (like the SU(3) gauge theory of the strong interaction). Note also that this term is a *pseudo* scalar, i.e., it breaks parity symmetry (which we know electrodynamics does not).

• The *a*₄-term is *not* gauge invariant. However, the continuity equation ensures that it modifies the Lagrangian only by a surface term under gauge transformations:

$$\tilde{A}_{\mu}j^{\mu} = (A_{\mu} - \partial_{\mu}\lambda)j^{\mu} = A_{\mu}j^{\mu} - (\partial_{\mu}\lambda)j^{\mu} = A_{\mu}j^{\mu} - \underbrace{\partial_{\mu}(\lambda j^{\mu})}_{\text{Surface term}}$$
(6.55)

(Here we used the continuity equation $\partial_{\mu} j^{\mu} = 0$.)

Consequently, the equations of motion must be gauge invariant despite the a_4 -term.

• It is easy to check that the quadratic Lorentz scalar $A_{\mu}A^{\mu}$ is not gauge invariant (not even up to a surface term); thus it is forbidden.

Note: Coincidentally, it is this term that would give the quantized excitations of the A-field a mass. Thus if you want massive gauge excitations (like the W^{\pm} - and Z-bosons of the weak interaction), you must find a way to smuggle the term $A_{\mu}A^{\mu}$ into your Lagrangian. This is what the \uparrow *Higgs mechanism* achieves.

iii | Thus we propose the Lagrangian density for Maxwell theory:

$$\mathcal{L} \equiv \mathcal{L}_{\text{Maxwell}}(A, \partial A) = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} A_{\mu} j^{\mu}$$
(6.56)

The prefactors have been chosen such that the Euler-Lagrange equations match the IME $(\rightarrow next step)$.

iv | Euler-Lagrange equations:

Details:
Problemset 7

There are four ($\mu = 0, 1, 2, 3$) Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial A_{\mu}} - \partial_{\nu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_{\nu} A_{\mu})} \right) = 0 \tag{6.57}$$



Straightforward calculations yield:

$$\frac{\partial \mathcal{L}}{\partial A_{\mu}} = -\frac{1}{c} j^{\mu} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} A_{\mu})} \stackrel{\circ}{=} \frac{1}{4\pi} F^{\mu\nu} \tag{6.58}$$

Hence the Euler-Lagrange equations are exactly the inhomogeneous Maxwell equations:

$$\partial_{\nu}F^{\mu\nu} = -\frac{4\pi}{c}j^{\mu} \tag{6.59}$$

 \rightarrow Eq. (6.56) is the correct Lagrangian density for Maxwell theory.

10 | <u>Coordinate-free notation:</u>

Remember the coordinate-free concepts introduced in Chapter 3: All tensor fields T_J^I are the chart-dependent components of chart-independent objects T (the actual tensor fields). This formalism allows us to reformulate the Maxwell equations in the language of differential geometry, without using coordinates altogether:

i | First, write gauge field

$$A := A_{\mu} \mathrm{d} x^{\mu} \tag{6.60}$$

and the field strength coordinate-free:

$$F := F_{\mu\nu} \,\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = \frac{1}{2} F_{\mu\nu} \underbrace{\left[\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} - \mathrm{d}x^{\nu} \otimes \mathrm{d}x^{\mu}\right]}_{=:\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} (``wedge product'')} . \tag{6.61}$$

We say that *A* is a 1-form and *F* is a 2-form.

ii | We can evaluate \uparrow *exterior derivative* of the gauge field:

$$dA \stackrel{\text{def}}{=} dA_{\nu} \wedge dx^{\nu} = \partial_{\mu}A_{\nu} dx^{\mu} \wedge dx^{\nu} = \frac{1}{2}F_{\mu\nu} dx^{\mu} \wedge dx^{\nu} = F$$
(6.62)

The exterior derivative d maps k-forms onto k + 1-forms.

iii | Now evaluate the exterior derivative of the field strength:

$$\mathrm{d}F \stackrel{\mathrm{def}}{=} \frac{1}{2} \partial_{\sigma} F_{\mu\nu} \,\mathrm{d}x^{\sigma} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu} \tag{6.63a}$$

$$= \frac{1}{6} \left(\partial_{\sigma} F_{\mu\nu} + \partial_{\nu} F_{\sigma\mu} + \partial_{\mu} F_{\nu\sigma} \right) \, \mathrm{d}x^{\sigma} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu} \tag{6.63b}$$

$$= \frac{1}{2} \sum_{\sigma < \nu < \mu} \left(\partial_{\sigma} F_{\mu\nu} + \partial_{\nu} F_{\sigma\mu} + \partial_{\mu} F_{\nu\sigma} \right) \, \mathrm{d}x^{\sigma} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\mu} \tag{6.63c}$$

(Here we used the antisymmetry of the wedge product in all factors.) Thus we find:

$$dF = 0 \quad \Leftrightarrow \quad \partial_{\sigma}F_{\mu\nu} + \partial_{\nu}F_{\sigma\mu} + \partial_{\mu}F_{\nu\sigma} = 0 \quad \stackrel{6.42}{\Leftrightarrow} \quad \partial_{\nu}\tilde{F}^{\mu\nu} = 0 \tag{6.64}$$

If the field strength is expressed in terms of the gauge field, the homogeneous Maxwell equations $\partial_{\nu} \tilde{F}^{\mu\nu} = 0$ are identities. In the coordinate-free notation of differential geometry, this identity follows from the fact that applying an exterior derivative twice produces the zero field:

$$\mathrm{d}F = \mathrm{d}\mathrm{d}A = 0 \quad \text{since} \quad \mathrm{d}^2 = 0 \tag{6.65}$$

The relation dF = 0 is known as a \uparrow *Bianchi identity*.



iv | Define the linear \uparrow *Hodge star operator* (here for a 4-dimensional Minkowski manifold):

$$\star (\mathrm{d}x^{\mu}) := \frac{1}{3!} \varepsilon^{\mu}{}_{\nu\rho\sigma} \left(\mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho} \wedge \mathrm{d}x^{\sigma} \right) \tag{6.66a}$$

$$\star (\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}) := \frac{1}{2!} \varepsilon^{\mu\nu}{}_{\rho\sigma} (\mathrm{d}x^{\rho} \wedge \mathrm{d}x^{\sigma}) \tag{6.66b}$$

$$\star (\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu} \wedge \mathrm{d}x^{\rho}) := \frac{1}{1!} \varepsilon^{\mu\nu\rho}{}_{\sigma} (\mathrm{d}x^{\sigma}) \tag{6.66c}$$

Note that the definition makes use of the metric tensor via pulling up/down indices of the Levi-Civita symbols. This implies in particular that any equation that uses the Hodge star depends on the geometry of spacetime (here flat Minkowski space).

v | The dual field-strength tensor (DFST) is the Hodge dual of the field-strength tensor (FST):

$$\star F = \frac{1}{2} F_{\mu\nu} \star (\mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}) \tag{6.67a}$$

$$= \frac{1}{4} F_{\mu\nu} \varepsilon^{\mu\nu}{}_{\rho\sigma} \left(\mathrm{d}x^{\rho} \wedge \mathrm{d}x^{\sigma} \right) \tag{6.67b}$$

$$=\frac{1}{2}\tilde{F}_{\rho\sigma}\left(\mathrm{d}x^{\rho}\wedge\mathrm{d}x^{\sigma}\right)\equiv\tilde{F}$$
(6.67c)

Beware: The Hodge star \star is not a multiplication symbol (as the notation on the right-hand side might suggest) but a linear operator that acts on the differential form to the right.

vi | The Hodge dual of the exterior derivative of the DFST yields:

$$\star \mathbf{d}(\star F) = \frac{1}{4} \varepsilon^{\mu\nu}{}_{\rho\sigma} \partial_{\pi} F_{\mu\nu} \star (\mathbf{d}x^{\pi} \wedge \mathbf{d}x^{\rho} \wedge \mathbf{d}x^{\sigma})$$
(6.68a)

$$= \frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\pi\rho\sigma}{}_{\alpha} \partial_{\pi} F^{\mu\nu} \left(\mathrm{d} x^{\alpha} \right)$$
(6.68b)

$$= \frac{1}{2} (\delta^{\pi}_{\mu} \eta_{\nu \alpha} - \delta^{\pi}_{\nu} \eta_{\mu \alpha}) \partial_{\pi} F^{\mu \nu} (\mathrm{d}x^{\alpha})$$
(6.68c)

$$= \eta_{\nu\alpha} \partial_{\mu} F^{\mu\nu} \left(\mathrm{d} x^{\alpha} \right) \tag{6.68d}$$

$$\stackrel{6.50b}{=} \frac{4\pi}{c} j_{\alpha} \left(dx^{\alpha} \right) \tag{6.68e}$$

Here we used a contraction identity for Levi-Civita symbols (over the two red pairs of indices).

vii | This motivates the definition of the coordinate-free current:

$$J := \frac{4\pi}{c} j_{\mu} \,\mathrm{d}x^{\mu} \tag{6.69}$$

viii | In conclusion, the Maxwell equations can be written without using a coordinate system as:

Homogeneous ME:
$$dF = 0$$
(6.70a)Inhomogeneous ME: $\star d(\star F) = J$ (6.70b)

- If one uses that $(\star)^2 = +1 \cdot 1$ on odd differential forms $(d(\star F) \text{ is a 3-form})$, Eq. (6.70b) can alternatively be written as $d(\star F) = \star J$. If one then defines the current not as a 1-form but as the dual 3-form, $J := \frac{4\pi}{c} j_{\mu} \star dx^{\mu}$, the inhomogeneous Maxwell equations take their simplest form: $d(\star F) = J$.
- Eq. (6.70) is the most general and elegant formulation of the Maxwell equations. In this form, the equations remain valid even in GENERAL RELATIVITY on curved space times. Then the Minkowski metric used in the definition of the Hodge star * (to pull the indices of the Levi-Civita symbols up/down) must be replaced by the dynamic, potentially curved metric of GENERAL RELATIVITY.



6.3. Noether theorem and the energy-momentum tensor

In the following, we consider first a generic (classical, relativistic) field theory, and specialize to electrodynamics later. This is to emphasize that most of the results in this chapter are not specific to electrodynamics.

Details: Chapter 1 of my QFT script [20]

1 | \triangleleft General transformation of field $\phi \mapsto \phi'$:

$$x \mapsto x' = x'(x)$$
 and $\phi(x) \mapsto \phi'(x') = \mathcal{F}(\phi(x))$ (6.71)

<u>Two</u> effects: coordinates <u>and</u> (values of the) field transformed These are atting transformations that shares physical $u'_{ij} = u'_{ij}(u)$ is not a (r

These are *active transformations* that change physics. x' = x'(x) is *not* a (passive) coordinate transformation; the frame of reference remains fixed in the following!

S Example 1: Homogeneous Lorentz transformations

The (active) homogeneous Lorentz transformation of a vector field A^{μ} reads

$$x^{\mu} \mapsto x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \quad \text{and} \quad A_{\nu}(x) \mapsto A'_{\mu}(x') = \underbrace{\Lambda^{\nu}_{\mu}A_{\nu}(x)}_{\mathcal{F}(A_{\mu}(x))}$$
(6.72)

whereas the Lorentz transformation of a *scalar field* ϕ reads

$$x^{\mu} \mapsto x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu} \quad \text{and} \quad \phi(x) \mapsto \phi'(x') = \underbrace{\phi(x)}_{\mathscr{F}(\phi(x))} .$$
 (6.73)

2 | \triangleleft <u>Infinitesimal</u> transformations (IT) ($|w_a| \ll 1$):

$$x'^{\mu} = x^{\mu} + w_a \,\delta^a x^{\mu}(x) \quad \text{and} \quad \phi'(x') = \phi(x) + w_a \,\delta^a \phi(x)$$
(6.74)

Here, w_a denotes infinitesimal parameters of the transformation (sum over *a* implied!) and we label different transformations by the labels *a*.

• Example 2: Homogeneous Lorentz transformations

Infinitesimal homogeneous Lorentz transformations take the form (Problemset 4)

$$\Lambda_{w} = \exp\left(-\frac{i}{2}w_{\alpha\beta}\mathcal{J}^{\alpha\beta}\right) \stackrel{|w_{\alpha\beta}| \ll 1}{\approx} \mathbb{1} - \frac{i}{2}w_{\alpha\beta}\mathcal{J}^{\alpha\beta}$$
(6.75)

(note that the $a = \alpha\beta$ are labels of generators that are not required to be tensor indices) with generators

$$\left(\mathcal{J}^{\alpha\beta}\right)^{\mu}{}_{\nu} = i\left(\eta^{\alpha\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\eta^{\beta\mu}\right). \tag{6.76}$$

With this it follows for the coordinates

$$w_{\alpha\beta}\,\delta^{\alpha\beta}x^{\mu} = x^{\prime\mu} - x^{\mu} = -\frac{i}{2}w_{\alpha\beta}(\mathcal{J}^{\alpha\beta})^{\mu}{}_{\nu}x^{\nu} = w_{\alpha\beta}\,\underbrace{\frac{1}{2}\left(\eta^{\alpha\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\eta^{\beta\mu}\right)x^{\nu}}_{\delta^{\alpha\beta}x^{\mu}} \tag{6.77}$$



so that

$$\delta^{\alpha\beta}x^{\mu} = \frac{1}{2} \left(\eta^{\alpha\mu}x^{\beta} - \eta^{\beta\mu}x^{\alpha} \right) \,. \tag{6.78}$$

Similar arguments yield $\delta^{\alpha\beta} A^{\mu} = \frac{1}{2} \left(\eta^{\alpha\mu} A^{\beta} - \eta^{\beta\mu} A^{\alpha} \right)$ for a vector field and $\delta^{\alpha\beta} \phi = 0$ for a scalar field.

3 Generator of IT:

$$\delta_w \phi(x) := \phi'(x) - \phi(x) \equiv -i w^a G_a \phi(x) \tag{6.79}$$

With (omit first line and refer to previous equation)

$$\phi'(x') = \phi(x) + w^{a} \,\delta_{a} \phi(x)$$
(6.80a)
= $\phi(x') - w^{a} (\delta_{a} x^{\mu}) \partial_{\mu} \phi(x') + w^{a} \delta_{a} \phi(x') + \mathcal{O}(w^{2})$ (6.80b)

(Here we replaced x by x' in the last term because this is a $\mathcal{O}(w^2)$ modification.)

it follows (replace x' by x; these are just labels!)

$$iG_a\phi = (\delta_a x^\mu)\partial_\mu\phi - \delta_a\phi \tag{6.81}$$

This function describes the infinitesimal change of the field at the same point.

S Example 3: Translations

- $\mathbf{i} \mid x^{\prime \mu} := x^{\mu} + w^{\mu} \equiv x^{\mu} + w^{\nu} \delta_{\nu} x^{\mu} \text{ with } \delta_{\nu} x^{\mu} = \delta_{\nu}^{\mu}$
- ii | $\delta_{\nu}\phi = 0$ (This is true for scalar and vector fields.)
- iii | $i G_{\mu} \phi = \delta^{\nu}_{\mu} \partial_{\nu} \phi 0$ and therefore

$$G_{\mu} = -i\,\partial_{\mu} \equiv P_{\mu} \tag{6.82}$$

- \rightarrow The "momentum operator" generates translations.
- **4** | So far the continuous transformations $\phi \mapsto \phi'$ were arbitrary.

Symmetry of the action
$$\Rightarrow S[\phi] = S[\phi']$$
 (6.83)

In principle, the action can vary by a surface term – equivalently, the Lagrangian density \mathcal{L} can vary by a 4-divergence $\partial_{\mu} K^{\mu}(\phi, x)$ – under the symmetry transformation (because such modifications do not affect the equations of motion). Here we consider for simplicity only the case where no such terms exist and the action is strictly invariant.

Then one can prove (see Chapter 1 of my QFT script [20] or Refs. [1,78]): $\stackrel{*}{\rightarrow}$

5 | *** Noether's (first) theorem:

For solutions ϕ of the equations of motion, the ** (canonical) (Noether) currents

$$j_{a}^{\mu} \stackrel{*}{=} \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\nu}\phi - \delta_{\nu}^{\mu}\mathcal{L} \right\} \delta_{a}x^{\nu} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta_{a}\phi$$
(6.84)

(associated to the infinitesimal transformations of coordinates $\delta_a x^{\nu}$ and fields $\delta_a \phi$) satisfy the continuity equations

$$\forall_a: \qquad \partial_\mu j_a^\mu = 0. \tag{6.85}$$

This means there is one conserved current j_a^{μ} for each generator *a* of the continuous symmetry.

6 | Conserved charge:

The currents Eq. (6.84) are called "conserved" because they describe the flow of a conserved ...

$$Q_a := \int_{\text{Space}} \mathrm{d}^{D-1}x \, j_a^0 \qquad \text{**} \text{ (Noether) charge} \tag{6.86}$$

There is one conserved charge Q_a for each generator a of the continuous symmetry. Indeed:

$$\frac{1}{c}\frac{\mathrm{d}\mathcal{Q}_a}{\mathrm{d}t} = \int_{\mathrm{Space}} \mathrm{d}^{D-1}x\,\partial_0 j_a^0 \stackrel{\mathbf{6.85}}{=} -\int_{\mathrm{Space}} \mathrm{d}^{D-1}x\,\partial_k j_a^k \stackrel{\mathrm{Gauss}}{=} -\int_{\mathrm{Surface}} \mathrm{d}\sigma_k j_a^k = 0 \qquad (6.87)$$

Here we assume that $j_a^k \equiv 0$ on the spatial boundaries—typically at infinity, i.e., the universe is closed. k = 1, 2, 3 denotes the spatial coordinates.

Note 6.1

The current Eq. (6.84) is called <u>canonical</u> current because it is not unique:

$$\tilde{j}_a^{\mu} := j_a^{\mu} + \partial_{\nu} B_a^{\mu\nu}$$
 with $B_a^{\mu\nu} = -B_a^{\nu\mu}$ arbitrary $\Rightarrow \partial_{\mu} \tilde{j}_a^{\mu} = 0$ (6.88)

This is particularly important for the energy-momentum tensor (\rightarrow *below*).

6.3.1. Application: The Energy-Momentum Tensor (EMT)

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7 | ⊲ Infinitesimal spacetime translations:

$$x^{\prime\mu} = x^{\mu} + w^{\mu} \quad \Rightarrow \quad \delta_{\nu} x^{\mu} = \delta_{\nu}^{\mu} \quad \text{and} \quad \delta_{\nu} \phi = 0$$
 (6.89)

& Translation-invariant action: S' = S (This includes translations in time!)



8 | <u>Conserved currents</u>: Eq. $(6.84) \rightarrow$

$$\Theta^{\mu}{}_{\nu} := \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\rho}\phi - \delta^{\mu}_{\rho}\mathcal{L} \right\} \underbrace{\delta_{\nu}x^{\rho}}_{\delta^{\rho}_{\nu}} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\nu}\phi - \delta^{\mu}_{\nu}\mathcal{L}$$
(6.90)

Note that the generator index *a* is in this case a proper Lorentz index ν so that we can pull it up, $\Theta^{\mu\nu} = \eta^{\nu\rho} \Theta^{\mu}{}_{\rho}$, and obtain:

** (Canonical) Energy-Momentum Tensor:

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial^{\nu}\phi - \eta^{\mu\nu}\mathcal{L}$$
(6.91)

with

$$\partial_{\mu}\Theta^{\mu\nu} = 0$$
 and four conserved charges $P^{\nu} := \frac{1}{c} \int d^3x \,\Theta^{0\nu}$. (6.92)

- Note that these quantities are only conserved for *solutions* of the Euler-Lagrange equations.
- P^ν is a 4-vector (show this!). Note that this is a non-trivial statement because d³x is not a Lorentz scalar and Θ^{0ν} not a 4-vector.
- The prefactor 1/c ensures that P⁰ has the same dimension as a conventional 4-momentum with p⁰ = E/c; note that Θ⁰⁰ has the dimension of an *energy density* because L has this dimension.
- **9** | Interpretation:
 - $\mathbf{i} \mid Energy (v = 0):$

$$cP^{0} = \int d^{3}x \,\Theta^{00} = \int d^{3}x \,\underbrace{\left\{\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L}\right\}}_{\text{Hamiltonian density}} = \underbrace{\int d^{3}x \,\mathcal{H}(\phi, \pi)}_{\text{Hamiltonian}} = H \qquad (6.93)$$

 \rightarrow The Hamiltonian is the component of a 4-vector and not Lorentz invariant! By contrast, the Lagrangian *is* Lorentz invariant (for relativistic field theories).

ii | Kinetic momentum (v = i):

$$P^{i} = \int d^{3}x \,\Theta^{0i} = \int d^{3}x \,\frac{\partial \mathcal{L}}{\partial \dot{\phi}}(-\partial_{i}\phi) = -\int d^{3}x \,\pi \,\partial_{i}\phi \tag{6.94}$$

 π is the *canonical* momentum conjugate to the field ϕ .

- **10** | The canonical EMT of electrodynamics:
 - i | $\triangleleft \underline{\text{Free}} (j^{\mu} = 0)$ electromagnetic field: $\mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$ \rightarrow Invariant under spacetime translations

Indeed, with $x'^{\mu} = x^{\mu} + w^{\mu}$ and the field transformation $A'_{\mu}(x) := A_{\mu}(x - w)$ it is

$$S_{\rm em}[A'] = \int d^4x \, \mathcal{L}_{\rm em}(A'(x), \partial A'(x)) = \int d^4x \, \mathcal{L}_{\rm em}(A(x-w), \partial A(x-w)) \tag{6.95a}$$

$$= \int d^4 y \,\mathcal{L}_{\rm em}(A(y), \partial A(y)) = S_{\rm em}[A] \tag{6.95b}$$

where we integrate over the full Minkowski spacetime $\mathbb{R}^{1,3}$, substituted $y^{\mu} = x^{\mu} - w^{\mu}$ and used $d^4x = d^4y$.



ii | \rightarrow Canonical EMT conserved: $\partial_{\mu}\Theta_{em}^{\mu\nu} = 0$ with

$$\Theta_{\rm em}^{\mu\nu} = \frac{\partial \mathcal{L}_{\rm em}}{\partial(\partial_{\mu}A_{\sigma})} \partial^{\nu}A_{\sigma} - \eta^{\mu\nu}\mathcal{L}_{\rm em} \stackrel{6.58}{=} \frac{1}{4\pi} F^{\sigma\mu}\partial^{\nu}A_{\sigma} + \frac{\eta^{\mu\nu}}{16\pi} F_{\sigma\rho}F^{\sigma\rho} \qquad (6.96)$$

Note that because the gauge field has multiple components A_{μ} , there is now an additional summation in the first term over these components (marked indices). This follows directly from a generalization of the proof of Noether's theorem for fields with multiple components.

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iii | <u>Problems:</u>

The canonical EMT $\Theta_{em}^{\mu\nu}$ has two problematic properties:

• Because of the term $\partial^{\nu} A_{\sigma}$, $\Theta_{em}^{\mu\nu}$ is gauge-dependent!

This is problematic because it means that we cannot hope to identify physical quantities like the energy density or the momentum density of the electromagnetic field with (the components) of this tensor.

• The canonical EMT is non-symmetric: $\Theta_{em}^{\mu\nu} \neq \Theta_{em}^{\nu\mu}!$

In GENERAL RELATIVITY, we will find that the right-hand side of the \rightarrow *Einstein field* equations (which determine how spacetime curves and evolves)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}$$
(6.97)

is given by the \rightarrow Hilbert energy-momentum tensor

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta(\mathcal{L}_{\text{Matter}})}{\delta g_{\mu\nu}}$$
(6.98)

where \mathcal{L}_{Matter} describes the Lagrangian density of all fields in the universe (except the metric tensor field). For example, \mathcal{L}_{Matter} contains the Maxwell Lagrangian \mathcal{L}_{em} ("matter" here includes every degree of freedom that has energy & momentum, i.e., also electromagnetic radiation).

Note that $T^{\mu\nu}$ is *symmetric* because the metric $g_{\mu\nu}$ is. Hence it cannot be identified with the canonical EMT $\Theta^{\mu\nu}$ in general (here for the example of Maxwell theory).

i! These problems are not specific to electrodynamics but typically affect all theories that are gauge theories and/or include non-scalar fields.

 \rightarrow How to solve these issues?

6.3.2. The Belinfante-Rosenfeld energy-momentum tensor (BRT)

We consider again first a generic field theory, and specialize to electrodynamics later.

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11 | Remember (Note 6.1) that the canonical EMT is not the only conserved EMT because

$$\tilde{\Theta}^{\mu\nu} := \Theta^{\mu\nu} + \partial_{\rho} K^{\rho\mu\nu} \quad \text{with} \quad K^{\rho\mu\nu} = -K^{\mu\rho\nu} \tag{6.99}$$

yields another EMT $\tilde{\Theta}^{\mu\nu}$ for any suitable tensor $K^{\rho\mu\nu}$.

 \rightarrow *Idea:* Find $K^{\rho\mu\nu}$ such that $\tilde{\Theta}^{\mu\nu} = \tilde{\Theta}^{\nu\mu}$ is symmetric (and hopefully gauge-invariant).



12 | Let us assume that our theory is also invariant under homogeneous Lorentz transformations (in addition to the spacetime translations needed for the conservation of the EMT).

Eq. (6.78)
$$\rightarrow \quad \delta^{\alpha\beta} x^{\mu} = \frac{1}{2} \left(\eta^{\alpha\mu} x^{\beta} - \eta^{\beta\mu} x^{\alpha} \right) .$$
 (6.100)

Assume that fields transform as $\delta^{\alpha\beta}\phi$.

For the following arguments, we do not need to fix whether our fields transform as scalar, vector, or even \rightarrow *spinor fields*.

Eq. (6.84) & Eq. (6.91) & Eq. (6.100) \rightarrow

Noether currents for homogeneous LTs:

$$L^{\mu\alpha\beta} \stackrel{\circ}{=} \frac{1}{2} \left(\Theta^{\mu\alpha} x^{\beta} - \Theta^{\mu\beta} x^{\alpha} \right) + \frac{1}{2} S^{\mu\alpha\beta}$$
(6.101)

with

** Spin current:
$$S^{\mu\alpha\beta} := -2 \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta^{\alpha\beta}\phi$$
 (6.102)

which satisfies $S^{\mu\alpha\beta} = -S^{\mu\beta\alpha}$.

(This follows because $\delta^{\alpha\beta}\phi = -\delta^{\beta\alpha}\phi$ as the generators of homogeneous LTs are antisymmetric.) The continuity equation reads

$$\partial_{\mu}L^{\mu\alpha\beta} = 0. ag{6.103}$$

Because homogeneous LTs describe *rotations* in space and time, the conserved current $L^{\mu\alpha\beta}$ can be identified as \uparrow *(canonical) angular momentum current*. The first part in Eq. (6.101) corresponds to the (canonical) *orbital angular momentum* while the second part $S^{\mu\alpha\beta}$ encodes the *intrinsic angular momentum* of the field (= its \downarrow *spin*). This immediately explains why for a scalar field with $\delta^{\alpha\beta}\phi = 0$, the spin current vanishes $S^{\mu\alpha\beta} = 0$.

13 | Eq. (6.92) & Eq. (6.103) \rightarrow

$$\partial_{\mu}S^{\mu\alpha\beta} \stackrel{\circ}{=} \Theta^{\alpha\beta} - \Theta^{\beta\alpha}$$
(6.104)

This means that a non-vanishing divergence in the spin current is responsible for the "non-symmetry" of the canonical EMT!

14 | Now define

$$K^{\rho\mu\nu} := -\frac{1}{2} \left(S^{\mu\nu\rho} + S^{\nu\mu\rho} - S^{\rho\nu\mu} \right)$$
(6.105)

 $\rightarrow K^{\rho\mu\nu} = -K^{\mu\rho\nu}$ (This follows from $S^{\mu\alpha\beta} = -S^{\mu\beta\alpha}$.)

With this we can finally define the ...

** Belinfante-Rosenfeld energy-momentum tensor (BRT):

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_{\rho} K^{\rho\mu\nu} := \Theta^{\mu\nu} - \frac{1}{2} \partial_{\rho} \left(S^{\mu\nu\rho} + S^{\nu\mu\rho} - S^{\rho\nu\mu} \right)$$
(6.106)

15 | It remains to be shown that $T^{\mu\nu}$ is always symmetric:

$$T^{\mu\nu} - T^{\nu\mu} \stackrel{6.104}{=} 0 \quad \textcircled{\circ} \tag{6.107}$$

It can be rigorously shown that the BRT is identical to the Hilbert EMT that shows up in GENERAL RELATIVITY as the source of gravity [79]. This is why the BRT gets its own symbol $T^{\mu\nu}$.

he<mark>oretical</mark> Physics



↓Lecture14 [30.01.24]

16 | The BRT of electrodynamics:

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i | Using $\mathcal{L}_{em} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$ and the transformation of a vector field (= spin-1)

$$\delta^{\alpha\beta}A_{\mu} = \frac{1}{2} \left(\delta^{\alpha}_{\mu}A^{\beta} - \delta^{\beta}_{\mu}A^{\alpha} \right)$$
(6.108)

in Eq. (6.102) yields the spin current:

$$S_{\rm em}^{\mu\alpha\beta} \stackrel{\circ}{=} \frac{1}{4\pi} \left(F^{\mu\alpha} A^{\beta} - F^{\mu\beta} A^{\alpha} \right) \tag{6.109}$$

ii | Eq. (6.96) & Eq. (6.106) & Eq. (6.109) \rightarrow

$$T_{\rm em}^{\mu\nu} \stackrel{\circ}{=} \frac{1}{4\pi} F^{\mu}_{\ \rho} F^{\rho\nu} - \eta^{\mu\nu} \mathcal{L}_{\rm em}$$

$$(6.110a)$$

$$= \frac{1}{4\pi} \left[F^{\mu}_{\ \rho} F^{\rho\nu} + \frac{\eta}{4} F^{\rho\sigma} F_{\rho\sigma} \right]$$
(6.110b)
$$= \left(\begin{array}{c} \varepsilon & | c \vec{\Pi}^T \end{array} \right)$$

$$\stackrel{\circ}{=} \left(\frac{c c n}{c \vec{\Pi} \mathbf{\Sigma}} \right)_{\mu\nu} \tag{6.110c}$$

To show this you have to use the Maxwell equations in vacuum: $\partial_{\nu}F^{\nu}_{\ \mu} = 0$. Components:

Energy density:
$$\mathcal{E} = \frac{1}{8\pi} (\vec{E}^2 + \vec{B}^2)$$
 (6.111a)

Momentum density:
$$\vec{\Pi} = \frac{1}{4\pi c} (\vec{E} \times \vec{B})$$
 (6.111b)

** Maxwell stress tensor:
$$\Sigma_{ij} = \frac{1}{4\pi} \left[\frac{\delta_{ij}}{2} (\vec{E}^2 + \vec{B}^2) - E_i E_j - B_i B_j \right]$$
 (6.111c)

;! Convince yourself that $T_{em}^{\mu\nu}$ is *symmetric* and *gauge invariant*. Note that we did not construct it to be gauge invariant, only to be symmetric! We got this as a bonus.

- iii | The conservation $\partial_{\mu}T^{\mu\nu} = 0$ of the BRT implies the following physical interpretations:
 - v = 0:

$$\partial_{\mu}T^{\mu 0} = \frac{1}{c}\frac{\partial\mathcal{E}}{\partial t} + c\nabla\cdot\vec{\Pi} = 0$$
(6.112)

 $\rightarrow \downarrow$ *Poynting's theorem* (in vacuum)

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \vec{S} = 0 \tag{6.113}$$



with

↓ Poynting vector:
$$\vec{S} = c^2 \vec{\Pi} = \frac{c}{4\pi} (\vec{E} \times \vec{B})$$
 (6.114)

Eq. (6.113) \rightarrow Poynting vector = Energy current density

This is simply the formal statement of *energy conservation* for the free electromagnetic field. As energy is the Noether charge for translations in time, it is of course no coincidence that the Poynting theorem follows from the time-component $\nu = 0$.

• v = i:

$$\partial_{\mu}T^{\mu i} = \frac{\partial \Pi_i}{\partial t} + \partial_k \Sigma_{ki} = 0$$
 (6.115)

- \rightarrow Conservation of momentum with ...
 - Π_i : *i*-momentum density
 - Σ_{ki} : *i*-momentum current density
- \rightarrow Maxwell stress tensor = Momentum current density

Note that are three momentum densities and corresponding current densities because there are three spatial momenta: i = x, y, z.

- iv | Some final remarks:
 - With the symmetric BRT one can define a gauge-invariant and conserved *angular momentum tensor*

$$M^{\rho\mu\nu} := T^{\rho\mu} x^{\nu} - T^{\rho\nu} x^{\mu}$$
(6.116)

with $\partial_{\rho} M^{\rho\mu\nu} = 0$ (show this!). The conserved Noether charges are

$$J^{\mu\nu} := \frac{1}{c} \int d^3x \, M^{0\mu\nu} = \frac{1}{c} \int d^3x \, \left(T^{0\mu} x^{\nu} - T^{0\nu} x^{\mu} \right) \tag{6.117}$$

which encodes the *total angular momentum* of the field. Indeed, for the spatial components one finds

$$J_{ij} := \int d^3x \, \left(\Pi_i x_j - \Pi_j x_i \right) \,. \tag{6.118}$$

Since Π_i is the momentum density, the three components $J_x \equiv J_{32}$, $J_y \equiv J_{13}$ and $J_z \equiv J_{21}$ can be identified as the total angular momentum \vec{J} of the field.

• If the electric current j^{μ} does *not* vanish (i.e., the field is not in vacuum), the BRT derived above is no longer conserved. Rather one finds

$$\partial_{\mu}T_{\rm em}^{\mu\nu} = -\frac{1}{c}F^{\nu\rho}j_{\rho} \tag{6.119}$$

which can be identified as the *Lorentz force density*. This is perfectly reasonable as an external (non-dynamic) current j^{μ} breaks the translation symmetry of the system in space and time on which the conservation of the BRT relies. Physically, the electromagnetic field is no longer a closed system because it can exchange momentum and energy with the charges described by j^{μ} . Only if one describes the charges as dynamic degrees of freedom (\rightarrow *next section*) and considers the total BRT

$$T^{\mu\nu} = T^{\mu\nu}_{\rm em} + T^{\mu\nu}_{\rm charges} \tag{6.120}$$

one would recover the conservation $\partial_{\mu}T^{\mu\nu} = 0$; this is then a statement about total energy and momentum conservation, including the energy and momentum of the charges.



6.4. Charged point particles in an electromagnetic field

 $|\mathbf{1}| \leq N$ charged point particles with charge q_i and mass m_i in a EM field A_{μ} :



Eq. (6.56) & Eq. (5.41) \rightarrow Relativistic action of the complete system:

$$S[\{x_k\}, A] = \int d^4x \left[\underbrace{-\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu}}_{S_{em}[A]} \underbrace{-\frac{1}{c^2} A_{\mu} j^{\mu}}_{S_{c}[\{x_k\}, A]} - \sum_{i=1}^{N} \underbrace{m_i c \int ds_i}_{S_{p}[x_i]}_{Particle i} \right]$$
(6.121)

Note that the Lagrangian is a Lorentz scalar! $S[{x_k}, A]$ is short for $S[x_1, ..., x_N, A]$. with current density

$$j^{\mu}(x) \stackrel{\textbf{6.18}}{=} \sum_{i} \rho_{i}(x) \frac{\mathrm{d}x_{i}^{\mu}}{\mathrm{d}t} = \sum_{i} \underbrace{q_{i}\delta(\vec{x}-\vec{x}_{i})}_{\text{Point particle}} \frac{\mathrm{d}x_{i}^{\mu}}{\mathrm{d}t}.$$
(6.122)

 $\mathbf{2} \mid \triangleleft \text{Coupling:}$

$$S_{c}[\{x_{k}\}, A] = -\frac{1}{c^{2}} \int d^{4}x \, A_{\mu}(x) j^{\mu}(x) \stackrel{\circ}{=} \sum_{i} \underbrace{\left\{-\frac{q_{i}}{c} \int A_{\mu}(ct, \vec{x}_{i}) \, dx_{i}^{\mu}\right\}}_{S_{c}[x_{i}, A]} \tag{6.123}$$

Here we used $\frac{dx_i^{\mu}}{dt}dt = dx_i^{\mu}$; the last integral is therefore a four-dimensional \checkmark *line integral* of the 4-vectorfield A_{μ} along the trajectory of particle *i*.

3 | Hamilton's principle:

$$\delta S[x_k, A] = 0 \quad \Leftrightarrow \quad \begin{cases} \frac{\delta S_{\rm em}[A]}{\delta A} + \frac{\delta S_{\rm c}[\{x_k\}, A]}{\delta A} = \frac{\delta S_j[A]}{\delta A} = 0\\ \forall_i : \frac{\delta S_{\rm c}[x_i, A]}{\delta x_i} + \frac{\delta S_{\rm p}[x_i]}{\delta x_i} = \frac{\delta S_A[\{x_k\}]}{\delta x_i} = 0 \end{cases}$$
(6.124)

4 | \triangleleft Gauge field variations δA :

Here we don't have to do anything because we already computed the Euler-Lagrange equations:

$$\frac{\delta S_j[A]}{\delta A} = 0 \quad \stackrel{\textbf{6.56 \& 6.58}}{\longleftrightarrow} \quad \partial_{\nu} F^{\nu\mu} \stackrel{\textbf{6.122}}{=} \frac{4\pi}{c} \sum_i q_i \delta(\vec{x} - \vec{x}_i) \frac{\mathrm{d}x_i^{\mu}}{\mathrm{d}t} \tag{6.125}$$

These are the inhomogeneous Maxwell equations with the N point particles as sources of the EM field. Note that this PDE system couples the particle coordinates $\{x_k^{\mu}\}$ to the EM field A^{μ} .

- **5** | \triangleleft Particle trajectory variations δx_i :
 - i | Eqs. (6.121) and (6.123) \rightarrow

$$S_A[\{x_k\}] = -\sum_i \int \left[m_i c \sqrt{\dot{x}_{i\mu} \dot{x}_i^{\mu}} + \frac{q_i}{c} A_\mu(x_i) \dot{x}_i^{\mu} \right] \mathrm{d}\lambda \tag{6.126}$$

Note that this action is again reparametrization invariant.

 $\stackrel{\circ}{\rightarrow}$ Euler-Lagrange equation for particle *i*:

$$\frac{\delta S_A[\{x_k\}]}{\delta x_i} = 0 \quad \stackrel{x^\mu \equiv x_i^\mu}{\longleftrightarrow} \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} \Big[\frac{m_i c \dot{x}_\mu}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} \Big] + \frac{q_i}{c} \Big[\dot{A}_\mu(x) - \dot{x}^\nu \frac{\partial A_\nu(x)}{\partial x^\mu} \Big] = 0 \quad (6.127)$$

ii | Choose proper-time parametrization $\lambda = \tau$:

$$m_{i}\frac{\mathrm{d}u_{\mu}}{\mathrm{d}\tau} + \frac{q_{i}}{c}\underbrace{\frac{\mathrm{d}A_{\mu}}{\mathrm{d}\tau}}_{\frac{\partial A_{\mu}}{\partial x^{\nu}}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}} - \frac{\partial A_{\nu}}{\partial x^{\mu}}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau} = 0$$
(6.128)

Thus we find as the EOM for particle *i* :

$$m_{i}\frac{\mathrm{d}u_{\mu}}{\mathrm{d}\tau} = \frac{q_{i}}{c}\underbrace{\left(\frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}}\right)}_{F_{\mu\nu}}u^{\nu}$$
(6.129)

Or in the form discussed previously in Chapter 5 (we restore the particle index *i*):

$$\frac{\mathrm{d}p_i^{\mu}}{\mathrm{d}\tau} = \frac{q_i}{c} F^{\mu}_{\ \nu} \left(x_i \right) u_i^{\nu} \tag{6.130}$$

with 4-momentum $p_i^{\mu} = m_i u_i^{\mu}$.

i! The field strength tensor is evaluated at the position of the particle at a given time.

iii | Compare Eqs. (5.6) and (6.130) \rightarrow 4-force:

$$K^{\mu} = \begin{pmatrix} \gamma_{v} \frac{\vec{F} \cdot \vec{v}}{c} \\ \gamma_{v} \vec{F} \end{pmatrix} = \frac{q_{i}}{c} F^{\mu}_{\ \nu} \gamma_{v} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}t}$$
(6.131)





 \rightarrow 3-force (we restore the particle index *i*):

$$\vec{F}_i \stackrel{\circ}{=} q_i \vec{E}_i + \frac{q_i}{c} (\vec{v}_i \times \vec{B}_i) \quad \checkmark \text{Lorentz force}$$

(6.132)

with $\vec{E}_i = \vec{E}(x_i)$, $\vec{B}_i = \vec{B}(x_i)$ and $\vec{v}_i = \frac{\mathrm{d}\vec{x}_i}{\mathrm{d}t}$.

- This result demonstrates that our concept of the relativistic 3-force introduced in Eq. (5.11) was reasonable: for a force due to an electromagnetic field, it exactly matches the Lorentz force.
- **6** | <u>Comments</u>:
 - Eqs. (6.125) and (6.130) together are the equations of motion of the composite system, i.e., the EM field and the N particles. Note that the system of differential equations is coupled: The dynamical positions of the particles determine the evolution of the EM field via Eq. (6.125), and the dynamical EM field determines the trajectories of the charged particles via Eq. (6.130).
 - This model of *N* charged particles interacting with and via an electromagnetic field is the culmination of our discussion of *relativistic mechanics* in Chapter 5 and *electrodynamics* in Chapter 6.
 - The theory Eqs. (6.125) and (6.130) is fully relativistic as the EOMs are manifestly Lorentz covariant (they are tensor equations).
 - Note that this model describes interactions between the N particles not directly via forces (as one would in Newtonian mechanics), but via coupling to the dynamic EM field. Thus a particle can *locally* affect the EM field due to its motion, the EM field then can propagate with the speed of light through space and affect the trajectory of any other particle within the lightcone of the first. There is no instantaneous interaction between the particles!
 - One can also consider the $\mu = 0$ component of Eq. (6.130). Then one finds with $p_i^0 = E_i/c$:

$$\frac{\mathrm{d}E_i}{\mathrm{d}t} \stackrel{\circ}{=} q_i \vec{E}_i \cdot \vec{v}_i \,. \tag{6.133}$$

This is just the statement that the change of energy for particle i is given by the distance it travels collinear with the electric field per time. This is no surprise: The Lorentz force Eq. (6.132) tells us that the force due to the magnetic field is always perpendicular to the direction of motion and therefore cannot not perform work on the particle.

- 7 | Corollary: Single particle in a static electromagnetic field:
 - i | The action follows from Eq. (6.126) with N = 1 as:

$$S_A[x] = \int d\lambda L(x^{\mu}, \dot{x}^{\mu}) = -\int \left[mc \sqrt{\dot{x}_{\mu} \dot{x}^{\mu}} + \frac{q}{c} A_{\mu}(x_i) \dot{x}^{\mu} \right] d\lambda$$
(6.134)

where A_{μ} is a fixed parameter (the static field configuration).

ii | \triangleleft Parametrization in coordinate time $\lambda = t$:

$$L(\vec{x}, \vec{v}) = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{c} \vec{A} \cdot \vec{v} - q \varphi$$
(6.135)

with $A_{\mu} = (\varphi, -\vec{A})$ (covariant!) and $\dot{\vec{x}} = \vec{v}$.



iii | <u>Canonical momentum:</u>

$$\vec{\pi} := \frac{\partial L}{\partial \vec{v}} \stackrel{\circ}{=} m \gamma_v \vec{v} + \frac{q}{c} \vec{A}$$
(6.136)

with mechanical momentum $\vec{p} = m \gamma_v \vec{v} \rightarrow$

$$\vec{p} = \vec{\pi} - \frac{q}{c}\vec{A} \tag{6.137}$$

- \vec{p} : Measurable momentum
- \rightarrow Mechanical momentum \vec{p} gauge-invariant
- \rightarrow Canonical momentum $\vec{\pi}$ not gauge-invariant
- iv | <u>Hamiltonian</u>:

$$H = \vec{\pi} \cdot \vec{v} - L \stackrel{\circ}{=} \underbrace{\frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}}_{E} + q\varphi \stackrel{5.26}{=} c\sqrt{\left(\pi - \frac{q}{c}\vec{A}\right)^2 + m^2c^2} + q\varphi \quad (6.138)$$

so that

$$E = H - q\varphi \tag{6.139}$$

E is gauge invariant \rightarrow *H* is *not* gauge invariant

v | Summary:

Gauge invariant
$$\begin{cases} E = H - q\varphi & E + q\varphi = H \\ \vec{p} = \vec{\pi} - \frac{q}{c}\vec{A} & \Leftrightarrow \vec{p} + \frac{q}{c}\vec{A} = \vec{\pi} \end{cases}$$
 Gauge dependent (6.140)

For more details on the aspect of the gauge-(in)variance of certain quantities, see Ref. [80]. Note that these subtleties are not specific to a relativistic treatment, they already appear in Newtonian mechanics (only the specific dependency of the Hamiltonian on the mechanical/canonical momentum and the functional form of the Lagrangian are relativistic).



6.5. Summary: The many faces of Maxwell's equations

Here is a compact overview over the many (physically equivalent) forms of Maxwell's equations that we encountered in this chapter:



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7. Relativistic Field Theories II: Relativistic Quantum Mechanics

Reminder

1 | The ↓ *Schrödinger equation* (SE)

$$i\hbar\partial_t\psi(t,\vec{x}) = \hat{H}\psi(t,\vec{x}) \tag{7.1}$$

is a linear field equation with \checkmark *Hamilton operator*

$$\hat{H} = \frac{\vec{p}^2}{2m} + V(\vec{x}) = -\frac{\hbar^2}{2m}\Delta + V(\vec{x})$$
(7.2)

and the complex-valued field $\psi : \mathbb{R}^{1,3} \to \mathbb{C}$.

It describes the time evolution of a single quantum particle with mass *m* in a potential $V(\vec{x})$ that is initially described by the wavefunction $\psi_0(\vec{x}) = \psi(0, \vec{x})$ at t = 0.

2 | The wavefunction has the interpretation

$$|\psi(t, \vec{x})|^2 = \langle \text{Probability to find particle at time } t \text{ at position } \vec{x} \rangle$$
 (7.3)

which necessitates the normalization condition

$$\forall_t : \|\psi(t)\|_2 := \int \mathrm{d}^3 x \, |\psi(t, \vec{x})|^2 = 1.$$
 (7.4)

Thus the wavefunction is an element of the Hilbert space $\psi \in L^2 \equiv L^2(\mathbb{R}^3, \mathbb{C})$ of squareintegrable functions.

The Hermiticity $\hat{H} = \hat{H}^{\dagger}$ of the Hamiltonian implies a unitary time evolution and thereby guarantees a conserved norm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi(t)\|_{2} = \int \mathrm{d}^{3}x \left[\psi^{*}\partial_{t}\psi + \psi\partial_{t}\psi^{*}\right] \stackrel{\mathbf{7.1}}{=} \frac{1}{i\hbar} \int \mathrm{d}^{3}x \left[\psi^{*}(\hat{H}\psi) - \psi(\hat{H}\psi)^{*}\right] \stackrel{\mathbf{7.6}}{=} 0, \quad (7.5)$$

where we used that for $\psi, \phi \in L^2$ and a Hermitian Hamiltonian

$$\int d^3x \,\phi^*(\hat{H}\psi) \stackrel{\text{def}}{=} \langle \phi | \hat{H}\psi \rangle \stackrel{\text{def}}{=} \langle \hat{H}^{\dagger}\phi | \psi \rangle \stackrel{\text{def}}{=} \int d^3x \,(\hat{H}^{\dagger}\phi)^*\psi \stackrel{\hat{H}=\hat{H}^{\dagger}}{=} \int d^3x \,\psi(\hat{H}\phi)^* \,.$$
(7.6)

- **3** | <u>Problem:</u> The SE is Galilei covariant but *not* Lorentz covariant! (recall) Problemset 1)
 - The SE is of first order in time but of second order in the spatial derivatives. This asymmetry already suggests that the equation cannot be Lorentz covariant: Time is treated differently than space in (non-relativistic) quantum mechanics.
 - We would like quantum mechanics to be described by a Lorentz covariant equation because we subscribed to *← Einstein's principle of special relativity* **SR** at the beginning of this course: All laws of physics must take the same form in all inertial systems (which are related by Lorentz transformations). This certainly includes quantum mechanics.

However, **SR** is just a (empirically motivated) *principle*, it is neither a law nor a theorem; there may be conceivable universes in which **SR** simply does not apply to the quantum realm – in which case the Schrödinger Eq. (7.1) would be a perfectly valid model.

As good physicists, we should seek for empirical evidence to settle the matter ...



4 | Evidence:

• First: The Schrödinger equation, published and studied by Erwin Schrödinger in a sequence of papers in 1926 [81-84] (so RELATIVITY was already known at the time), was (and is) a highly successful theory that describes a plethora of microscopic phenomena remarkably well. Examples are the \checkmark double-slit experiment, \checkmark quantum tunneling effects, and, of course, the \checkmark spectrum of the hydrogen atom:

The Hamilton operator for the relative electron-proton system of the hydrogen atom is

$$\hat{H} = -\frac{\hbar^2}{2\mu} \nabla^2 - \frac{e^2}{|\vec{x}|}$$
(7.7)

with reduced mass $\mu = m_e m_p / (m_e + m_p)$. The discrete part of the spectrum of the operator \hat{H} can be computed exactly (E_R is the \checkmark Rydberg energy),

$$E_n = -\frac{E_R}{n^2}$$
 with principal quantum number $n \in \{1, 2, ...\}$, (7.8)

and determines the hydrogen spectrum:



The transitions between the levels of the hydrogen spectrum can be measured by spectroscopy (\downarrow *Lyman series* [85], \downarrow *Balmer series* [86], ...; these observations have been made around 1900). The explanation of these spectral lines by the non-relativistic Schrödinger equation is the crown jewel of quantum mechanics, and one of the most remarkable advances of 20th century physics.

However, it's not all sunshine and roses. It was already known at the end of the 19th century (due to advances in spectroscopy [87]) that the spectral lines of various atomic species (including hydrogen) had a ↓ *fine-structure*. Expressed in terms of the energy levels of the hydrogen atom, this means that some of the degenerate eigenstates of Eq. (7.7) are actually *not exactly* degenerate:




Note that this was known to Schrödinger when he published his equation in 1926; he writes in Ref. [84] (p. 132–133):

Im Anschluß an die zuletzt erwähnten physikalischen Probleme, [..], möchte ich nun doch die vermutliche relativistisch-magnetische Verallgemeinerung der Grundgleichungen [..] hier ganz kurz mitteilen, wenn ich es auch vorerst nur fur das Einelektronenproblem und nur mit der allergrößten Reserve tun kann. Letzteres aus zwei Grunden. Erstens beruht die Verallgemeinerung vorläufig auf rein formaler Analogie. Zweitens führt sie, wie schon in der ersten Mitteilung erwähnt wurde, im Falle des Keplerproblems zwar formal auf die Sommerfeldsche Feinstrukturformel und zwar mit "halbzahligem" Azimutal- und Radialquant, was heute allgemein als korrekt angesehen wird; allein es fehlt noch die zur Herstellung numerisch richtiger Aufspaltungsbilder der Wasserstofflinien notwendige Ergänzung, die im Bohrschen Bilde durch den Goudsmit-Uhlenbeckschen Elektronendrall geliefert wird.

Note that Schrödinger was very much aware that his equation lacked Lorentz covariance and viewed (and constructed) it as a *non-relativistic approximation* of a truly "relativistic quantum mechanics" (which he didn't know how to formulate consistently).

He also makes this clear in the introduction of Ref. [83] (p. 439):

Wesentlich größeres Interesse wird natürlich die (hier noch nicht darchgeführte) Anwendung auf den Zeemaneffekt bieten. Diese erscheint mir unlöslich geknüpft an eine korrekte Formulierung des relativistischen Problems in der Sprache der Wellenmechanik, weil bei vierdimensionaler Formulierung das Vektorpotential von selbst dem skalaren ebenbürtig an die Seite tritt. Schon in der ersten Mitteilung wurde erwähnt, daß das relativistische Wasserstoffatom sich zwar ohne weiteres behandeln läßt, aber zu "halbzahligen" Azimutalquanten, also zu einem Widerspruch mit der Erfahrung führt. Es mußte also noch "etwas fehlen". Seither habe ich [..] gelernt, was fehlt: in der Sprache der Elektronenbahnentheorie der Drehimpuls des Elektrons um seine Achse, der ihm ein magnetisches Moment verleiht.

• We can also make a back-of-the-envelope calculation to estimate whether relativistic effects could be the root cause for the discrepancy between the non-relativistic Schrödinger equation and the observed fine-structure:

In a *classical* approximation, kinetic and potential energy are of the same order:

Kinetic energy
$$\frac{1}{2}mv^2 \sim \frac{e^2}{r}$$
 Potential energy. (7.9)

Because the system is quantum, momentum and position obey the \downarrow *Heisenberg uncertainty* relation $\Delta p \Delta r \sim \hbar$. In the energy eigenstates of an interacting quantum system (like an atom) we typically have $\Delta p \sim p$ and $\Delta r \sim r$, and in our semi-classical approximation it is



 $p \sim mv$, so that

$$v \sim \frac{e^2}{mvr} \sim \frac{e^2}{\hbar c}c = \alpha c = \text{Fine-structure constant} \times c \approx \frac{c}{137}$$
. (7.10)

The semi-classical velocity of the electron v is therefore much smaller than the speed of light c; this explains why the non-relativistic Schrödinger equation is so successful (and your course non non-relativistic quantum mechanics is no waste of time). However, the observed fine-structure splitting of spectral lines is indeed very small, so it is reasonable that relativistic effects can have small but measurable effects in atomic physics.

The situation is therefore similar to that of Newtonian mechanics before we made it relativistic: We have a very successful Galilei covariant theory that, however, shows signs of being the lowvelocity/energy approximation of another, presumably relativistic theory.

(Note that *historically* the situation is very different, though: While Newtonian mechanics, born in the 17th century, had to wait more than 200 years to be "made relativistic", the development of relativistic quantum mechanics was very fast: Non-relativistic quantum mechanics was established in 1925/26 – and just two years later, in 1928, Paul Dirac published the correct equation describing relativistic electrons: the \rightarrow *Dirac equation* [88].)

 \rightarrow Are there relativistic field equations which allow for a probabilistic interpretation?

7.1. The Klein-Gordon equation

The Klein-Gordon equation has been studied by Klein [89] and Gordon [90] in 1926 as a possible relativistic version of the Schrödinger equation. Schrödinger and Fock found the equation independently as well.

- $1 \mid \triangleleft Complex \text{ scalar field: } \phi : \mathbb{R}^{1,3} \to \mathbb{C}$
 - → Most general *quadratic* (superposition principle!) and *Lorentz covariant* Lagrangian density:

$$\mathcal{L}_{\mathrm{KG}}(\phi,\partial\phi) = (\partial^{\mu}\phi)(\partial_{\mu}\phi^{*}) - M^{2}\phi\phi^{*}$$
(7.11)

 $M = \frac{mc}{\hbar} \in \mathbb{R}$: arbitrary parameter (*m* will be the mass of the particle)

- Note that $M = \frac{mc}{\hbar} = \frac{2\pi}{\lambda}$ has the dimension of an inverse length; here $\lambda = \frac{h}{mc}$ is the \leftarrow Compton wavelength Eq. (5.77).
- One can also derive the non-relativistic Schrödinger equation from a Lagrangian density (→ *below*):

$$\mathcal{L}_{SE}(\psi,\partial\psi) = i\hbar\psi^*\partial_t\psi - \frac{\hbar^2}{2m}(\nabla\psi^*)(\nabla\psi) - V(x)\psi^*\psi$$
(7.12)

This is of course not a Lorentz scalar (you cannot write this combining only tensors).

2 | Euler-Lagrange equations:

Trick: Consider ϕ and ϕ^* as *independent* fields; let ϕ^* be the complex conjugate of ϕ at the end.

$$\frac{\partial \mathcal{L}_{\mathrm{KG}}}{\partial \phi^*} - \partial_\mu \frac{\partial \mathcal{L}_{\mathrm{KG}}}{\partial (\partial_\mu \phi^*)} = 0 \quad \Rightarrow \quad -M^2 \phi - \partial_\mu \partial^\mu \phi = 0 \tag{7.13}$$

The Euler-Lagrange equations for the field ϕ yield the complex conjugate Klein-Gordon equation.



 \rightarrow

$$(\partial^2 + M^2)\phi(x) = 0$$
 ** Klein-Gordon equation (7.14)

The Klein-Gordon equation (KGE) is the simplest relativistic wave equation.

The non-relativistic Schrödinger equation follows along the same lines from Eq. (7.12):

$$\frac{\partial \mathcal{L}_{SE}}{\partial \psi^*} - \partial_\mu \frac{\partial \mathcal{L}_{SE}}{\partial (\partial_\mu \psi^*)} = 0 \quad \Rightarrow \quad i\hbar \partial_t \psi - V\psi + \frac{\hbar^2}{2m} \nabla^2 \psi = 0 \tag{7.15}$$

The Euler-Lagrange equations for ψ^* yield the complex conjugate of the Schrödinger equation.

3 | Lorentz symmetry of the KGE:

The KGE is manifest Lorentz covariant. However, it is instructive (and useful for our derivation of the Dirac equation \rightarrow *later*) to check its invariance manually. To this end, we view Lorentz transformations as *active* transformations, mapping solutions to different solutions. This is equivalent to the *passive* viewpoint where the coordinate system is transformed instead:

- i | \triangleleft Coordinate transformation: $\bar{x} = \Lambda x$ & Field transformation: $\bar{\phi}(\bar{x}) = \phi(x)$ We write $\bar{x} = \Lambda x$ for $\bar{x}^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$.
- ii $| \langle \phi(x) \rangle$ with $(\partial^2 + M^2)\phi(x) = 0$ for all x That is, $\phi(x)$ is a solution of the KGE.
- iii | $\stackrel{\circ}{\rightarrow} \bar{\phi}(x) := \phi(\Lambda^{-1}x)$ is a new solution:

Use the chain rule in the first step twice:

$$(\eta^{\mu\nu}\partial_{\mu}\partial_{\nu} + M^2)\bar{\phi}(x) = [\eta^{\mu\nu}(\Lambda^{-1})^{\sigma}{}_{\mu}\partial_{\sigma}(\Lambda^{-1})^{\rho}{}_{\nu}\partial_{\rho} + M^2]\phi(\Lambda^{-1}x)$$
(7.16a)

Use invariance of the metric Eq.
$$(4.21)$$
 (7.16b)

$$= (\eta^{\sigma\rho}\partial_{\sigma}\partial_{\rho} + M^2)\phi(\Lambda^{-1}x)$$
(7.16c)

$$= (\partial^2 + M^2)\phi(\Lambda^{-1}x) \stackrel{\phi \text{ solution}}{=} 0 \tag{7.16d}$$

Here $\partial_{\sigma}\phi(\Lambda^{-1}x)$ must be read as $\partial_{\sigma}\phi(y)|_{y=\Lambda^{-1}x}$, i.e., we compute the derivative of the function ϕ with respect to its argument y and then plug in the value $\Lambda^{-1}x$.

4 | <u>Conserved current:</u>

 $i \mid \triangleleft$ Global phase rotations:

$$\phi'(x) = e^{i\alpha}\phi(x) \quad \text{for} \quad \alpha \in [0, 2\pi)$$
(7.17)

with infinitesimal generator $|\alpha| = |w| \ll 1$

$$\phi'(x) = \phi(x) + iw\phi(x) \equiv \phi(x) + w\,\delta\phi(x) \quad \Rightarrow \quad \delta\phi = i\phi \tag{7.18}$$

Note that this is an "internal symmetry" that has nothing to do with spacetime; thus $\delta x = 0$. For the complex conjugate field ϕ^* one finds analogously $\delta \phi^* = -i\phi^*$.

 \rightarrow Continuous symmetry:

$$\mathcal{L}_{\mathrm{KG}}(\phi,\partial\phi) = \mathcal{L}_{\mathrm{KG}}(\phi',\partial\phi') \tag{7.19}$$

If the Lagrangian density is invariant, the action is trivially invariant!

ii | Noether theorem Eq. (6.85) \rightarrow Conserved Noether current density Eq. (6.84):

$$j_{\rm KG}^{\,\mu} \stackrel{\circ}{=} i(\partial^{\mu}\phi)\phi^* - i(\partial^{\mu}\phi^*)\phi \tag{7.20}$$

Note that if one treats ϕ and ϕ^* independent fields, one has to sum over the two fields in the evaluation of the Noether current; this then yields the real-valued current density above.

 \rightarrow Noether charge density:

$$\rho_{\mathrm{KG}}(x) := j_{\mathrm{KG}}^{0}(x) = \frac{i}{c} \left(\dot{\phi} \phi^* - \dot{\phi}^* \phi \right) \quad \text{with} \quad \rho_{\mathrm{KG}}(x) \in \mathbb{R} \tag{7.21}$$

 \rightarrow Conserved Noether charge:

$$Q = \int \mathrm{d}^3 x \,\rho_{\mathrm{KG}}(x) = \frac{i}{c} \int \mathrm{d}^3 x \,\left(\dot{\phi}\phi^* - \dot{\phi}^*\phi\right) \tag{7.22}$$

Important: $\rho_{\text{KG}}(x) \leq 0$ is *not* positive-definit! \rightarrow

$$\rho_{\rm KG}(x) \ cannot \ be \ interpreted \ as \ a \ probability \ density!$$
(7.23)

- To sum up:
 - The inner product (= positive-definite, symmetric sesquilinear form) on $L^2(\mathbb{R}^{1,3},\mathbb{C})$

$$\langle \phi | \psi \rangle_{L^2} := \int \mathrm{d}^3 x \, \phi^* \psi \tag{7.24}$$

is not conserved under the time-evolution of the KGE.

- The *indefinite* symmetric sesquilinear form (which is not an inner product!)

$$\langle \phi | \psi \rangle_{\text{KG}} := \frac{i\hbar}{2mc^2} \int d^3x \, \left(\phi^* \dot{\psi} - \dot{\phi}^* \psi \right) \tag{7.25}$$

is conserved under the time-evolution of the KGE. But because it is not positive-(semi)definite, we cannot interpret the induced "norm" as a probability.

The prefactor $\frac{\hbar}{2mc^2}$ is chosen such that it has the dimension of a time (because $\frac{\hbar}{mc} \propto \lambda$ has the dimension of a length). Then the square of the fields (= wavefunctions) has the dimension of one over a volume – which is the conventional dimension of wavefunctions. The factor $\frac{1}{2}$ is chosen to simplify expressions later.

• Compare this to the conserved current for the same phase rotation symmetry that follows for the Schrödinger field Eq. (7.12) with $\delta \psi = i \psi$ and $\delta \psi^* = -i \psi^*$:

$$j_{\rm SE}^{\,\mu} \equiv \begin{pmatrix} \hbar c \rho_{\rm SE} \\ \hbar \vec{j}_{\rm SE} \end{pmatrix} \stackrel{7.12}{=} \begin{cases} \hbar c \psi^* \psi \,, & \mu = 0 \\ \frac{i \hbar^2}{2m} \left[(\nabla \psi^*) \psi - (\nabla \psi) \psi^* \right] & \mu = i = 1, 2, 3 \end{cases}$$
(7.26)

(Recall that you must sum over the fields ψ and ψ^* .)

This is the positive-definite probability density you already know from quantum mechanics,

$$\rho_{\rm SE}(x) = \psi^*(x)\psi(x) = |\psi(x)|^2 \ge 0, \qquad (7.27)$$



and the \downarrow probability current density

$$\vec{j}_{\rm SE} = \frac{i\hbar}{2m} \left[(\nabla\psi^*)\psi - (\nabla\psi)\psi^* \right].$$
(7.28)

In this context, Noether's theorem ensures probability conservation:

$$\partial_{\mu} j_{\rm SE}^{\mu} = 0 \quad \Leftrightarrow \quad \partial_t \rho_{\rm SE} + \nabla \cdot \vec{j}_{\rm SE} = 0.$$
 (7.29)

5 | Solutions: (for the free Klein-Gordon field)

i | The KG Eq. (7.14) is a wave equation:

$$\left[\frac{1}{c^2}\partial_t^2 - \nabla^2 + \frac{m^2c^2}{\hbar^2}\right]\phi(t,\vec{x}) = 0$$
(7.30)

 \rightarrow Solution space spanned by plane waves:

$$\phi(t,\vec{x}) = e^{\frac{1}{\hbar}(\vec{p}\cdot\vec{x}-Et)}$$
(7.31)

Plug this ansatz into Eq. (7.30) \rightarrow Dispersion relation:

$$-\frac{E^2}{c^2\hbar^2} + \frac{\vec{p}^2}{\hbar^2} + \frac{m^2c^2}{\hbar^2} \stackrel{!}{=} 0$$
(7.32)

$$E = \pm \sqrt{\vec{p}^2 c^2 + m^2 c^4}$$
(7.33)

- This is the relativistic energy-momentum relation Eq. (5.26).
- i! There are *two* solutions for each 3-momentum \vec{p} , one of which has *negative* energy E < 0 (if we interpret the prefactor of t as the energy as usual). This is a consequence of the quadratic nature of the KGE (as compared to the SE), and therefore a direct consequence of its relativistic covariance.
- At the time of its inception, the negative energy solutions of the KGE could not be interpreted properly. This (together with the fact that its conserved "norm" cannot be interpreted as a probability and it fails to predict the fine-structure of the hydrogen atom correctly, *→ below*) lead to its dismissal as a relativistic wave equation for quantum wave functions. It only became clear later that the negative energy solutions herald the existence of *antiparticles*. Only in modern *↑ relativistic quantum field theories* [where the KGE reappears as the equation of motion of (free scalar) quantum fields, see Chapter 2 of my script on QFT [20]] this "feature" can be cast into a consistent framework: The negative energy solutions are interpreted as eigenmodes of antiparticles with *positive* energies (and norms). If the particles are charged, their antiparticles have opposite charge; then the conserved Noether charge Eq. (7.22) is interpreted as *charge conservation* (and not probability conservation).
- ii | As usual, one can "normalize" the plane wave solutions Eq. (7.31) if one considers a finite system with volume $V = L^3$. Then one finds the "orthonormal" solution basis of the KGE:





$$\phi_{\vec{k}}^{(\pm)}(t,\vec{x}) = N_{\vec{k}} e^{i(\vec{k}\cdot\vec{x}\mp\omega_k t)} \text{ with } \dots$$

$$Dispersion: \qquad \omega_k = \sqrt{\vec{k}^2 c^2 + m^2 c^4 / \hbar^2}$$

$$Momentum: \qquad \vec{p} = \hbar \vec{k} \in \hbar \frac{2\pi}{L} \mathbb{Z}^3$$

$$Normalization: \qquad N_k = \sqrt{\frac{mc^2}{V\hbar\omega_k}}$$

$$(7.34a)$$

• It is straightforward to check that these states are "orthonormal" with respect to the Klein-Gordon sesquilinear form Eq. (7.25):

$$\langle \psi_{\vec{k}}^{(\alpha)} | \psi_{\vec{k}'}^{(\beta)} \rangle_{\mathrm{KG}} \stackrel{\circ}{=} \alpha \, \delta_{\alpha,\beta} \delta_{\vec{k},\vec{k}'} \quad \text{with} \quad \alpha,\beta \in \{\pm\}.$$
(7.35)

Note that the (-) states have negative "norm".



↓Lecture15 [06.02.24]

6 | Coupling to a static EM field:

The KGE can be coupled to the gauge field of electrodynamics. This is necessary to described charged particles (in particular: the hydrogen atom). Note that in the following the gauge field is a *parameter* and not a dynamic degree of freedom.

- i | Goal: Construct Lagrangian density that is ...
 - ... a Lorentz scalar.
 - ... quadratic in ϕ .
 - ... gauge invariant under the gauge transformation $A'_{\mu} = A_{\mu} \partial_{\mu}\lambda$.
 - ... couples ϕ and A_{μ} in a non-trivial way.

Without additional tools, this is a tough job!

ii | \triangleleft Gauge transformation $A'_{\mu} = A_{\mu} - \partial_{\mu}\lambda(x)$

Let us assume that the KG field transforms under the gauge transformation as follows:

$$\phi'(x) := e^{iQ\lambda(x)}\phi(x) \quad \text{with the } * U(1) \text{ charge } Q = \frac{q}{\hbar c} \in \mathbb{R} \,. \tag{7.36}$$

q: electric charge of the particle described by the wavefunction ϕ

- It is reasonable to assume that the KG field must transform via phase factors because we already know [recall Eq. (7.19)] that the KG Lagrangian is invariant under *global* phase transformations $\lambda(x) = \text{const.}$ Our hope is that we can "extend" this symmetry for arbitrary non-constant $\lambda(x)$.
- The charge Q is a property of the field and quantifies how it transforms under gauge transformations; it essentially plays the role of the electric charge of the particle described by ϕ ; e.g., for an electron we would set $Q = \frac{-e}{\hbar c} < 0$.

The additional division by $\hbar c$ is necessary for dimensional reasons: $[\lambda] = L[\varphi]$ with $A^{\mu} = (\varphi, \vec{A})$; therefore $[\lambda q] = L[\varphi q] = L[E] = \frac{ML^3}{T^2}$ and it is $[\hbar c] = \frac{ML^3}{T^2}$ as well. In natural units (where $\hbar = 1 = c$, Q = q is simply the electric charge.

The term "U(1) charge" highlights that the gauge transformation e^{iQλ(x)} ∈ U(1) is a U(1) gauge transformation; the charge is the generator of this Lie group.

iii | <u>Problem</u>:

Derivatives transform complicated under gauge transformations:

$$\partial_{\mu}\phi'(x) \stackrel{\circ}{=} e^{iQ\lambda(x)} \left[iQ(\partial_{\mu}\lambda)\phi(x) + \partial_{\mu}\phi(x) \right]$$
(7.37)

 \rightarrow It is hard to combine derivatives of fields to construct gauge-invariant terms!

Solution:

Define the ...

** (Gauge) Covariant derivative:
$$D_{\mu} := \partial_{\mu} + i Q A_{\mu}$$

 \rightarrow Lorentz vector (thus we can it use to construct Lorentz scalars!)

(7.38)



The covariant derivative has the following useful property:

$$D'_{\mu}\phi' = \left[\partial_{\mu} + iQA_{\mu} - iQ(\partial_{\mu}\lambda)\right]e^{iQ\lambda}\phi \stackrel{\circ}{=} e^{iQ\lambda}D_{\mu}\phi \tag{7.39}$$

 $\rightarrow D_{\mu}\phi$ transforms like ϕ under gauge transformations. [and not as ugly as Eq. (7.37)!]

This is useful because it allows us to combine derivatives into gauge-invariant terms.

iv Using the covariant derivative, we can now construct the following general Lagrangian density that satisfies our four requirements above:

$$\mathcal{L}_A(\phi,\partial\phi) = (D^{\mu}\phi)(D_{\mu}\phi)^* - M^2\phi\phi^*$$
(7.40)

Please appreciate the ingenuity of the term $(D^{\mu}\phi)(D_{\mu}\phi)^*$: It is *Lorentz invariant* because we pair the indices correctly, and it is *gauge invariant* because we pair $(D^{\mu}\phi)$ with its complex conjugate $(D_{\mu}\phi)^*$ (which is sufficient because $D^{\mu}\phi$ gauge-transforms like ϕ).

This Lagrangian density is gauge-invariant by construction in the sense that

$$\mathcal{L}_{A}(\phi,\partial\phi) = \mathcal{L}_{A'}(\phi',\partial\phi') \quad \text{or} \quad \mathcal{L}(\phi,D\phi) = \mathcal{L}(\phi',D'\phi').$$
(7.41)

- A comparison of the free Klein-Gordon Lagrangian Eq. (7.11) and the new one Eq. (7.40) reveals that we simply made the substitution ∂_μ → D_μ, i.e., we replaced partial derivatives by covariant derivatives (which depend on the gauge field). This trick is not specific to the Klein-Gordon field and yields gauge-invariant theories in general. This procedure is called ↑ *minimal coupling*.
- Note that the transformation Eq. (7.36) is a *local* phase rotation of the KG-field. In Eq. (7.17) we considered a *global* phase rotation and identified it as a continuous symmetry of the KG Lagrangian L_{KG}. You can check that the new *local* transformation does not leave L_{KG} invariant, but it does leave L_A invariant if A^μ transforms together with φ as defined above. The transition from L_{KG} (with a global symmetry) to L_A (with a local version of the same symmetry) is called *gauging the symmetry*. You can use this line of reasoning to "invent" the electromagnetic gauge field: If you start from a global continuous symmetry and demand that it becomes a local symmetry, you have to pay for it by introducing a new field: the gauge field.

v | Klein-Gordon equation in a static EM field:

The Euler-Lagrange equations of \mathcal{L}_A yield: Eq. (6.6) $\xrightarrow{\text{Eq. (7.40)}}$

$$(D^2 + M^2)\phi(x) = 0 (7.42)$$

with $D^2 = D_\mu D^\mu$ and $M = \frac{mc}{\hbar}$.

In the form Eq. (7.42) both Lorentz covariance and gauge invariance are manifest (because we use the covariant derivative). If we expand everything, we loose these features but obtain a less abstract (but more complicated) form of the PDE:

$$\left[\frac{1}{c^2}\left(\partial_t + i\mathcal{Q}c\varphi\right)^2 - \left(\nabla - i\mathcal{Q}\vec{A}\right)^2 + \frac{m^2c^2}{\hbar^2}\right]\phi(t,\vec{x}) = 0$$
(7.43)

Here we used $A_{\mu} = (\varphi, -\vec{A})$ (covariant!).



vi | Example: Hydrogen atom

Goal: Describe the electron of the hydrogen atom in the static EM field generated by the proton in terms of the KGE; i.e., we interpret the KG field ϕ naïvely as the wavefunction of the electron. Our hope is that the energy spectrum of this relativistic theory explains the observed fine-structure splitting.

a | \triangleleft Coulomb potential (of proton with charge e > 0) \rightarrow

Choose a gauge where
$$\varphi(x) = \frac{e}{|\vec{x}|}$$
 and $\vec{A} = \vec{0}$ (7.44)

 $\stackrel{\circ}{\rightarrow}$ With electron charge $Q = \frac{-e}{\hbar c} < 0$ one finds:

$$\left[\frac{1}{c^2}\left(i\partial_t + \frac{e^2}{\hbar|\vec{x}|}\right)^2 + \nabla^2 - \frac{m^2c^2}{\hbar^2}\right]\phi(t,\vec{x}) = 0$$
(7.45)

b | \triangleleft Ansatz $\phi(t, \vec{x}) = \tilde{\phi}(\vec{x})e^{-\frac{i}{\hbar}Et} \rightarrow$ "Stationary" Klein-Gordon equation:

$$\left[c^{2}\hbar^{2}\Delta + \left(E + \frac{e^{2}}{|\vec{x}|}\right)^{2} - m^{2}c^{4}\right]\tilde{\phi}(\vec{x}) = 0$$
(7.46)

Note that this PDE is *quadratic* in the energy E (and not linear, like the time-independent Schrödinger equation).

c | One can use a clever mapping to the non-relativistic Schrödinger equation to solve for $\tilde{\phi}(\vec{x})$ and determine the energies *E* for which solutions exist:

$$\stackrel{*}{\to} \quad E_{n,l} = \frac{mc^2}{\sqrt{1 + \frac{\alpha^2}{(n-\delta_l)^2}}} \quad \text{with} \quad \delta_l = l + \frac{1}{2} - \sqrt{\left(l + \frac{1}{2}\right)^2 - \alpha^2} \,. \tag{7.47}$$

Here n = 1, 2, ... is the \downarrow principal quantum number and l = 0, 1, 2, ... is the \downarrow orbital angular momentum quantum number. $\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ is the fine-structure constant.

- d | <u>Comments:</u>
 - The spectrum Eq. (7.47) predicts a splitting of the *l*-degeneracy; recall that this degeneracy is perfect in the non-relativistic hydrogen atom [cf. Eq. (7.8)]. Unfortunately, the spectrum Eq. (7.47) does *not* match observations! The reason is that the Klein-Gordon equation does not know about the *electron spin*. Schrödinger and his contemporaries were aware of this solution and its problems (this shines through in the quotes at the beginning of this chapter). This failure to predict the fine-structure correctly led to the dismissal of the Klein-Gordon equation and motivated Paul Dirac to search for another equation (→ *next section*).
 - Today we know that the Klein-Gordon equation is *not wrong*: It simply does not apply to particles with non-zero spin (and the electron in the hydrogen atom happens to have spin s = 1/2). However, it *does* apply to spin-0 particles like ↑ *kaons* (K mesons, bound states of two quarks), ↑ *pions* (pi mesons), and the ↑ *Higgs boson* (the latter being the only *elementary* particle with zero spin). But since we cannot build hydrogen atoms out of these particles, the significance of the above solution remains limited.

7 | <u>First-order formulation:</u>

Here we consider again the free KGE (without EM field) for simplicity.

i | KGE = *Second-order* PDE in time



Problem: $\phi(t = 0, \vec{x})$ does *not* specify the state of the system completely [unlike for the Schrödinger equation one also needs $\dot{\phi}(t = 0, \vec{x})$ to pick out a unique solution $\phi(t, \vec{x})$].

Recall: Every higher-order differential equation can be recast as a first-order differential equation with multiple components.

 \rightarrow <u>Goal</u>: Rewrite the KGE in the first-order form

$$i\hbar\partial_t \Phi = \hat{H}_{\rm KG} \Phi \quad \text{with} \quad \Phi = \begin{pmatrix} \phi_+\\ \phi_- \end{pmatrix}.$$
 (7.48)

Downside: In this form, the KGE is no longer manifest Lorentz covariant.

ii | Define

$$\phi_{\pm} := \frac{1}{2} \left(\phi \pm \frac{i\hbar}{mc^2} \partial_t \phi \right) \tag{7.49}$$

so that

$$\phi = \phi_{+} + \phi_{-}$$
 and $\frac{i\hbar}{mc^{2}}\partial_{t}\phi = \phi_{+} - \phi_{-}$. (7.50)

iii | Define the 2×2 differential operator

$$\hat{H}_{\text{KG}} := \begin{pmatrix} \hat{H}_0 + mc^2 & \hat{H}_0 \\ -\hat{H}_0 & -\hat{H}_0 - mc^2 \end{pmatrix} = \hat{H}_0 \otimes (\sigma^z + i\sigma^y) + mc^2 \sigma^z$$
(7.51)

with $\hat{H}_0 = -\frac{\hbar^2}{2m} \nabla^2$ the free particle Hamiltonian and the Pauli matrices

$$\sigma^{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
 (7.52)

 \hat{H}_{KG} is a linear operator on the Hilbert space $L^2 \otimes \mathbb{C}^2$ of two-component square-integrable functions. Note that $\hat{H}_{\text{KG}}^{\dagger} = \hat{H}_0 \otimes (\sigma^z - i\sigma^y) + mc^2\sigma^z \neq \hat{H}_{\text{KG}}$ is non-Hermitian with respect to the conventional inner product on $L^2 \otimes \mathbb{C}^2$:

$$\langle \Phi | \Psi \rangle_{L^2 \otimes \mathbb{C}^2} = \int d^3 x \, \Phi^{\dagger}(x) \Psi(x) = \int d^3 x \, \left(\phi_+^* \psi_+ + \phi_-^* \psi_- \right) \,.$$
 (7.53)

iv | Check that the differential equation in first-order Schrödinger form

$$i\hbar\partial_t \Phi = \hat{H}_{\mathrm{KG}} \Phi \quad \Leftrightarrow \quad \begin{cases} i\hbar\partial_t \phi_+ = (\hat{H}_0 + mc^2)\phi_+ + \hat{H}_0\phi_- \\ i\hbar\partial_t \phi_- = -\hat{H}_0\phi_+ - (\hat{H}_0 + mc^2)\phi_- \end{cases}$$
(7.54)

is equivalent to the KGE:

a | Indeed, the *difference* of the two equations yields

$$-\frac{\hbar^2}{mc^2}\partial_t\chi = (\hat{H}_0 + mc^2)\phi + \hat{H}_0\phi \quad \Leftrightarrow \quad \frac{1}{c^2}\partial_t\chi - \nabla^2\phi + \frac{m^2c^2}{\hbar^2}\phi = 0 \quad (7.55)$$

where we defined $\phi := \phi_+ + \phi_-$ and $\chi := \frac{mc^2}{i\hbar}(\phi_+ - \phi_-)$.

b | By contrast, the *sum* of the two equation yields

$$mc^2 \partial_t \phi = (\hat{H}_0 + mc^2)\chi - \hat{H}_0\chi \quad \Leftrightarrow \quad \partial_t \phi = \chi.$$
 (7.56)

c | Combining Eq. (7.55) and Eq. (7.56) returns the KGE:

$$\frac{1}{c^2}\partial_t^2 \phi - \nabla^2 \phi + \frac{m^2 c^2}{\hbar^2} \phi = 0.$$
 (7.57)

v | If one defines the

** Klein-Gordon adjoint
$$\bar{\Phi} := \Phi^{\dagger} \sigma^{z} = (\phi_{+}^{*}, -\phi_{-}^{*})$$
, (7.58)

one can express the Klein-Gordon sesquilinear form Eq. (7.25) as

$$\langle \Phi | \Psi \rangle_{\mathrm{KG}} := \int \mathrm{d}^3 x \, \bar{\Phi}(x) \Psi(x) \stackrel{7.49}{=} \frac{i\hbar}{2mc^2} \int \mathrm{d}^3 x \, \left(\phi^* \dot{\psi} - \dot{\phi}^* \psi \right) \stackrel{7.25}{=} \langle \phi | \psi \rangle_{\mathrm{KG}} \,. \tag{7.59}$$

Remember that this is not a proper inner product because it is not positive-definite.

vi | If one defines additionally for an operator A on $L^2 \otimes \mathbb{C}^2$ the

$$\stackrel{*}{*} Klein-Gordon \ adjoint \quad \bar{A} := \sigma^{z} A^{\dagger} \sigma^{z} , \qquad (7.60)$$

it follows $\overline{A\Phi} = \overline{\Phi}\overline{A}$ and $\overline{\overline{A}} = A$, and thereby

$$\langle \Phi | A\Psi \rangle \stackrel{7.59}{=} \langle \bar{A}\Phi | \Psi \rangle.$$
 (7.61)

vii | It is easy to verify that the KG Hamiltonian is "Klein-Gordon Hermitian", namely

$$\hat{H}_{\rm KG} \stackrel{7.51}{=} \hat{H}_{\rm KG} \tag{7.62}$$

because $\sigma^z \sigma^y \sigma^z = -\sigma^y$.

viii | With this machinery, we have now a new method to check that the time-evolution generated by the KGE leaves the KG sesquilinear form invariant:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \phi | \psi \rangle_{\mathrm{KG}} \stackrel{7.59}{=} \frac{\mathrm{d}}{\mathrm{d}t} \langle \Phi | \Psi \rangle_{\mathrm{KG}} \tag{7.63a}$$

$$= \langle \Phi | \dot{\Psi} \rangle_{\rm KG} + \langle \dot{\Phi} | \Psi \rangle_{\rm KG} \tag{7.63b}$$

$$\stackrel{7.54}{=} \frac{1}{i\hbar} \langle \Phi | \hat{H}_{\rm KG} \Psi \rangle_{\rm KG} - \frac{1}{i\hbar} \langle \hat{H}_{\rm KG} \Phi | \Psi \rangle_{\rm KG}$$
(7.63c)

$$\stackrel{7.61}{=} \frac{1}{i\hbar} \left(\langle \Phi | \hat{H}_{\rm KG} \Psi \rangle_{\rm KG} - \langle \Phi | \hat{H}_{\rm KG} \Psi \rangle_{\rm KG} \right) = 0 \tag{7.63d}$$

We already knew this from Noether's theorem, but it is always nice to derive such statements in various ways.

8 | <u>Non-relativistic limit:</u>

i | Goal: Derive a non-relativistic approximation of the Klein-Gordon equation

$$\left[\frac{1}{c^2}\partial_t^2 - \nabla^2 + \frac{m^2 c^2}{\hbar^2}\right]\phi(t, \vec{x}) = 0.$$
(7.64)

ii | \triangleleft Kinetic energy: $E_{kin} = E - mc^2 = \sqrt{\vec{p}^2 c^2 + m^2 c^4} - mc^2 \approx \frac{1}{2}mv^2 + \mathcal{O}(\beta^4)$ (Note that both E_{kin} and E are non-negative!) \triangleleft Ansatz:

$$\phi_{\pm}(t,\vec{x}) = \tilde{\phi}_{\pm}(\vec{x})e^{\mp \frac{i}{\hbar}Et} = \underbrace{\tilde{\phi}_{\pm}(\vec{x})e^{\mp \frac{i}{\hbar}E_{\rm kin}t}}_{=:\hat{\phi}_{\pm}(t,\vec{x})}e^{\mp \frac{i}{\hbar}mc^{2}t}$$
(7.65)

 $\hat{\phi}(t, \vec{x})$ contains only the time evolution due to the kinetic energy, excluding the rest energy.



iii | If we use that

$$\partial_t^2 \hat{\phi}_{\pm} = -\frac{E_{\rm kin}^2}{\hbar^2} \hat{\phi} \,, \tag{7.66}$$

we can make the following approximation in the non-relativistic limit $E_{kin} \ll mc^2$:

$$\partial_t^2 \phi_{\pm} = e^{\pm \frac{i}{\hbar}mc^2t} \left\{ \partial_t^2 \hat{\phi}_{\pm} \pm \frac{2imc^2}{\hbar} \partial_t \hat{\phi}_{\pm} - \left(\frac{mc^2}{\hbar}\right)^2 \hat{\phi}_{\pm} \right\}$$
(7.67a)

$$= -e^{\mp \frac{i}{\hbar}mc^{2}t} \left\{ \pm \frac{2imc^{2}}{\hbar} \partial_{t}\hat{\phi}_{\pm} + \left(\frac{mc^{2}}{\hbar}\right)^{2} \left[1 + \left(\frac{E_{\rm kin}}{mc^{2}}\right)^{2}\right]\hat{\phi}_{\pm} \right\}$$
(7.67b)

$$\approx -e^{\pm \frac{i}{\hbar}mc^{2}t} \left\{ \pm \frac{2imc^{2}}{\hbar} \partial_{t}\hat{\phi}_{\pm} + \left(\frac{mc^{2}}{\hbar}\right)^{2} \hat{\phi}_{\pm} \right\}$$
(7.67c)

iv | Eq. (7.67c) in Eq. (7.64) yields:

$$e^{\pm \frac{i}{\hbar}mc^{2}t} \left[\pm \frac{2im}{\hbar} \partial_{t} + \frac{m^{2}c^{2}}{\hbar^{2}} + \nabla^{2} - \frac{m^{2}c^{2}}{\hbar^{2}} \right] \hat{\phi}_{\pm}(t, \vec{x}) = 0$$
(7.68)

And finally:

$$\pm i\hbar\partial_t\hat{\phi}_{\pm}(t,\vec{x}) = -\frac{\hbar^2}{2m}\nabla^2\hat{\phi}_{\pm}(t,\vec{x})$$
(7.69)

This is the Schrödinger equation for a free particle.

Note that the "negative energy solutions" ϕ_{-} lead to the *time-inverted* Schrödinger equation.

7.2. The Dirac equation

The Dirac equation was published by Paul Dirac in [88], only two years after Schrödinger published the Schrödinger equation.

1 | <u>Goal:</u>

The Klein-Gordon equation has a few undesirable quirks:

It's conserved U(1) current has no positive-definite density and therefore cannot be interpreted as a
probability current. Conversely, the conventional norm on L² is not conserved. In the first-order
formulation, this corresponds to a non-Hermitian Hamiltonian.

 \rightarrow Can we construct a relativistic field equation with a conserved positive-definite density that gives rise to a norm and a Hermitian Hamiltonian?

• In its manifest Lorentz covariant formulation, the KGE is of second order in time, so that we must provide both the wavefunction and its time derivative as initial data.

 \rightarrow Can we construct a relativistic field equation which is first order in time (just like the Schrödinger equation)?

- For each momentum there is are two solutions: one with positive and one with negative energy.
 - \rightarrow Can we get rid of the negative energy solutions?

The Dirac equation succeeds in solving the first two issues – but not the last one, i.e., there will still be negative energy solutions.



2 Observation:

To reach our goals we must equip our "toolbox" of tensor calculus with additional building blocks. As it turns out, there is another type of field (besides the tensor fields we introduced in Chapter 3) that plays an important role in quantum mechanics: \uparrow *spinor fields*.

Remember: Vector fields under rotations: $\vec{\phi}'(\vec{x}) = R\vec{\phi}(R^{-1}\vec{x})$

 \rightarrow In general, a field $\phi(x) \in \mathbb{C}^n$ can transform under homogeneous Lorentz transformations as

$$\phi'_{a}(x) = M_{ab}(\Lambda)\phi_{b}(\Lambda^{-1}x) \qquad a = 1, \dots, n$$
 (7.70)

where

$$M(\Lambda')M(\Lambda)\phi(\Lambda^{-1}\Lambda'^{-1}x) \stackrel{!}{=} M(\Lambda'\Lambda)\phi((\Lambda'\Lambda)^{-1}x)$$
(7.71)

is a *n*-dimensional representation of the (proper orthochronous) Lorentz group $SO^+(1, 3)$.

- Regarding groups and their representations:
 Problemset 1.
- More explicitly: The tensor fields (of various rank) we know so far allow only for the description of particles with *integer spin* $S = 0, 1, 2, \cdots$ (spin = internal angular momentum). What we are missing are fields that can describe particles with *half-integer spin* $S = \frac{1}{2}, \frac{3}{2}, \cdots$; these are the spinor fields.

The reason why this is crucial for relativistic quantum mechanics in particular has to do with the fact that multiplying wave functions by a global phase does not change the state. In mathematical parlance we are dealing with \uparrow projective Hilbert spaces and \uparrow projective representations of symmetries. Thus if you are interested what rotations SO(3) do to the quantum state of your system, you must study all projective representations of SO(3). It turns out that these can be identified with the "conventional" (= linear) representations of another group: SU(2) (the so called \uparrow *double cover* of SO(3)). And you know that the irreducible representations of SU(2) are labeled by "spin quantum numbers" $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ In general, the double covers of SO(n) are called \uparrow spin groups Spin(n), and similarly, the double cover of the proper orthochronous Lorentz group SO⁺(1, 3) is the group Spin(1, 3) \simeq $SL(2, \mathbb{C})$ (the group of complex 2 \times 2 matrices with determinant one). It turns out that the irreducible representations of this group can be labeled by *two* numbers (m, n) with m, n = $0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ The spinor representations we are interested in (the ones missing from our discussion of tensor fields) are the ones for which m + n is half-integer. Conversely, the $(\frac{1}{2}, \frac{1}{2})$ representation is our well-known 4-vector representation A^{μ} and the (0,0) representation is that of a scalar like ϕ .

3 We want a first-order relativistic field equation \rightarrow Ansatz:

$$(\partial^{\mu}\partial_{\mu} + \text{const})\phi = 0 \quad \Rightarrow \quad (i \blacksquare^{\mu}\partial_{\mu} + \text{const})\phi = 0 \tag{7.72}$$

We do not yet know what \blacksquare is (only that it cannot be a derivative). The *i* anticipates wave-like solutions for real \blacksquare .

A covariant equation of the form $\partial_{\mu}\phi = 0$ or $\partial_{\mu}A^{\mu} = 0$ would of course also be possible; their solutions, however, are either too simple or do not match observations.

- 4 | Then (combine 2 & 3)
 - i \triangleleft Coordinate transformation $x' = \Lambda x$ & Field transformation $\phi'(x') = M(\Lambda)\phi(x)$
 - ii | $\triangleleft \phi$ with $(i \blacksquare^{\mu} \partial_{\mu} + \text{const})\phi(x) = 0$ for all x

That is, $\phi(x)$ is a solution of the equation we want to construct.



iii | When is $\phi'(x) = M(\Lambda)\phi(\Lambda^{-1}x)$ is a new solution?

We want the equation to be Lorentz covariant; this means that the Lorentz group must be (part of) its invariance group: Lorentz transformations map solutions to new solutions.

$$(i \blacksquare^{\mu} \partial_{\mu} + \text{const})\phi'(x) = [i \blacksquare^{\mu} (\Lambda^{-1})^{\nu}{}_{\mu} \partial_{\nu} + \text{const}] M(\Lambda)\phi(\Lambda^{-1}x) \stackrel{!}{=} 0 \quad (7.73)$$

Multiply with $M^{-1}(\Lambda)$:

$$\Leftrightarrow \quad [i \underbrace{M^{-1}(\Lambda) \blacksquare^{\mu} M(\Lambda) (\Lambda^{-1})^{\nu}{}_{\mu}}_{\stackrel{!}{=} \blacksquare^{\nu}} \partial_{\nu} + \text{const}] \phi(\Lambda^{-1}x) \stackrel{!}{=} 0 \qquad (7.74)$$

 $\rightarrow \mathbf{I}^{\mu} \equiv \gamma^{\mu}$ must be $n \times n$ -matrices with

$$M^{-1}(\Lambda)\gamma^{\mu}M(\Lambda) = \Lambda^{\mu}_{\ \nu}\gamma^{\nu}$$
(7.75)

The γ -matrices "translate" the "spinor"-representation $M(\Lambda)$ into the "vector"-representation Λ and vice versa.

5 | *Question:* How to find appropriate γ^{μ} and $M(\Lambda)$ that satisfy Eq. (7.75)?

Remember: $SO^+(1, 3)$ is a Lie group (Recall \bigcirc Problemset 4):

$$\Lambda = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}\right] \stackrel{\omega \ll 1}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}$$
(7.76a)

$$M(\Lambda) = \exp\left[-\frac{i}{2}\omega_{\alpha\beta}\,\mathscr{S}^{\alpha\beta}\right] \stackrel{\omega \ll 1}{\approx} \mathbb{1} - \frac{i}{2}\omega_{\alpha\beta}\,\mathscr{S}^{\alpha\beta} \tag{7.76b}$$

 $\omega_{\alpha\beta}$ antisymmetric tensor \rightarrow 3 rotations (angles) + 3 boosts (rapidities)

It is $(\mathcal{J}^{\alpha\beta})_{\mu\nu} = i(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu})$; these 4 × 4 matrices $\mathcal{J}^{\alpha\beta}$ generate the 4-vector representation $(\frac{1}{2}, \frac{1}{2})$, i.e., the 4 × 4-matrices Λ . By contrast, the $n \times n$ -matrices $\delta^{\alpha\beta}$ generate the spinor-representation $M(\Lambda)$ [we will find $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$]. The generators are antisymmetric in the spacetime indices.

• Infinitesimal form of Eq. (7.75):

$$\left[\gamma^{\mu}, \mathscr{S}^{\alpha\beta}\right] \stackrel{\circ}{=} \left(\mathscr{J}^{\alpha\beta}\right)^{\mu}{}_{\nu}\gamma^{\nu} \stackrel{\circ}{=} i\left(\eta^{\alpha\mu}\gamma^{\beta} - \eta^{\beta\mu}\gamma^{\alpha}\right) \tag{7.77}$$

• $\triangleleft \mathcal{J}^{\alpha\beta}$ (\bigcirc Problemset 4) \rightarrow Lie-algebra of Lorentz group $(J = \mathscr{S}, \mathscr{J})$:

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] \stackrel{\circ}{=} i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}) \tag{7.78}$$

The Lie algebra defines the structure of the Lie group by exponentiation and is therefore the same for all representations, recall Eq. (4.63).

6 | Solution to Eq. (7.75) via Dirac's trick [88]: $\triangleleft \gamma^{\mu}$ such that

 $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} \mathbb{1}_{n \times n} \quad \text{** Dirac algebra}$

with the \checkmark *anticommutator* $\{X, Y\} = XY + YX$.

(7.79)



- Matrices $\gamma^{\mu} = (\gamma^0, \gamma^1, \gamma^2, \gamma^3)$ that satisfy Eq. (7.79) are called $\stackrel{*}{*}$ Dirac matrices or $\stackrel{*}{*}$ Gamma matrices.
- This is the 16-dimensional *Clifford algebra* $C\ell_{1,3}(\mathbb{C})$.

Then

$$\mathscr{S}^{\mu\nu} := \frac{i}{4} \left[\gamma^{\mu}, \gamma^{\nu} \right] \tag{7.80}$$

satisfies the Lorentz algebra Eq. (7.78) and Eq. (7.77).

Check this by plugging Eq. (7.80) into Eq. (7.78) and Eq. (7.77) and using Eq. (7.79)!

 \rightarrow Problem of solving Eq. (7.75) has been reduced to finding 4 matrices γ^{μ} that satisfy Eq. (7.79).

- 7 | Representations of Eq. (7.79):
 - At least n = 4-dimensional (Think of the γ^μ as Majorana modes and construct ladder operators → 2 modes.)
 - All 4-dimensional representations are unitarily equivalent (Actually, they constitute the *unique* irrep of the Dirac algebra which is 4-dimensional.)
 - We use the Weyl representation (sometimes called *chiral representation*):

$$\gamma^{\mathbf{0}} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \text{ and } \gamma^{i} = \begin{pmatrix} 0 & \sigma^{i} \\ -\sigma^{i} & 0 \end{pmatrix} \quad i = 1, 2, 3$$
 (7.81)

- Recall the Pauli matrices Eq. (7.52):

$$\sigma^{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
(7.82)

- Other common choices are the *t Dirac representation* and the *Majorana representation*.

• Henceforth: $\Lambda_{\frac{1}{2}} \equiv M(\Lambda)$

It turns out that these are two "copies" of a spin- $\frac{1}{2}$ projective representation: $\Lambda_{\frac{1}{2}}$ corresponds to the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation of SL(2, \mathbb{C}). Since $n + m = \frac{1}{2}$, this is a spinor representation, i.e., a projective representation of the Lorentz group SO⁺(1, 3). The fact that it is the sum of two such representations makes it *reducible*. The wavefunction $\Psi(x)$ has therefore n = 4 components and is a *spinor field* (and not a tensor field).

8 | Setting const = $-M = -\frac{mc}{\hbar}$ (which has dimension of an inverse length), we find:

$$(i\gamma^{\mu}\partial_{\mu} - M)\Psi = 0 \quad \text{** Dirac equation}$$
 (7.83)

Here, $\Psi(x)$ is a ** (bi)spinor-field:

$$\Psi: \mathbb{R}^{1,3} \to \mathbb{C}^4 = \mathbb{C}^2 \oplus \mathbb{C}^2.$$
(7.84)

Introduce the * Feynman slash notation: $\phi := \gamma^{\mu} O_{\mu}$ (Here, O_{μ} stands for any object with a 4-vector index.)



With the slash notation, the Dirac equation can be written as:

$$(i\,\partial \!\!\!/ - M)\Psi = 0 \tag{7.85}$$

The Dirac equation is engraved in a plaque on the floor of Westminster Abbey next to Isaac Newton's tomb (they abbreviate $\gamma \cdot \partial = \gamma^{\mu} \partial_{\mu}$ and are in natural units $\hbar = 1 = c$ where M = m):



(Photograph from https://cerncourier.com/a/paul-dirac-a-genius-in-the-history-of-physics.)

9 | The components $\Psi_a(x)$ (a = 1, 2, 3, 4) satisfy the KGE:

$$0 = (-i\gamma^{\mu}\partial_{\mu} - M)(i\gamma^{\nu}\partial_{\nu} - M)\Psi \stackrel{7.79}{=} (\partial^2 + M^2)\Psi$$
(7.86)

On the right hand side of Eq. (7.86) there is an identity $\mathbb{1}_{4\times 4}$ that we omit.

- The Dirac differential operator is the "square root" of the Klein-Gordon differential operator.
- i! Although Ψ has as many components as the EM gauge field A^μ, we do not write these components as Ψ^μ, but either simply as Ψ (and think of it as a four-dimensional column vector), or as Ψ_a with spinor index a = 1, 2, 3, 4. The purpose of this notational difference is to denote the different ways the fields transform under Lorentz transformations:

$$A^{\prime \mu} = \Lambda^{\mu}_{ \nu} A^{\nu} \quad \text{versus} \quad \Psi_{a}^{\prime} = (\Lambda_{\frac{1}{2}})_{ab} \Psi_{b} \quad \text{or simply} \quad \Psi^{\prime} = \Lambda_{\frac{1}{2}} \Psi \,. \tag{7.87}$$

Note that $\Lambda \equiv \Lambda^{\mu}_{\nu}$ and $\Lambda_{\frac{1}{2}} = M(\Lambda)$ are *not* the same 4×4 matrices!

10 | Dirac adjoint:

We would like to find a Lagrangian density for the Dirac equation; since this must be a Lorentz scalar, we ask the question:

How to form Lorentz scalars from Dirac spinors?

 $\mathbf{i} \mid \text{First try: } \Psi^{\dagger} \Psi$

$$\Psi^{\dagger}\Psi^{\prime} = \Psi^{\dagger} \underbrace{\Lambda_{\frac{1}{2}}^{\dagger}\Lambda_{\frac{1}{2}}}_{\neq 1} \Psi \neq \Psi^{\dagger}\Psi$$
(7.88)

 $\Lambda_{\frac{1}{2}}$ is *not* unitary because $\mathscr{S}^{\mu\nu}$ is not Hermitian for boosts ($\mu = 0$ and $\nu = 1, 2, 3$).

This is a consequence of the \uparrow *non-compactness* of the Lorentz group due to boosts.

ii | Define

$$\bar{\Psi} := \Psi^{\dagger} \gamma^{\mathbf{0}} \quad \text{** Dirac adjoint} \tag{7.89}$$



 $\stackrel{\circ}{\rightarrow} \bar{\Psi}'\Psi' = \bar{\Psi}\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\Psi = \bar{\Psi}\Psi \Rightarrow \text{Lorentz scalar}$ Use Eq. (7.80) and Eq. (7.76b) and the Dirac algebra to show this!

11 | Lagrangian:

With these tools, it is reasonable to propose the following Lagrangian density:

$$\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(i\gamma^{\mu}\partial_{\mu} - M)\Psi = \bar{\Psi}(i\partial - M)\Psi$$
(7.90)

 $\stackrel{\circ}{\rightarrow}$ Euler-Lagrange equations = Dirac equation

• Note that in explicit index notation, the Lagrangian density reads

$$\mathcal{L}_{\text{Dirac}} = i \Psi_a \gamma^{\mu}_{ab} (\partial_{\mu} \Psi_b) - M \Psi_a \Psi_a \tag{7.91}$$

where sums over pairs of spinor indices are implied.

The Euler-Lagrange equations follow again by treating Ψ_a and $\overline{\Psi}_a$ as independent fields:

$$0 \stackrel{!}{=} \frac{\partial \mathscr{L}_{\text{Dirac}}}{\partial \bar{\Psi}_a} - 0 \qquad \qquad = i\gamma^{\mu}_{ab}(\partial_{\mu}\Psi_b) - M\Psi_a = \left[(i\not\partial - M)\Psi\right]_a \qquad (7.92a)$$

$$0 \stackrel{!}{=} \frac{\partial \mathscr{L}_{\text{Dirac}}}{\partial \Psi_a} - \partial_\mu \frac{\partial \mathscr{L}_{\text{Dirac}}}{\partial (\partial_\mu \Psi_a)} = -M \bar{\Psi}_a - i (\partial_\mu \bar{\Psi}_b) \gamma_{ba}^\mu \stackrel{\circ}{=} \left[\overline{(i \not \partial - M) \Psi} \right]_a$$
(7.92b)

Note that the two equations are Dirac adjoints of each other.

• Let us check that \mathcal{L}_{Dirac} is a Lorentz scalar:

$$\mathcal{L}'_{\text{Dirac}} = \bar{\Psi}' \left(i \gamma^{\mu} \partial'_{\mu} - M \right) \Psi'$$
(7.93a)

$$=\Psi\Lambda_{\frac{1}{2}}^{-1}\left(i\gamma^{\mu}\Lambda_{\mu}^{\nu}\partial_{\nu}-M\right)\Lambda_{\frac{1}{2}}\Psi\tag{7.93b}$$

$$= \bar{\Psi} \left(i \Lambda_{\frac{1}{2}}^{-1} \gamma^{\mu} \Lambda_{\frac{1}{2}} \Lambda_{\mu}^{\nu} \partial_{\nu} - M \right) \Psi$$
(7.93c)

$$\stackrel{IS}{=} \bar{\Psi} \left(i \Lambda^{\mu}{}_{\rho} \gamma^{\rho} \Lambda^{\nu}{}_{\mu} \partial_{\nu} - M \right) \Psi \tag{7.93d}$$

$$=\Psi\left(i\gamma^{\nu}\partial_{\nu}-M\right)\Psi=\mathcal{L}_{\text{Dirac}}$$
(7.93e)

Here we used the following fact:

The gamma matrices transform *not* like Lorentz vectors: $\gamma'^{\mu} = \gamma^{\mu}$.

(7.94)

This is good because otherwise the Dirac equation would be different in different inertial systems. This also means that slashed quantities (like $\partial = \gamma^{\mu} \partial_{\mu}$) are *not* Lorentz scalars. Think of it like this: they do *not* have a Lorentz index, but they *do* have a two spinor indices (which we don't write) because they are *matrices*. To get rid of these indices, you must pair them with the indices of spinor fields. That is, slashed quantities become Lorentz scalars if put between two Dirac spinors like in the Dirac Lagrangian: $\bar{\Psi}\partial \Psi$ *is* a scalar field.

12 | <u>Conserved current:</u>

Now that we have a Lagrangian, it is just a straightforward application of Noether's theorem to obtain the conserved current associated to global phase rotations:



$i \mid \triangleleft$ Global phase rotations:

Eq. (7.90) is clearly invariant under global phase rotations of the spinors:

$$\Psi'(x) = e^{i\alpha}\Psi(x) \quad \text{for} \quad \alpha \in [0, 2\pi)$$
(7.95)

with infinitesimal generator $|\alpha| = |w| \ll 1$

$$\Psi'(x) = \Psi(x) + iw\Psi(x) \equiv \Psi(x) + w\,\delta\Psi(x) \quad \Rightarrow \quad \delta\Psi = i\Psi \tag{7.96}$$

 \rightarrow Continuous symmetry:

$$\mathcal{L}_{\text{Dirac}}(\Psi, \partial \Psi) = \mathcal{L}_{\text{Dirac}}(\Psi', \partial \Psi')$$
(7.97)

ii | Noether theorem $6.85 \rightarrow$ Conserved current density:

A straightforward calculation yields:

$$j_{\text{Dirac}}^{\mu} \stackrel{6.84}{=} -\frac{\partial \mathcal{L}_{\text{Dirac}}}{\partial (\partial_{\mu} \Psi_{a})} \delta \Psi_{a} \stackrel{7.91}{=} \bar{\Psi}_{b} \gamma_{ba}^{\mu} \Psi_{a} = \bar{\Psi} \gamma^{\mu} \Psi \,.$$
(7.98)

$$j_{\text{Dirac}}^{\mu} = \bar{\Psi} \gamma^{\mu} \Psi \quad \text{with} \quad \partial_{\mu} j_{\text{Dirac}}^{\mu} = 0$$
 (7.99)

Since the Lagrangian density \mathcal{L}_{Dirac} is a Lorentz scalar, this Noether current must be a 4-vector. We can check this explicitly:

$$j_{\text{Dirac}}^{\prime\mu} = \bar{\Psi}^{\prime}\gamma^{\mu}\Psi^{\prime} = \bar{\Psi}\Lambda_{\frac{1}{2}}^{-1}\gamma^{\mu}\Lambda_{\frac{1}{2}}\Psi \stackrel{7.75}{=} \Lambda^{\mu}{}_{\nu}\bar{\Psi}\gamma^{\nu}\Psi = \Lambda^{\mu}{}_{\nu}j_{\text{Dirac}}^{\mu}.$$
 (7.100)

iii | Conserved Noether charge:

$$Q = \int d^3x \, \bar{\Psi} \gamma^0 \Psi = \int d^3x \, \underbrace{\Psi^{\dagger} \Psi}_{\geq 0} \geq 0 \tag{7.101}$$

 \rightarrow

Conserved norm on
$$L^2 \otimes \mathbb{C}^4$$
: $\|\Psi\|^2 := \int d^3x \ \Psi^{\dagger} \Psi$ (7.102)

i! The positive-definite density Ψ[†]Ψ = Ψ
 ¹Ψγ⁰Ψ is the time-component of a 4-vector and therefore *not* Lorentz invariant. However, the Noether charge Q is a Lorentz scalar so that the norm is Lorentz invariant: ||Ψ'|| [±] ||Ψ||.

Note that not all Noether charges are Lorentz scalars. The total field momentum Eq. (6.92), for example, is a 4-vector; similarly, the total field angular momentum Eq. (6.117) is a tensor of rank 2. However, it can be shown that the Noether charges of *internal* symmetries (like the U(1) symmetry considered here) are necessarily Lorentz scalars (\uparrow *Coleman-Mandula theorem* [91]).

Let us prove $Q'_a = Q_a$ in the case where the Noether current j^{μ}_a has no other Lorentz index (and the internal group generators commute with the generators of Lorentz transformations):



a | We consider an infinitesimal Lorentz transformation.

Coordinates transform according to Eq. (6.78),

$$\delta^{\alpha\beta}x^{\mu} = \frac{1}{2} \left(\eta^{\alpha\mu}x^{\beta} - \eta^{\beta\mu}x^{\alpha} \right) , \qquad (7.103)$$

and, as a 4-vector, the *components* of the current transform in the same way:

$$\delta^{\alpha\beta}j^{\mu}_{a} = \frac{1}{2} \left(\eta^{\alpha\mu}j^{\beta}_{a} - \eta^{\beta\mu}j^{\alpha}_{a} \right) \stackrel{\circ}{=} j^{\nu}_{a} \left(\partial_{\nu}\delta^{\alpha\beta}x^{\mu} \right) \,. \tag{7.104}$$

(The labels *a* of the internal symmetry do not mix under this transformation because the internal symmetry is assumed to commute with Lorentz transformations.)

The generator of Lorentz transformations acts then according to Eq. (6.81) on the current *field*

$$-iG^{\alpha\beta}j^{\mu}_{a}(x) = \delta^{\alpha\beta}j^{\mu}_{a} - (\partial_{\nu}j^{\mu}_{a})\delta^{\alpha\beta}x^{\nu}.$$
(7.105)

In the following we suppress the indices $\alpha\beta$ whenever possible.

b | It is easy to check that $\partial_{\nu} \delta x^{\nu} = 0$; furthermore, we know that $\partial_{\nu} j_a^{\nu} = 0$ from the Noether theorem. Together, this allows us to write the action of infinitesimal Lorentz transformations on the current as a 4-divergence:

$$-iGj_a^{\mu}(x) = \underbrace{(\partial_{\nu}j_a^{\nu})}_{=0} \delta x^{\mu} + j_a^{\nu}(\partial_{\nu}\delta x^{\mu}) - (\partial_{\nu}j_a^{\mu})\delta x^{\nu} - j_a^{\mu}\underbrace{(\partial_{\nu}\delta x^{\nu})}_{=0}$$
(7.106a)
$$= \partial_{\nu}\left(j_a^{\nu}\delta x^{\mu} - j_a^{\mu}\delta x^{\nu}\right) .$$
(7.106b)

Here we used that $\delta j_a^{\mu} = j_a^{\nu} (\partial_{\nu} \delta x^{\mu}).$

c | We finally obtain for the infinitesimal Lorentz transformation of the Noether charge:

$$-iGQ_a = \int d^3x \, (-iGj_a^0) \tag{7.107a}$$

$$= \int \mathrm{d}^{3}x \,\partial_{\nu} \left(j_{a}^{\nu} \delta x^{0} - j_{a}^{0} \delta x^{\nu} \right) \tag{7.107b}$$

$$= \int \mathrm{d}^3x \,\partial_i \left(j_a^i \delta x^0 - j_a^0 \delta x^i \right) \tag{7.107c}$$

Gauss's theorem

$$= \int_{\partial} \mathrm{d}\sigma_i \left(j_a^i \delta x^0 - j_a^0 \delta x^i \right) = 0 \tag{7.107d}$$

In the last step we used that on the surface ∂ (typically spatial infinity) all fields vanish (for wavefunctions in L^2 this is clearly true).

Thus any Noether charge derived from internal symmetries transforms as a Lorentz scalar. In particular, the Dirac norm $\|\Psi\|$ is invariant under Lorentz transformations of the bispinor fields $\Psi(x)$.

13 | <u>Hamiltonian</u>:

i | Since the Dirac equation is first order in time, we can easily bring it into Schrödinger form and identify the Hamiltonian as the generator of time translations:

Eq. (7.83)
$$\Leftrightarrow \left[i\hbar\gamma^{0}\partial_{t}+i\hbar c\gamma^{i}\partial_{i}-mc^{2}\right]\Psi=0$$
 (7.108)

Use $(\gamma^0)^2 = \mathbb{1} \rightarrow$

$$i\hbar\partial_t\Psi = \left[-i\hbar c\gamma^0\gamma^i\partial_i + \gamma^0 mc^2\right]\Psi$$
(7.109)

ii | Let us define the new matrices:

$$\beta := \gamma^{\mathbf{0}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_i := \gamma^{\mathbf{0}} \gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \quad i = 1, 2, 3$$
(7.110)

with $\beta^2 = \mathbb{1} = \alpha_i^2$ and $\{\alpha_i, \alpha_j\} = 0 = \{\alpha_i, \beta\}$ for $i \neq j$, and in particular

$$\beta^{\dagger} = \beta$$
 and $\alpha_i^{\dagger} = \alpha_i$. (7.111)

i! Note that the spatial gamma matrices are *anti-Hermitian*: $(\gamma^i)^{\dagger} = -\gamma^i$.

iii | With these matrices we can define the ...

** Dirac Hamiltonian:

$$\hat{H}_{\text{Dirac}} = -i\hbar c \,\vec{\alpha} \cdot \nabla + \beta \, mc^2 = c \,\vec{\alpha} \cdot \vec{p} + \beta \, mc^2$$
(7.112)

with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and the \checkmark momentum operator $\vec{p} = -i\hbar\nabla$.

 \rightarrow The Dirac Hamiltonian is <u>Hermitian</u>: (With respect to the standard inner product on $L^2 \otimes \mathbb{C}^4$):

$$\hat{H}_{\text{Dirac}}^{\dagger} = c \,\vec{\alpha}^{\dagger} \cdot \vec{p}^{\dagger} + \beta^{\dagger} \,mc^2 = c \,\vec{\alpha} \cdot \vec{p} + \beta \,mc^2 = \hat{H}_{\text{Dirac}}$$
(7.113)

Here we use that the momentum operator is self-adjoint (Hermitian) for (a dense subset of) functions in $L^2(\mathbb{R}^3, \mathbb{C})$:

$$\langle \psi | \vec{p}\phi \rangle = \int d^3x \, \psi^*(-i\hbar\nabla\phi) = \int d^3x \, (-i\hbar\nabla\psi)^*\phi = \langle \vec{p}\psi | \phi \rangle \,. \tag{7.114}$$

We used partial integration and $\lim_{|\vec{x}|\to\infty} \phi(\vec{x}) = 0 = \lim_{|\vec{x}|\to\infty} \psi(\vec{x})$ for admissible functions.

iv | The Dirac equation then takes the Schrödinger form

$$i\hbar\partial_t\Psi(x) = \hat{H}_{\text{Dirac}}\Psi(x)$$
 (7.115)

In this form its Lorentz covariance is no longer manifest.

v | Eq. (7.102) conserved → \triangleleft Inner product on $L^2 \otimes \mathbb{C}^4$:

$$\langle \Psi | \Phi \rangle := \int d^3 x \, \Psi^{\dagger}(t, \vec{x}) \Phi(t, \vec{x}) \quad \text{with} \quad \|\Psi\| = \sqrt{\langle \Psi | \Psi \rangle}$$
 (7.116)

This inner product is constant under the evolution of the Dirac equation:

Eq. (7.113) & Eq. (7.115)
$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \langle \Psi | \Phi \rangle \stackrel{\circ}{=} 0$$
 (7.117)

• This generalizes our previous finding in Eq. (7.102) about the conserved norm.



• That the inner product is constant is straightforward to show:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\Psi|\Phi\rangle = \int \mathrm{d}^{3}x \left[\Psi^{\dagger}\dot{\Phi} + \dot{\Psi}^{\dagger}\Phi\right] \tag{7.118a}$$

$$\stackrel{7.115}{=} \frac{1}{i\hbar} \int d^3x \left[\Psi^{\dagger} \left(\hat{H}_{\text{Dirac}} \Phi \right) - \left(\hat{H}_{\text{Dirac}} \Psi \right)^{\dagger} \Phi \right]$$
(7.118b)

$$\stackrel{7.113}{=} \frac{1}{i\hbar} \int \mathrm{d}^{3}x \left[\Psi^{\dagger} \left(\hat{H}_{\mathrm{Dirac}} \Phi \right) - \Psi^{\dagger} \left(\hat{H}_{\mathrm{Dirac}} \Phi \right) \right] = 0 \qquad (7.118c)$$

14 | Conclusion:

Let us summarize our findings and compare them to the Klein-Gordon equation:

	Klein-Gordon Equation	Dirac Equation		
	$(\partial^2 + M^2)\phi = 0$	$(i\partial - M)\Psi = 0$		
Time derivative	second order	first order		
Function space	$L^2(\mathbb{R}^{1,3},\mathbb{C})$	$L^2(\mathbb{R}^{1,3},\mathbb{C}^2\oplus\mathbb{C}^2)$		
Wavefunction	Complex scalar field $\phi(x)$	Complex bispinor field $\Psi(x)$		
Conserved form	$i\int \mathrm{d}^3x \left(\phi_1^*\dot{\phi}_2-\dot{\phi}_1^*\phi_2\right)$	$\int \mathrm{d}^3 x \ \Psi_1^\dagger \Psi_2$		
Positive definite?	×	\checkmark		
Hermitian Hamiltonian?	×	✓		

 \rightarrow What about the *eigenenergies* and *eigenstates* of \hat{H}_{Dirac} ?

7.2.1. Free-particle solutions of the Dirac equation

15 | Eq. (7.86): Solutions of the Dirac equation satisfy the Klein-Gordon equation component-wise: $\xrightarrow{\text{Eq. (7.34)}}$ Ansatz:

$$\Psi^{\pm}(x) = \psi^{\pm}(p)e^{\pm \frac{i}{\hbar}px}$$
 with $p^{0} = \frac{E}{c} = \sqrt{\vec{p}^{2} + m^{2}c^{2}} > 0$ (7.119)

with complex-valued four-component ** bispinor

$$\psi^{\pm}(p) \equiv \begin{pmatrix} \psi_L^{\pm} \\ \psi_R^{\pm} \end{pmatrix} \in \mathbb{C}^4 \simeq \mathbb{C}^2 \oplus \mathbb{C}^2 .$$
(7.120)

- We set $p^0 > 0$ for both positive (+) and negative (-) energy/frequency solutions and change the sign of p in the exponent (to simplify the discussion below).
- Note that $px = p_{\mu}x^{\mu} = Et \vec{p} \cdot \vec{x}$.
- **16** | Eq. (7.119) in Eq. (7.83) yields:

$$(\pm \gamma^{\mu} p_{\mu} - mc) \psi^{\pm}(p) = \begin{pmatrix} -mc & \pm p\sigma \\ \pm p\bar{\sigma} & -mc \end{pmatrix} \begin{pmatrix} \psi_{L}^{\pm} \\ \psi_{R}^{\pm} \end{pmatrix} = 0$$
(7.121)

with $p\sigma = p_{\mu}\sigma^{\mu}$ and $\sigma^{\mu} = (\mathbb{1}, \sigma^{x}, \sigma^{y}, \sigma^{z})$ and $\bar{\sigma}^{\mu} = (\mathbb{1}, -\sigma^{x}, -\sigma^{y}, -\sigma^{z})$.

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- 17 | Mathematical facts (check these!):
 - $(p\sigma)(p\bar{\sigma}) \stackrel{\circ}{=} p^2 = m^2 c^2$
 - Eigenvalues of $p\sigma$ and $p\bar{\sigma}$: $p^0 \pm |\vec{p}| \rightarrow$ for $p^0 > 0$ and $m \neq 0$ positive spectrum $\rightarrow p\sigma$ and $p\bar{\sigma}$ are *invertible* and the positive square roots $\sqrt{p\sigma}$ and $\sqrt{p\bar{\sigma}}$ are Hermitian.
- 18 $| \triangleleft \psi_L^{\pm} \equiv \sqrt{p\sigma} \, \xi^{\pm}$ with arbitrary, normalized $[(\xi^{\pm})^{\dagger} \xi^{\pm} = 1] \, ** \text{ spinor } \xi^{\pm} \in \mathbb{C}^2$:

Eq. (7.121)
$$\Rightarrow -mc\sqrt{p\sigma}\xi^{\pm}\pm p\sigma\psi_R^{\pm}=0$$
 (7.122)

Use $\sqrt{p\sigma}\sqrt{p\bar{\sigma}} = mc$:

$$\psi_R^{\pm} = \pm \frac{mc}{\sqrt{p\sigma}} \xi^{\pm} = \pm \sqrt{p\bar{\sigma}} \xi^{\pm}$$
(7.123)

 $\rightarrow \psi_L^{\pm}$ and ψ_R^{\pm} are now parametrized by the spinor $\xi^{\pm} \in \mathbb{C}^2$ (which is unconstrained!).

The second equation in Eq. (7.121) yields the same solution.

19 | <u>Solutions:</u>

Let us adopt the more conventional notation

$$\begin{array}{ccc} \xi^+ \mapsto \xi & & \psi^+ \mapsto u \\ \xi^- \mapsto \eta & & \psi^- \mapsto v \end{array}$$
(7.124)

and choose the spinor basis ξ^s , η^s ($s = \uparrow, \downarrow$) with

$$\xi^{\uparrow}, \eta^{\uparrow} = \begin{pmatrix} 1\\ 0 \end{pmatrix} \text{ and } \xi^{\downarrow}, \eta^{\downarrow} = \begin{pmatrix} 0\\ 1 \end{pmatrix}.$$
 (7.125)

Then linearly independent solutions of the free Dirac equation can be written as:

$$\Psi_{\vec{p},s}^{+}(x) = \underbrace{\begin{pmatrix} \sqrt{p\sigma}\xi^{s} \\ \sqrt{p\bar{\sigma}}\xi^{s} \end{pmatrix}}_{u^{s}(\vec{p})} e^{-\frac{i}{\hbar}px} \quad \text{(positive energy solutions)}$$
(7.126a)
$$\Psi_{\vec{p},s}^{-}(x) = \underbrace{\begin{pmatrix} \sqrt{p\sigma}\eta^{s} \\ -\sqrt{p\bar{\sigma}}\eta^{s} \end{pmatrix}}_{v^{s}(\vec{p})} e^{+\frac{i}{\hbar}px} \quad \text{(negative energy solutions)}$$
(7.126b)

with $p^{\mu} = (p^0, \vec{p}), p^0 = \sqrt{\vec{p}^2 + m^2 c^2} > 0$ and $s = \uparrow, \downarrow$.

 \rightarrow Four linearly independent solutions for each 3-momentum \vec{p} (± and s = 1, 2). You can easily check that Eq. (7.126) form an orthogonal eigenbasis of the Dirac Hamiltonian:

$$\hat{H}_{\text{Dirac}}\Psi^{\pm}_{\vec{p},s} \stackrel{\circ}{=} \pm E_{\vec{p}}\Psi^{\pm}_{\vec{p},s}$$
 with spectrum $E_{\vec{p}} = \sqrt{p^2c^2 + m^2c^4}$. (7.127)

Their orthogonality follows with the identities

$$[u^{r}(\vec{p})]^{\dagger}u^{s}(\vec{p}) \stackrel{\circ}{=} \frac{2}{c}E_{\vec{p}}\delta^{rs}, \quad [v^{r}(\vec{p})]^{\dagger}v^{s}(\vec{p}) \stackrel{\circ}{=} \frac{2}{c}E_{\vec{p}}\delta^{rs}, \quad [u^{r}(\vec{p})]^{\dagger}v^{s}(-\vec{p}) \stackrel{\circ}{=} 0.$$
(7.128)

 \rightarrow The Dirac equation still has *negative-energy solutions*.

(7.129)



20 | Interpretation:

• The negative energy solutions are not problematic as long as we consider a single particle (electron) without interactions (this is also why we can apply the Dirac equation to describe the hydrogen atom, → *below*). However, in reality the electron couples to a dynamic electromagnetic field and therefore could emit a photon (thereby lowering its energy). If the negative energy eigenstates really exist, there is no reason why this process should terminate; as a consequence, no stable electrons should exist.

Dirac writes in Ref. [92]:

It is true that in the case of a steady electromagnetic field we can draw a distinction between those solutions [..] with E positive and those with E negative and may assert that only the former have a physical meaning (as was actually done when the theory was applied to the determination of the energy levels of the hydrogen atom), but if a perturbation is applied to the system it may cause transitions from one kind of state to the other. In the general case of an arbitrarily varying electromagnetic field we can make no hard-and-fast separation of the solutions of the wave equation into those referring to positive and those to negative kinetic energy. Further, in the accurate quantum theory in which the electromagnetic field also is subjected to quantum laws, transitions can take place in which the energy of the electron changes from a positive to a negative value even in the absence of any external field, the surplus energy [..] being spontaneously emitted in the form of radiation. [..] Thus we cannot ignore the negative-energy states without giving rise to ambiguity in the interpretation of the theory.

Dirac suggested a "fix" for this problem [92]: Because the electron is a fermion, it obeys the Pauli exclusion principle. Thus one can imagine that (for some reason) all the negative energy states *are already occupied* by electrons. The electrons we see can then only occupy the positive energy states and cannot decay to states of arbitrarily low energy. This construct is know as the \uparrow *hole theory* because creating a "hole" in this \uparrow *Dirac sea* of electrons with negative energy can be viewed as an excitation with *positive* energy. Dirac's holes are of course a precursor to what we know today as \uparrow *antiparticles*. (Dirac didn't think of it this way, he conjectured that the holes in his sea of electrons are the *protons*!)

 However, Dirac's interpretation is not how we deal with the negative-energy solutions today: Within the modern framework of ↑ *relativistic quantum field theories*, the four single-particle wave functions are associated (through "second" quantization of the Dirac field and the construction of a fermionic ↑ *Fock space*) to two particle types, both with *positive* energy and two internal spin-¹/₂ states:

	Туре	Momentum	Spin	Energy	Charge
$\Psi^+_{\vec{p},\uparrow}$:	fermion	$+\vec{p}$	$+\frac{1}{2}$	$+E_{\vec{p}}$	+1
$\Psi^+_{\vec{p},\downarrow}$: :	fermion	$+\vec{p}$	$-\frac{1}{2}$	$+E_{\vec{p}}$	+1
$\Psi^{\vec{p},\uparrow}$:	antifermion	$+\vec{p}$	$-\frac{1}{2}$	$+E_{\vec{p}}$	-1
$\Psi^{\vec{p},\downarrow}$:	antifermion	$+\vec{p}$	$+\frac{1}{2}$	$+E_{\vec{p}}$	-1

Here "Spin" refers to the \checkmark spin-polarization quantum number $m_z = \pm \frac{1}{2}$.

 \rightarrow Take home message:

Relativistic quantum mechanics predicts spin and antiparticles.

(7.131)



The negative energy solutions (and therefore the existence of antiparticles) are a necessary feature of relativistic quantum mechanics (more precisely: relativistic quantum field theories, via the \uparrow *CPT-theorem*).

By contrast, the fact that particles can have an internal angular momentum (spin), and that this angular momentum can take half-integer values $S = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ is not a relativistic feature *per se*: Spin enters quantum mechanics the moment one considers *spatial rotations* and its representations on the Hilbert space. Because these can be \uparrow *projective*, one is forced to study the irreducible linear representations of SU(2) – the double cover of the rotation group SO(3) – which happen to be labeled by the "spin quantum numbers" $S = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ Now, since the rotation group is a subgroup of the homogeneous Lorentz group, SO(3) \subset SO⁺(1, 3), the moment a quantum theory is relativistic [i.e., features a representation of SO⁺(1, 3)], spin enters the stage automatically. However, you can describe quantum particles with spin *without* making quantum mechanics relativistic.

• The Dirac equation applies to all spin- $\frac{1}{2}$ fermions. The most prominent example is of course the electron e^- and its associated antiparticle, the positron e^+ . However, all other elementary fermions, namely leptons (like the muon/antimuon, the tau/antitau and the neutrinos) and the six quark/antiquark pairs, are described by the Dirac equation as well.

7.2.2. The relativistic hydrogen atom

21 | Dirac equation with a static EM field:

To couple the Dirac field Ψ in a gauge- and covariant way to a static EM field A^{μ} , we use the same trick as for the Klein-Gordon equation:

 \leftarrow Minimal coupling Eq. (7.38) \rightarrow

$$\partial_{\mu} \mapsto D_{\mu} = \partial_{\mu} + iQA_{\mu} \quad \Rightarrow \quad \not \partial \mapsto \not D = \not \partial + iQA = \gamma^{\mu}\partial_{\mu} + iQ\gamma^{\mu}A_{\mu} \quad (7.132)$$

For an electron it is $Q = -\frac{e}{\hbar c}$ with e > 0.

 \rightarrow

$$(i \not\!\!D - M)\Psi = 0 \tag{7.133}$$

In this form, the Dirac equation is manifest Lorentz- and gauge invariant.

We can expand Eq. (7.133) to obtain a less abstract (but more convoluted) expression:

$$i\gamma^{\mu}\partial_{\mu} - Q\gamma^{\mu}A_{\mu} - M]\Psi = 0$$
(7.134a)

$$\Leftrightarrow \quad \left[i\hbar\gamma^{0}\partial_{t} + i\hbar c\gamma^{i}\partial_{i} - q\gamma^{0}\varphi + q\gamma^{i}A_{i} - mc^{2}\right]\Psi = 0 \tag{7.134b}$$

$$\Leftrightarrow \quad \left[i\hbar\partial_t + i\hbar c\,\vec{\alpha}\cdot\nabla - q\,\varphi + q\,\vec{\alpha}\cdot\vec{A} - \beta\,mc^2\right]\Psi = 0 \tag{7.134c}$$

Here we used $Q = \frac{q}{\hbar c}$, $M = \frac{mc}{\hbar}$, and $A_{\mu} = (\varphi, -\vec{A})$; q is the charge of the particle. In Schrödinger form the Dirac equation reads then:

$$i\hbar\partial_t\Psi = \left[-i\hbar c\,\vec{\alpha}\cdot\nabla + q\,\varphi - q\,\vec{\alpha}\cdot\vec{A} + \beta\,mc^2\right]\Psi\tag{7.135a}$$

$$i\hbar\partial_t \Psi = \underbrace{\left[c\,\vec{\alpha}\cdot\left(\vec{p}-\frac{q}{c}\vec{A}\right) + q\,\varphi + \beta\,mc^2\right]}_{\hat{H}_{\text{Dirac},A}}\Psi \tag{7.135b}$$



22 | Choose the Coulomb potential (of the proton)

$$\varphi(x) = \frac{e}{|\vec{x}|}$$
 and $\vec{A} = \vec{0}$ (7.136)

and set q = -e (charge of the electron) \rightarrow

$$i\hbar\partial_t\Psi = \left[-i\hbar c\,\vec{\alpha}\cdot\nabla - \frac{e^2}{|\vec{x}|} + \beta\,mc^2\right]\Psi\tag{7.137}$$

With the ansatz $\Psi(t, \vec{x}) = \psi(\vec{x})e^{-\frac{i}{\hbar}Et}$ one obtains the time-independent eigenvalue problem

$$\left[-i\hbar c\,\vec{\alpha}\cdot\nabla - \frac{e^2}{|\vec{x}|} + \beta\,mc^2 - E\right]\psi(\vec{x}) = 0 \quad \text{with} \quad \psi = \begin{pmatrix}\psi_L\\\psi_R\end{pmatrix}: \ \mathbb{R}^3 \to \mathbb{C}^4.$$
(7.138)

Note that β (unlike α_i) is an off-diagonal block matrix that mixes the two spinors ψ_+ and ψ_- ; this complicates the solution. However, one can solve Eq. (7.138) *exactly* and compute the eigenvalues E and eigenstates $\psi(\vec{x})$.

23 | <u>Solution</u>: $\xrightarrow{*}$ Eigenenergies (including the rest energy of the electron):

$$E_{n,j} = mc^2 \left\{ 1 + \frac{\alpha^2}{\left[n - j - \frac{1}{2} + \sqrt{\left(j + \frac{1}{2}\right)^2 - \alpha^2} \right]^2} \right\}^{-\frac{1}{2}}$$
(7.139)

with

- \downarrow principal quantum number n = 1, 2, ...
- \checkmark total angular momentum quantum number $j = \frac{1}{2}, \frac{3}{2}, \dots, n \frac{1}{2}$
- \oint fine-structure constant $\alpha \approx \frac{1}{137}$

The principal quantum number n = 1, 2, ... constrains the allowed *orbital* angular momentum to l = 0, 1, ..., n - 1. The allowed *total* angular momentum is then given by the usual rules of angular momentum addition: $|l - \frac{1}{2}| \le j \le |l + \frac{1}{2}|$ (in integer steps, $s = \frac{1}{2}$ is the electron spin). So for example n = 1 allows only for l = 0 and therefore $j = \frac{1}{2}$; this is the $1S_{1/2}$ orbital and the ground state of the hydrogen atom. For n = 2 one finds again l = 0 with $j = \frac{1}{2}$ (the $2S_{1/2}$ orbital) but also l = 1 with $j = \frac{1}{2}$ and $j = \frac{3}{2}$ (the $2P_{1/2}$ and $2P_{3/2}$ orbitals – which are no longer degenerate because $E_{2,1/2} \ne E_{2,3/2}$).

This result explains why in the hydrogen spectrum the degeneracy of the $2S_{1/2}$ and $2P_{3/2}$ orbitals is lifted whereas the $2S_{1/2}$ orbital remains degenerate with the $2P_{1/2}$ orbital (\leftarrow *fine-structure*).

The Dirac equation explains the fine-structure of the hydrogen atom ©.

(7.140)

<u>Note:</u> You may have encountered the following Hamiltonian for the hydrogen atom with added *relativistic corrections*:

$$\hat{H}_{\rm rel} = \underbrace{\frac{\vec{p}^2}{2m} - \frac{e^2}{r}}_{\substack{\rm Non-rel, \\ \rm hydrogen \ atom}} - \underbrace{\frac{1}{2mc^2} \left(\frac{p^2}{2m}\right)^2}_{\substack{\rm Rel. \ kinetic \ energy}} + \underbrace{\frac{e^2}{2m^2c^2} \frac{\vec{L} \cdot \vec{S}}{r^3}}_{\substack{\rm Spin-orbit \ coupling}} - \underbrace{\frac{e^2\hbar^2}{8m^2c^2} \Delta\left(\frac{1}{r}\right)}_{\substack{\rm Darwin \ term}} \,.$$
(7.141)

Relativistic corrections

This Hamiltonian can reproduce the fine-structure as well. It has several drawbacks, though:



- It is only an *approximation*.
- It is hard to solve (perturbation theory!).
- The Schrödinger equation $i\hbar\partial_t\psi = \hat{H}_{rel}\psi$ is not manifestly Lorentz covariant.
- The relativistic corrections are *ad hoc* and seemingly independent of each other.

Luckily, Eq. (7.141) does not have to appear out of thin air; one can show via a complicated derivation (\uparrow *Foldy-Wouthuysen transformation*) that it is indeed the non-relativistic limit [with corrections in order $(v/c)^2$] of the Dirac equation Eq. (7.138) in a Coulomb potential (without the rest energy mc^2 of the electron).

7.2.3. The electron *g*-factor

Besides the fine structure, there is one other "mystery" that the relativistic treatment of the electron in terms of the Dirac equation finally explains: The non-classical ratio between the electrons internal magnetic moment and its spin.

24 | \triangleleft Dirac electron in homogeneous magnetic field $\vec{B} = \nabla \times \vec{A} \ (\varphi = 0)$:

Eq. (7.135b)
$$\xrightarrow{\Psi=\psi e^{-\frac{i}{\hbar}Et}} \left[c\,\vec{\alpha}\cdot\left(\vec{p}+\frac{e}{c}\vec{A}\right)+\beta\,mc^2-E\right]\psi=0 \qquad (7.142)$$

with bispinor

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \colon \mathbb{R}^3 \to \mathbb{C}^4 \,. \tag{7.143}$$

Using Eq. (7.110) we can write this equation in terms of the two spinors:

$$(-c\vec{\sigma}\vec{\pi} - E)\psi_L + mc^2\psi_R = 0 \tag{7.144a}$$

$$\left(+c\vec{\sigma}\vec{\pi} - E\right)\psi_R + mc^2\psi_L = 0 \tag{7.144b}$$

Here we used $\vec{\pi} = \vec{p} + \frac{e}{c}\vec{A}$ and introduced $\vec{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$.

We can now use one of the two equations to decouple the system:

$$(c\vec{\sigma}\vec{\pi} + E) (c\vec{\sigma}\vec{\pi} - E) \psi_R + (mc^2)^2 \psi_R = 0$$
 (7.145a)

$$\Leftrightarrow \quad c^{2}(\vec{\sigma}\,\vec{\pi})^{2}\psi_{R} - \left[E^{2} - (mc^{2})^{2}\right]\psi_{R} = 0 \tag{7.145b}$$

25 $\stackrel{\circ}{\rightarrow}$ Non-relativistic approximation:

We can use $E^2 - (mc^2)^2 = (E - mc^2)(E + mc^2) \approx 2mc^2\tilde{E}$ with $\tilde{E} = E - mc^2$ to find a non-relativistic approximation of Eq. (7.145b):

$$\frac{1}{2m}(\vec{\sigma}\vec{\pi})^2\psi_R = \tilde{E}\psi_R \tag{7.146}$$

Last, use the Pauli algebra $\sigma^i \sigma^j = \delta^{ij} + i \varepsilon^{ijk} \sigma^k$ and $B_k = \varepsilon_{ijk} (\partial_i A_j)$ to show that $(\vec{\sigma} \vec{\pi})^2 \stackrel{\text{algebra}}{=} \vec{\pi}^2 + \frac{\hbar e}{c} \vec{\sigma} \cdot \vec{B}$. We end up with the non-relativistic, time-independent Schrödinger equation of a charged particle in a magnetic field with a spin-dependent Zeeman term:

$$\left[\underbrace{\frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^{2}}_{\text{Particle in mag. field}} + \underbrace{\frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}}_{\text{Zeeman effect}}\right]\psi_{R} = \tilde{E}\psi_{R}$$
(7.147)

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 \rightarrow Potential energy of electron in magnetic field:

$$E_{\text{mag}} \stackrel{\text{def}}{=} -\vec{\mu} \cdot \vec{B} \stackrel{7.147}{=} \frac{e\hbar}{2mc} \vec{\sigma} \cdot \vec{B}$$
(7.148)

26 $| \rightarrow$ Magnetic moment (operator) of the electron:

$$\vec{\mu}_e = -\frac{e\hbar}{2mc}\vec{\sigma} = g_e \frac{\mu_B}{\hbar}\vec{S}$$
(7.149)

with \checkmark spin operator $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ and \checkmark Bohr magneton $\mu_B = \frac{e\hbar}{2mc}$ and

$$\stackrel{*}{*} Electron g - factor \quad g_e = -2.$$
(7.150)

27 | <u>Comments:</u>

• What makes Eq. (7.149) with $g_e = -2$ remarkable is that it is *not* what one would expect if the magnetic moment would be caused by a charge flying along a tiny orbit with angular momentum \vec{S} . Indeed, a straightforward classical calculation yields for the relation between magnetic moment and (orbital) angular momentum \vec{L} :

$$\vec{\mu}_L = g_L \frac{\mu_B}{\hbar} \vec{L} \quad \text{with} \quad g_L = -1 \,. \tag{7.151}$$

So, quite surprisingly, the Dirac equation predicts that the internal angular momentum (= spin) produces "twice as much" magnetic moment as one would naïvely expect.

That this really is the case can be easily measured: Just apply a magnetic field to hydrogen atoms and observe how strongly their spectral lines split as a function of the magnetic field strength (\uparrow *anomalous Zeeman effect*). This effect had already been experimentally observed at the end of the 19th century [93,94]. Since it was unknown at the time that electrons had spin, certain line splittings could not be explained (therefore "anomalous"). The fact that the Dirac equation explains both – the electron spin and its "non-classical" *g*-factor – is therefore a remarkable feature of relativistic quantum mechanics.

• If one measures the electron g-factor really, really precisely, one finds [95]

$$g_e = -2.00231930436118(27) \,. \tag{7.152}$$

You may notice that this is not *exactly* -2 but a tiny bit off. One cannot explain this deviation with the Dirac equation because it stems from "virtual particles" that modify how the electron interacts with the EM field (and the Dirac equation is a single-particle wave equation). It is therefore remarkable that modern theories *can* explain this deviation perfectly (up to error bars), but for this one needs the machinery of \uparrow *relativistic quantum field theory*.

Part II. General Relativity



↓Lecture16 [09.04.24]



8. Limitations of SPECIAL RELATIVITY

8.1. Reminder: SPECIAL RELATIVITY

1 <u>SPECIAL RELATIVITY in a nutshell:</u>

• ← *Inertial frames* [Section 1.1]

There exists a special class of infinitely extended reference frames (equipped with Cartesian coordinates) in which the law of inertia holds (**IN** = the trajectories of free particles are straight lines that are traversed with constant velocity). All inertial frames move relative to each other with constant velocities:



• ← *Einstein's principle of (special) relativity* **SR** [Section 1.3]

The laws of physics (orange boxes in the sketch below) have the same form in all inertial systems. This extends Galilei's principle of relativity which makes this claim only for the realm of mechanics. The modifier "special" emphasizes that the principle makes only claims about the special class of inertial systems:



We characterized **SR** previously as follows: *No experiment can distinguish between inertial frames.* This description can be misleading, so let me prevent any misconceptions: What we mean is that there is no local physical experiment that you can perform in a sealed box (at rest in some inertial system) that allows you to figure out in which inertial system your box is at rest. In this way you probe the *form of physical laws* (e.g., whether there is an additional Coriolis term in your equation of motion or not) and thus probe the validity of **SR** as formulated above.



The statement above does *not* mean that there is no operational way to label specific inertial systems. For example, we can define the (approximate) inertial frame in which the center of Earth is at rest and, for comparison, another inertial frame in which the \uparrow cosmic microwave background (CMB) has no dipole structure (the latter has a velocity of roughly 360 km s⁻¹ wrt. the former). Clearly there are experiments to decide whether you are in one or the other (measure the CMB dipole and/or the velocity of Earth relative to you). This does *not* violate **SR** though, because all phenomena you observe in these frames of reference are described correctly by the same equations (e.g. you can use the same Maxwell equations to describe the CMB radiation in both inertial frames). This is also why the existence of the global inertial frame labeled by a CMB without dipole is not in conflict with SPECIAL RELATIVITY. There is a difference between physical states and physical laws; **SR** only makes claims about the latter.

• ← Lorentz transformations [Section 1.5]

The coordinate transformations that map the record of physical events from one inertial system to another are given by Lorentz transformations (more generally: Poincaré transformations). (Proper orthochronous) Lorentz transformations form a group SO⁺(1, 3) and are parametrized by a three-dimensional rotation and a three-dimensional boost velocity. They linearly map the spacetime coordinates (t, \vec{x}) of an event in one inertial system K to the spacetime coordinates $(t', \vec{x'})$ of the same event in another inertial system K'.

A pure boost in *x*-direction has the form:

$$\Lambda(K \xrightarrow{v_x} K'): \begin{cases} ct' = \gamma \left(ct - \frac{v_x}{c} x\right) \\ x' = \gamma (x - v_x t) \\ y' = y \\ z' = z \end{cases} \text{ with Lorentz factor}$$

$$(8.1)$$

• ← Constancy of the speed of light SL [Section 1.5]

Lorentz transformations are characterized (and differ from Galilei transformations) by the existence of a *finite maximum velocity* v_{max} . Experience tells us to identify this velocity with the speed of light c. Lorentz transformations then imply that this maximum velocity is the same for all inertial observers (\leftarrow relativistic addition of velocities):

Experiments $\rightarrow v_{\text{max}} < \infty \Leftrightarrow$ Lorentz transformations (8.2a)

 $v_{\text{max}} = \infty \quad \Leftrightarrow \quad \text{Galilei transformations}$ (8.2b)

• ← *Tensor calculus* [Chapter 3 and Chapter 4]

Combining the principle of relativity with the assertion that Lorentz transformations translate between inertial systems implies that the laws of nature must be expressed as equations that are forminvariant under Lorentz transformations (\leftarrow Lorentz covariance). The Lorentz covariance of a theory can be quite tedious to show and even more tedious to ensure when constructing it from scratch. (Recall Maxwell equations in their conventional form!) This is why we prefer equations in which the Lorentz covariance is *manifest*. To achieve this, we developed tensor calculus as a "toolbox" to construct Lorentz covariant equations from Lorentz scalars, vectors, tensors,

For example, the equation of motion for a charged particle in an electromagnetic field reads



and transforms as follows:

$$\frac{\bar{p}^{\mu} = \Lambda^{\mu}{}_{\nu}p^{\nu}}{\bar{d}r^{\mu} = \frac{q_{i}}{c}F^{\mu}{}_{\nu}u^{\nu}} \xrightarrow{\bar{F}^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda_{\nu}{}^{\beta}F^{\alpha}{}_{\beta}} \xrightarrow{\text{Inertial system } \bar{K}} \underbrace{\frac{\bar{d}p_{i}^{\mu}}{\bar{d}\tau} = \frac{q_{i}}{c}\bar{F}^{\mu}{}_{\nu}\bar{u}^{\nu}_{i}}_{(8.3)}$$

...

2 | <u>Problems:</u>

Despite the undeniable success of SPECIAL RELATIVITY, it's not just sunshine and roses:

• What about gravitation?

In our discussion of SPECIAL RELATIVITY we explicitly avoided the phenomenon of gravitation (we will see below why). This makes SPECIAL RELATIVITY clearly incomplete (and *special*) as a description of nature (which, on very large scales, is dominated by the gravitational force) and asks for a more *general* theory.

Why are inertial coordinate systems special?

SPECIAL RELATIVITY describes physics with respect to a particular class of reference frames (inertial frames) in a particular class of coordinates (Cartesian coordinates). Only in these coordinate frames the laws of nature take their "simplest" form and the Lorentz transformation only translates between these special coordinate systems. However, in our very general discussion of differential geometry (Chapter 3) we established the notion of "geometric objects" that are independent of coordinates. We also interpreted coordinates as mathematical auxiliary structures to label events, and denied their physical existence ("coordinates do not exist"). SPECIAL RELATIVITY does not live up to this rather fundamental claim with its focus on inertial coordinate systems. Shouldn't there be a formulation of physics in which coordinates play no role at all?

• What is the origin of inertia?

Remember Newton's bucket (p. 14)? It's punchline was to argue for the existence of an entity ("absolute space") which determines whether an object is accelerated or not. SPECIAL RELATIVITY, of course, disposes of Newton's absolute space wrt. to which position and velocity can be measured (no *ether*!). The existence of such, however, was never implied by the bucket experiment anyway, which asks about the absolute notion of *acceleration*. And SPECIAL RELATIVITY is silent about the origin of inertia and what determines whether the water in Newton's bucket is concave or flat (we simply assumed that inertial frames exist, we neither asked where they come from nor what makes them inertial in the first place). This situation is clearly unsatisfactory.

$\mathbf{3} \mid$ Non-problems:

Sometimes one hears that *acceleration* is a problem for SPECIAL RELATIVITY. This is not so:

Accelerated motion

SPECIAL RELATIVITY of course describes accelerated objects perfectly well. Recall our concept of 4-acceleration in Section 5.1, the relativistic equation of motion in Eq. (5.6), and the validity of the proper time integral for arbitrary time-like trajectories in Eq. (2.25). Note that these equations are only valid in inertial frames, though.

While our equations were given in inertial systems (where, according to Einstein's principle of relativity, the laws of physics take the same *and simplest* form), SPECIAL RELATIVITY



can describe the physics in accelerated non-inertial frames as well (e.g. using the concept of instantaneous rest frames). In such non-inertial coordinate systems the physical laws do *not* take their simplest forms and can look messy (in particular, one cannot Lorentz transform into these frames). This, however, does not mean that we cannot describe what happens in such systems. As an example recall the relativistic rocket of \bigcirc Problemset 6. (For details see Chapter 6 in MISNER *et al.* [3] and also Einstein's original work [96].)

8.2. The special role of gravity

Let us now focus on the problem of incorporating gravity into SPECIAL RELATIVITY. It is important to understand why the gravitational force poses a fundamental problem for the framework of SPECIAL RELATIVITY (and is not just a technical inconvenience).

Note on nomenclature:

In English, there are two terms with slightly different meaning (if we take Merriam-Webster as a reference):

Gravity: the gravitational attraction of the mass of the Earth, the moon, or a planet for bodies at or near its surface

Gravitation: a force manifested by acceleration toward each other of two free material particles or bodies or of radiant-energy quanta

This distinction has no counterpart in German as far as I can tell (perhaps "Schwerkraft" vs. "Gravitation"?). Given that even the English literature does not seem to be consistent, I will use these two terms interchangeably. Their context will suffice to establish semantic clarity.

- 4 | Recall Newton's law of universal gravitation:
 - i | \triangleleft Mass distribution $\rho(\vec{x}) \rightarrow$

$$\nabla^2 \phi(\vec{x}) = 4\pi G \rho(\vec{x}) \rightarrow \text{Gravitational potential } \phi(\vec{x})$$
 (8.4)

 $G \approx 6.67430 \times 10^{-11} \,\mathrm{m^3 s^{-2}/kg}$: Gravitational constant

Please appreciate the smallness of G (and therefore the weakness of gravity) as compared to the human-scale units in which it is given. Gravity is, if compared to the other three fundamental forces, by far (really really really far) the weakest force. It is a fundamental unsolved problem of physics why this is so.

 \rightarrow Equation of motion of test mass (e.g. a satellite):

$$m_{\rm I}\vec{\vec{r}} = -m_{\rm G}\nabla\phi(\vec{r}) \tag{8.5}$$

 $m_{\rm I}$: inertial mass $m_{\rm G}$: gravitational mass

We will discuss the relation of $m_{\rm I}$ and $m_{\rm G}$ later (Section 9.1).

ii | Example: Static point mass $M_G \gg m_G$ as source in origin (e.g. Earth):

$$\phi(\vec{x}) = -\frac{GM_{\rm G}}{|\vec{x}|} \quad \Rightarrow \quad m_{\rm I}\ddot{\vec{r}} = \vec{F}_{\rm G} = -G\frac{m_{\rm G}M_{\rm G}}{r^2}\hat{r} \tag{8.6}$$

If the source is dynamic as well, \vec{r} is the relative distance vector between the two masses and $m_{\rm I}$ must be replaced by the reduced mass of the two bodies.



iii | Observation: Equations [especially Eq. (8.4)] are not Lorentz covariant!

You can check that they are \leftarrow *Galilei invariant*, recall Eq. (1.18).

This is no surprise: We already know from our discussions in Section 6.4 that in RELATIVITY, classical forces can only act *locally*, and not at a distance. Interactions between distant objects must be mediated by dynamical degrees of freedom (a "field") to obey the speed limit for information propagation imposed by Lorentz symmetry. But Newton's gravitational potential ϕ is static and not dynamic!

5 | <u>Problem:</u> "Action at a distance" (Gravitational force acts instantaneously and has no dynamics.)

Isaac Newton writes in a letter to Bentley in 1692 [97]:

It is inconceivable, that inanimate brute Matter should, without the Mediation of something else, which is not material, operate upon, and affect other Matter without mutual Contact, as it must be, if Gravitation in the Sense of Epicurus, be essential and inherent in it. And this is one Reason why I desired you should not ascribe innate Gravity to me. That Gravity should be innate, inherent and essential to Matter, so that one Body may act upon another at a Distance through a Vacuum, without the Mediation of any thing else, by and through which their Action and Force may be conveyed from one to another, is to me so great an Absurdity, that I believe no Man who has in philosophical Matters a competent Faculty of thinking, can ever fall into it. Gravity must be caused by an Agent acting constantly according to certain Laws; but whether this Agent be material or immaterial, I have left to the Confederation of my Readers.

Thus even Newton himself was not entirely satisfied with his law of universal gravitation (which describes an action at a distance) and anticipated some entity that mediates the force.

6 | First try: Make gravitational potential a *dynamic* field:

Poisson equation Eq. (8.4)
$$\rightarrow$$

Wave equation:
$$\partial^2 \phi(t, \vec{x}) = \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\phi = -4\pi G \rho(t, \vec{x})$$
 (8.7)

 \rightarrow Gravity propagates with the speed of light \bigcirc

$\stackrel{*}{\rightarrow}$ <u>Problems:</u>

For a detailed study of a fully specified scalar theory of gravity:
Problemset 1 (also Exercise 7.1 in MISNER *et al.* [3]). See also Ref. [98].

• Electromagnetic field cannot couple to gravity \rightarrow No bending of light \otimes

Today it is a well tested fact that light follows a curved trajectory in strong gravitational fields $(\Rightarrow later)$. Thus any theory that does not couple the EM field to gravity must be incorrect.

Here is a quick-and-dirty explanation why a theory of the form Eq. (8.7) fails to couple to the electromagnetic field in a relativistic setting:

Since ϕ is assumed to be a scalar field, for Eq. (8.7) to be Lorentz covariant, ρ must be a scalar as well. In a relativistic theory, energy and (inertial) mass are equivalent ($E_0 = mc^2$). If we assume that gravitational mass and inertial mass are equivalent (\Rightarrow *later*), this implies that energy (density) must be a source of gravity. The problem is that the energy (density) (of any theory) is the 00-component of the \leftarrow *energy-momentum tensor* T^{00} (this is the charge density associated to the Noether current that comes from translation symmetry in time); in particular, the energy density is *not* a scalar and therefore cannot be used as a source on the right-hand side of Eq. (8.7). The only scalar we can construct from the energy-momentum tensor is the * *Laue scalar* $T = T^{\mu}_{\mu} = \eta_{\mu\nu}T^{\mu\nu}$, i.e., the trace of the EMT. Thus a



simplistic but fully Lorentz covariant form of scalar gravity is

$$\partial^2 \phi = -\frac{4\pi G}{c^2} T \,. \tag{8.8}$$

(For a complete theory one also needs a Lorentz covariant analog of Eq. (8.5) which determines the motion of matter in dependence of the gravitational field ϕ . This equation is not relevant for the following argument.)

If the EM field couples to gravity, it must also be a source of gravity. This coupling is then described by the EMT of Maxwell theory Eq. (6.110) (in its symmetric, gauge-invariant form). The problem is that the trace of this particular EMT vanishes identically, $T_{\rm em} = (T_{\rm em})^{\mu}{}_{\mu} = 0$ (check this!), so that the scalar gravitational field and the EM field do not "feel" each other. In particular, there is no bending of light in the vicinity of massive bodies.

• Wrong value for perihelion precession (even with a wrong sign) 🙁

The \uparrow perihelion precession of Mercury deviates measurably from its Newtonian value (which is caused by perturbations by other planets). For Einstein, this anomaly served as a "litmus test" on his quest to generalize SPECIAL RELATIVITY, and his first application of GENERAL RELATIVITY was the successful explanation of Mercury's anomalous perihelion precession [14] (\rightarrow later). Thus any theory that does not predict the correct value for the perihelion precession cannot be correct.

Historically, this first approach [Eq. (8.7)] to patch up Newton's theory and make it consistent with SPECIAL RELATIVITY goes back to the Finnish physicist GUNNAR NORDSTRÖM. He quickly dismissed Eq. (8.7) because of fundamental problems (especially its linearity, \rightarrow below). He then proposed another (non-linear) scalar theory of gravity (\uparrow Nordström's theory of gravitation) which circumvented the most glaring issues but still failed to predict the bending of light (for the same fundamental reason sketched above) and produced the wrong value for the perihelion precession (even with a wrong sign!). Nonetheless, the theory merits consideration because it led EINSTEIN and ADRIAAN FOKKER to a groundbreaking realization [99]: Properly reformulated, the scalar field could be interpreted as a local "stretching" of the Minkowski metric. For the first time there was a clear formal link between a relativistic theory of gravity and a geometric deformation of spacetime, where the shape of the latter is determined by the distribution of mass and energy.

For a historical account of Nordström's gravity and its role in the genesis of GENERAL RELATIV-ITY see Refs. [100, 101].

7 | Second try: Make potential a *vector field*:

Since scalar gravity fails to match observations, a natural next step would be to consider a vector field and treat gravity analogous to Maxwell theory. This is also reasonable insofar as the gravitational potential of a point mass in Newton's theory and the Coulomb potential of a point charge in Maxwell's theory share the same form. For example, we can take Eq. (6.121) as a blueprint and propose an analogous Lagrangian for a vector gravitational field:

Eq. (6.121) $\rightarrow \triangleleft$ Vector field ϕ^{μ} & particle with trajectory $y^{\mu}(\lambda)$:

$$S_{G}[y,\phi] \stackrel{?}{=} \underbrace{\frac{+1}{16\pi G} \int d^{4}x \, G_{\mu\nu} G^{\mu\nu}}_{\text{Gravitational field}} \underbrace{-mc \int \sqrt{\dot{y}^{\mu} \dot{y}_{\mu}} d\lambda}_{\text{Relativistic particle}} \underbrace{-\frac{m}{c} \int \phi_{\mu} \dot{y}^{\mu} d\lambda}_{\text{with "charge" }m}$$
(8.9)

with $\dot{y}^{\mu} = \frac{\mathrm{d}y^{\mu}}{\mathrm{d}\lambda}$ and the "gravitational field strength tensor" $G_{\mu\nu} := \partial_{\mu}\phi_{\nu} - \partial_{\nu}\phi_{\mu}$.

• Note the sign difference compared to Eq. (6.121)! This ensures that equal charges (= masses) *attract* each other.



• The Lagrangian for the relativistic particle differs from the one given in Exercise 7.2 in MISNER *et al.* [3]; the two are equivalent and lead to the same equations of motion.

$\stackrel{*}{\rightarrow}$ <u>Results:</u>

For details see Exercise 7.2 in MISNER et al. [3]; see also Ref. [98].

- No bending of light 🙁
- Wrong perihelion precession 😳
- Gravitational waves have negative energy 🙁
- **8** | Third try: Make potential a *tensor field*:

At that point, desperation starts to kick in. But since scalar and vector fields failed miserably, we have no other choice: add another index and consider a tensor field. Interestingly, this makes it rather straightforward to write down a Lorentz-covariant modification of Eq. (8.8) [or Eq. (8.4)] where we no longer must butcher the EMT by taking a trace:

 \triangleleft Symmetric tensor field $\phi^{\mu\nu} = \phi^{\nu\mu}$:

$$\partial^2 \phi^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu} \tag{8.10}$$

The EMT on the right is the symmetric BRT of whatever matter occupies space (Section 6.3.2).

 $\stackrel{*}{\rightarrow}$ <u>Results</u>:

For details:
Problemset 1 (also Exercise 7.3 in MISNER et al. [3])

- Light is bent around gravitational potentials ©
- Gravitational waves have positive energy ©
- Describes perihelion precession not correctly ②
- Theory not self-consistent 🙁

Notes:

- Eq. (8.10) will describe the *linearized* version of the correct field equations of GENERAL RELATIVITY (the \rightarrow *Einstein field equations*) with $\phi^{\mu\nu}$ essentially the (small) deviation of the metric tensor from flat Minkowski space.
- That the linear tensor theory of gravity Eq. (8.10) is not self-consistent follows if one completes the theory with dynamic matter (which is the source of the gravitational field, but also influenced by the latter). Then one can show that this system of differential equations as no solution.
- As we will discuss below, the deficiency of this theory is its *linearity* (in the gravitational field); this is the root cause for its inconsistency and wrong predictions. And here comes a fascinating insight: One can show [102] that if one systematically fixes the inconsistencies of this theory, it becomes inevitably non-linear and one eventually ends up with the *correct* equations of GENERAL RELATIVITY (which we will find much later via a different route)!
- **9** | So far, all our tentative theories of relativistic gravity failed (none of them describe observations correctly and they even suffer from intrinsic inconsistencies). There is a simple argument why this must be so, and why the correct theory *must* be more complicated:
 - i | The source (= charge) of gravity is, by definition, the gravitational mass $m_{\rm G}$.

This is a physically vacuous statement.



- ii | A relativistic theory of gravity must be a field theory with a dynamical field.This is necessary so that gravity does not propagate faster than the speed of light.
- Since the field is dynamical, it has a non-vanishing energy density.
 Recall that energy is the Noether charge of time translations and therefore generates the time evolution (think of the Hamiltonian in quantum mechanics).
- iv | As a relativistic theory it must obey the mass-energy equivalence: $E_0 = m_I c^2$. We write m_I to emphasize that SPECIAL RELATIVITY only knows about *inertial* mass.
- Experiments tell us that inertial and gravitational mass are the same: $m_{\rm I} = m_{\rm G}$. We will discuss this, and the closely related → *equivalence principle*, in detail below.
- vi | Thus a gravitational field has a non-vanishing density of gravitational mass $m_{\rm G}$.

Please appreciate how strange this is! If an analog statement were true for Maxwell theory (which it is not), electromagnetic waves would be *electrically charged*, and other electromagnetic waves could scatter off them!

vii | Excitations of the gravitational field are sources of the gravitational field.

This means that a relativistic theory of gravity must allow for self-interaction. In particular, it *cannot* feature a \checkmark *superposition principle* and the field equations must be *non-linear*.

 \rightarrow

The field theory of gravity must be non-linear and allow for self-interactions.

- All of the above theories are linear in the gravitational field; hence they are bound to fail!
- This argument also clarifies the fundamental difference between relativistic theories of *gravity* and *electrodynamics* (both of which are classical field theories that mediate forces): The EM field is also dynamical and carries energy, hence, via the mass-energy equivalence and the equivalence of inertial and heavy mass, it is a source of *gravity*. But the mass/energy carried by the EM field is not the source of the *EM field* (electrical charge is). Thus Maxwell theory does not close the "vicious circle" from above and can be both *relativistic* and *linear*.
- If you want an even more boiled down version:

Gravity is special in RELATIVITY, because SPECIAL RELATIVITY has something to say about (inertial) mass ($E_0 = m_1 c^2$) and the latter is – via the \rightarrow equivalence principle – the source of gravity.


↓Lecture17 [16.04.24]

10 | Yet another problem:

Besides the formal complications encountered above, there are also less formal yet fundamental lines of reasoning that suggest that the phenomenon of gravitation and the premises of SPECIAL RELATIVITY are incompatible:

- i | Experimental facts:
 - Gravity cannot be shielded. Contrary to all other forces (which have negative charges), there is no negative mass.
 - Gravity is typically *inhomogeneous*.
 - In a gravitationally homogeneous universe there are no planets and we wouldn't exist.
 - In free fall, gravity is exactly countered by the inertial force.

We will discuss this in more detail later (\rightarrow *equivalence principle*).

ii | For the machinery of SPECIAL RELATIVITY to work, we need inertial frames. Can we find inertial frames in the presence of gravity?

Thought experiment:

 $a \mid \triangleleft$ Laboratory on the surface of Earth:



 \rightarrow Not an inertial system \odot

The problem is that we cannot simply shroud our lab by some magic material that shields the gravitational force. By contrast, this *can* be done for the electromagnetic field (\checkmark *Faraday cage*, \uparrow *Mu-metal*). Note that this is not a technical problem, it is a fundamental one!

b $| \triangleleft$ Interior of orbital space station:



 \rightarrow Approximate inertial system \bigcirc

The space station is equivalent to a free falling laboratory, where the gravitational force is canceled exactly by inertia. What makes a space station so convenient is that it also has orbital velocity so that it "falls around Earth" and therefore can be used much longer than a free falling lab that eventually crashes on the surface.



This is a situation where it actually makes sense to use the terms "gravity" and "gravitation" differently: In the space station, there is no *gravity* (the astronauts float), but there is a *gravitational field*! (The latter is just canceled by the inertial force due to free fall.) This situation is different from being in a spaceship far away from Earth in interstellar space where no gravitational forces can be measured. (Although these two situations cannot be distinguished from within a small space station/spaceship, \rightarrow *later*.)

 $c \mid \triangleleft$ Very large orbital space station:



↑ *Tidal forces* \rightarrow Not an inertial system \bigcirc

When we extend the size of the space station, the inhomogeneity of the gravitational acceleration becomes noticeable and the "inertial test" **IN** fails. Inhomogeneous gravitational fields therefore constrain the size (both in space and time) of reference systems that satisfy the properties of an inertial system. Hence our assumption that inertial systems cover all of spacetime (and therefore can describe arbitrary physical phenomena, not just local ones) is invalidated by the presence of inhomogeneous gravitational fields.

Note that there is another way to detect the inhomogeneous gravitational field and make the system non-inertial: Stay in the small spacecraft and wait longer. At some point you will notice that the two balls drift apart—even when they are only centimeters apart. This shows that the approximate inertial system really must be small in space *and time*.

 $d \mid \triangleleft$ Two small orbital space stations:



 \rightarrow Local inertial systems accelerated wrt. each other \otimes

If you imagine that these small inertial systems overlap on their boundaries, you could ask how to transform the coordinates of an event in this overlap from one of these systems into the other. Because these systems are accelerated wrt. each other, this transformation cannot be linear, in particular it cannot be a Lorentz transformation! There seems to be something missing; what determines this transformation?



iii | This thought experiment leads us to the following (troubling) conclusion:

Inertial systems can only exist *locally* in an inhomogeneous gravitational field. How to transform between these local inertial systems is unclear (\rightarrow *later*).

→ Extended phenomena cannot be described by SPECIAL RELATIVITY! "Extended" here means "on the scale of gravitational inhomogeneities."

11 | In a nutshell:

SPECIAL RELATIVITY cannot ...

- ... describe the gravitational field itself.
- ... describe physics in inhomogeneous gravitational fields.

How to fix this? \rightarrow GENERAL RELATIVITY

8.3. ‡ The gravitational redshift and curved spacetime

Quite surprisingly, one can derive some predictions of GENERAL RELATIVITY without knowledge of the detailed theory. One is the \rightarrow gravitational redshift of light, which, again without the usage of heavy math, implies that GENERAL RELATIVITY must describe a *curved* spacetime.

The following is based on Sections 7.2 and 7.3 of MISNER et al. [3] and Section 2.1 of CARROLL [4].

1 Gravitational redshift:

EINSTEIN already concluded in 1908 that light leaving a gravitational potential must be redshifted [96]. The following he showed in 1911 [103], that is, years before he finalized GENERAL RELATIVITY.

 $i \mid \triangleleft$ Particle of rest mass *m* in (homogeneous) gravitational potential:

In the following, we assume that *inertial* and *gravitational* mass are equal: $m_{\rm I} = m = m_{\rm G}$.





$$E_{\rm tot} = m_{\rm I}c^2 + m_{\rm G}gh = m(c^2 + gh)$$
(8.11)

iii | Step 2: Assume electron annihilates into a photon with energy:

$$E_{\downarrow} = E_{\text{tot}} \tag{8.12}$$

- iv | Step 3: Let the photon propagate upwards by $h \rightarrow \text{New photon energy } E_{\uparrow}$:
 - Possibility 1: Photon is *not* affected by gravity → E_↑ = E_↓ > mc² ×
 This immediately leads to a violation of energy conservation because the photon can now be used to recreate the particle *plus some kinetic energy* that wasn't there before.
 - Possibility 2: Photon is *redshifted* by gravitational field such that $E_{\uparrow} = mc^2 \checkmark$ This is the only possibility consistent with energy conservation, i.e., the photon must loose energy just as a particle would when climbing the potential.
- **v** | Thus we find for the photon energies:

$$E_{\downarrow} = \frac{E_{\uparrow}}{c^2} \left(c^2 + gh \right) = E_{\uparrow} \left(1 + \frac{gh}{c^2} \right)$$
(8.13)

vi $| < ** Redshift parameter <math>z := \Delta \lambda / \lambda = (\lambda_{\uparrow} - \lambda_{\downarrow}) / \lambda_{\downarrow}$ The redshift z measures the relative change in wavelength $\Delta \lambda$ wrt. a reference wavelength λ .

 \rightarrow Gravitational redshift: (Use the photon energy $E = hv = hc/\lambda$.)

$$1 + z = \frac{\lambda_{\uparrow}}{\lambda_{\downarrow}} = \frac{E_{\downarrow}}{E_{\uparrow}} = 1 + \frac{gh}{c^2}$$
(8.14)

Using the \uparrow *Mößbauer effect*, ROBERT POUND and GLEN REBKA verified this prediction in 1960 with their famous \uparrow *Pound-Rebka experiment* [104, 105].

2 | Schild's argument:

The following reasoning goes back to ALFRED SCHILD [106] (see also references in [3]) and demonstrates that a relativistic theory of gravity cannot be formulated on a flat Minkowski space-time:

- i | Assumptions:
 - There exists an extended inertial frame *K* attached to Earth's center. (We relax our definition and allow particles to be accelerated near earth.)
 - In this frame, proper time and lengths are given by the Minkowski metric.
 - There is some gravitational field (of unspecified nature) that matches observations. (This implies the gravitational redshift derived above.)
- ii | Thought experiment:
 - **a** $| \triangleleft$ Two observers $O_{\downarrow,\uparrow}$ at height $z_{\downarrow,\uparrow}$ with $z_{\uparrow} = z_{\downarrow} + h$ at rest in K

oreti



b | Observer O_{\downarrow} emits a light signal with wavelength λ_{\downarrow}

 \rightarrow Time for one wavelength: $\Delta t_{\downarrow} = \lambda_{\downarrow}/c$

Note that since both observers are at rest in K, their proper times and the coordinate time of K coincide.

 ${\bf c} \mid \ {\rm Observer} \ O_{\uparrow}$ receives the signal with wavelength λ_{\uparrow}

 \rightarrow Time for one wavelength: $\Delta t_{\uparrow} = \lambda_{\uparrow}/c$

- $\mathbf{d} \mid \text{ Redshift} \rightarrow \lambda_{\uparrow} > \lambda_{\downarrow} \rightarrow \Delta t_{\uparrow} > \Delta t_{\downarrow}$
- **e** | But in the Minkowski diagram of the (imagined) global inertial frame, the experiment looks as follows:



$\rightarrow \Delta t_{\uparrow} = \Delta t_{\downarrow} 4$

Note that the only important aspect for the contradiction is that the two world lines of the start and end of one wavelength are *congruent* in Minkowski space. That is, we do not need to know how gravity affects the trajectory of light (maybe it is bent). The only important thing is that both trajectories are bent in the same way, which is to be expected in a static scenario where the gravitational field does not change.

iii | Conclusion:

In the presence of gravity, the trajectories of light signals in spacetime must be congruent (if they are straight: parallel)—but at the same time their distance in time direction must change! This is impossible in the flat (pseudo-)Euclidean geometry of Minkowski space; but it *is* possible in a curved spacetime. As we will see \rightarrow *later*, the tendency of initially parallel "straight lines" (\rightarrow *geodesics*) to approach or recede from another is exactly what characterizes a curved space(time):





 \rightarrow Spacetime must be curved!

3 | We can summarize:

A Lorentz covariant theory of gravity cannot be formulated on Minkowski space.

This already suggests that we will need the more general machinery of differential geometry, introduced in Chapter 3, to model spacetime not as flat Minkowski space, but as a more general, curved pseudo-Riemannian manifold.



9. Conceptual Foundations

9.1. Einstein's equivalence principle

The wording of the equivalence priciples are paraphrased from CARROLL [4].

1 | <u>Remember:</u>

There are two concepts of mass in Newtonian physics:

Inertial mass
$$m_{I}$$
: $\vec{F} = m_{I}\vec{a}$ (9.1a)

Gravitational mass
$$m_{\rm G}$$
: $\vec{F} = -m_{\rm G} \nabla \phi$ (9.1b)

Strictly speaking, two gravitational masses must be conceptually distinguished: The *passive* gravitational mass is the charge that couples to the gravitational field via Eq. (9.1b). The *active* gravitational mass is the source of the gravitational potential $\phi = -GM_G/r$. However, given that Newton's third law is valid (action equals reaction), the situation is completely symmetric and these two masses can be identified. Thus, in the following we only distinguish between inertial and gravitational mass.

 \rightarrow Gravitational acceleration:

$$a_{\rm G} = \frac{m_{\rm G}}{m_{\rm I}} \underbrace{\frac{GM_{\oplus}}{r^2}}_{g} \tag{9.2}$$

It has been long known (since Galileo Galilei) that the gravitational acceleration is *independent* of the material of the body (if one can ignore air resistance): *All bodies fall at the same rate*.

Experience:

$$\frac{m_{\rm G}}{m_{\rm I}} = {\rm const} \rightarrow {\rm Choose units appropriately:} \frac{m_{\rm G}}{m_{\rm I}} = 1$$
 (9.3)

In classical mechanics, this is just an observation; it is neither explained nor necessary for its consistency.

2 | The Eötvös experiment [107]: (See also the later publication [108].)

While there have been earlier experiments that quantitatively tested the equivalence of inertial and gravitational mass, the experiment by Hungarian physicist ROLAND EÖTVÖS made huge improvements in precision. The experiment was used by Einstein as an argument for his \rightarrow equivalence principle.





 \bigcirc Problemset 1 $\xrightarrow{\circ}$

Torque:
$$\tau \approx \underbrace{m_G a_x l}_{\neq 0} \left(\frac{m_I}{m_G} - \frac{m'_I}{m'_G} \right)$$
 (9.4a)

$$\tau = 0 \quad \Leftrightarrow \quad \underbrace{\frac{m_{\rm I}}{m_{\rm G}}}_{\text{Material A}} = \underbrace{\frac{m'_{\rm I}}{m'_{\rm G}}}_{\text{Material B}} = \text{const}$$
 (9.4b)

 \rightarrow Result by Eötvös [107]:

$$\frac{\delta m}{m} = \frac{m_{\rm I} - m_{\rm G}}{m_{\rm I}} < 3 \times 10^{-9} \tag{9.5}$$

The latest (2022) and most precise results testing the equivalence of inertial and gravitational mass come frome the satellite-based MICROSCOPE experiment [109]; it improved the upper bound for a violation of the equivalence to $\delta m/m < 10^{-15}$. Recent experiments also demonstrated the equivalence for antimatter [110].

 \rightarrow Experimental fact:



(9.6)

This trivial sounding assertion (when have you ever distinguished between these two masses?) has profound consequences: Recall that SPECIAL RELATIVITY is concerned with the concept of *inertia* (e.g. by using inertial systems); in particular, the (rest) mass that shows up in the mass-energy equivalence $E_0 = mc^2$ is the inertial mass m_I of the system. The equivalence above now links this mass to the gravitational mass, and therefore asserts the the inert bodies of SPECIAL RELATIVITY must be affected by (and be sources of) gravity. But SPECIAL RELATIVITY had nothing to say about gravity! Quite to the contrary: as discussed in Section 8.2, the theory cannot accomodate gravity in a consistent way.

3 | Classical mechanics does not explain Eq. (9.6). However, if we take Eq. (9.6) for granted, a homogeneous gravitational field vanishes in an accelerated frame:

 $\triangleleft N$ particles of (inertial = gravitational) mass m_k in gravitational field:

$$\underbrace{m_k \frac{\mathrm{d}^2 \vec{x}_k}{\mathrm{d}t^2}}_{\text{Inertia}} = \underbrace{m_k \vec{g}}_{\text{Gravity}} + \sum_{l \neq k} \vec{F}_{kl} (\vec{x}_k - \vec{x}_l) \quad \text{with } k = 1, \dots, N.$$
(9.7)



< Coordinate transformation into free-falling frame:

$$t' = t$$
 and $\vec{x}'_k = \vec{x}_i - \frac{1}{2}\vec{g}t^2$. (9.8)

This coordinate transformation is *non-linear*, in particular, it is not a Galilei transformation!

 $\stackrel{\circ}{\rightarrow}$ Equations of motion in the free-falling coordinate system:

$$m_k \frac{\mathrm{d}^2 \vec{x}'_k}{\mathrm{d}{t'}^2} = \sum_{l \neq k} \vec{F}_{kl} (\vec{x}'_k - \vec{x}'_l) \quad \text{with } k = 1, \dots, N.$$
(9.9)

 \rightarrow No graviy in the free-falling frame!

This matches our experience and can be illustrated with the following though experiment:



4 Caveat: Only true for *homogeneous* gravitational fields: $m_k \vec{g}$.

What about inhomogeneous gravitational fields?

Note that the inhomogeneity of gravity is essential for planets and stars to form; it is the root cause for complexity in the world that is neccessary for life to exist. This is not a slight inconvenience we can sweep under the rug!

 \rightarrow Small enough regions look homogeneous:



 \rightarrow Gravity can be compensated *locally* in an accelerated frame.

i! This implies that there is no transformation to a *global* free-falling reference frame in which inhomogeneous gravitational fields vanish. Thus acceleration and gravity are only equivalent *locally*; globally, they are physically distinct. In particular, this means that the phenomenon of gravity is not just "acceleration in disguise." As mentioned previously, accelerated coordinate systems (and bodies) are something that SPECIAL RELATIVITY can handle. If gravity and acceleration were equivalent globally, SPECIAL RELATIVITY would be sufficient to describe gravity and there was no need for GENERAL RELATIVITY.



5 | Given all these facts, it is reasonable to proclaim the following principle:

§ Postulate 6: Weak equivalence principle WEP (Universality of free fall)

In small enough regions of spacetime, the motion of *freely-falling particles in a gravitational field* and *free particles in a uniformly accelerated frame* are the same.

(This formulation formalizes the idea sketched in the upper panel of the sketch above.)

Equivalently:

For every event, there is a local reference frame, covering small enough regions of spacetime in its vicinity, such that gravity has no effect on the motion of arbitrary particles in this frame and the law of inertia holds.

(This formulation formalizes the idea sketched in the lower panel of the sketch above.)

6 | Einstein's generalization:

The local equivalence of gravity and accelerated frames is true for *all* physical phenomena (and not only classical mechanics).

Einstein was aware of the Eötvös experiment and was convinced that the equivalence of inertial and gravitational mass hinted at a deep relationship between inertia (acceleration) and gravitation. He wrote in 1907 [96] (highlights are mine):

Bisher haben wir das Prinzip der Relativität, [..], nur auf beschleunigungsfreie Bezugssysteme angewendet. Ist es denkbar, daß das Prinzip der Relativität auch für Systeme gilt, welche relativ zueinander beschleunigt sind? [..]

Wir betrachten zwei Bewegungssysteme Σ_1 und Σ_2 . Σ_1 sei in Richtung seiner X-Achse beschleunigt, und es sei γ die (zeitlich konstante) Größe dieser Beschleunigung. Σ_2 sei ruhend; es befinde sich aber in einem homogenen Gravitationsfelde, das allen Gegenständen die Beschleunigung $-\gamma$ in Richtung der X-Achse erteilt.

Soweit wir wissen, unterscheiden sich die physikalischen Gesetze in bezug auf Σ_1 nicht von denjenigen in bezug auf Σ_2 ; es liegt dies daran, daß alle Körper im Gravitationsfelde gleich beschleunigt werden. Wir haben daher bei dem gegenwärtigen Stande unserer Erfahrung keinen Anlaß zu der Annahme, daß sich die Systeme Σ_1 und Σ_2 in irgendeiner Beziehung voneinander unterscheiden, und wollen daher im folgenden die völlige physikalische Gleichwertigkeit von Gravitationsfeld und entsprechender Beschleunigung des Bezugssystems annehmen.



Pictorially, Einstein claims that *any type* of local experiment cannot distinguish gravity from acceleration (here, for example, some quantum mechanical scattering process):



7 | We can formalize this as follows:

§ Postulate 7: Einstein's equivalence principle EEP

In small enough regions of spacetime, the laws of physics reduce to those of SPECIAL RELATIVITY: It is impossible to detect the existence of a gravitational field by means of local (non-gravitational) experiments.

Equivalently:

For every event, there is a local reference frame, covering small enough regions of spacetime in its vicinity, such that gravity has no effect *on any (non-gravitational) experiment* in this frame and the law of inertia holds.

- The **EEP** implies the **WEP**.
- Excluding non-gravitational experiments means that the intrinsic gravitational energy of our experiment does not contribute significantly to its mass (see **SEP** below). Note that we do not require that gravitational experiments (using large masses) *can* locally distinguish between gravity and acceleration; the **WEP** does simply not constraint such experiments.
- It is important to appreciate the profound implications of this principle for doing physics in a gravitational field: It asserts that as long as your laboratory is small (compared to the inhomogeneties of the gravitational field) and free-falling (e.g. a space station in orbit), SPECIAL RELATIVITY is sufficient to describe *all experiments* that you can conduct in this lab. In particular, the fact that SPECIAL RELATIVITY cannot describe gravity is not important because in the free-falling lab there is none. This means that everything we discussed last term remains valid—and therefore useful—*locally*. Thus gravity does not completely invalidate SPECIAL RELATIVITY, it only restricts its domain of validity to local, free-falling inertial frames. I hope you are happy to hear that!
- More precisely, for every event (point in spacetime) there is an *equivalence class* of local inertial frames (related by boosts), equiped with inertial coordinate systems (related by translations and rotations), in all of which SPECIAL RELATIVITY holds good. The coordinate transformations between these systems are given by Lorentz transformations. (You can check



the existence of such frames in our Newtonian calculation above by adding a term $\vec{v}t$ to the transformation of the position coordinates.)

- In our mathematical framework of differential geometry, the equivalence class of local inertial systems at a spacetime point will be identified with the *← tangent space* of the spacetime manifold at that point.
- 8 | Concerning gravitational laws of physics:

In our definition of the **EEP**, we excluded experiments that depend on the gravitational interaction itself (e.g., use objects with considerable intrinsic gravitational energy). This exclusion follows SCHRÖDER [2], whereas other authors like CARROLL [4] include the (unkown) gravitational laws of physics in the **EEP**.

For us, it makes then sense to define an extension of the **EEP** as follows:

§ Postulate 8: Strong equivalence principle SEP

The **EEP** is valid for all laws of physics, including the gravitational laws.

- GENERAL RELATIVITY satisfies the SEP (and thereby the EEP and the WEP).
- The reason to separate the **EEP** from the **SEP** is that alternatives to GENERAL RELATIVITY can *satisfy* the **EEP** (and the **WEP**) but *violate* the **SEP**. This alternatives can be metric theories like GENERAL RELATIVITY with additional fields; see Ref. [111] for details.
- In particular, the **SEP** requires that the universality of free fall (**WEP**) also holds for large bodies like planets (not just small test particles) with significant amounts of gravitational self-energy. More precisely, the **SEP** demands that the rest mass $E_{\rm grav}/c^2$ that comes from the gravitational self-energy $E_{\rm grav}$ accelerates just like any other rest mass in an external gravitational field.

Note that the validity of the **SEP** cannot be deduced from typical experiments that test the **WEP** because these experiments use small test masses with a gravitational self-energy that is way too small to detect any violation of the **SEP** (because gravity is such a weak force). One needs to use planet-sized objects to draw conclusions about the **SEP** (\rightarrow *next*).

- Using reflectors on the moon (left by Apollo 11 in 1969), lunar laser ranging (LLR) can be used to experimentally test the SEP since the fractions of gravitational self-energy of moon and earth are different enough to modify moon's orbit measurably if the SEP was violated. To date there is no evidence of such a violation to high precision [112, 113], hence we will assume that the SEP holds.
- **9** | If SPECIAL RELATIVITY explains *everything* you can do in a local, free-falling laboratory, at which point does gravity enter the picture? Well, the hitch is that not all physical processes *can* be restricted to a single, local inertial frame:





< Meteroid traversing the inhomogeneous gravitational field of earth:

How to model the trajectory?

Imagine you start in a local inertial system where you know the initial data (position, velocity) of the meteroid. Since SPECIAL RELATIVITY is valid in this small patch, you can use the known equations of relativistic mechanics to compute the trajectory of the meteroid. However, at some point, the meteroid will leave the local inertial system and enter another one. To proceed with your application of relativistic mechanics, you need to know the coordinate transformation that maps the coordinates of the final position and velocity in the first inertial system to the coordinates of the next.

But a priori these inertial system are unrelated, in particular, they can be accelerated with respect to one another (recall the two small space stations in Section 8.2). To proceed with your application of relativistic mechanics, you need this coordinate transformation! And this is where gravity hides: The gravitational field (here generated by earth) and the pattern of local coordinate transformations are one and the same thing! This is what is meant by *gravity becoming a geometric property of spacetime*.

The gravitational field is the (dynamical) structure that determines which local frames of reference are inertial or, equivalently, how to transform from one local inertial frame to the next.

↓Lecture 18 [23.04.24]



9.2. General relativity and covariance, background independence

The equivalence principle **EEP** is the foundation of GENERAL RELATIVITY; it motivates both the metrization of gravity (Section 9.4 and Chapter 12) and the minimal coupling of matter to gravity (Chapter 11). However, there are additional principles that are conceptually important to understand and were historically important for the genesis of GENERAL RELATIVITY as well:

10 | <u>Motivation:</u>

• No global inertial systems anymore \rightarrow More general coordinate charts needed!

As we have seen, the description of gravity forces us to give up the restriction to formulate physical models within the distinguished family of infinitely extended inertial coordinate systems. Hence we must formulate our physical theories in a way that is valid for arbitrary coordinate charts, and allows for arbitrary coordinate transformations between them.

Einstein was not satisfied with the distinguished role of inertial systems in SPECIAL RELA-TIVITY. After all, RELATIVITY is all about the *relativity of states of motion*, i.e., only motion of systems with respect to one another are of physical significance – and no class of states of motion should be distinguished. SPECIAL RELATIVITY clearly does not live up to this rigorous form of relativity as it singles out inertial frames as special. Einstein's ultimate goal was to make all states of motion (including accelerated motion) "equivalent." GENERAL RELATIVITY *does not achieve this goal!* Even in GENERAL RELATIVITY, inertial motion is physically distinct from accelerated motion; the new thing is that mass and energy determine which states of motion *are* inertial.

We are therefore in the strange (and confusing) situation, that Einstein's original motivation to seek out equations that have "the same form" in all coordinate systems does not achieve its goal, but nevertheless is the correct way forward (see \rightarrow *next* point). We will also see that "having the same form" means something different in SPECIAL RELATIVITY than in GENERAL RELATIVITY because the former is formulated on a fixed background (Minkowski space) and the latter not (\rightarrow *background independence*) – and this changes what it means for two equations to have "the same form." The situation is quite convoluted and we can disentangle it not until the end of this course.

• Chapter 3: Physics describes relations of geometric entities ("Coordinates don't exist.")

 \rightarrow Coordinates should play no role in the formulation of physical models!

Recall our motivation in Chapter 3 for the introduction of tensor fields: We realized that coordinates are mathematical artifacts that we use to label events in spacetime. The essence of physical laws should clearly be independent of the labeling scheme we choose. Thus we should strive for a formulation of physical models (which, hopefully, capture physical laws) that is independent of coordinates, or at least makes it manifest that physical predictions do not depend on the choice of coordinates.

Note that this argument is very different from Einstein's hope to extend the principle of special relativity **SR** by "equalizing" more states of motion. The argument is way more fundamental, has nothing to say about states of motion, and, in some sense, is almost tautological. It's only physical content is the rather uncontroversial statement that "coordinates do not exist as physical entities."



11 | This motivates the following definition:

* Definition 1: General covariance GC (Coordinates don't exist)

An equation is said to be *** generally covariant* if it is forminvariant under arbitrary (differentiable) coordinate transformations.

\rightarrow Tensor equations are automatically generally covariant.

Generally covariant equations have an alternative *** coordinate-free* formulation in terms of geometric objects on a manifold (** differential forms*):

Examples:

• \triangleleft Two vector fields $A = A^{\mu}\partial_{\mu}$ and $B = B^{\mu}\partial_{\mu}$:

$$\left\{\begin{array}{c} \phi_{A} = \phi_{B} \\ \vec{A} = \vec{B} \\ \vec{A} = \vec{B} \\ \vec{B}^{i} = \dots \\ \vec{B}^{i} = \dots \\ \vec{B}^{i} = \dots \\ \vec{B}^{i} = \dots \end{array}\right\} \Leftrightarrow \underbrace{A^{\mu} = B^{\mu}}_{Manifestly} \Leftrightarrow \underbrace{A = B}_{Coordinate}$$
(9.10)
$$\underbrace{A^{\mu} = B^{\mu}}_{Covariant} \Leftrightarrow \underbrace{A = B}_{free}$$

- While mathematicians often prefer the coordinate-free notation, in physics, the coordinate-dependent, manifestly covariant notation is more widespread. This has to do with how physics is done: While coordinates do not exist *a priori*, physicists typically make them exist in their labs because measurements always use some form of reference system. The generally covariant equations are more useful in that regard because they can be specialized to any coordinate system most convenient for an experiment.
 - \rightarrow In this course we will only use the manifestly covariant notation.
- To decided whether an equation remains form invariant under arbitrary coordinate transformations, you must first know how the elementary fields of the equation transform. This is why the *non*-manifest notation is so cumbersome: In addition to the equation(s), you must figure out (or specify) how the different fields transform. (Recall the non-tensorial form of the Maxwell equations and how cumbersome it was to check their Lorentz covariance [see Eq. (6.34)]!)

This makes the benefit of the *manifest* notation clear: First, by convention, the tensor notation A^{μ} implies that the transformation of the field is $\bar{A}^{\mu} = \frac{\partial \bar{x}^{\mu}}{\partial x^{\nu}} A^{\nu}$, and second, because of the rules of tensor calculus, checking the general covariance of a (valid) tensor equation is trivial.

• < Inhomogeneous Maxwell equations on arbitrary spacetime (Section 11.3):

$$\underbrace{F^{\mu\nu}_{;\nu} = -\frac{4\pi}{c}J^{\mu}}_{\text{Manifestly covariant}} \Leftrightarrow \underbrace{d(\star F) = \star J}_{\text{Coordinate-free}}$$
(9.11)

Remember that ; v denotes the \leftarrow *covariant derivative* Eq. (3.79) which implicitly depends on the metric of spacetime. In the coordinate-free notation, the metric is hidden in the definition of the \uparrow *Hodge star operator* \star .



12 We can now use this definition to formulate our physical insight that the equations that describe physical laws must not single out specific coordinate systems:

In Einstein's words [21]:

Die Gesetze der Physik müssen so beschaffen sein, daß sie in bezug auf beliebig bewegte Bezugssysteme gelten. (p. 772)

Die allgemeinen Naturgesetze sind durch Gleichungen auszudrücken, die für alle Koordinatensysteme gelten, d.h. die beliebigen Substitutionen gegenüber kovariant (allgemein kovariant) sind. (p. 776)

 \rightarrow

§ Principle 1: General relativity principle GRP

Models of laws of nature must take the same form in *all* coordinate systems; i.e., they must be expressed in terms of generally covariant equations.

Here the "must take the form" means that it must be *possible* to formulate them in a coordinateindependent way; if this were not the case, the theory (and its prediction) would implicitly depend on (and single out) a specific coordinate system. Note that there is nothing wrong in formulating such a theory in a way that is not *generally covariant*.

For example, Maxwell equations in their conventional (non-tensorial) form are not generally covariant, they are only Lorentz covariant. This is not a problem, though, because these equations are just a specialization of Eq. (9.11) to a particular class of coordinate systems (namely inertial systems). If you (naïvely) apply these specialized equations in a non-inertial frame (such as a laboratory on the surface of earth!), you can get incorrect results (\bigcirc Problemset 3 and Ref. [114, 115]).

13 | What is the physical content of **GRP**?

GRP, while being important for the formulation of physical models in general and being strictly satisfied in GENERAL RELATIVITY, is neither specific nor fundamental to and for GENERAL RELATIVITY. For example, the Maxwell equations in the manifestly covariant form of Eq. (9.11) satisfy the **GRP** on the fixed background of Minkowski space and have nothing to do with GENERAL RELATIVITY.

 \rightarrow

The principle of general relativity GRP has (almost) no physical content.

The "almost" referes to the fact that the principle asserts that there are no distinguished coordinate systems that exist as physically independent structures.

• The relativity postulate **GRP**, and its mathematical manifestation as general covariance **GC** have been criticized already in 1917 by KRETSCHMANN [116]:

[Man] vergegenwärtigt sich, daß alle physikalischen Beobachtungen letzten Endes in der Feststellung rein topologischer Beziehungen ("Koinzidenzen") zwischen räumlichzeitlichen Wahrnehmungsgegenständen besteht und daher durch sie unmittelbar kein Koordinatensystem vor irgend einem anderen bevorrechtigt ist, so wird man zu dem Schlusse gezwungen, daß jede physikalische Theorie ohne Änderung ihres-beliebigendurch Beobachtungen prüfbaren Inhaltes mittels einer rein mathematischen und mit



höchstens mathematischen Schwierigkeiten verbundenen Umformung der sie darstellenden Gleichungen mit jedem beliebigen-auch dem allgemeinsten-Relativitätspostulate in Einklang gebracht werden kann.

• To drive the point home: One can also formulate good old non-relativistic Newtonian mechanics in a generally covariant form (it's quite ugly, though)! See the original literature [117] and MISNER *et al.* [3] (Box 12.4 and §12.5):

Any physical theory originally written in a special coordinate system can be recast in geometric, coordinate-free language. Newtonian theory is a good example [..]. Hence, as a sieve for separating viable theories from nonviable theories, the principle of general covariance is useless.

- For a detailed account on the role general covariance plays in GENERAL RELATIVITY (and historically played in its inception), see Ref. [118].
- 14 We can summarize the relation of SR, EEP, and GRP as follows:



- Both **SR** and **EEP** make claims about the equivalence (indistinguishability) of certain states of motion. These are physical claims about reality that can be assessed by experiments. Note that to check whether they are false or true you do not even know how to work with mathematical equations. It's a simple matter of collecting the results of experiments (recall the \leftarrow *Michelson-Morley experiment*). It is this physical content that makes **SR** and **EEP** the foundations of SPECIAL RELATIVITY and GENERAL RELATIVITY, respectively.
- By contrast, **GRP** makes no such claims about reality. **GRP** does *not* claim that all states of motion are indistinguishable (they are not, even in GENERAL RELATIVITY you can tell local inertial frames and accelerated frames apart); the principle only claims that all fundamental theories of physics should have a formulation that can be applied by all possible observers. **GRP** is therefore more a statement about physical models than about reality.
- The sketch makes it clear that the **EEP** is actually more similar to the **SR** (in the role it plays for GENERAL RELATIVITY) than the **GRP** is. In that sense "principle of general relativity" is kind of a misnomer.
- **15** There is another important concept that (contrary to **GC** / **GRP**) distinguishes GENERAL RELA-TIVITY from other theories and must itself be distinguished from **GC** / **GRP**:



* Definition 2: Background independence **BI** (No prior geometry)

Physical models that do <u>not</u> contain the geometry of spacetime as an absolute element are called *** background independent*. This implies that the geometry of spacetime emerges dynamically as *solutions* of the theory.

(Counter-)examples:

• ✓ GENERAL RELATIVITY is (and historically was the first example of) a background independent theory (→ *below*):

$$S_{\text{Einstein-Hilbert}}[g] = \frac{c^3}{16\pi G} \int d^4x \,\sqrt{g}R \tag{9.12}$$

Here R is the $\rightarrow Ricci \, scalar$ that depends in a complicated way on the metric tensor field $g_{\mu\nu}(x)$. \sqrt{g} is short for $\sqrt{|\det[g_{\mu\nu}(x)]|}$ (Minkowski metric: $\sqrt{\eta} = 1$).

• X Maxwell theory is *not* background independent (recall Eq. (6.56)):

$$S_{\text{Maxwell}}[A] = \int d^4x \sqrt{g} \left(-\frac{1}{16\pi} g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} F^{\mu\nu} \right)$$
(9.13)

with field-strength tensor $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$.

- Note that you do *not* extremize this action wrt. the metric g; the metric (e.g. Minkowski metric $g = \eta$) is given as a *parameter* (absolute element) of the theory. Hence it plays the role of a static background.
- Note also that the Maxwell equations (and their Lagrangian) *are* generally covariant: they are tensorial expressions that descibe geometric objects on a manifold. That does not prevent them to have a tensor field (the metric) as an absolute element.

16 Beware:

GENERAL RELATIVITY is most likely the first (and possibly last) generally covariant and backgroundindependent theory that you will encounter in your university courses. Thus it is important to mention a peculiarity that, if ignored, can lead to lots of confusion when studying such theories:

 $i \mid \triangleleft$ Trajectory of particle in spacetime:



Because of general covariance (GC) there is always a coordinate system in which the (spatial) coordinates of an object are constant in time.

 \rightarrow One cannot infer from coordinates whether an object is moving!



(There is no absolute notion of "motion" in GENERAL RELATIVITY; the only motion that makes sense is motion wrt. some reference object, \rightarrow *next step*.)

ii | \triangleleft Two objects at rest in some coordinate system:



In a background-independent theory (**BI**) the metric is a dynamic degree of freedom. Therefore two objects a and b can have *constant coordinates* in space while their distance varies over time! Note that the coordinates are completely independent of the metric in general.

 \rightarrow One cannot infer from coordinates whether distances of objects change!

We can sum this up as follows:

Coordinates have no physical meaning in GENERAL RELATIVITY; they are simply (arbitrary) labels of events.

• Please make sure you grasp this statement fully (we will see explicit examples later when we understand GENERAL RELATIVITY better):

If I tell you in SPECIAL RELATIVITY that (in some inertial frame) two test particles have constant spatial coordinates x^i and y^i , you immediately know their relative velocity and distance: $\vec{v}_{rel} = \dot{\vec{x}} - \dot{\vec{y}} = \vec{0}$ and $\Delta l^2 = -(x^{\mu} - y^{\nu})^2 = |\vec{x} - \vec{y}|^2$.

By contrast, if I tell you in GENERAL RELATIVITY that two test particles have (in some coordinate system) constant spatial coordinates x^i and y^i , this tells you *nothing* about their distance; not even whether it is constant or varies in time! This information is hidden in the values of the metric field, not in the coordinates.

• This is important, for example, when studying the effects of gravitational waves (\rightarrow *later*).

17 | Summary:

- Every reasonable fundamental theory has a generally covariant formulation.
- A generally covariant theory does not need to be background independent.
- GENERAL RELATIVITY is background independent and generally covariant.

• We will return to the question of general covariance, background independence (and diffeomorphism invariance) → *later* when we know more about GENERAL RELATIVITY. At this point it is only important that you know the conceptual difference between the terms *general*



covariance **GC** and *background independence* **BI** and the requirement of the *principle of general relativity* **GRP**.

• One sometimes hears that general covariance GC is a distinguishing feature of GENERAL RELATIVITY (as explained above, it is not). Sometimes even the very concepts of general covariance GC and background independence BI are confused. This confusion is partially rooted in history because Einstein himself didn't separate the two concepts clearly. MISNER *et al.* explain [3] (p. 431):

Mathematics was not sufficiently refined in 1917 to cleave apart the demands for "no prior geometry" and for a "geometric, coordinate-independent formulation of physics." Einstein described both demands by a single phrase, "general covariance." The "no-prior-geometry" demand actually fathered general relativity, but by doing so anonymously, disguised as "general covariance," it also fathered half a century of confusion.

• For more details on the relation between the concepts of background independence, general covariance, and diffeomorphism invariance see Ref. [119] (and references therein).

9.3. Mach's principle (an its failure in GENERAL RELATIVITY)

Mach's principle is not a logical postulate of GENERAL RELATIVITY and mostly of historical (and perhaps philosophical) importance. However, it is also conceptually interesting because at a fist glance once might conclude (as Einstein did), that GENERAL RELATIVITY actually satisfies the principle. It is rather subtle (and instructive) why this is not so:

18 | Recall \leftarrow Newton's bucket:



Question: Rotation with respect to what determines the shape of the water surface?

NEWTON: Absoute space!

Note that the experiment already suggests that this "absolute space" must have certain symmetries since the experiment cannot distinguish specific points nor specific states of uniform motion. SPECIAL RELATIVITY tells us that the correct symmetry group of spacetime is the Poincaré group. So from our modern perspective, Newton's answer must be read as follows: *The experiment demonstrates the independent existence of an entity which determines the local inertial systems*. We may call this entity "spacetime."

19 | The austrian physicist ERNST MACH fervently disagreed with Newton [120]:

Der Versuch Newton's mit dem rotirenden Wassergefäss lehrt nur, dass die Relativdrehung des Wassers gegen die Gefässwände keine merklichen Centrifugalkräfte weckt, dass dieselben aber durch die Relativdrehung gegen die Masse der Erde und die übrigen Himmelskörper geweckt werden. Niemand kann sagen, wie der Versuch verlaufen würde, wenn die Gefässwände immer dicker und massiger, zuletzt mehrere Meilen dick würden. Es liegt nur der eine Versuch



vor, und wir haben denselben mit den übrigen uns bekannten Thatsachen, nicht aber mit unsern willkürlichen Dichtungen in Einklang zu bringen.

MACH denied NEWTON's notion of an independent entity responsible for inertia and proposed that the large-scale structure of matter in the cosmos determines the local inertial systems instead (\uparrow *relationalism*):

§ Principle 2: Mach's principle MP

Local inertial frames are determined by the cosmic motion and distribution of matter.

- MACH never formulated his principle precisely, which leaves room for interpretation and personal taste. This is why there are various readings of MP in the literature, not all equivalent. The above phrasing is a rather strict version of the principle.
- Here is an alternative way to illustrate the point by STEVEN WEINBERG [121] (p. 17):

There is a simple experiment that anyone can perform on a starry night, to clarify the issues raised by Mach's principle.

First stand still, and let your arms hang loose at your sides. Observe that the stars are more or less unmoving, and that your arms hang more or less straight down. Then pirouette. The stars will seem to rotate around the zenith, and at the same time your arms will be drawn upward by centrifugal force. It would surely be a remarkable coincidence if the inertial frame, in which your arms hung freely, just happened to be the reference frame in which typical stars are at rest, unless there were some interaction between the stars and you that determined your inertial frame.

Put this way, the situation is quite puzzling indeed and Mach's principle doesn't seem far fetched at all.

• EINSTEIN was responsible for coining the term "Mach's principle" and was influenced by it during his construction of GENERAL RELATIVITY. At first, he believed that in his new theory of gravity the principle was indeed satisfied. He writes in a letter to Mach in 1913 [122]:

Dieser Tage haben Sie wohl meine neue Arbeit über Relativität und Gravitation erhalten, die nach unendlicher Mühe und quälendem Zweifel nun endlich fertig geworden ist. Nächstes Jahr bei der Sonnenfinsternis soll sich zeigen, ob die Lichtstrahlen an der Sonne gekrümmt werden, ob m. a. W. die zugrunde gelegte fundamentale Annahme von der Aequivalenz von Beschleunigung des Bezugssystems einerseits und Schwerefeld andererseits wirklich zutrifft.

Wenn ja, so erfahren Ihre genialen Untersuchungen über die Grundlagen der Mechanik – Planck's ungerechtfertigter Kritik zum Trotz – eine glänzende Bestätigung. Denn es ergibt sich mit Notwendigkeit, dass die Trägheit in einer Art Wechselwirkung der Körper ihren Ursprung hat, ganz im Sinne Ihrer Überlegungen zum Newton'schen Eimer-Versuch.

(You may wonder how Einstein could write this letter in 1913 when he finalized GENERAL RELATIVITY in November of 1915. Einstein refers to his paper with Marcel Grossmann published in 1913 [123] in which they established the "Entwurftheorie", a precursor of GENERAL RELATIVITY that already included most of the pieces needed [but not yet the correct field equations].)

20 | So here is the case MACH (& early EINSTEIN) vs. NEWTON:



• Newton:

Space exists as an independent entity and determines locally which frames are inertial.

• Mach:

Space emerges from the relations between matter and does not exist independently. Hence the distribution of matter in the universe completely determines the local inertial systems.

Who is right according to GENERAL RELATIVITY?

- 21 Answer: Both ... (in a sense, though Newton is more correct)
 - NEWTON'S conclusion was correct: Because of locality (constancy of the speed of light) the matter distribution of the cosmos (the fixed stars) cannot immediately influence the local inertial frame; there must be a mediator, some "background" that is here right now. GENERAL RELATIVITY tells us what this is: the metric tensor field that determines the geometry of spacetime.
 - MACH was right insofar as it is indeed not a coincidence that the local inertial frame on earth is at rest with respect to the fixed stars. There *is* a relation, although not a direct and immediate one. GENERAL RELATIVITY tells us that the large-scale distribution of matter (and energy) in the universe determines the (large-scale) metric of spacetime, which, in turn, determines the local inertial systems everywhere. But there is a hitch: the metric is not uniquely determined by the mass distribution. Thus the metric (and therefore spacetime) carries independent degrees of freedom. There is more than matter in the world, spacetime is a real entity!

Notes:

- Today we know that there are solutions of the Einstein field equations (e.g. the ↑ *Gödel universe* [124]) that violate Mach's principle explicitly [125].
- MACH, in his critique of NEWTON's bucket experiment, asked (rhetorically) what would happen if the walls of the bucket would become very thick and massive. His point was that it is not excluded that at some point the rotation of the bucket *would* influence the shape of the water. GENERAL RELATIVITY tells us that this is so indeed, because a very massive bucket affects the geometry of spacetime. This is known as the → *Lense-Thirring effect* [126, 127] (also know as ↑ *frame-dagging*) and has been experimentally confirmed (not with a massive bucket, of course, but with earth) [128, 129]. However, for the reasons explained above, this effect does not make GENERAL RELATIVITY comply with Mach's principle in the strict sense.
- Because of the many different versions of MP floating around, for some the case is still not closed (at least for philosophers of science, it seems). For doing physics with GENERAL RELATIVITY, MP is irrelevant.
- MACH advocated a relational view of space(time): Only relations between the degrees of freedom of matter are observable. There is no independent meaning of, say, an electron being *here now*. It is interesting to realize that this relational view might very well be true (and in accordance with GENERAL RELATIVITY) if one accepts that the metric field is just another collection of degrees of freedom which can be in relations (coincide or interact) with other degrees of freedom. For example, an electron being *here now* might simply mean that an excitation of the field that describes the electron coincides/interacts with a particular degree of freedom of the metric field.

22 | Conclusion:

The controversy about the **MP** essentially boils down to the question whether spacetime has independent degrees of freedom (and therefore exists in a physical sense):



In its strict version, the **MP** denies spacetime this independent role. By contrast, GENERAL RELATIVITY grants spacetime independent degrees of freedom because the \rightarrow *Einstein field* equations only constrain the \rightarrow *Einstein tensor* but not the metric directly (\rightarrow *Gravitational waves*):

GENERAL RELATIVITY *violates* Mach's principle **MP** because matter *influences* the geometry of spacetime but does not *determine* it uniquely.

This situation is exemplified by gravitational waves: When in 2015 the interferometers of LIGO detected a gravitational wave passing earth, the spacetime geometry in our vicinity changed by a very tiny bit. However, the mass distribution in the vicinity of earth didn't change at all. So while the geometry of spacetime certainly is *influenced* by earths mass, it is not uniquely *determined* by it. LIGO therefore measured directly the dynamics of the degrees of freedom the existence of which MP denies.

9.4. Overview and Outline

Now that we know the conceptual starting point of GENERAL RELATIVITY, and argued that more general spacetimes than flat Minkowski space are needed to accomodate gravity, we can reveal the gist of GENERAL RELATIVITY and sketch the plan for the remainder of this course:

i! You are not required to fully grasp the *how* and *why* of the following statements. Understanding the details is the objective of this course. However, I think that it is useful to start off with a rough picture of what we want to accomplish because otherwise one is easily swamped by the details along the way.

GENERAL RELATIVITY in a Nutshell

• Ontology:

Spacetime \equiv 4D differentiable manifold *M* Gravitational field \equiv pseudo-Riemannian metric *g* with signature (1, 3)

→ Spacetime is a ← 4D Lorentzian manifold

- Note that we only fix the dimensionality of M (and thereby its *local* topology) but not its *global* topology (i.e., whether it is simply \mathbb{R}^4 , a sphere, a torus, or something even more fancy). Thus, for example, GENERAL RELATIVITY makes no a priori statement about the finiteness of the universe. (Asking about the *local* topology is like asking where space and time come from—and GENERAL RELATIVITY is silent about that. A reasonable theory of quantum gravity must address this question!)
- At this stage it is sufficient to interpet the points $E \in M$ of the manifold as points in spacetime and therefore as (equivalence classes of) events. However, we will see that this interpretation is problematic (\rightarrow *Hole argument*) because of the diffeomorphism invariance of GENERAL RELATIVITY. It is thus questionable whether points of the manifold (and thereby the manifold itself) can be associated to any existing entity. An entity that certainly does exist, however, is the metric field.



• Equivalence principle:

The **EEP** is built right into the mathematical framework of GENERAL RELATIVITY:

For every point with coordinates y, there exists a coordinate transformation φ_y with $\bar{x} = \varphi_y(x)$ such that:

$$\bar{g}^{\mu\nu}(\bar{x}) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}(x) \overset{x \approx y}{\approx} \eta^{\mu\nu} \quad \text{and} \quad \bar{\partial}_{\rho} \bar{g}^{\mu\nu}(\bar{x}) \overset{x \approx y}{\approx} 0 \tag{9.14}$$

with Minkowski metric

$$\eta^{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{pmatrix}_{\mu\nu}.$$
(9.15)

 \rightarrow *Locally* inertial coordinates \checkmark (\bigcirc Problemset 2)

(9.16)

- i! In the presence of gravity, there is *no* coordinate transformation that brings the metric into Minkowski form *everywhere* on the spacetime manifold. Conversely, if this *is* possible, spacetime is flat Minkowski space and you were doing SPECIAL RELATIVITY all along (perhaps in curvilinear coordinates).
- Note that metrization of gravity is not a *mathematical corollary* of the **EEP** (the latter is a physical principle, not a rigorous mathematical statement). However, the **EEP** is most naturally incorporated into a mathematical framework where gravity is described by a metric because the gist of the **EEP** is that all (local) physical phenomena are affected by gravity in the same way. This is exactly what happens if gravity is identified with the geometry of spacetime!
- At every point, the basis {∂_ρ} of the tangent space forms a so called *** local Lorentz frame*.
 You can choose such a basis for all points of spacetime. However, in general there is no coordinate system that induces this basis everywhere; you have to use multiple charts to patch together spacetime.
- Important fields:

All degrees of freedom (some gauge, some physical) of GENERAL RELATIVITY are stored in the metric tensor field. From the metric, one can then derive other fields that play important roles in the formulation of the theory:





Note that the Einstein tensor is non-linear in the metric an contains (up to) second-order derivatives.

• Einstein field equations (EFE): (here without cosmological constant)

The centerpiece of GENERAL RELATIVITY is a tensorial partial differential equation that determines the metric tensor field in dependence of the energy momentum tensor of matter:

$$\underset{\text{tensor}}{\text{Einstein}} \rightarrow \underbrace{G_{\mu\nu}}_{\text{"Geometry"}} = -\kappa \underbrace{T_{\mu\nu}}_{\text{"Matter"}} \leftarrow \underbrace{\text{Energy-momentum}}_{\text{tensor}} \tag{9.17}$$

i! "Matter" refers here to all degrees of freedom that carry energy and/or momentum. This includes bodies with rest mass but also electromagnetic radiation etc.

 \triangleleft Eq. (9.17): Non-linear, second-order PDE for $g_{\mu\nu}$:

GENERAL RELATIVITY describes the geometry of space as a dynamical field that evolves "in time" according to a highly nontrivial PDE: \rightarrow

GENERAL RELATIVITY = ** Geometrodynamics

(9.18)

The nonlinearity makes Eq. (9.17) hard to solve, even in vacuum were the right-hand side vanishes.

- Spacetime geometry is dynamical \rightarrow Background independence \checkmark
- Tensor equation \rightarrow General covariance (no preferred coordinate system) \checkmark
- Mass distribution determines metric determines local inertial frames
 However: Fixing G_{μν} leaves some degrees of freedom of g_{μν} unconstrained!
 → Boundary conditions required for unique solution
 - \rightarrow Mach's principle is not satisfied (but partially survives in spirit) $X/\sqrt{}$
- Recall Eq. (8.10) and our discussion that followed (also ● Problemset 1). Eq. (9.17) is structurally similar but fixes the problem of linearity because the Einstein tensor is a non-linear function of the metric.
- Physics with gravity:

Once gravity is described by the metric, one must generalize the other relativistic theories (mechanics, electrodynamics, ...) into a generally covariant form that couples to the metric. This generalization is a priori not unique because matter can couple in various ways to the fields derived from the metric.

However, the **EEP** severely restricts the couplings that are allowed and leads to a "recipe" how the Lorentz covariant equations of SPECIAL RELATIVITY must be rewritten to match the principles of GENERAL RELATIVITY (\rightarrow "Comma-Goes-to-Semicolon Rule", Minimal coupling):

The **GRP** demands physical theories to be specified by *tensor equations*. The **EEP** restricts the possible couplings of matter and metric.

 \rightarrow

- Mechanics: (with $u^{\mu} = \frac{dx^{\mu}}{d\tau}$ the 4-velocity)

$$m \underbrace{\frac{Du^{\mu}}{D\tau}}_{\substack{\text{Absolute} \\ \text{derivative}}} = m \underbrace{\frac{d^2 x^{\mu}}{d\tau^2}}_{\substack{4-\text{accel.}}} + m \underbrace{\prod_{\alpha\beta} \frac{dx^{\alpha}}{d\tau} \frac{dx^{\beta}}{d\tau}}_{\text{New!}} = \underbrace{K^{\mu}}_{\substack{4-\text{force}}}$$
(9.19)



- \rightarrow Free particle: $K^{\mu} = 0 \rightarrow$ Geodesic equation ("straight lines" in spacetime)
- Electrodynamics: ⊲ Inhomogeneous Maxwell equations: (cf. Eq. (6.50))

$$\underbrace{F^{\mu\nu}_{,\nu}}_{\mu\nu} = -\frac{4\pi}{c} j^{\mu} \rightarrow \underbrace{F^{\mu\nu}_{;\nu}}_{Covariant} = -\frac{4\pi}{c} J^{\mu} \qquad (9.20)$$

The covariant derivative contains the connection Γ and therefore the metric g. This is how the electromagnetic field is affected by the gravitational field (e.g., bent in the vicinity of heavy masses). Conversely, the EM field gives rise to the energy momentum tensor $T_{\rm em}^{\mu\nu}$ [$\leftarrow Eq.$ (6.110)] and thereby contributes to the right-hand side of Eq. (9.17).

 \rightarrow Energy-momentum tensor $T_{\mu\nu}$ is dynamical

A generally relativistic theory of matter, interacting with and via gravity, is then described by a *coupled, non-linear, higher-order system of partial differential equations* (where $T_{\mu\nu}$ depends on the dynamical variables of the matter theory).

 \rightarrow Hard to solve in general \rightarrow Approximations needed!

Outline of this course

Here is our approach for this course to study these various aspects of GENERAL RELATIVITY:

• Step 1 (Section 9.4): How to describe non-Euclidean manifolds mathematically?

In Chapter **3** we introduced the concept of differentiable manifolds and introduced the concept of a (pseudo-)Riemannian metric to measure lengths of curves on the manifold. To formulate GENERAL RELATIVITY mathematically, we need to revisit and extend this toolbox of tensor calculus.

In particular, we will study two (at first independent) structures that can be put on a differentiable manifold:

\rightarrow Affine connection	\rightarrow	Determines parallel transport, straight lines, and curvature
← Riemannian metric	\rightarrow	Determines lengths, shortest lines, and angles

As it turns out, when you are given a Riemannian metric, there is a unique way to construct an affine connection. This means that once you are given a spacetime manifold with a (pseudo-) Riemannian (Lorentzian) metric, all concepts in the list above are well-defined. This is then the framework we will use: The spacetime of GENERAL RELATIVITY is a Lorentzian manifold and the degrees of freedom (field) of the theory is the Lorentzian metric itself.

• Step 2 (Chapter 11): How to formulate relativistic theories on non-Euclidean spacetimes?

In the first part of this course, we first established the tenets of SPECIAL RELATIVITY (Lorentz symmetry) and then incorporated them successively in known theories of physics (point mechanics in Chapter 5, electrodynamics in Chapter 6, quantum mechanics in Chapter 7). Now that we established the tenets of GENERAL RELATIVITY (the spacetime metric is not necessarily the Minkowski metric but an arbitrary Lorentzian metric), we must again reformulate our theories to comply with this new insight. Recall p. 17 in the introduction:



This will lead us to generally covariant formulations of relativistic mechanics in Section 11.2 and electrodynamics in Section 11.3 (we will skip quantum mechanics this time, but this is also possible). Thanks to the combination of **EEP** and **GRP** (and tensor calculus), the recipe to go from the equations of SPECIAL RELATIVITY to that of GENERAL RELATIVITY will be very simple.

• Step 3 (Chapter 12): How to determine the geometry of spacetime dynamically?

Up until this point we simply declared that the metric of spacetime is an arbitrary Lorentzian metric and studied the effects on physics given such a metric. The core idea of GENERAL RELATIVITY (and maybe the most important insight of Albert Einstein) was that this metric was not part of the laws of nature but just another degree of freedom that had to be dynamically determined. This means that there is no a priori geometry of spacetime, a principle known as \leftarrow background independence. The equations that dynamically determine the metric are the \rightarrow Einstein field equations Eq. (9.17); they are the centerpiece of GENERAL RELATIVITY and determine the geometry of spacetime, given the distribution of mass and energy and some boundary conditions. We will derive these equations via an action principle from a Lagrangian.

Together with Step 2, this completes the framework of GENERAL RELATIVITY.

• Step 4 (Chapter 13): What does GENERAL RELATIVITY predict?

If we combine the results of Step 2 and Step 3 we obtain a self-contained, background independen framework to describe physics: Matter determines the geometry of spacetime (Step 3) and, conversely, this geometry determines how matter evolves (Step 2). This interplay makes for beautiful but mathematically complicated models. Thus, to study the predictions of GENERAL RELATIVITY, we typically resort to simplified approaches:

- Consider a static, inhomogeneous distribution of large masses (e.g. the sun). Using the Einstein field equations from Step 3 (and reasonable boundary conditions), calculate the geometry of spacetime induced by this distribution. Then use the results of Step 2 to determine the evolution of small test particles on this curved spacetime (without taking their backaction on spacetime into account). This approach leads to a variety of phenomena, e.g., the slowing down of clocks close to large masses (Section 13.2.5), the perihelion precession of Mercury (Section 13.2.1), the bending of light (Section 13.2.2), etc.
- Consider the Einstein field equations in vacuum, i.e., without any matter (or energy). Because the EFEs are non-linear (recall Section 8.2) and the geometry of spacetime is not uniquely determined by the distribution of mass and energy, this situation is not as boring and trivial as it sounds (its actually very complicated). But even in the weak field limit (where one drops the self-interactions) one finds something interesting: gravitational waves (Section 13.4).
- Consider an idealized universe that is homogeneously filled with matter and energy (and potentially dark matter and dark energy). If one calculates the solutions of the EFEs in such a scenario, one obtains the (approximate) spacetime geometry of the whole universe. This leads into the field of ↑ *relativistic cosmology* and to the current standard model of cosmology, known as "ACDM". This is where one finds the possibility of an expanding universe and its origin, the Big Bang; this is also where the cosmological constant becomes important.

↓Lecture 19 [30.04.24]



10. Mathematical Tools II: Curvature

Here we continue our discussion of differential geometry in Chapter 3. We study two structures on a differentiable manifold that are particularly important for GENERAL RELATIVITY: \rightarrow connections and the \leftarrow Riemannian metric (the latter we already know). Since most of our results are not specific to GENERAL RELATIVITY, we mostly consider general D dimensional manifolds, and only specialize to the case of D = 3 + 1 spacetime dimensions later.

• The mathematical framework of GENERAL RELATIVITY is ↑ *Riemannian geometry*, i.e., the field of differential geometry that studies differentiable manifolds equipped with a Riemannian (pseudo-)metric. The field was kickstared in 1854 by German mathematician BERNHARD RIEMANN with his inaugural lecture in Göttingen titled "*Über die Hypothesen, welche der Geometrie zu Grunde liegen*" [130]. In the audience was CARL FRIEDRICH GAUB, who had also picked the topic for Riemann's habilitation (Gauß died one year later).

<u>Fun fact:</u> In his 1854 lecture, Riemann speculated that the material bodies might determine the metric of space; many years before Einstein worked out GENERAL RELATIVITY (see Part III, Paragraph 3 in Ref. [130], highlights are mine):

Die Frage über die Gültigkeit der Voraussetzungen der Geometrie im Unendlichkleinen hängt zusammen mit der Frage nach dem innern Grunde der Maßverhältnisse des Raumes. Bei dieser Frage, welche wohl noch zur Lehre vom Raume gerechnet werden darf, kommt [..] zur Anwendung, daß bei einer diskreten Mannigfaltigkeit das Prinzip der Maßverhaltnisse schon in dem Begriffe dieser Mannigfaltigkeit enthalten ist, bei einer stetigen aber anders woher hinzukommen muß. Es muß also entweder das dem Raume zugrunde liegende Wirkliche eine diskrete Mannigfaltigkeit bilden, oder der Grund der Maßverhaltnisse außerhalb, in darauf wirkenden bindenden Kräften gesucht werden.

He continues ...

Die Entscheidung dieser Fragen kann nur gefunden werden, indem man von der bisherigen durch die Erfahrung bewährten Auffassung der Erscheinungen, wozu Newton den Grund gelegt, ausgeht und diese durch Tatsachen, die sich aus ihr nicht erklären lassen, getrieben allmählich umarbeitet; [..].

... and closes:

Es führt dies hinüber in das Gebiet einer andern Wissenschaft, in das Gebiet der Physik, welches wohl die Natur der heutigen Veranlassung nicht zu betreten erlaubt.

Not only did he sketch the route Einstein would take half a century later, he even seemed intrigued exploring it himself.

• The mathematical field of geometry was conceived in ancient times as a formalization of observable facts about physical space and culminated in the axiomatization of ↓ *Euclidean geometry*. One of the facts/axioms of Euclidean geometry is the ** parallel postulate*:

If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.

For two millenia (!) it was suspected that this axiom can be derived from the other four axioms of Euclidean geometry (so that it doesn't deserve the title "axiom" after all). Finally, GAUB (and



contemporaries) recognized that the parallel postulate cannot be proven from the other four; it is an independent axiom that can be modified to define consistent geometries that differ from Euclid's! The result is * *non-Euclidean geometry* which comes in two flavours, \uparrow *elliptic geometry* and \uparrow *hyperbolic geometry*:



The realization by mathematicians that there are many consistent geometries opened a new question for physics: *Are we sure that the geometry of space really is Euclidean?* The answer of GENERAL RELATIVITY is: No, on large scales space is only *approximately* Euclidean, and it can be very non-Euclidean in regimes of strong gravitational fields.

10.1. Summary: What we know and what comes next

- 1 | Concepts we already know:
 - ← *Differentiable manifolds* (Section 3.1):

A *D*-dimensional manifold is locally homeomorphic (continuously isomorphic) to \mathbb{R}^{D} (it locally "looks like" Euclidean space). A continuous, invertible function that maps a region of the manifold to a subset of \mathbb{R}^{D} is called a *(coordinate) chart*. A collection of overlapping charts that covers the whole manifold is called an *atlas*. If the transition functions that map between different coordinates in regions where two charts overlap are all differentiable (smooth) on \mathbb{R}^{D} , the manifold is a \leftarrow *differentiable (smooth) manifold*. On a differentiable manifold we can talk about the differentiation of functions defined on the manifold. In physics we consider almost exclusively such manifolds:



• ← *Tangent and cotangent spaces* (Section 3.3):

Given a differentiable manifold (which is not a vector space in general!), there is a canonical way to associate a vector space to every point of the manifold: the \leftarrow tangent space $T_p M$. Mathematically, it is the vector space of directional derivative operators that act on smooth functions on that point. Given a coordinate chart, the directional derivatives along the coordinates (evaluated at $p \in M$) induce a basis $\{\partial_i|_p\}$ of the tangent space $T_p M$ (different



coordinates lead to different bases). In addition, to every vector space there is an associated *dual space* spanned by the linear forms on the vector space; thus there is a dual tangent space: the \leftarrow cotangent space T_n^*M . It is spanned by the dual basis $\{dx_n^i\}$ of differential forms:



Vector space of directional derivatives with evaluation at $p \in M$. Spanned by coordinate basis derived from given chart.

(10.1)

With the dual basis (we often drop the subscript p)

$$dx_p^i(\partial_j|_p) := \delta_j^i = \left. \frac{\partial x^i}{\partial x^j} \right|_p \tag{10.2}$$

we can define the

Cotangent space
$$T_p^*M$$
 at $p \in M$ = span $\left\{ dx_p^i \mid i = 1, \dots, D \right\}$ (10.3)

• \leftarrow Tensor fields (Sections 3.2 to 3.4):

Since there are canonical vector and covector spaces associated to every point of the manifold, we can consider (reasonably smooth) functions that map every point of the manifold to a tensor product of p vectors and q covectors; we call such functions \leftarrow *tensor fields* of rank (p,q). They are "geometric objects" in that they are independent of coordinate charts; physical quantities (like the electromagnetic field) must be represented by such fields. Once we have chosen a coordinate chart, we can encode these fields in terms of their *components* wrt. the coordinate basis on tangent and cotangent space. The coordinate independence of tensor fields translates then into a particular transformation law for their components:

 \triangleleft Coordinate transformation $\bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x})$ $\rightarrow (p,q)$ -Tensor (field) $T : \Leftrightarrow$

$$\underbrace{\bar{T}^{i_{1}\dots i_{p}}_{j_{1}\dots j_{q}}(\bar{x})}_{=:\frac{\partial\bar{x}^{i_{1}}}{\partial x^{m_{1}}}\cdots\frac{\partial\bar{x}^{i_{p}}}{\partial x^{m_{p}}}\right]}_{=:\frac{\partial\bar{x}^{I}}{\partial x^{M}}}\underbrace{\left[\frac{\partial x^{n_{1}}}{\partial\bar{x}^{j_{1}}}\cdots\frac{\partial x^{n_{q}}}{\partial\bar{x}^{j_{q}}}\right]}_{=:\frac{\partial\bar{x}^{N}}{\partial\bar{x}^{J}}}\underbrace{T^{m_{1}\dots m_{p}}_{n_{1}\dots n_{q}}(x)}_{=T^{M}_{N}(x)} \tag{10.4}$$

(Einstein sum convention = Sums over pairs of up- and down indices are implied.)



Examples:

$(0, 0)$ -Tensor \equiv Scalar:	$\Phi(\bar{x}) = \Phi(x)$	(10.5a)
$(1,0)$ -Tensor \equiv Contravariant vector:	$\bar{A}^{i}(\bar{x}) = \frac{\partial \bar{x}^{i}}{\partial x^{k}} A^{k}(x)$	(10.5b)
$(0, 1)$ -Tensor \equiv Covariant vector:	$\bar{B}_i(\bar{x}) = \frac{\partial x^k}{\partial \bar{x}^i} B_k(x)$	(10.5c)
$(1, 1)$ -Tensor \equiv (Mixed) Tensor:	$\bar{T}^{i}_{\ j}\left(\bar{x}\right) = \frac{\partial \bar{x}^{i}}{\partial x^{k}} \frac{\partial x^{l}}{\partial \bar{x}^{j}} T^{k}_{\ l}\left(x\right)$	(10.5d)

• *← Riemannian metric* (Section 3.5):

A Riemannian metric is a (0, 2) tensor field with a few additional properties (symmetry and non-degeneracy) so that it defines a (pseudo-)inner product on the tangent space at every point of the manifold. A differentiable manifold equipped with such a metric is called a \leftarrow *Riemannian manifold*. On a Riemannian manifold we can measure angles between tangent vectors and lengths of curves:

Riemannian (pseudo-)metric
$$ds^2 := \begin{cases} Symmetric \\ non-degenerate \\ (0, 2)-tensor field \end{cases}$$
 (10.6)

More formally:

$$ds^{2}: M \ni p \mapsto \underbrace{\left(ds_{p}^{2}: T_{p}M \times T_{p}M \to \mathbb{R}\right)}_{\text{Bilinear & symmetric & non-degenerate}} \in T_{p}^{*}M \otimes T_{p}^{*}M$$
(10.7)

with coordinate representation

$$ds_p^2 = \sum_{i,j=1}^{D} g_{ij}(x) \, dx^i \otimes dx^j \equiv g_{ij}(x) \, dx^i \, dx^j$$
(10.8)

where $g_{ij} = g_{ji}$ (symmetry) and $g = \det(g_{ij}) \neq 0$ (non-degeneracy).

A Riemannian metric allows us to define the following geometric concepts:

- Angle between two vectors $A = A^i \partial_i, B = B^i \partial_i \in T_p M$:

$$\langle A, B \rangle \equiv \mathrm{d}s_p^2(A, B) = g_{ij}(p)A^i B^j \equiv ||A||_p ||B||_p \cos\theta \tag{10.9}$$

with the norm on $T_p M$

$$||A||_p := \sqrt{\mathrm{d}s_p^2(A,A)} = \sqrt{g_{ij}(p)A^iA^j} \,. \tag{10.10}$$

- Length of curve $\gamma : [a, b] \to M$:

$$L[\gamma] := \int_{a}^{b} \sqrt{g_{ij}(\gamma(t))} \frac{\mathrm{d}\gamma^{i}(t)}{\mathrm{d}t} \frac{\mathrm{d}\gamma^{j}(t)}{\mathrm{d}t} \,\mathrm{d}t = \int_{a}^{b} \underbrace{\|\dot{\gamma}(t)\|_{\gamma(t)}}_{\text{"Velocity"}} \mathrm{d}t \,. \tag{10.11}$$



• ← Pulling indices up and down (Section 3.5):

A symmetric, non-degenerate bilinear form defines a canonical isomorphism between a vector space and its dual. A special case is a Riemannian metric which provides us with a isomorphism between tangent and cotangent spaces at every point of the manifold. In tensor calculus, this isomorphism is applied by "pulling indices up and down" with the metric:

Pulling *down*:
$$T^{i_1} \dots \square \dots \square j_p \square \dots \square \square \dots \square j_1 \dots \square j_q := g_{ik} T^{i_1} \dots \square \dots \square \square \square \dots \square j_1 \dots \square j_q$$
 (10.12a)

where g^{ij} is the *inverse metric* defined via $g^{ik}g_{kj} \stackrel{!}{=} \delta^i_j$.

• ← *Christoffel symbols and covariant derivatives* (Section 3.6):

We realized that the partial derivatives of tensor fields are *not* tensor fields themselves (this only works for scalars). This motivated the introduction of a "patched up derivative," the so called \leftarrow *covariant derivative* that transforms again like a tensor. To define the covariant derivative, we needed a set of (non-tensorial) functions called \leftarrow *Christoffel symbols* that were defined by a given Riemannian metric:

 \downarrow Covariant derivative wrt. k

Scalar:
$$\Phi_{;k} := \Phi_{,k}$$
 (10.13a)

Contravariant vector:
$$A^{i}_{;k} := A^{i}_{,k} + \Gamma^{i}_{kl}A^{l}$$
 (10.13b)

Covariant vector:
$$B_{i:k} := B_{i,k} - \Gamma^l_{ik} B_l$$
 (10.13c)

with $\Phi_{k} \equiv \partial_{k} \Phi$ etc. and the \leftarrow Christoffel symbols (of the second kind)

$$\Gamma^{i}_{kl} := \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right) \,. \tag{10.14}$$

i! In the following two sections we will revisit, motivate and study the concept of a covariant derivative in more detail. We will also see where the Christoffel symbols come from and which role they play geometrically on the manifold.

So if you were not satisfied with the way the covariant derivative and the Christoffel symbols appeared out of thin air in Section 3.6: Now comes the proper introduction!

• ← *Manifest covariance* (Section 3.6):

The whole point of our endeavor was to find a mathematical toolbox that allows us to write down equations that are guaranteed to be form-invariant under arbitrary coordinate transformations. These equations describe relations between geometric objects on a manifold, such that their content is independent of the chosen coordinate chart. This toolbox is called \leftarrow *tensor calculus* and consists of rules how to combine/construct tensors (e.g. via multiplication, contraction of indices, covariant derivatives, ...) to form generally covariant equations. The general covariance of tensorial equations is *manifest* because their mere structure guarantees general covariance:

$$T^{I}{}_{J}(x) = 0 \qquad \stackrel{\text{Coordinate trafo: } \bar{x} = \varphi(x)}{\bar{T}^{I}{}_{J}(\bar{x}) = \frac{\partial \bar{x}^{I}}{\partial x^{M}} \frac{\partial x^{N}}{\partial \bar{x}^{J}} T^{M}{}_{N}(x)} \qquad \bar{T}^{I}{}_{J}(\bar{x}) = 0 \qquad (10.15)$$



2 Plan:



• Section 10.2:

Introduce and study \rightarrow *connections* and the concept of parallel transport and curvature.

• Section 10.3:

Use a Riemannian metric to derive a special connection: the \rightarrow *Levi-Civita connection*. Study properties of this special connection: Riemannian curvature and geodesic curves.

10.2. Affine connections

- An *affine connection* is an additional structure on a differentiable manifold (no metric needed!) that allows for the definition of the following concepts:
 - Parallel transport
 - Covariant derivatives
 - Autoparallel curves
 - Curvature

"Additional" means that it is not intrinsic or canonical to a manifold; *you* can add a connection to obtain more structure. It also implies that typically there are many connections to choose from.

• Terminology:

In modern differential geometry, the term "connection" has a rather broad meaning. Generally speaking, a connection is a structure that allows one to "parallel transport" objects along curves on a manifold. The most straightforward objects to move around are vectors taken from the tangent spaces of the manifold; this type of connection is called an \rightarrow *affine connection*, and it is this variety we use in GENERAL RELATIVITY.

However, you can also (artificially) attach other spaces to every point of a manifold (e.g., Lie groups like U(1)). Then you can ask how objects of these spaces are parallel transported around the manifold. This gives rise to other types of connections that are particularly important in modern formulations of gauge theories (\uparrow gauge connections). The gauge field A^{μ} of electrodynamics is an example of a U(1) gauge connection; it transports U(1) phases around (not tangent vectors) and is therefore not an affine connection.

In the following we will often drop the "affine" and simply talk about "connections." Keep in mind, however, that we only consider affine connections in this chapter (and this course).

1 | \triangleleft Differentiable *D*-dimensional manifold *M*; vector field $A = A^i \partial_i$; scalar field Φ :

$\partial_k \Phi \rightarrow \checkmark \operatorname{covariant} \operatorname{vector} \operatorname{field}$	$(\leftarrow Eq. (3.19))$	(10.16a)
--	---------------------------	----------

 $\partial_k A^i \rightarrow \not no \text{ tensor field } (\leftarrow Eq. (3.73))$ (10.16b)



This is a problem because we often need derivatives of tensors to formulate physical models; and since these equations must be generally covariant (**GRP** !), we need them to transform as tensors!

- 2 | Problem:
 - i $\triangleleft \triangleleft$ Directional derivative of $A = A^i \partial_i$ along a curve $\gamma(\lambda)$ with $\gamma(0) = p \in M$:

$$\frac{\mathrm{d}A(\gamma(\lambda))}{\mathrm{d}\lambda}\Big|_{\lambda=0} \stackrel{?}{=} \lim_{\delta\lambda\to 0} \frac{A(\gamma(\delta\lambda)) - A(\gamma(0))}{\delta\lambda} \equiv \lim_{\delta\lambda\to 0} \underbrace{\frac{4 \,\mathrm{Undefined!}}{\delta\lambda}}_{\delta\lambda} \tag{10.17}$$

Note that $A(q) \in T_q M$ and $A(p) \in T_p M$, i.e., these values of the vector field belong to *different* vector spaces. Hence their difference is completely undefined!

ii | We can of course try to work with the *components* of the vector field wrt. a given chart instead. Since $A^i \in \mathbb{R}$, the following expression is at least well-defined:

$$\frac{\mathrm{d}A^{i}(\gamma(\lambda))}{\mathrm{d}\lambda}\Big|_{\lambda=0} = \lim_{\delta\lambda\to0} \frac{A^{i}(q) - A^{i}(p)}{\delta\lambda}$$
(10.18)

Unfortunately this does not solve the problem, because these components are given wrt. to *different*, *coodinate-dependent* bases on T_qM and T_pM , respectively:

$$A(q) = A^{i}(q)\partial_{i}|_{q} \quad \text{with} \quad \text{span}\left\{\partial_{i}|_{q}\right\} = T_{q}M \tag{10.19}$$

$$A(p) = A^{i}(p)\partial_{i}|_{p} \quad \text{with} \quad \text{span}\left\{\partial_{i}|_{p}\right\} = T_{p}M \tag{10.20}$$

iii | To understand why this is a problem, imagine you fix the basis $\{\partial_i|_p\}$ of $T_p M$; this does not fix the basis $\{\partial_i|_q\}$ of $T_q M$ because choosing different (curvilinear) coordinates can be used to modify the induced basis $\{\partial_i|_q\}$ without changing $\{\partial_i|_p\}$:



As a consequence, the components $A^i(q)$ can be modified arbitrarily without changing the vector field A itself. Thus the difference $A^i(q) - A^i(p)$, and thereby the directional derivative above, do not encode a geometric, coordinate independent object! Mathematically, this is reflected in the *non-tensorial* transformation of the difference under arbitrary coordinate transformations

$$\bar{A}^{i}(q) - \bar{A}^{i}(p) = \frac{\partial \bar{x}^{i}(q)}{\partial x^{k}} A^{k}(q) - \frac{\partial \bar{x}^{i}(p)}{\partial x^{k}} A^{k}(p) \neq \frac{\partial \bar{x}^{i}(p)}{\partial x^{k}} \left[A^{k}(q) - A^{k}(p) \right]$$
(10.21)

This explains why partial derivatives of the form $\partial_k A^i$ (which are simply directional derivatives along coordinate axes) fail to transform as tensors.



3 | <u>Idea:</u>

The problem is conceptually most transparent in Eq. (10.17) which is mathematically undefined. However, if we could make it well-defined, we would immediately obtain a geometric, coordinateindependent object. To make the difference between the two vectors well-defined, they must live in the same tangent space, though.

Our only way out is to assume that we are given some function $\Gamma_{p\to q} : T_pM \to T_qM$ that establishes a correspondence between the two nearby tangent spaces by "parallel transporting" vectors between them. We then could "parallel transport" A(p) from T_pM to T_qM like so: $\Gamma_{p\to q}(A(p)) \in T_qM$. With this new vector, the difference is mathematically well-defined:

$$\frac{\mathrm{D}A}{\mathrm{D}\lambda} := \lim_{\delta\lambda\to 0} \underbrace{\frac{\epsilon T_q M}{A(q) - \Gamma_{p\to q}(A(p))}}_{\delta\lambda} \quad \text{or} \quad \frac{\mathrm{D}A^i}{\mathrm{D}\lambda} := \lim_{\delta\lambda\to 0} \frac{A^i(q) - A^i(p \xrightarrow{\Gamma} q)}{\delta\lambda}$$
(10.22)

We use the capital letter D to indicate that the difference in the numerator of the difference quotient has been modified by (and depends on) Γ .

$\rightarrow \Gamma_{p \rightarrow q}$ is an ** affine connection

(This is not yet very rigorous, we will specify our idea more formally below.)

 As already mentioned, the interpretation of an affine connection Γ is that it formalizes the notion of "parallel translating" or "parallel transporting" tangent vectors along curves on the manifold from one tangent space to another. It is important to realize that the notion of "parallel transport" is mathematically subtle and not trivial. It must be carefully defined and can lead to quite surprising results when considering curved manifolds:



Note that the (intuitive) parallel transport on the Euclidean plane (left) is independent of the path along which the vector is transported. By contrast, intuitively transporting vectors on a sphere (right) yields different results depending on the chosen path. The fact that there is no unique "parallel vector" to a given vector, but that the notion of parallelelism depends on the path taken, is the hallmark of \rightarrow *curvature*.

To be clear: the failure to produce a tensorial object from the directional derivative of a vector field is a fundamental and not a technical issue. We were neither too naïve when performing the derivative in Eq. (10.17), nor will our "solution" Eq. (10.22) render it magically tensorial. It is *impossible* to define a tensorial derivative on a manifold without specifying an additional structure (namely an affine connection Γ).

4 | Motivation:

To understand the properties of parallel transport (and thereby an affine connection Γ) better, we consider the simple example of parallel transporting a vector in the affine space $M = E^2 = \mathbb{R}^2$ (the Euclidean plane), described in curvilinear polar coordinates:



i $| \triangleleft M = E^2$ with Cartesian coordinates $\vec{x} = (x, y)$ and Cartesian basis $\{\partial_x, \partial_y\}$: \triangleleft "Constant" vector field



If you are given a differentiable manifold without additional structure, it does not make sense to ask whether a vector field "is constant." For example, if we consider $M = E^2$ as a manifold (forgetting about its Euclidean metric and affine structure), it does not make sense to call the vector field A "constant"; its *components* are constant wrt. to the basis induced by a specific coordinate system. However, a coordinate-independent statement like A(p) = A(q)for all $p, q \in M$ is nonsensical because $A(p) \in T_p M$ and $A(q) \in T_q M$, and there is no canonical isomorphism connecting $T_p M$ and $T_q M$; without an affine connection Γ , these are completely unrelated vector spaces and we do not know how to compare vectors at different points on the manifold (there is no concept of "parallel" vectors).

ii | \triangleleft Coordinate transformation $(x, y) = \varphi^{-1}(r, \theta)$ to polar coordinates:

$$x = r\cos\theta \tag{10.24a}$$

$$y = r\sin\theta \tag{10.24b}$$

 $\stackrel{\circ}{\rightarrow}$ Induced basis change on tangent spaces ($\leftarrow Eq.$ (3.5)):

$$\partial_r = \cos\theta \,\partial_x + \sin\theta \,\partial y \tag{10.25a}$$

$$\partial_{\theta} = -r\sin\theta \,\partial_x + r\cos\theta \,\partial_y \tag{10.25b}$$

 $\stackrel{\circ}{\rightarrow}$ Components of vector field:

$$A = A^{x}\partial_{x} + A^{y}\partial_{y} = A^{r}\partial_{r} + A^{\theta}\partial_{\theta}$$
(10.26)

with (no longer constant!)

$$A^{r}(r,\theta) = A^{x}\cos\theta + A^{y}\sin\theta \qquad (10.27a)$$

$$A^{\theta}(r,\theta) = \frac{1}{r} (A^{y} \cos \theta - A^{x} \sin \theta). \qquad (10.27b)$$

iii | \triangleleft Two infinitesimally separated points $p, q \in E^2$ with coordinates

$$u(p) = (r, \theta)$$
 and $u(q) = (r + \delta r, \theta + \delta \theta)$ (10.28)


and associated vectors $(A^r = A^r(p) \text{ and } A^{\theta} = A^{\theta}(p))$

$$A(p) = A^{r} \partial_{r} + A^{\theta} \partial_{\theta}, \qquad (10.29a)$$
$$A(q) = [A^{r} + \delta A^{r}] \partial_{r} + [A^{\theta} + \delta A^{\theta}] \partial_{\theta}, \qquad (10.29b)$$



Eq. (10.27) $\xrightarrow{\circ}$ (via Taylor expansion)

$$\delta A^{r} = r A^{\theta} \delta \theta \tag{10.30a}$$

$$\delta A^{\theta} = -\frac{1}{r} (A^{\theta} \delta r + A^{r} \delta \theta) \tag{10.30b}$$

If we now *declare* the vector field *A* to be *constant*, the variations Eq. (10.30) must be "fake" in the sense that they are caused by our choice of curvilinear coordinates rather an "intrinsic" variation of the vector field itself.

- \rightarrow This choice specifies an \rightarrow *affine connection*.
- iv Now that we *specified* which changes of the components of vector fields (in our coordinate system) are considered to be "fake", i.e., artifacts of the coordinates, we can define the "real" changes of arbitrary vector fields (which then can be non-constant wrt. our specific notion of parallel vectors) as their "complete" variation corrected by the "fake" variation δA^i :

 \triangleleft Arbitrary ("non-constant") vector field with $B^{i}(p) = B^{i}(r, \theta)$

 $\xrightarrow{\text{Eq. (10.30)}} \text{"True change" due to "intrinsic" variation of the vector field:}$

$$[B^{r}(q) - B^{r}(p)] - \delta B^{r} = \frac{\partial B^{r}}{\partial r} \delta r + \left(\frac{\partial B^{r}}{\partial \theta} - rB^{\theta}\right) \delta \theta$$
(10.31a)

$$[B^{\theta}(q) - B^{\theta}(p)] - \delta B^{\theta} = \left(\frac{\partial B^{\theta}}{\partial r} + \frac{1}{r}B^{\theta}\right)\delta r + \left(\frac{\partial B^{\theta}}{\partial \theta} + \frac{1}{r}B^{r}\right)\delta\theta \quad (10.31b)$$

The idea is to use such "corrected" differences in the numerator of a difference quotient like Eq. (10.22) to define a derivative of the vector field that transforms like a tensor.

That is, we define

$$A^{i}(p \xrightarrow{\Gamma} q) = A^{i}(p) + \delta A^{i}(p).$$
(10.32)



5 Generalization:

Drawing from the example and the form of the particular connection Eq. (10.30), we can select reasonable properties that an general affine connection should satisfy (in terms of components):

- (i) $A^i(p \xrightarrow{\Gamma} q)$ is *linear* in $A^i(p)$.
- (ii) The variation $\delta A^i(p)$ is *linear* in the first-order variation δx^i of coordinates.

We can satisfy both conditions if the variation has the general form (the minus is convention)

$$\delta A^{i}(p) = -\Gamma^{i}_{\ kl}(p)A^{k}(p)\delta x^{l}$$
(10.33)

 \rightarrow

$$A^{i}(p \xrightarrow{\Gamma} q) = A^{i}(p) + \delta A^{i}(p) = \left[\delta^{i}_{k} - \Gamma^{i}_{kl}(p) \,\delta x^{l}\right] A^{k}(p) \tag{10.34}$$

with some undetermined set of coefficients Γ_{kl}^{i} that completely specify the affine connection (in the particular coordinates chosen):

 $\Gamma^{i}_{kl}(x)$: ** (coefficients of the) affine connection Γ (in x)

Example:

From Eq. (10.30) and Eq. (10.33) it follows for the coefficients of the affine connection of the Euclidean plane, expressed in polar coordinates (\bigcirc Problemset 2):

$$\Gamma^{r}_{kl}(r,\theta) = \begin{pmatrix} 0 & 0\\ 0 & -r \end{pmatrix}_{kl} \quad \text{and} \quad \Gamma^{\theta}_{kl}(r,\theta) = \begin{pmatrix} 0 & \frac{1}{r}\\ \frac{1}{r} & 0 \end{pmatrix}_{kl}.$$
 (10.35)

6 | Interpretation:

The affine connection establishes a connection (hence the name) between tangent spaces at different points on the manifold by establishing a notion of *parallelism*:

$$\underbrace{\overbrace{A(p)}^{\epsilon T_p M}}_{A(p)} \xrightarrow{\text{Infinitesimal}} \underbrace{\overbrace{\Gamma_{p \to q}(A(p))}^{\epsilon T_q M}}_{F_{p \to q}(A(p))} = A^i(p \xrightarrow{\Gamma} q) \partial_i|_q \qquad (10.36)$$

$$= [A^i(p) + \delta A^i(p)] \partial_i|_q$$

$$= [\delta^i_k - \Gamma^i_{kl} \delta x^l] A^k(p) \partial_i|_q$$





We say: $\Gamma_{p \to q}(A(p))$ is the vector in q that is *parallel* to A(p) in p.

7 <u>*** Absolute derivative:</u>

We can now express the absolute derivative using the connection:

$$\frac{DA^{i}}{D\lambda} \stackrel{10.22}{:=} \lim_{\delta\lambda \to 0} \frac{\overline{A^{i}(\gamma(\lambda + \delta\lambda)) - A^{i}(\gamma(\lambda))} - \delta A^{i}}{\delta\lambda} \stackrel{10.33}{=} \frac{dA^{i}}{d\lambda} + \Gamma^{i}_{kl} A^{k} \frac{dx^{l}}{d\lambda}$$
(10.37)

We want the absolute derivative to transform as a contravariant vector:

$$\frac{\mathbf{D}\bar{A^{i}}}{\mathbf{D}\lambda} \stackrel{!}{=} \frac{\partial \bar{x}^{i}}{\partial x^{k}} \frac{\mathbf{D}A^{k}}{\mathbf{D}\lambda}$$
(10.38)

A straightforward but cumbersome calculation shows [recall Section 3.6] that this is the case if and only if the connection coefficients transform as follows:

$$\bar{\Gamma}^{i}_{kl} \stackrel{\circ}{=} \underbrace{\frac{\partial \bar{x}^{i}}{\partial x^{m}} \frac{\partial x^{n}}{\partial \bar{x}^{k}} \frac{\partial x^{o}}{\partial \bar{x}^{l}} \Gamma^{m}_{no}}_{\text{Tensor}\checkmark} + \underbrace{\frac{\partial \bar{x}^{i}}{\partial x^{p}} \frac{\partial^{2} x^{p}}{\partial \bar{x}^{k} \partial \bar{x}^{l}}}_{\text{No tensor}\checkmark}$$
(10.39)

 $\rightarrow \Gamma^{i}_{kl}$ does *not* transform as a tensor!

- i! For a given manifold M, there are *infinitely many* choices for an affine connection Γ .
- i! The definition Eq. (10.37) makes sense for any contravariant vector Aⁱ that is defined (and differentiable) along the curve γ(λ) [for example, a particle trajectory x^μ(λ)]. Although we considered a vector field Aⁱ in our discussion, it is *not* necessary for Aⁱ to be defined *in the neighborhood* of the trajectory γ(λ); i.e., partial derivatives ∂_j Aⁱ do not need to be defined for the definition of the absolute derivative Eq. (10.37). This is why we distinguish between the *absolute derivative* and the → *covariant derivative*.
- The additional term that makes the transformation of the connection coefficients non-tensorial is needed to compensate for a corresponding non-tensorial term from the total (non-covariant) derivative $\frac{dA^{i}}{d\lambda}$.
- Every set of fields Γⁱ_{kl} that transforms according to Eq. (10.39) can be used to define a connection (and therefore a notion of what "parallel" means on a manifold). This definition allows for more solutions than the specific type of connection that we used for our motivation, namely connections derived from declaring a given vector field as "constant." Interestingly, not all connections can be constructed in this way (the ones that can are actually quite boring because they do not have → *curvature*), and in Section 10.3 we will find a recipe to construct a special connection from every Riemannian metric.

8 | <u>Torsion:</u>

In general, the connection coefficients are *not* symmetric in their lower two indices. \rightarrow

$$\Gamma^{i}_{kl} = \underbrace{\frac{1}{2} \left(\Gamma^{i}_{kl} + \Gamma^{i}_{lk} \right)}_{\Gamma^{i}_{(kl)}} + \underbrace{\frac{1}{2} \underbrace{\left(\overline{\Gamma^{i}_{kl}} - \overline{\Gamma^{i}_{lk}} \right)}_{\Gamma^{i}_{[kl]}}}_{\Gamma^{i}_{[kl]}}$$
(10.40)



oret

Eq. (10.39) (Note that the non-tensorial part in Eq. (10.39) is symmetric in k and l!)

$$\bar{S}^{i}_{\ kl} = \frac{\partial \bar{x}^{i}}{\partial x^{m}} \frac{\partial x^{n}}{\partial \bar{x}^{k}} \frac{\partial x^{o}}{\partial \bar{x}^{l}} S^{m}_{\ no}$$
(10.41)

- \rightarrow Antisymmetric part S^{i}_{kl} of connection is a tensor: ** Torsion tensor
 - i! This is not true for the *symmetric* part.
 - GENERAL RELATIVITY is based on the assumption that the affine connection of spacetime is torsion-free. Hence it is sufficient to focus on symmetric, torsion-free connections to formulate the theory.
 - Interpretation:

On a manifold with torsion, infinitesimal parallelograms do not close:



To see this, consider two infinitesimal vectors δx_1^i and δx_2^i at some point $p \in M$. Then parallel transport δx_1^i along δx_2^i to produce $\delta \tilde{x}_1^i$ and vice versa:

$$\delta \tilde{x}_{1}^{i} = \delta x_{1}^{i} - \Gamma_{kl}^{i} (\delta x_{1}^{k}) (\delta x_{2}^{l}), \qquad (10.42a)$$

$$\delta \tilde{x}_{2}^{i} = \delta x_{2}^{i} - \Gamma^{i}_{\ kl} (\delta x_{2}^{k}) (\delta x_{1}^{l}) \,. \tag{10.42b}$$

The amount by which this infinitesimal parallelogram does not close is:

$$\Delta^{i} := (\delta x_{1}^{i} + \delta \tilde{x}_{2}^{i}) - (\delta x_{2}^{i} + \delta \tilde{x}_{1}^{i}) = (\delta x_{1}^{i} - \delta \tilde{x}_{1}^{i}) - (\delta x_{2}^{i} - \delta \tilde{x}_{2}^{i})$$

$$\stackrel{10.42}{=} \left(\Gamma^{i}_{\ kl} - \Gamma^{i}_{\ lk}\right) (\delta x_{1}^{k}) (\delta x_{2}^{l}) \stackrel{\text{def}}{=} S^{i}_{\ kl} (\delta x_{1}^{k}) (\delta x_{2}^{l}).$$
(10.43)

Non-vanishing torsion therefore implies:

$$\Delta^{i} = S^{i}_{\ kl} \left(\delta x_{1}^{k}\right) \left(\delta x_{2}^{l}\right) \neq 0 \quad \Leftrightarrow \quad S^{i}_{\ kl} \left(\delta x_{1}^{k}\right) \left(\delta x_{2}^{l}\right) \neq S^{i}_{\ kl} \left(\delta x_{2}^{k}\right) \left(\delta x_{1}^{l}\right) \tag{10.44}$$

 \rightarrow The direction of paths matters: First going along δx_1^k and then parallel to δx_2^l leads to a different point than doing the opposite. (Similar to the motion of a screw, which is different for clockwise and counterclockwise rotation.)

- It is possible to extend GENERAL RELATIVITY by allowing the torsion of spacetime to be non-zero (and dynamic as well) [131, 132]. In such theories, the \downarrow spin of particles becomes the source of torsion, just as their mass is the source of \rightarrow curvature. Such theories can predict additional forces between spinful particles, see Ref. [133] for a review.
- Since torsion is "just another tensor field" (which is not true for the symmetric part of the connection), it is reasonable to keep a geometric theory of gravity slim and assume torsion to vanish. If the theory matches observations, we didn't produce unnecessary clutter by dragging torsion along (\checkmark Occam's razor); however, if there happen to be phenomena that cannot be explained, we can still "patch" the theory by adding new (tensor) fields (that might play the role of torsion). In any case, there is no experimental evidence to date that makes a torsion field necessary.



 \rightarrow Henceforth we consider only torsion-free connections:

$$\Gamma^{i}_{\ kl} = \Gamma^{i}_{\ lk}$$

9 | Locally geodesic coordinate systems:

Since we know how the coefficients of a connection transform, we can ask whether there are special coordinate systems in which the connection looks particularly simple:

Details:
Problemset 2

i | Goal:

Show that for every point $p \in M$ there is a coordinate system in which the connection coefficients *in this point* vanish:

$$\forall p \in M \exists \text{ Chart } u \text{ with } u(p) = x_0 : \Gamma^i_{kl}(x_0) = 0 \quad \forall_{ikl}$$
(10.45)

- u: ** Locally geodesic coordinate system
- ii | First, show the alternative form of the transformation: (recall Eq. (3.75))

$$\bar{\Gamma}^{i}_{\ kl} \stackrel{a}{=} \frac{\partial \bar{x}^{i}}{\partial x^{m}} \frac{\partial x^{n}}{\partial \bar{x}^{k}} \frac{\partial x^{o}}{\partial \bar{x}^{l}} \Gamma^{m}_{\ no} - \frac{\partial x^{m}}{\partial \bar{x}^{l}} \frac{\partial x^{p}}{\partial \bar{x}^{k}} \left(\frac{\partial^{2} \bar{x}^{i}}{\partial x^{p} \partial x^{m}} \right)$$
(10.46)

This follows from Eq. (10.39) by differentiating $\frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^j} = \delta^i_j$.

iii | \triangleleft Coordinates v with $v(p) = 0 \in \mathbb{R}^D$ (in general it is $\Gamma_{kl}^i(0) \neq 0$ in this chart)

 \rightarrow Coordinate transformation $\bar{x} = \varphi(x) = u \circ v^{-1}(x)$ in vicinity of $p \in M$:

$$\bar{x}^{i} = x^{i} + \frac{1}{2}C^{i}_{\ kl}(0) x^{k}x^{l} + \dots$$
 (10.47)

with (*w.l.o.g.*) symmetric coefficients $C^{i}_{kl} = C^{i}_{lk}$. iv $| \rightarrow$ Partial derivatives at u(p) = 0 = v(p):

$$2\pi i$$

$$\frac{\partial \bar{x}^{i}}{\partial x^{m}}\Big|_{x=0} = \delta^{i}_{m} \quad \text{and} \quad \frac{\partial^{2} \bar{x}^{i}}{\partial x^{p} \partial x^{m}}\Big|_{x=0} = C^{i}_{pm}(0) \quad (10.48)$$

 $\xrightarrow{\text{Eq. (10.46)}} \bar{\Gamma}^{i}_{kl} \stackrel{\text{e}}{=} \Gamma^{i}_{kl} - C^{i}_{kl}$

 $\mathbf{v} \mid \bar{\Gamma}^{i}_{kl}(0) \stackrel{!}{=} 0 \text{ and } \bar{\Gamma}^{i}_{kl} = \bar{\Gamma}^{i}_{lk} \text{ (torsion-free!)} \rightarrow C^{i}_{kl}(0) := \Gamma^{i}_{kl}(0)$

Notes:

- i! Note that we only showed that the connection coefficients can be made zero *in a single point*; in general one cannot find a coordinate system where the coefficients vanish everywhere. This also implies that in general the *derivatives* $\partial_m \Gamma^i_{kl}(0)$ do *not* vanish in *p*.
- In locally geodesic coordinates, the absolute derivative Eq. (10.37) is simply the "normal" total derivative. As a consequence, in the context of Riemannian manifolds, the coordinate lines are local geodesics ("shortest paths", → *later*) hence the name.
- The above argument fails for connections with non-vanishing torsion $S_{kl}^i \neq 0$ since the latter transforms as a tensor and cannot be zeroed by a coordinate transformation (unless it vanishes in all coordinates).



• The fact that locally geodesic coordinates exist at every point will be the foundation for the implementation of Einstein's equivalence principle **EEP** in the mathematical framework of GENERAL RELATIVITY. Physically, these coordinates will be identified with the free falling, local inertial frames.



↓Lecture 20 [07.05.24]

10.2.1. Covariant derivatives

10 | The definition Eq. (10.37) of the \rightarrow *absolute derivative* did not require $A^i(\lambda)$ to be defined in a neighborhood of the curve $\gamma(\lambda)$. However, if $A^i(\lambda) \equiv A^i(\gamma(\lambda))$ is defined on the whole manifold (or at least in a neighborhood of the curve), we can define a more useful derivative:

$$\frac{\mathrm{d}A^{i}}{\mathrm{d}\lambda} = \frac{\partial A^{i}}{\partial x^{k}} \frac{\mathrm{d}x^{k}}{\mathrm{d}\lambda} \quad \Rightarrow \quad \frac{\mathrm{D}A^{i}}{\mathrm{D}\lambda} \stackrel{10.37}{=} \left(\frac{\partial A^{i}}{\partial x^{k}} + \Gamma^{i}_{mk}A^{m}\right) \frac{\mathrm{d}x^{k}}{\mathrm{d}\lambda} \equiv A^{i}_{;k} \frac{\mathrm{d}x^{k}}{\mathrm{d}\lambda} \quad (10.49)$$

 $\rightarrow *$ Covariant derivative of a contravariant vector:

$$\left\{\begin{array}{c}
D_{k}A^{i} \\
\nabla_{k}A^{i} \\
A^{i}_{;k} \\
\overrightarrow{A^{i}_{;k}} \\
\overrightarrow{A^{i}_{rk}} \\
\overrightarrow{A^{i}_{rk}}$$

$\stackrel{\circ}{\rightarrow} A^{i}_{;k}$ is (1, 1)-tensor

Proof: Via the \leftarrow *quotient theorem* or by straightforward calculation using Eq. (10.39) (\leftarrow *Section 3.6*).

11 | Covariant derivative of a scalar:

$$\Phi_{;k} := \Phi_{,k} \tag{10.51}$$

$\stackrel{\circ}{\rightarrow} \Phi_{;k}$ is (0, 1)-tensor [Proof: Eq. (3.19)]

That the partial derivatives of scalar fields encode geometric objects, and there is no need to use the additional structure of a connection, is a consequence of the fact that scalar fields map to \mathbb{R} and not $T_p M$. Note that it makes sense to talk about a *constant* scalar field $\phi(p) = \phi(q)$ for all $p, q \in M$ without referring to a particular coordinate system or specifying an additional structure!

12 One demands that the \downarrow *Leibniz product rule* is valid for covariant derivatives:

$$(A^{i}B_{i})_{;k} \stackrel{!}{=} A^{i}_{;k}B_{i} + A^{i}B_{i;k}$$
(10.52)

 \rightarrow Covariant derivative of *covariant* vector:

$$B_{i;k} := B_{i,k} - \Gamma^{m}_{\ ik} B_{m} \tag{10.53}$$

Cf. Eq. (10.50): Different summation indices and different sign!

 $\stackrel{\circ}{\rightarrow} B_{i:k}$ is (0, 2)-tensor



Proof. First we note that

$$A^{i}_{;k}B_{i} + A^{i}B_{i;k} \stackrel{10.52}{=} (A^{i}B_{i})_{;k} \stackrel{10.51}{=} (A^{i}B_{i})_{,k} = A^{i}_{,k}B_{i} + A^{i}B_{i,k}$$
(10.54)

since $A^i B_i$ is a scalar. With the definition Eq. (10.50) it follows

$$A^{i}B_{i;k} \stackrel{\circ}{=} A^{i}\left(B_{i,k} - \Gamma^{m}_{\ ik}B_{m}\right) . \tag{10.55}$$

Since this must be true for arbitrary A^i , Eq. (10.53) follows.

13 | Covariant derivatives of higher-rank tensors:

The above structure can be generalized to tensors of arbitrary rank:

$$T^{ik...}_{rs...;l} := T^{ik...}_{rs...,l} \underbrace{+\Gamma^{i}_{ml} T^{mk...}_{rs...} + \dots}_{\forall upper indices} \underbrace{-\Gamma^{m}_{rl} T^{ik...}_{ms...} - \dots}_{\forall lower indices}$$

(10.56)

Example:

Covariant derivatives of rank-2 tensors:

$$T^{ik}_{;l} = T^{ik}_{,l} + \Gamma^{i}_{ml} T^{mk} + \Gamma^{k}_{ml} T^{im} \rightarrow (2, 1) \text{-tensor}$$
(10.57a)
$$T_{ik\cdot l} = T_{ik\cdot l} - \Gamma^{m}_{il} T_{mk} - \Gamma^{m}_{k\cdot l} T_{im} \rightarrow (0, 3) \text{-tensor}$$
(10.57b)

$$T^{i}_{k;l} = T^{i}_{k,l} + \Gamma^{i}_{ml}T^{m}_{\ k} - \Gamma^{m}_{\ kl}T^{i}_{\ m} \to (1,2)$$
-tensor (10.57c)

For a proof, see SCHRÖDER [2] (p. 53).

10.2.2. Parallel vector fields and autoparallel curves

14 $| \triangleleft \text{Vector field } A = A^i \partial_i \& \text{ curve } \gamma$:



• Given a connection Γ , Eq. (10.58) is a first-order differential equation for A^i . By solving it for a given initial value of $A^i (\lambda = 0)$, one can reconstruct a parallel vector field on the curve γ .

• For higher-rank tensors, one defines parallelism along a curve analogously:

$$\frac{\mathbf{D}T^{ik\dots}}{\mathbf{D}\lambda} \stackrel{!}{=} 0 \tag{10.59}$$

15 | Autoparallel curve: Generalization of a straight line in \mathbb{R}^D :

Straight line: Curve that "keeps its direction constant."

We cannot characterize a straight line as "the shortest curve between two points" because we do not have a metric, only a connection!

 \triangleleft Curve γ with parametrization $\gamma^{\mu}(\lambda)$ (in some chart)

 γ is $\ast \ast$ *autoparallel* : \Leftrightarrow Tangent field $A = A^i \partial_i := \frac{d\gamma^i}{d\lambda} \partial_i$ is \leftarrow *parallel* along γ :



Eq. (10.58)

$$\frac{\mathrm{d}^{2}\gamma^{i}}{\mathrm{d}\lambda^{2}} + \Gamma^{i}_{\ kl} \frac{\mathrm{d}\gamma^{k}}{\mathrm{d}\lambda} \frac{\mathrm{d}\gamma^{l}}{\mathrm{d}\lambda} = 0 \quad \Rightarrow \quad \gamma \text{ is } \overset{*}{*} autoparallel$$
(10.60)

- i! If a parametrization of a curve satisfies the DGL Eq. (10.60), the curve is autoparallel and the given parametrization is called *** affine*. Since Eq. (10.60) is *not* reparametrization invariant (→ *below*), there are other (non-affine) parametrizations of the same autoparallel curve that do *not* satisfy Eq. (10.60). Every autoparallel curve has such an affine parametrization (which is unique up to affine transformations).
- Once we have a metric and a compatible connection (→ *Section 10.3*), the autoparallel curves will be identical to the curves of *shortest length* (→ *geodesics*).
- Let us assume that an affine parametrization of an autoparallel curve satisfies Eq. (10.60). Now consider a reparametrization $\mu = f(\lambda)$ given by some strictly monotone function f.

The new parametrization is then $\tilde{\gamma}^i(\mu) = \tilde{\gamma}^i(f(\lambda)) := \gamma^i(\lambda)$ and satisfies the DGL

$$\frac{\mathrm{d}^{2}\tilde{\gamma}^{i}}{\mathrm{d}\mu^{2}} + \Gamma^{i}_{kl}\frac{\mathrm{d}\tilde{\gamma}^{k}}{\mathrm{d}\mu}\frac{\mathrm{d}\tilde{\gamma}^{l}}{\mathrm{d}\mu} \stackrel{\circ}{=} h(\mu)\frac{\mathrm{d}\tilde{\gamma}^{i}}{\mathrm{d}\mu} \quad \text{with} \quad h(\mu) = -\frac{\mathrm{d}^{2}\mu}{\mathrm{d}\lambda^{2}}\left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)^{-2} \,. \tag{10.61}$$

The definition of h is equivalent to the DGL

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2} + h(\mu) \left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)^2 = 0.$$
(10.62)

oret



If λ is an affine parameter, the transformation f yields *another* affine parameter μ if and only if $h(\mu) \equiv 0$, i.e.,

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2} = 0\,,\tag{10.63}$$

which is solved by reparametrizations of the *affine* form $\mu = f(\lambda) = a\lambda + b$. That is, affine parametrizations are unique up to affine *re*parametrizations.

• This problem does not affect the definition of a parallel vector field because Eq. (10.58) *is* reparametrization invariant.

10.2.3. The curvature tensor

Now that we have a formal concept of the parallel transport of vectors from one tangent space to another, we can ask whether the result of such a transport depends only on the final destination, or whether the path of the transport also plays a role. The answer will be that, for a generic connection, parallel transport indeed is path dependent, and that this path dependence is a manifestation of the intrinsic *curvature* of the manifold (more precisely: its connection).

16 $| \triangleleft$ Parallel transport of vector $A = A^i \partial_i$ from q to q' via different paths γ_1 and γ_2 :



 \rightarrow It is easier (and sufficient) to study an infinitesimal parallelogram.

$\mathbf{17} \mid \blacktriangleleft \operatorname{Path} p \xrightarrow{p_1} p':$

The first parallel transport along δx_1 yields:

$$A^{i}(p \xrightarrow{\delta x_{1}} p_{1}) \stackrel{10.34}{=} \underbrace{A^{i}(p) + \delta_{1}A^{i}(p)}_{\equiv A^{i} + \delta_{1}A^{i}} = A^{i}(p) - \Gamma^{i}_{kl}(p)A^{k}\delta x_{1}^{l}$$
(10.64)

The subsequent parallel transport along δx_2 yields:

$$A^{i}(p \xrightarrow{\delta x_{1}} p_{1} \xrightarrow{\delta x_{2}} p') = A^{i}(p \xrightarrow{\delta x_{1}} p_{1}) + \delta_{2}A^{i}(p \xrightarrow{\delta x_{1}} p_{1})$$
(10.65a)
$$= A^{i} + \delta_{1}A^{i} - \Gamma^{i}_{nm}(p_{1}) \left[A^{n} + \delta_{1}A^{n}\right] \delta x_{2}^{m}$$
(10.65b)

$$= A^{*} + \delta_{1}A^{*} - \Gamma_{nm}(p_{1}) \left[A^{*} + \delta_{1}A^{*}\right] \delta x_{2}^{*}$$
(10.65)

Our goal is to express everything in the initial point p. \rightarrow

$$\Gamma^{i}_{nm}(p_1) \approx \Gamma^{i}_{nm}(p) + \partial_l \Gamma^{i}_{nm}(p) \,\delta x_1^l \tag{10.66}$$



(Since we consider an *infinitesimal* parallelogram, we only need linear variations of all quantities.) With this expansion, we find for the parallel vector in p':

$$A^{i}(p \xrightarrow{\delta x_{1}} p_{1} \xrightarrow{\delta x_{2}} p') \stackrel{\circ}{=} A^{i} \underbrace{-\Gamma^{i}_{kl} A^{k} \delta x_{1}^{l}}_{\delta_{1} A^{i}(p)} \underbrace{-\Gamma^{i}_{nm} A^{n} \delta x_{2}^{m}}_{\delta_{2} A^{i}(p)} + \Gamma^{i}_{nm} \Gamma^{n}_{kl} A^{k} \delta x_{1}^{l} \delta x_{2}^{m} - \partial_{l} \Gamma^{i}_{nm} A^{n} \delta x_{1}^{l} \delta x_{2}^{m} + \mathcal{O}\left((\delta x)^{3}\right)$$

$$(10.67)$$

In this expression, all connection coefficients and fields are evaluated in *p*!

18
$$| \triangleleft \underline{\text{Path } p \xrightarrow{p_2} p'}: \text{Same expression with } \delta x_1 \leftrightarrow \delta x_2:$$

$$A^{i}(p \xrightarrow{\delta x_{2}} p_{2} \xrightarrow{\delta x_{1}} p') \stackrel{\circ}{=} A^{i} \underbrace{-\Gamma^{i}_{kl} A^{k} \delta x_{2}^{l}}_{\delta_{2} A^{i}(p)} \underbrace{-\Gamma^{i}_{nm} A^{n} \delta x_{1}^{m}}_{\delta_{1} A^{i}(p)} + \Gamma^{i}_{nm} \Gamma^{n}_{kl} A^{k} \delta x_{2}^{l} \delta x_{1}^{m} - \partial_{l} \Gamma^{i}_{nm} A^{n} \delta x_{2}^{l} \delta x_{1}^{m} + \mathcal{O}\left((\delta x)^{3}\right)$$

$$(10.68)$$

19 \rightarrow Path dependence:

$$\Delta A^{i} := A^{i} \left(p \xrightarrow{\delta x_{1}} p_{1} \xrightarrow{\delta x_{2}} p' \right) - A^{i} \left(p \xrightarrow{\delta x_{2}} p_{2} \xrightarrow{\delta x_{1}} p' \right)$$

$$= \left\{ \begin{array}{l} \text{Change of } A^{i} \text{ after parallel transport along} \\ \text{closed path } p \rightarrow p_{1} \rightarrow p' \rightarrow p_{2} \rightarrow p. \end{array} \right\}$$

$$\text{Drop } \mathcal{O} \left((\delta x)^{3} \right) \text{ terms.}$$

$$\equiv R^{i}_{klm} A^{k} \delta x_{1}^{m} \delta x_{2}^{l} \qquad (10.69)$$

$$\text{nture tensor}$$

with the ** curvature tensor

$$R^{i}_{\ klm} \stackrel{\circ}{=} \partial_{l} \Gamma^{i}_{\ km} - \partial_{m} \Gamma^{i}_{\ kl} + \Gamma^{i}_{\ nl} \Gamma^{n}_{\ km} - \Gamma^{i}_{\ nm} \Gamma^{n}_{\ kl} .$$
(10.70)

Although Γ^i_{kl} is no tensor, this particular combination is a (1, 3)-tensor (Proof: $\rightarrow next$).

20 | Covariant derivatives are defined by an infinitesimal parallel transport. As parallel transport is path pendent, the subsequent application of two covariant derivatives in different directions cannot be commutative. Indeed:

$$A_{k[;l;m]} \equiv A_{k;l;m} - A_{k;m;l} \stackrel{\circ}{=} R^{i}_{\ klm} A_{i} \qquad \stackrel{*}{*} Ricci identity \tag{10.71}$$

 \rightarrow Covariant derivatives of tensors are *not* commutative (in general)!

[Eq. (10.71) is valid in this form only for torsion-free connections.]

 $A_{k;l;m}$ is (0, 3)-tensor $\stackrel{\leftarrow Quotient \ theorem}{\longrightarrow} R^i_{\ klm}$ is (1, 3)-tensor \checkmark

• Alternatively, you can prove the tensorial transformation of $R^i_{\ klm}$ manually using the expression Eq. (10.70) and the transformation of the connection coefficients Eq. (10.39) and partial derivatives Eq. (3.5).



• Compare the non-commutativity of the covariant derivative of tensors with the commutativity of conventional partial derivatives:

$$A_{k[,l,m]} \equiv A_{k,l,m} - A_{k,m,l} = \partial_m \partial_l A_k - \partial_l \partial_m A_k = 0.$$
(10.72)

21 | <u>Notes:</u>

• The curvature tensor can be interpreted geometrically as follows:



Since curvature is the property that vectors parallel transported around infinitesimal loops change their direction, one can encode all features of curvature in an object that tells you how an arbitrary vector is transformed if transported around any infinitesimal parallelogram in the ml-plane. This object is the curvature tensor, and from this perspective it is clear that it must be of rank four (two indices to specify the plane, two for the transformation of the vector).

• (A manifold with) a connection is called *flat* iff the curvature tensor is identically zero everywhere: $R^{i}_{klm}(p) \equiv 0$. In particular, this means (for a torsion-free connection) that in a *neighborhood* of every point on the manifold (and not just the point itself!) you can find a coordinate system in which the connection coefficients vanish identically (i.e., these neighborhoods behave like flat Euclidean space).

In summary, the following statements are equivalent:

- The curvature tensor vanishes identically.
- The manifold is flat.
- Parallel transport is path-independent.
- Covariant derivatives are commutative.
- Whether a space is curved or not is a property of its *connection* and not of its *topology*! For example, here are two topologically equivalent (\uparrow *homeomorphic*) tori ("donuts"):





The left one is defined by identifying opposite edges with each other and inherits the connection of the Euclidean plane. The right torus is embedded in 3D Euclidean space and inherits the metric of \mathbb{R}^3 and its induced connection. Both spaces are topological tori, but the left one is flat whereas the right one is not [as illustrated by the path(in)dependence of parallel transport].

So if someone asks you whether a torus is flat or curved, the correct answer is that this is an undefined question unless a particular connection is specified! (Interestingly, this is not true for the two-dimensional sphere S^2 . While there are many connections you can assign to a 2D sphere, none of them is flat! This is a corollary of the \uparrow *Gauss-Bonnet theorem* or, alternatively, the \uparrow *hairy ball theorem*.)

10.3. Affine connections on Riemannian manifolds

We already know the benefits of a Riemannian manifold (M, g), i.e., a manifold equipped with a (pseudo-)Riemannian metric g. In the previous section, we studied another type of structure that lives on a manifold: a connection Γ . In this section we bring both (a priori independent) concepts together by asking whether, among all possible connections, there are *distinguished* ones on a Riemannian manifold. This will lead us to a connection that can be constructed directly from the metric and plays a central role in GENERAL RELATIVITY.

10.3.1. The LEVI-CIVITA connection

1 Motivation:

In Euclidean space, the parallel transport of two vectors does not change their inner product (in particular, their norm/length remains constant):



 \rightarrow It makes sense to generalize this property to general Riemannian manifolds with a connection.

2 | < Riemannian manifold (M, g) with (pseudo-)Riemannian metric $g_{ii}(x)$

A connection Γ is called a * *metric-compatible* $:\Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\langle A,B\rangle \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{ik}A^{i}B^{k}) \stackrel{\mathrm{10.51}}{=} \frac{\mathrm{D}}{\mathrm{D}\lambda}(g_{ik}A^{i}B^{k}) \stackrel{\mathrm{!}}{=} 0$ (10.73) along any curve $\gamma(\lambda)$ for all *parallel* vector fields A and B along γ .

Recall that for a *scalar* the total and absolute derivative are identical.



A and B parallel vector fields: $\frac{DA^i}{D\lambda} = 0 = \frac{DB^k}{D\lambda} \rightarrow$

Eq. (10.73)
$$\Leftrightarrow \frac{\mathrm{D}g_{ik}}{\mathrm{D}\lambda} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \forall_{i,k,l} : g_{ik;l} \stackrel{!}{=} 0$$
 (10.74)

Use the Leibniz product rule Eq. (10.52) to show this.

 $\rightarrow g_{ij}(x)$ is covariantly constant

3 | <u>Eq. (10.57b)</u> →

$$\partial_l g_{ik} - \Gamma^m_{\ \ il} g_{mk} - \Gamma^m_{\ \ kl} g_{im} \stackrel{!}{=} 0 \tag{10.75}$$

Since Eq. (10.74) holds for arbitrary indices, we also have equations with cyclic permutations:

$$\partial_k g_{li} - \Gamma^m_{\ \ lk} g_{mi} - \Gamma^m_{\ \ ik} g_{lm} \stackrel{!}{=} 0,$$
 (10.76a)

$$-\partial_i g_{kl} + \Gamma^m_{\ ki} g_{ml} + \Gamma^m_{\ li} g_{km} \stackrel{!}{=} 0.$$
 (10.76b)

Adding up the three equations yields

$$\Gamma_{i(kl)} \equiv \Gamma^{m}_{(kl)} g_{mi} \stackrel{!}{=} \frac{1}{2} \left(\partial_{l} g_{ik} + \partial_{k} g_{li} - \partial_{i} g_{kl} \right) + \frac{1}{2} \left(S^{m}_{\ li} g_{mk} + S^{m}_{\ ki} g_{ml} \right) \\ = \frac{1}{2} \left(\partial_{l} g_{ik} + \partial_{k} g_{li} - \partial_{i} g_{kl} \right) + S_{(kl)i}$$
(10.77)

with torsion $S_{li}^{m} = \Gamma_{li}^{m} - \Gamma_{il}^{m}$ and the symmetrized coefficient $\Gamma_{(kl)}^{m} := \frac{1}{2} \left(\Gamma_{kl}^{m} + \Gamma_{lk}^{m} \right)$ and torsion tensor $S_{(kl)i} := \frac{1}{2} \left(S_{kli} + S_{lki} \right)$.

If we assume a torsion-free connection, it is $\Gamma_{i(kl)} = \Gamma_{ikl}$ and $S_{(kl)i} = 0$ so that

$$\Gamma_{ikl} = \frac{1}{2} \left(\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl} \right) \,. \tag{10.78}$$

These are the connection coefficients of the unique Levi-Civita connection.

4 Use symmetry $\Gamma^{i}_{kl} = \Gamma^{i}_{lk}$ (torsion-free!) and definition $\Gamma_{ikl} := g_{im}\Gamma^{m}_{kl}$ Eqs. (10.75) and (10.76)

** Christoffel symbols (of the first kind)	$\Gamma_{ikl} \stackrel{\circ}{=} \frac{1}{2} \left(\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl} \right)$	(10.79a)
** Christoffel symbols (of the second kind)	$\Gamma^{i}_{\ kl} = \frac{1}{2}g^{im}\left(\partial_{l}g_{mk} + \partial_{k}g_{ml} - \partial_{m}g_{kl}\right)$	(10.79b)

i! You *cannot* pull indices up/down inside partial derivatives because the metric itself depends on the coordinates. For example: $g^{im}\partial_l g_{mk} \neq \partial_l (g^{im}g_{mk}) = \partial_l \delta^i_k = 0$.

This torsion-free, metric-compatible connection is unique and called the Levi-Civita connection:

Christoffel symbols Γ^{i}_{kl} = Connection coefficients of the ** *Levi-Civita connection*

• In GENERAL RELATIVITY, we only work with the Levi-Civita connection; i.e., when we use the symbols Γ^i_{kl} , we always refer to the Christoffel symbols Eq. (10.79) (and not to generic coefficients of a [metric-compatible] connection, \rightarrow *below*).



- For a given metric, there are many compatible connections (→ *next*). However, if we demand *in addition* that the connection is symmetric (= torsion-free), there is only one possible choice: the Levi-Civita connection (↑ *Fundamental theorem of Riemannian geometry*).
- The Christoffel symbols are sometimes written as [132, 133]

$$\begin{cases} i\\kl \end{cases} = \frac{1}{2}g^{im}\left(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}\right) .$$
 (10.80)

(Einstein used an "upside down" version of this notation in his original work on GENERAL RELATIVITY, e.g., in Ref. [12].)

Then it follows from Eq. (10.77) that a *general* metric-compatible connection can be written as

$$\Gamma^{i}_{kl} = \Gamma^{i}_{(kl)} + \Gamma^{i}_{[kl]} = \begin{cases} i\\kl \end{cases} + \underbrace{\frac{1}{2} \left(S^{i}_{kl} - S^{i}_{lk} + S^{i}_{kl} \right)}_{=:-K^{i}_{kl}}, \quad (10.81)$$

with $\Gamma^{i}_{[kl]} = \frac{1}{2}S^{i}_{kl}$; the tensor K^{i}_{kl} is known as \uparrow contorsion tensor ("Verdrehungstensor"). The torsion-free Levi-Civita connection is the special case where

$$\Gamma^{i}_{\ kl} = \begin{cases} i\\kl \end{cases}. \tag{10.82}$$

Because we use only the torsion-free Levi-Civita connection in GENERAL RELATIVITY, we don't make use of this notation and only write Γ^i_{kl} .

5 | Interpretation:

For the special case of a 2D manifold embedded in 3D Euclidean space, the Levi-Civita connection can be geometrically interpreted as follows:



i! This illustration is based on an embedding of the manifold into an ambient Euclidean space (which induces a metric on the manifold). Note, however, that the Levi-Civita connection is *intrinsically* defined and does not require such an embedding.



↓Lecture 21 [14.05.24]

- **6** Corollaries:
 - Working with a metric-compatible connection has the benefit that one can pull indices up and down *within* a covariant derivative:

$$T_{i;k} = (g_{im}T^m)_{;k} = \underbrace{g_{im;k}}_{=0} T^m + g_{im}T^m_{;k} \stackrel{10.74}{=} g_{im}T^m_{;k}$$
(10.83)

• The inverse metric is also covariantly constant:

$$g^{ik}_{\ ;l} = 0$$
 (10.84)

To show this, note that $\delta_{i;l}^i = 0$ [Eq. (10.57b)] and use the Leibniz product rule:

$$0 = \delta_{j;l}^{i} = (g^{ik}g_{kj})_{;l} = g^{ik}_{;l}g_{kj} + g^{ik}g_{kj;l} \stackrel{10.74}{=} g^{ik}_{;l}g_{kj}.$$
(10.85)

- 7 | Local inertial coordinates: (Details:) Problemset 2)
 - i | \triangleleft Levi-Civita connection in \leftarrow *locally geodesic coordinates* at *p* ∈ *M*: (For simplicity, we assume that the point *p* has the coordinates *u*(*p*) = 0.)

$$\partial_l g_{ik}(0) \stackrel{10.75}{=} \underbrace{\Gamma^m_{il}(0)}_{=0} g_{mk} + \underbrace{\Gamma^m_{kl}(0)}_{=0} g_{im} = 0 \tag{10.86}$$

 \rightarrow In these coordinates, the metric tensor is constant in linear order:

$$g_{ij}(x) = g_{ij}(0) + \frac{1}{2} \partial_{\alpha} \partial_{\beta} g_{ij}(0) x^{\alpha} x^{\beta} + \mathcal{O}(x^3)$$
(10.87)

ii | $\triangleleft Affine \text{ coordinate transformation: } \bar{x}^i = M^i_{\ j} x^j + b^i \xrightarrow{\text{Eq. (10.39)}} \bar{\Gamma}^i_{\ kl} = 0$

Note that under affine/linear coordinate transformations, the connection coefficients transform like tensors! In particular, if the connection coefficients vanish in one (geodesic) coordinate system, they vanish in all coordinates that can be reached by affine transformations; i.e., geodesic coordinates are not unique!

 \rightarrow Use linear transformation to bring metric of signature (r, s) into the form

$$\bar{g}_{ij}(0) = \operatorname{diag}(\underbrace{+1,\ldots,+1}_{\times r},\underbrace{-1,\ldots,-1}_{\times s}).$$
(10.88)

That this is possible follows from \uparrow *Sylvester's law of inertia*: First, use the symmetry of the metric to diagonalize the matrix $g_{ij}(0)$ by an orthogonal transformation, then use another non-singular transformation to normalize the eigenvalues to ± 1 .

iii | \triangleleft Special case (r = 1, s = 3) = Lorentzian manifold \rightarrow

Metric in * locally inertial coordinates: $\bar{g}_{\mu\nu}(\bar{x}) \stackrel{\bar{x}\to 0}{\approx} \eta_{\mu\nu} + \frac{1}{2} \bar{\partial}_{\alpha} \bar{\partial}_{\beta} \bar{g}_{\mu\nu}(0) \bar{x}^{\alpha} \bar{x}^{\beta}$ (10.89)



- In words: For every point of a Lorentzian manifold there exist coordinate systems such that the metric in this point takes the Minkowski form $\eta_{\mu\nu}$ and is constant in linear order; we call such charts *locally inertial coordinates*.
- Recall that Lorentz transformations are linear and leave the Minkowski metric invariant
 [← Eq. (4.21)]. This implies that locally inertial coordinates are also not unique: You can
 use arbitrary Lorentz transformations without changing the structure of Eq. (10.89).

8 | <u>Useful relations:</u>

Here we list a few identities that will be useful for many calculations in GENERAL RELATIVITY. You prove these relations in \bigcirc Problemset 2.

• The trace of the Christoffel symbols simplifies to

$$\Gamma^{i}_{\ ki} \stackrel{\circ}{=} \frac{1}{2} g^{im} g_{im,k} \,. \tag{10.90}$$

• With the determinant of the metric $g = det(g_{im})$, the *inverse* metric can be written as

$$g^{im} \stackrel{\circ}{=} \frac{1}{g} \frac{\partial g}{\partial g_{im}} \,. \tag{10.91}$$

• With Eqs. (10.90) and (10.91), the trace of the Christoffel symbols takes the simple form

$$\Gamma^{i}_{\ ki} = \frac{1}{2g}g_{,k} = \left(\ln\sqrt{\pm g}\right)_{,k} , \qquad (10.92)$$

such that $\pm g > 0$.

<u>Note:</u> In GENERAL RELATIVITY it is $det(g_{\mu\nu}) < 0$ (because of the Lorentzian signature) and we redefine $g := -det(g_{\mu\nu}) > 0$ to simplify expressions.

• The other trace of the Christoffel symbols can also be written in a compact form:

$$g^{kl}\Gamma^i_{\ kl} \stackrel{\circ}{=} -\frac{1}{\sqrt{g}}\left(\sqrt{g}g^{im}\right)_{,m} . \tag{10.93}$$

• It is straightforward to show the following useful identity:

$$g_{ik}(g^{kl})_{,m} \stackrel{\circ}{=} -(g_{ik})_{,m} g^{kl}$$
 (10.94)

• The *state covariant divergence* of a contravariant vector field is defined as one would expect:

$$A^{i}_{;i} \stackrel{10.92}{=} A^{i}_{,i} + A^{l} (\ln \sqrt{g})_{,l} = \frac{1}{\sqrt{g}} \left(\sqrt{g} A^{i} \right)_{,i}$$
(10.95)

• For the covariant divergence of an *antisymmetric* (2, 0)-tensor there is a similar expression:

$$A^{ik}_{;k} \stackrel{\circ}{=} \frac{1}{\sqrt{g}} \left(\sqrt{g} A^{ik} \right)_{,k} \quad \text{with} \quad A^{ij} = -A^{ji} \,.$$
 (10.96)

• Eq. (10.95) can be used to rewrite the covariant Laplacian (divergence of a gradient) of a scalar:

$$\Delta \phi \equiv \phi^{;i}_{;i} = \frac{1}{\sqrt{g}} \left(\sqrt{g} g^{ik} \phi_{,k} \right)_{,i} \,. \tag{10.97}$$

The differential operator Δ maps scalar functions onto scalar functions and is known as \uparrow *Laplace-Beltrami operator*.



- Generalized divergence theorem:
 - i | \triangleleft Coordinate transformation $\bar{x} = \varphi(x)$

 \rightarrow *D*-dimensional (oriented) volume element (more precisely: volume form) transforms as (\leftarrow *Eq.* (3.39))

$$\mathrm{d}^{D}\bar{x} = \det\left(\frac{\partial\bar{x}}{\partial x}\right)\mathrm{d}^{D}x \tag{10.98}$$

with \checkmark Jacobian determinant det $\left(\frac{\partial \bar{x}}{\partial x}\right)$.

ii | The determinant of the metric transforms in the opposite way ($\leftarrow Eq. (3.54)$):

$$\sqrt{\bar{g}} = \left| \det \left(\frac{\partial x}{\partial \bar{x}} \right) \right| \sqrt{g} \tag{10.99}$$

(Note the absolute value of the Jacobian determinant!)

iii | Hence the product of metric determinant and (oriented) volume element transforms like a pseudo scalar:

$$\sqrt{\bar{g}} \,\mathrm{d}^D \bar{x} = \mathrm{sign} \left[\mathrm{det} \left(\frac{\partial \bar{x}}{\partial x} \right) \right] \sqrt{g} \,\mathrm{d}^D x \;.$$
 (10.100)

Here sign $\left[\det\left(\frac{\partial \bar{x}}{\partial x}\right)\right]$ denotes the sign of the Jacobian determinant, which encodes whether the coordinate transformation is orientation preserving (+1) or not (-1). This makes $\sqrt{g} d^{D}x$ transform like a *pseudo* scalar.

If we are only interested in non-oriented volume elements, or restrict ourselves to orientation-preserving coordinate transformations, Eq. (10.100) simplifies to a true scalar transformation:

$$\sqrt{\bar{g}} \,\mathrm{d}^D \bar{x} = \sqrt{g} \,\mathrm{d}^D x \ . \tag{10.101}$$

This subtlety will not be important in the following and we use Eq. (10.101) henceforth.

iv | Eq. (10.101) is the reason why integrals over scalar quantities $\bar{\phi}(\bar{x}) = \phi(x)$ are forminvariant under arbitrary coordinate transformations if we use the "modified" volume element $\sqrt{g} d^{D}x$ for integration:

$$\int \underbrace{\frac{\mathrm{d}^{N} x \sqrt{g(x)}}{\mathrm{Scalar}}}_{\mathrm{Scalar}} \underbrace{\frac{\phi(x)}{\mathrm{Scalar}}}_{\mathrm{Scalar}} \stackrel{\bar{x}=\varphi(x)}{=} \int \mathrm{d}^{N} \bar{x} \sqrt{\bar{g}(\bar{x})} \,\bar{\phi}(\bar{x}) \tag{10.102}$$

• Using the covariant divergence Eq. (10.95) and the modified volume element Eq. (10.101), we find the generalized form of the divergence theorem

$$\int_{V} \mathrm{d}^{D} x \sqrt{g} A^{i}{}_{;i} \stackrel{\mathbf{10.95}}{=} \int_{V} \mathrm{d}^{D} x \,\partial_{i} \left(\sqrt{g} A^{i}\right) \stackrel{\mathrm{Gauss}}{=} \oint_{\partial V} \mathrm{d}\sigma_{i} \sqrt{g} A^{i} , \qquad (10.103)$$

where ∂V is the surface of V and $d\sigma_i$ denotes the D - 1-dimensional surface element.



10.3.2. The RIEMANN curvature tensor

Now that we identified the special Levi-Civita connection (which can be computed from the metric), we can also express its curvature tensor (then called *Riemann* curvature tensor) in terms of the metric as well: Detailed calculations:
Problemset 3

9 $| \triangleleft$ Locally geodesic coordinates LG:

$$\{R_{iklm}\}^{\mathrm{LG}} = \{g_{ia}R^{a}_{\ klm}\}^{\mathrm{LG}} \stackrel{\mathrm{10.70}}{=} g_{ia}\left(\partial_{l}\Gamma^{a}_{\ km} - \partial_{m}\Gamma^{a}_{\ kl}\right)$$
(10.104)

Recall that the connection coefficients - but not their derivatives - vanish in these coordinates!

10 Now use the explicit form of the Levi-Civita connection to find an expression in terms of the metric:

$$\{R_{iklm}\}^{\text{LG }10.79} \stackrel{10.79}{=} \frac{1}{2} \left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right)$$
(10.105)

- Recall that $g_{ij,k} = 0$ in locally geodesic coordinates [$\leftarrow Eq. (10.86)$].
- This expressions tells us that curvature prevents us from finding coordinates in which the *second* derivatives of the metric vanish.
- **11** | In general coordinates, the expression becomes more complicated:

$$R_{iklm} \stackrel{\circ}{=} \left\{ R_{iklm} \right\}^{\mathrm{LG}} + g_{ab} \left(\Gamma^{a}_{\ kl} \Gamma^{b}_{\ im} - \Gamma^{a}_{\ km} \Gamma^{b}_{\ il} \right)$$
(10.106)

To show this, start from Eqs. (10.70) and (10.79) and use Eqs. (10.75) and (10.94).

12 | Algebraic identities:

• Eqs. (10.105) and (10.106) \rightarrow

$$R_{iklm} = -R_{kilm}, \quad R_{iklm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}$$
(10.107)

In words: the Riemann tensor is *antisymmetric* in the first two and last two indices, but *symmetric* if both pairs of indices are swapped.

• ** First/Algebraic BIANCHI identity:

The cyclic sums of Riemann tensors vanish identically:

$$R_{i\langle klm\rangle} \equiv R_{iklm} + R_{ilmk} + R_{imkl} \stackrel{\circ}{=} 0$$

(10.108)

The same is true for the cyclic sums of arbitrary triples of indices.

The relations Eqs. (10.107) and (10.108) are *identities*, i.e., their validity follows directly from the definition of the Riemann curvature tensor, independent of the specific metric. This means that a Riemann tensor in D-dimensions has less independent components as the naïve count D^4 suggests.

For example, on the D = 4-dimensional spacetime of GENERAL RELATIVITY, at most 20 (and not $4^4 = 256$) numbers are needed to specify R_{iklm} in every point of the spacetime manifold (\bigcirc Problemset 3). [Beware: This does not mean that there are 20 *physical* degrees of freedom in GENERAL RELATIVITY! R_{iklm} is still a tensor and can be modified by arbitrary coordinate transformations without changing its physical content. We will see \rightarrow *later* that GENERAL RELATIVITY has a large gauge group (\rightarrow *diffeomorphism invariance*) so that there are way less physical degrees of freedom than the 20 alluded to above.]

13 | *** Second/Differential* BIANCHI *identity*:

The cyclic sums of covariant derivatives of the Riemann tensor vanish identically:

$$R^{a}_{\ k\langle lm;n\rangle} \equiv R^{a}_{\ klm;n} + R^{a}_{\ kmn;l} + R^{a}_{\ knl;m} \stackrel{\circ}{=} 0$$
(10.109)

Proof. A neat trick to prove tensor relations is to choose a coordinate system in which their derivation is simple, and then use the tensor character of the involved objects to infer the validity of the relation in general coordinates.

Both the Riemann tensor and covariant derivatives are particularly simple in locally geodesic coordinates:

$$\left\{R^{a}_{\ klm;n}\right\}^{\text{LG 10.70}} \stackrel{\text{10.70}}{=} \Gamma^{a}_{\ km,l,n} - \Gamma^{a}_{\ kl,m,n} \ . \tag{10.110}$$

Adding up the cyclic permutations of this expression yields:

$$\left\{ R^{a}_{\ k\langle lm;n\rangle} \right\}^{\text{LG}} = \left\{ R^{a}_{\ klm;n} \right\}^{\text{LG}} + \left\{ R^{a}_{\ kmn;l} \right\}^{\text{LG}} + \left\{ R^{a}_{\ knl;m} \right\}^{\text{LG}}$$
(10.111a)
$$= \Gamma^{a}_{\ km,l,n} - \Gamma^{a}_{\ kl,m,n} + \Gamma^{a}_{\ kn,m,l} - \Gamma^{a}_{\ km,n,l} + \Gamma^{a}_{\ kl,n,m} - \Gamma^{a}_{\ kn,l,m}$$
(10.111b)

Now, since $R^a_{k\langle lm;n\rangle}$ is a tensor and vanishes in one coordinate system, it vanishes in *all* coordinate systems (because tensor components transform linearly under coordinate transformations); thus $R^a_{k\langle lm;n\rangle} = 0$ and we are done.

Notes:

• Remember that commutators [A, B] = AB - BA satisfy the \downarrow *Jacobi identity*:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0.$$
(10.112)

But the \leftarrow *Ricci identity* Eq. (10.71) relates the curvature tensor (not necessarily a Riemannian one, but the connection must be torsion-free) to the commutator of covariant derivatives:

$$A_{k[;l;m]} = A_a R^a_{\ klm} \,. \tag{10.113}$$

Using this, one can derive the second (and also the first) Bianchi identity from the Jacobi identity; see NAKAHARA [134] (p. 269).

14 | Derived tensors:

The following tensors can be derived from the Riemann tensor and will play an important role in the formulation of GENERAL RELATIVITY:

i | The only non-trivial contraction of the Riemann tensor sums one index of the first pair with one index of the second pair (all other contractions vanish due to symmetries):

** RICCI tensor:
$$R_{kl} := R^a_{\ kla} = -R^a_{\ kal}$$
 (10.114)

ii | The Ricci tensor is symmetric:

$$R_{kl} = R_{lk} \tag{10.115}$$



To show this, contract the first Bianchi identity Eq. (10.108),

$$R^{a}_{\ kla} + R^{a}_{\ lak} + R^{a}_{\ akl} = 0, \qquad (10.116)$$

and use $R^a_{\ akl} = 0$ due to the antisymmetry of the Riemann tensor.

 \rightarrow In D = 4 dimensions, the Ricci tensor has 10 algebraically independent components.

iii | We can contract the Ricci tensor to obtain a curvature scalar:

$$\overset{*}{*} \operatorname{Ricci} scalar: \quad R := g^{ab} R_{ab} = R^{a}_{\ a}$$
 (10.117)

iv | ** Contracted BIANCHI identity:

Ricci tensor and -scalar obey an identity that derives from the second Bianchi identity:

$$R^{a}_{\ n;a} = \frac{1}{2}R_{;n} \tag{10.118}$$

Proof. To show this, contract the differential Bianchi identity Eq. (10.109) over *a* and *m*:

$$R_{kl;n} - R_{kn;l} - R_{k\ nl;a}^{\ a} = 0.$$
(10.119)

Tracing out k and l (recall that our connection is metric-compatible, i.e., we are allowed to pull indices up/down inside covariant derivatives) yields:

$$0 = g^{kl} R_{kl;n} - g^{kl} R_{kn;l} - g^{kl} R_{k}^{\ a}_{nl;a}$$
(10.120a)

$$\stackrel{\text{10.114}}{=} R_{;n} - R^{l}_{n;l} - R^{la}_{nl;a}$$
(10.120b)

$$\stackrel{10.114}{=} R_{;n} - R^{l}_{n;l} - R^{a}_{n;a}$$
(10.120c)

$$= R_{;n} - 2R^a_{n;a} \,. \tag{10.120d}$$

• As preparation for GENERAL RELATIVITY, we define another tensor using the Ricci tensor, Ricci scalar, and metric:

** EINSTEIN *tensor:*
$$G_{ij} := R_{ij} - \frac{1}{2}g_{ij}R$$
 (10.121)

For D = 4 on a Lorentzian manifold, this tensor will be used as the left-hand side of the \rightarrow *Einstein field equations*.

vi | The form of Eq. (10.121) is structurally similar to the contracted Bianchi identity. Indeed, Eq. (10.118) immediately implies:

Eq. (10.118)
$$\Rightarrow G^{a}_{i;a} = 0$$
 (10.122)

• Eq. (10.122) will be crucial for the consistency of the → *Einstein field equations* with energy momentum conservation.

• For D = 4 one can show that the Einstein tensor $G_{\mu\nu}$ (besides the metric tensor $g_{\mu\nu}$) is the *only* rank-2 tensor with vanishing (covariant) divergence that one can construct from the metric and its first and second derivatives [135, 136]. This result is known as \uparrow *Lovelock's theorem* and states under which conditions the field equations of GEN-ERAL RELATIVITY (including the cosmological constant) are *unique* (\Rightarrow *later*). The uniqueness of $G_{\mu\nu}$ and Lovelock's theorem impose important constraints on possible extensions (or modifications) of GENERAL RELATIVITY.

10.3.3. Geodesics

In Section 10.2 we defined "straight lines" as curves that keep their direction constant, and formalized this notion as \leftarrow *autoparallel curves*. Now that we have a metric at hand, we can also define "straight lines" as the *shortest* curves connecting two points. We will show now that these two concepts coincide for the metric-compatible, torsion-free Levi-Civita connection induced by the metric:

15 | \triangleleft Length of curve γ connecting two points P_2 and $P_2 [\leftarrow Eq. (3.55)]$:

$$L[\gamma] = \int_{\gamma} \mathrm{d}s = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{g_{ij} \dot{x}^i \dot{x}^j}$$
(10.123)

Here, $x^i(\lambda_{1/2})$ are the coordinates of $P_{1/2}$ in some chart. The right expression is independent of both the parametrization $x^i(\lambda)$ of the curve and the coordinate system.

To see the latter, recall that for a coordinate transformation $\bar{x} = \varphi(x)$ it is

$$\frac{\mathrm{d}\bar{x}^{i}}{\mathrm{d}\lambda} = \frac{\partial\bar{x}^{i}}{\partial x^{m}}\frac{\mathrm{d}x^{m}}{\mathrm{d}\lambda} \quad \text{and} \quad \bar{g}_{ij} = \frac{\partial x^{k}}{\partial\bar{x}^{i}}\frac{\partial x^{l}}{\partial\bar{x}^{j}}g_{kl} \,. \tag{10.124}$$

Remember that the directional derivatives $\dot{x}^i \partial_i$ along a curve are vectors in the tangent space $T_p M$ and transform accordingly. Thus, in the expression Eq. (10.123), the total derivative wrt. λ is important! In contrast to the special coordinate transformations of SPECIAL RELATIVITY (Lorentz transformations), the coordinates x^i themselves do *not* transform as tensors (they transform like $\bar{x} = \varphi(x)$, which is non-linear in general).

16 | "Straight line" from P_1 to $P_2 \equiv$ Shortest curve γ^* (** *Geodesics*) from P_1 to P_2

i! Strictly speaking, we will not study *globally shortest* curves, but curves that *locally extremize* the length functional Eq. (10.123). For now, you can think of geodesics as "shortest curve" connecting two points, but keep in mind that this is not necessarily true (\rightarrow comments below).

 \rightarrow Extremize length over curves starting at P_1 and terminating at P_2 :





$$\mathfrak{L}_{\chi}(x,\dot{x}) := \chi(\underbrace{g_{kl}(x)\dot{x}^k\dot{x}^l}_{=:\gamma})$$
(10.126)

For example: $\chi(x) = \sqrt{x}$ yields the integrand of Eq. (10.123) as Lagrangian.

 \rightarrow More general variation principle:

$$\delta \int_{P_1}^{P_2} \mathrm{d}\lambda \,\mathfrak{L}_{\chi}(x, \dot{x}) = 0 \tag{10.127}$$

Depending on χ , this "action" is no longer reparametrization invariant in general.

18 \mid \rightarrow Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\frac{\partial \mathfrak{L}_{\chi}}{\partial \dot{x}^{i}} \right) - \frac{\partial \mathfrak{L}_{\chi}}{\partial x^{i}} = 0 \quad \Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda} \left(\chi'(y) 2g_{ik} \dot{x}^{k} \right) - \chi'(y) \frac{\partial g_{kl}}{\partial x^{i}} \dot{x}^{k} \dot{x}^{l} = 0 \quad (10.128)$$

19 $| \triangleleft$ Parametrization with $y = g_{ij}(x)\dot{x}^i\dot{x}^j \equiv ||\dot{x}||_x^2 \stackrel{!}{=} 1 = \text{const}$

This choice fixes an *affine parametrization* $\lambda = s$ of the curve γ where the "velocity" $\|\dot{x}\|_x$ is constant. Since we require $\|\dot{x}\|_x = 1$, the "time" λ is equal to the length s of the curve from the start to $x^i(\lambda)$ (up to a constant offset).

Later, on the (pseudo-Riemannian) Lorentzian manifolds of GENERAL RELATIVITY, we will also consider *space-like* geodesics with y < 0; for such curves, you must add an additional minus in the square root of Eq. (10.123) and choose y = -1 = const instead. The rest of the derivation is then completely analogous.

 $\rightarrow \chi'(y) = \text{const} \neq 0 \text{ (strict monotonic!)} \rightarrow$

Eq. (10.128)
$$\Leftrightarrow g_{ik}\ddot{x}^k + g_{ik,l}\dot{x}^k\dot{x}^l - \frac{1}{2}g_{kl,i}\dot{x}^k\dot{x}^l = 0$$
 (10.129)

Note that this differential equation is independent of χ !

Eq. (10.129)
$$\Leftrightarrow g_{ik}\ddot{x}^k + \underbrace{\frac{1}{2}(g_{il,k} + g_{ik,l} - g_{kl,i})}_{\Gamma_{ikl}}\dot{x}^k\dot{x}^l = 0.$$
 (10.130)

20 | Identify Christoffel symbols Eq. (10.79) $\stackrel{\circ}{\rightarrow}$

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{\ kl} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \frac{\mathrm{d}x^l}{\mathrm{d}\lambda} = 0 \qquad \text{** Geodesic equation} \tag{10.131}$$

Solutions of this DGL are called ****** (affinely parametrized) Geodesics.

- 21 | <u>Notes:</u>
 - i! We derived the Geodesic equation by a variational principle *extremizing* the length between two points. This means that geodesics are not necessarily the *shortest* curves between two points. Ignoring the peculiarities of *pseudo*-Riemannian metrics for now (→ Section 11.1),



Geodesics are only *locally* the shortest connections between close by points, but not necessarily *globally*. Put differently: Every shortest path connecting two points is a geodesic, but not every geodesic connecting two points is a shortest path.

An example is a great circle on a sphere connecting two points $(\rightarrow below)$, say the north pole and a point on the equator. The great circle satisfies the geodesic equation everywhere, and is therefore a geodesic. The shortest path connecting the two points is part of the great circle (and therefore also a geodesic). But the "long way around" is certainly not the shortest path (but still a geodesic, as it is also part of the great circle).

- With our derivation we showed that the curves (Geodesics) that solve the geodesic equation Eq. (10.131) not only extremize the *length* Eq. (10.123), but the more general class of "actions" defined by the "Lagrangian" Eq. (10.126). This will be useful \rightarrow *later* when we study the classical mechanics of points on the Lorentzian manifolds of GENERAL RELATIVITY.
- As already discussed previously (in Section 10.2.2), Eq. (10.131) is not invariant under arbitrary but only *affine* reparametrizations $\mu = a\lambda + b$. The geodesic equation therefore not only picks out the locally shortest (more precisely: extremal) curves on the manifold, but selects also a particular way to parametrize them (namely a parameter that is proportional to the length of the curve, i.e., an *affine parametrization*).
- The geodesic equation is a second-order differential equation. As such it has a unique solution $x^i(\lambda)$ for any point p of the manifold and tangent vector in $v_p = v_p^i \partial_i \in T_p M$; in coordinates:

$$\begin{array}{c} x^{i}(0) := x_{p}^{i} \\ \dot{x}^{i}(0) := v_{p}^{i} \end{array} \xrightarrow{\text{Eq. (10.131)}} \quad \text{Geodesic } x^{i}(\lambda) \text{ through } p \text{ in direction } v_{p}.$$
 (10.132)

This is reminiscent of classical mechanics where, given some potential $V(\vec{x})$, Newton's law determines a unque trajectory for every initial position \vec{x}_0 and initial velocity \vec{v}_0 of a test particle by solving the second-order differential equation

$$n\vec{x}'' + \nabla V(\vec{x}) = 0.$$
 (10.133)

However, there is a subtle difference between Eq. (10.133) and Eq. (10.131):

Solutions of Newton's equation of motion are *not* invariant under affine reparametrizations in general. That is, if $\vec{x}(t)$ is a solution of Eq. (10.133), the rescaled trajectory $\vec{y}(t) := \vec{x}(\alpha t)$ is no longer a solution (check this!). Note that the effect of the time rescaling α is to scale the initial velocity: $\vec{y}'(0) = \alpha \vec{x}'(0) = \alpha \vec{v}_0$. Physically, this makes sense: If you throw a ball in the *same direction* with *different velocity*, its trajectory will look *different* in a generic potential.

In conclusion, the solutions of Eq. (10.133) form a family of curves through every point, with many different curves going off in the same direction:



Compare this to the geodesic Eq. (10.131):



Given an (affinely parametrized) geodesic $x^i(\lambda)$, which shoots of from $x^i(0)$ in direction $\dot{x}^i(0) = v_p^i$, the reparametrized curve $y^i(\lambda) := x^i(\alpha\lambda)$ is again a solution (check this!). This new curve has again a rescaled tangent vector at p ("initial velocity"), namely $\dot{y}^i(0) = \alpha \dot{x}^i(0) = \alpha v_p^i$. But the two curves $x^i(\lambda)$ and $y^i(\lambda)$ trace out the *same* curve on the manifold, only with a different parametrization ("speed").

The affine reparametrization symmetry of the geodesic equation therefore leads to a *unique* geodesic shooting off in every direction $v_p \in T_p M$ at every point $p \in M$. Rescaling v_p produces the same geodesic, only with a different parametrization (left sketch):



Note that geodesics emanating from a point can meet and cross each other at other points of the manifold (this depends on the curvature, and therefore the metric).

An example is the sphere (right sketch); its geodesics are great circles. At every point of the sphere there is a unique great circle for every direction. But two great circles shooting off in different directions eventually cross again at the antipode of the point where the started from.

You may wonder: If we know all (unparametrized/projected) geodesics through all points in all directions, do we then know the metric of the manifold? This question is actually of physical significance in GENERAL RELATIVITY, where the geodesics of spacetime correspond to the trajectories of free falling bodies (\rightarrow *later*). In the language of GENERAL RELATIVITY, the question then asks whether one can reconstruct the metric of spacetime by observing enough free falling bodies (asteroids, stars, etc.).

In its strictest sense, the answer to the question is negative. This is easy to see: Consider \mathbb{R}^2 and equip this manifold with (1) the Euclidean metric δ_{ij} , and (2) the Minkowski metric η_{ij} . Since both metrics are constant, their Christoffel symbols vanish identically and the solutions of the geodesic Eq. (10.131) are all straight lines for both metrics. On says that the two metrics are \uparrow geodesically equivalent.

However, in general it turns out that this is a quite subtle question to answer, see Ref. [137]. Note that one must carefully distinguish between *unparametrized* geodesics (you know only the traces of geodesics on the manifold), and *(affinely) parametrized* geodesics (where you know also the lengths along the traces). Despite the example above, it turns out that *generic* metrics *can* be characterized by their geodesics (even unparametrized ones); i.e., two metrics being geodesically equivalent is not the norm but the exception.

Imagine you are given a Riemannian manifold and a machine that, input two nearby points on the manifold, spits out the affinely parametrized geodesic through these points (i.e., a curve with "distance ticks" on it). Using this device, you can reconstruct the Levi-Civita connection on the manifold (i.e., you can use it to parallel transport tangent vectors) via a geometric construction known as ↑ *Schild's ladder* [↑ Misner *et al.* [3] (§10.2, pp. 248–249)].

Fun fact: There is also a science fiction novel called *Schild's Ladder* [138] by the Australian mathematician and Hugo Award winning author GREG EGAN. If you are a fan of hard, mindbending science fiction à la LEM, ASIMOV and HEINLEIN (and not afraid to encounter



concepts from your physics courses in a work of fiction), you might give his novels a try.

• On a Riemannian manifold with a generic metric-compatible connection, that is not necessarily the torsion-free Levi-Civita connection, the coefficients Γ^i_{kl} in Eq. (10.131) are still the Christoffel symbols (which no longer equal the connection). So the geodesic equation on such a manifold still reads (now with the alternative notation for Christoffel symbols to distinguish them from the connection coefficients):

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \begin{cases} i\\kl \end{cases} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \frac{\mathrm{d}x^l}{\mathrm{d}\lambda} = 0.$$
(10.134)

This equation determines the "shortest lines" (geodesics) on the manifold.

By contrast, the "straightest lines" (autoparallels) are determined by the autoparallel equation Eq. (10.60):

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \Gamma^i_{kl} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \frac{\mathrm{d}x^l}{\mathrm{d}\lambda} \stackrel{\mathbf{10.81}}{=} \frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \left[\begin{cases} i\\kl \end{cases} - K^i_{kl} \right] \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \frac{\mathrm{d}x^l}{\mathrm{d}\lambda} = 0 \,. \tag{10.135}$$

Here we used the general form of a metric-compatible connection Eq. (10.81) with the \leftarrow contorsion tensor $K^i_{\ kl}$. Introducing the symmetric part $K^i_{\ (kl)} \stackrel{\circ}{=} \frac{1}{2}(S_l^{\ i}_k + S_k^{\ i}_l)$ of the contorsion tensor yields (for reference see e.g. [139])

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\lambda^2} + \begin{cases} i\\kl \end{cases} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \frac{\mathrm{d}x^l}{\mathrm{d}\lambda} = K^i_{(kl)} \frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \frac{\mathrm{d}x^l}{\mathrm{d}\lambda}.$$
(10.136)

The geodesic equation and the autoparallel equation are therefore equivalent if and only if the symmetric part $K^{i}_{(kl)}$ of the contorsion tensor vanishes (a sufficient, but not necessary, condition is that the torsion S^{i}_{kl} vanishes).

In conclusion, knowing all the geodesics on a manifold only conveys information about the symmetric part of the connection; the *geodesics* know nothing about torsion (but autoparallels do, at least partially). Thus, for a generic metric-compatible connection, there is a difference between "shortest lines" (geodesics) and "straightest lines" (autoparallels).

In GENERAL RELATIVITY, where we only use the torsion-free Levi-Civita connection, we do not have to make this distinction, so that autoparallels and geodesics are the same.

• If the metric $g_{ij}(x)$ is *independent* of a coordinate x^i , Eq. (10.128) implies for the allowed choice $\chi(x) = x/2$

$$p_i := g_{ik} \dot{x}^k = \text{const} \,. \tag{10.137}$$

This "constant of motion" corresponds to the \checkmark *cyclic variable* x^i and can be used to simplify the solution of the geodesic equation.

22 | <u>Geodesic deviation:</u>

Details: Problemset 3

i | \triangleleft Continuous family of nearby (non-crossing) geodesics $\gamma_s^i(t)$:





Define two vectors fields:

$$T^{i} := \frac{\partial \gamma_{s}^{i}(t)}{\partial t}$$
 ("Velocity") and $S^{i} := \frac{\partial \gamma_{s}^{i}(t)}{\partial s}$ ("Deviation"). (10.138)

< Relative acceleration of nearby geodesics:

$$A^{i} := \frac{D^{2}S^{i}}{Dt^{2}} \stackrel{\text{10.49}}{=} T^{n} \left(T^{m}S^{i}_{;m} \right)_{;n} .$$
 (10.139)

The covariant acceleration A^i measures whether two infinitesimally close geodesics "attract" or "repel" each other.

ii | Using the ← *geodesic equation* and the ← *Ricci identity*, one finds:

Eqs. (10.71) and (10.131) \rightarrow

$$\frac{D^2 S^i}{Dt^2} \stackrel{\circ}{=} R^i{}_{jkl} T^j T^k S^l \qquad \text{** Geodesic deviation equation} \tag{10.140}$$

Proof:
Problemset 3

(Note that the geodesic equation can be written as $T^k T^i_{:k} = 0.$)

 \rightarrow Curvature makes parallel geodesics attract/repel each other!

iii | But this looks very much like gravity (more precisely: the *tidal effects* of gravity):



(Note that this sketch is a projection of geodesics from *spacetime* to *space*.)

 \rightarrow Reasonable approach to a *geometric* theory of gravity:

- Free-falling bodies follow *geodesics* in spacetime: → *Chapter* 11
- Masses create *curvature* of spacetime: → *Chapter* 12

↓Lecture 22 [28.05.24]



11. Classical physics on curved spacetime

Our mathematical toolbox is now fully equipped to formulate GENERAL RELATIVITY. In this chapter, we start by assuming a spacetime metric as given, and study how relativistic mechanics and electrodynamics can be formulated on this (curved) spacetime. Where the metric actually comes from will be discussed in the next Chapter 12.

11.1. Spacetime

1 | Setting the stage:

Here are some facts:

- We live in 3 spatial and 1 time dimension.
 For an argument why 3 + 1-dimensional spacetimes are special, recall Section 4.4.
- The **EEP** requires the existence of *← locally inertial coordinates* (*←* Section 10.3.1). Recall that in such coordinates the metric locally looks like the Minkowski metric.
- \rightarrow Spacetime is a \leftarrow 4D Lorentzian manifold:

Spacetime = 4D Lorentzian manifold (M, g)with pseudo-Riemannian metric g of signature (1, 3)

- Henceforth all manifolds are of this type. We indicate this by using Greek indices μ, ν, ... = 0, 1, 2, 4 for tensors; Latin indices i, j, ... = 1, 2, 3 are now reserved for the spatial components of tensors.
- With the metric g we can measure lengths of curves on the spacetime manifold and norms of and angles between vectors in the tangent bundle. There is also a lot of bonus structure: The metric defines a Levi-Civita connection, which, in turn, defines concepts like parallel transport, covariant derivatives, and curvature.
- Note that the *global topology* of *M* is not specified by GENERAL RELATIVITY, e.g., whether *M* is compact in all or some dimensions. For example, the universe could be periodic in one or more spatial dimensions, i.e., it could be a *torus*. While currently there are no observations that indicate a non-trivial topology, such topologies are also not conclusively ruled out and subject to ongoing research [140]. (Note that even assuming a completely flat universe which is consistent with observations does not rule out non-trivial topologies; recall the flat torus in Section 10.2.3.)
- **2** | <u>Geodesics on Lorentzian manifolds:</u>

In Section 10.3.3 we considered generic (pseudo-)Riemannian manifolds. We are now interested in D = 4-dimensional Lorentzian manifolds of signature (1, 3) ("Spacetime"). This comes along with a few peculiarities concerning geodesics on this spacetime:



i | <u>Null cones:</u>

 \triangleleft Tangent space $T_p M$ with basis $\{\partial_{\mu}\}$ induced by \leftarrow *locally inertial coordinates*:



 \rightarrow The * *null cone* is the subset of tangent vectors $v = v^{\mu} \partial_{\mu} \in T_p M$ with

$$\|v\|_{p}^{2} = \mathrm{d}s_{p}^{2}(v, v) = \underbrace{\eta_{\mu\nu}v^{\mu}v^{\nu}}_{\text{locally inertial}} \stackrel{!}{=} 0.$$
(11.1)

- That the null vectors of the Minkowski metric η form a cone was discussed in Section 1.6.
- Recall [← *Eq.* (4.16)] that all other vectors with strictly positive (negative) Minkowski norm are called ← *time-like* (← *space-like*). We adopt this nomenclature for vectors in the tangent spaces of Lorentzian manifolds.
- We call the cone "*null* cone" and not "*light* cone" because the latter term is reserved for a similar but distinct structure on the manifold (→ *below*).
- \rightarrow A Lorentzian metric induces a "null cone texture" on the manifold (\rightarrow *below*).

This means that you can think of a Lorentzian manifold as being covered with little null cones that vary smoothly from point to point (not only their orientation, but also their "opening angle" can vary!). The null cones live in the tangent spaces and indicate which directions on the manifold are time-like, light-like (null), or space-like.

ii | Classification of geodesics:

 \triangleleft Geodesic $\gamma^{\mu}(t)$ in an arbitrary coordinate system and parametrization

We can use the null cone structure to classify geodesics on a Lorentzian manifold. To this end, consider the sign of the norm (squared) of the "velocity vector" of a geodesic:

< Sign of norm of tangent at geodesics:

$$\operatorname{sign} \|\dot{\gamma}(t)\|_{\gamma(t)}^2 = \operatorname{sign} \left[g_{\mu\nu}(\gamma(t))\dot{\gamma}(t)^{\mu}\dot{\gamma}(t)^{\nu} \right]$$
(11.2)

 $\stackrel{\circ}{\rightarrow}$ Eq. (11.2) is ...

• ... independent of the coordinate system.

The tangent vectors $\dot{\gamma}(t)^{\mu}$ contracted with the metric tensor yield a scalar.

• ... constant along the geodesic.

It is easy to check by straightforward calculation that the norm of the tangent vector is



constant along a geodesic:

$$\frac{\mathrm{d}\|\dot{\gamma}\|_{\gamma}^{2}}{\mathrm{d}\lambda} = g_{\mu\nu,\sigma}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\dot{\gamma}^{\sigma} + 2g_{\mu\nu}\dot{\gamma}^{\mu}\ddot{\gamma}^{\nu} \tag{11.3a}$$

$$= 2g_{\mu\nu,\sigma}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\dot{\gamma}^{\sigma} - g_{\nu\sigma,\mu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}\dot{\gamma}^{\sigma} + 2g_{\mu\nu}\dot{\gamma}^{\mu}\ddot{\gamma}^{\nu}$$
(11.3b)

$$=2\dot{\gamma}^{\mu}\left(g_{\mu\nu,\sigma}\dot{\gamma}^{\nu}\dot{\gamma}^{\sigma}-\frac{1}{2}g_{\nu\sigma,\mu}\dot{\gamma}^{\nu}\dot{\gamma}^{\sigma}+g_{\mu\nu}\ddot{\gamma}^{\nu}\right)$$
(11.3c)

$$\stackrel{10.129}{=} 0$$
 (11.3d)

 $\rightarrow \|\dot{\gamma}\|_{\gamma}^2 = \text{const along a geodesic } \gamma.$

This of course immediately follows from our observation that geodesics are autoparallel curves, together with the metric-compatibility of the Levi-Civita connection.

• ... invariant under reparametrizations.

The independence of the sign on the parametrization of the curve is easy to show if one remembers that a reparametrization $\tilde{\gamma}(\tau) = \gamma(t)$ is given by a *strictly monotone* function $\tau = \tau(t)$:

$$\operatorname{sign}\left[\frac{\mathrm{d}\gamma(t)_{\mu}}{\mathrm{d}t}\frac{\mathrm{d}\gamma(t)^{\mu}}{\mathrm{d}t}\right] = \operatorname{sign}\left[\frac{\mathrm{d}\tilde{\gamma}(\tau)_{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}\tilde{\gamma}(\tau)^{\mu}}{\mathrm{d}\tau}\underbrace{\left(\frac{\mathrm{d}\tau}{\mathrm{d}t}\right)^{2}}_{>0}\right] = \operatorname{sign}\left[\frac{\mathrm{d}\tilde{\gamma}(\tau)_{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}\tilde{\gamma}(\tau)^{\mu}}{\mathrm{d}\tau}\right].$$
(11.4)

Note that the norm of the "velocity vector" itself (without the sign) *does* depend on the parametrization! This makes sense if you think of the parameter as time: Changing how you measure time of course changes how you measure velocity.

 \rightarrow sign $\|\dot{\gamma}\|_{\nu}^2$ characterizes geodesics:

$$\begin{array}{l} \gamma \text{ time-like} \\ \gamma \text{ light-like (or null)} \\ \gamma \text{ space-like} \end{array} \right\} \quad :\Leftrightarrow \quad \text{sign } \|\dot{\gamma}\|_{\gamma}^{2} = \begin{cases} +1 \\ 0 \\ -1 \end{cases}$$
(11.5)

Hence there are three types of geodesics on a Lorentzian manifold.

- We adopt the same nomenclature also for spacetime curves that are *not* geodesics. In this case, the claim that the sign is constant along the curve is not (necessarily) the consequence of some dynamical law, but simply a feature of a particular curve.
- On the *D* = 4-dimensional spacetime of GENERAL RELATIVITY, the time-like geodesics correspond to possible trajectories of free-falling bodies (also: possible time axes). The light-like geodesics are the trajectories of, well, light rays. Space-like geodesics are the analog of "straight lines" in space.
- There is a subtlety regarding light-like/null geodesics: Since their "velocity" vanishes (by definition), their length Eq. (10.123) vanishes as well. As a consequence, we cannot use their length s as an affine parameter λ. To see what goes wrong, note that for χ(y) = √y setting y = ||yµ|_ν² = 0 in Eq. (10.128) is undefined (division by zero).

Luckily, this is only a technical inconvenience. Recall that in our setting, the equations for autoparallel curves Eq. (10.60) and geodesics Eq. (10.131) are identical. While the *norm* $\|\dot{\gamma}\|_{\nu}^2$ of a null vector vanishes, the *vector* itself $\dot{\gamma}^i$ is a perfectly normal vector in



the tangent space (courtesy of $g_{\mu\nu}$ being a *pseudo*-metric). We then can simply fall back to the autoparallel equation Eq. (10.60) to describe null geodesics. The only difference is then that the affine parameter of light-like solutions of Eq. (10.60) (or, equivalently, Eq. (10.131)) cannot be interpreted as the length along the geodesic anymore.

iii | Light cones:

 \triangleleft Point/Event $E \in M$; Draw all null geodesics emanating from E

 $\rightarrow **$ *Light cone* of *E*:



Notes:

• Null geodesics remain null everywhere, i.e., their tangent vectors at every point lie on the null cone of the corresponding tangent space. Since the metric is Lorentzian (but otherwise arbitrary) the null cones can point in "different directions" at different points, so that the light cone can be warped and deformed.

In summary: The *null* cones live in the tangent spaces attached to the manifold, the *light* cone lives on the manifold itself and warps according to the local null cones (and thereby the metric).

- Note how all null cones on the future light cone point "inward", whereas all null cones on the past light cone point "outward". They act like unidirectional "pores" in a membrane that allow time-like trajectories (not necessarily geodesics) to *leave* the past light cone and *enter* the future light cone (but not to other way around).
- All time-like geodesics through *E* stay within its past- and future light cone. Conversely, all space-like geodesics remain outside of this light cone.

Note that because of curvature in the metric [\leftarrow Eq. (10.140)], geodesics can "attract" each other; in particular, two time-like geodesics emanating from a common event might cross again at another event! (Example: Imagine two satellites orbiting earth on the same orbit in opposite directions. Both are falling freely and – according to GENERAL RELATIVITY– follow geodesics in spacetime. But they periodically meet each other, i.e., their geodesics cross in spacetime repeatedly.)

• Not every time-/light-/space-like *curve* is a time-/light-/space-like *geodesic*! Here is an example of a completely light-like curve on Minkowski space (in inertial coordinates):

$$\gamma^{\mu}(\lambda) = (\lambda, \cos(\lambda), \sin(\lambda), 0) .$$
(11.6)



Indeed:

$$\|\dot{\gamma}\|_{\gamma}^{2} = \eta_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu} = 1 - \left[\sin^{2}(\lambda) + \cos^{2}(\lambda)\right] = 0.$$
(11.7)

On Minkowski space, all geodesics are straight lines in inertial coordinates (because the Christoffel symbols vanish in them); the helical curve above clearly isn't a straight line, i.e., it is no geodesic but still *null* everywhere.

• The null cone texture (also called a ↑ *cone field*) induces a ← *partial order of events*, which encodes a ← *causality structure* on the spacetime manifold (recall Section 1.6 for the case of Minkowski space). Up to a local (conformal) deformation of time- and length scales, this structure is essentially *equivalent* to the Lorentzian metric [141]! This suggests the intriguing possibility that the null cone texture (equivalently: the causal structure of events) might be the truly fundamental field of GENERAL RELATIVITY, and the Lorentzian metric is just a convenient tool to encode it.

(Note that a local "stretching" of the metric by a strictly positive scalar field, $\Omega(x)g_{\mu\nu}(x)$, does neither alter the null cone texture nor angles between tangent vectors, thus it is a \uparrow *conformal transformation*. This is why one says that the null cone texture determines the \uparrow *conformal class* of the Lorentzian metric.)

For more details on Lorentzian manifolds, null cones, light cones, and the causal structure of spacetime, see the monograph [142].

 By comparison, in flat Minkowski space all geodesics are straight lines and never cross twice:



Note how the *null cone* (which lives in the tangent space) of the reference event coincides with its *light cone* (which lives on the manifold). Mathematically, Minkowski space $\mathbb{R}^{1,3}$ is not just a Riemannian manifold (with Minkowski metric η) but also an \checkmark *affine space*; this allows for a natural embedding of its tangent spaces into the manifold itself. Minkowski space is therefore a rather "degenerate" case of a generic spacetime and is not well suited to carve out the essential features of GENERAL RELATIVITY.

• Remember that there are locally inertial coordinates for every *point* of the manifold where (1) the Christoffel symbols vanish and (2) the metric has the Minkowski form (Section 10.3.1). This concept can be generalized:

For every *geodesic*, there is a coordinate system (defined in a "tube" around the geodesic) such that on the geodesic, the metric takes the Minkowski form and the Christoffel



symbols vanish (and so do the first derivatives of the metric). Such coordinates are called \uparrow *Fermi normal coordinates* [143] and are useful for a freely falling observer to describe physics along (and close to) its time-like geodesics (which is then the time-axis of these coordinates).

iv | Extremal properties of geodesics:

We defined geodesics by a variational principle Eq. (10.125). Hence they *extremize* their Riemannian length *locally*. Since null geodesics have vanishing length, we focus here on time-like and space-like geodesics.

 \triangleleft Length of time/space-like geodesics γ :

Proper time:
$$L_{\text{Time}}[\gamma] = \int_{\gamma} \underbrace{\sqrt{+g_{\mu\nu}dx^{\mu}dx^{\nu}}}_{\sim c^2 dt^2 - (dx^2 + dy^2 + dz^2) > 0}$$
 (11.8a)

Proper distance:
$$L_{\text{Space}}[\gamma] = \int_{\gamma} \underbrace{\sqrt{-g_{\mu\nu} dx^{\mu} dx^{\nu}}}_{\sim (dx^2 + dy^2 + dz^2) - c^2 dt^2 > 0}$$
 (11.8b)

Recall that the metric has signature (1, 3) = (+, -, -, -). The expressions below the integrals are valid approximately in locally inertial coordinates.

Local variations (in locally inertial coordinates; geodesic *w.l.o.g.* along coordinate axis):



Time-like geodesics are local *maxima* of *proper time*. *Space-like* geodesics are local *saddle points* of *proper distance*.

3 | Proper time:

Which quantity corresponds to the time interval $\Delta \tau$ (*proper time*) measured by an ideal clock in GENERAL RELATIVITY?

Requirements:

• Correspondence principle:

GENERAL RELATIVITY must reduce to SPECIAL RELATIVITY if the spacetime manifold is flat Minkowski space: $(M, g) = (\mathbb{R}^4, \eta) = \mathbb{R}^{1,3}$.



Remember that we established in Section 2.4 that the time measured by a clock moving along an arbitrary *time-like* trajectory $\gamma^{\mu} : [\lambda_a, \lambda_b] \to \mathbb{R}^{1,3}$ in Minkowski space is given by [Eqs. (2.25) and (4.14)]

$$\Delta \tau[\gamma] = \frac{1}{c} \int_{\lambda_a}^{\lambda_b} d\lambda \sqrt{\eta_{\mu\nu} \dot{\gamma}^{\mu} \dot{\gamma}^{\nu}} \,. \tag{11.9}$$

This expression is valid in global inertial coordinate systems.

• General covariance:

Following **GRP**, the expression for $\Delta \tau$ must be a geometric property of the trajectory γ that depends on the metric of the spacetime manifold, but not on the chosen coordinates and/or parametrization of the curve.

These conditions suggest the following definition of the propert time in GENERAL RELATIVITY:

 \triangleleft Clock following *time-like* trajectory $\gamma : [\lambda_a, \lambda_b] \rightarrow M$ on arbitrary spacetime (M, g):

;! γ is not required to be a geodesic.

 \rightarrow ** *Proper time* measured by this clock:

$$\Delta \tau[\gamma] := \frac{1}{c} \int_{\lambda_a}^{\lambda_b} d\lambda \sqrt{g_{\mu\nu}(\gamma(\lambda))} \dot{\gamma}^{\mu} \dot{\gamma}^{\nu} \equiv \frac{1}{c} \int_{\gamma} \sqrt{g_{\mu\nu}(x)} dx^{\mu} dx^{\nu}$$
(11.10)

- Note that, because γ is a *time-like* curve by assumption, the expression under the squareroot is always strictly positive.
- That Eq. (11.10) is the correct expression for the reading of ideal clocks following arbitrary time-like trajectories on arbitrary spacetimes is reasonable, but it is not a "mathematical necessity" it is a *prediction* of GENERAL RELATIVITY that can be experimentally assessed by its physical implications (→ *later*).

This is actually a rather subtle point: What *is* an ideal clock? The only sensible thing to do is to *declare* any dynamic physical process that counts time according to Eq. (11.10) as an ideal clock. That ideal clocks measure Eq. (11.10) becomes then a tautology and physically vacuos. That physical systems *exist* that (up to some limiting acceleration) measure Eq. (11.10) as predicted by GENERAL RELATIVITY is not, however. Atomic clocks, for instance, turn out to be rather good and robust approximations of ideal clocks, whereas pendulum clocks are very sensitive to accelerations and quickly deviate from Eq. (11.10). This deviation, however, is not a feature of time itself, but a consequence of the particular dynamical law governing the motion of a pendulum under accelerated motion. Conversely, to verify that atomic clocks do not suffer from such effects, and therefore are good proxies for measuring proper time, one can check whether their readings match the predictions of GENERAL RELATIVITY for $\Delta \tau$ in various situations, e.g., in the presence of gravitational fields (\Rightarrow *later*). There are also more direct, operational procedures (using light rays and freely falling test particles) to assess how closely a physical process resembles an ideal clock [144] (\Rightarrow *below*).

• Here is an analogy to demystify clocks: The voltage U_{ab} between two points \vec{r}_a and \vec{r}_b is given by the line integral of the electric field $\vec{E}(\vec{r})$:

$$U_{ab} = -\int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{r} .$$
(11.11)



A \checkmark voltmeter is a measurement device that exploits electrodynamic processes to measure U_{ab} , and thereby a particular property of the electromagnetic field. Voltmeters are no magical devices that, by decree, always measure the quantity Eq. (11.11) (this would be an *ideal* voltmeter, which, unfortunately, you cannot buy). A "good" voltmeter is a device that exploits a physical process such that its output correlates with Eq. (11.11) for a wide range of voltages; however, if you exceed its voltage ratings, this is no longer true and the readings are no longer reliable.

Similarly, clocks are measurement devices that exploit some physical process to produce outputs that correlate with the quantity Eq. (11.10), and thereby measure a property of the metric field $g_{\mu\nu}(x)$. An ideal clock does so for all curves γ in all conceivable metric fields $g_{\mu\nu}$; a "good" clock (like an atomic clock) does so approximately under a wide variety of circumstances, while a "bad" clock (like a pendulum clock) has only a very narrow range of applicability (e.g., unaccelerated trajectories).

- According to our discussion above, time-like geodesics correspond to trajectories of clocks along which they run *fastest*. This generalizes our discussion of the twin "paradox" in Section 2.4, where we concluded that the twin staying home (in an inertial system, we ignore the gravitational effects of Earth) ages quicker than the one following an accelerated trajectory with his rocket. In our new reading, the earth-bound twin follows a *geodesic* in Minkowski space; by contrast, the rocket-twin follows a *non-geodesic* time-like curve in Minkowski space.
- In specific coordinate systems, the integral Eq. (11.10) can look simpler.
 - For example, one can always choose a coordinate system \hat{x}^{μ} with the clock fixed in the origin $\vec{0}$ (recall the discussion in Section 9.2 about the role of coordinates in GENERAL RELATIVITY). In such coordinates, the proper time integral simplifies to

$$\Delta \tau[\gamma] = \frac{1}{c} \int_{x_a^0}^{x_b^0} d\hat{x}^0 \sqrt{\hat{g}_{00}(\hat{x}^0, \vec{0})} \equiv \int_{\tau_a}^{\tau_b} d\tau , \qquad (11.12)$$

so that the proper time interval is given by

$$d\tau = \frac{1}{c} \sqrt{\hat{g}_{00}} \, d\hat{x}^0 \quad \text{with} \quad \hat{g}_{00} > 0 \,. \tag{11.13}$$

But why stop there? Nothing prevents you from locally "stretching" and "squeezing" the time coordinate $d\tilde{x}^0 := \sqrt{\hat{g}_{00}} d\hat{x}^0$ to absorb the time-dependence of the metric so that

$$\mathrm{d}\tau = \frac{1}{c}\mathrm{d}\tilde{x}^0\,,\tag{11.14}$$

and thereby

$$\Delta \tau[\gamma] = \frac{1}{c} \int_{\tilde{x}_a^0}^{\tilde{x}_b^0} \mathrm{d}\tilde{x}^0 \,. \tag{11.15}$$

Such coordinates can be systematically constructed (↑ *proper reference frames*) for clocks on arbitrary time-like trajectories ("observers"); see MISNER *et al.* [3] (§13.6, pp. 327–332) for details.

Note that the evaluation of $\Delta \tau$ is not simplified by Eq. (11.15) in general because you must know the integral boundries $\tilde{x}_{a,b}^0$ in these coordinates (which is tantamount to knowing $\Delta \tau$).

4 | <u>Radar coordinates:</u>

In GENERAL RELATIVITY, coordinates are mathematical artifices that are used to catalog events, while preserving their local causal relations ("continuity"). In contrast to the inertial coordinate



systems of SPECIAL RELATIVITY, there is *no operational meaning* associated to most coordinates! To find special coordinate charts that have a physical interpretation (at least in some region of spacetime) one can proceed the other way around: Define an operational procedure that assigns four numbers to events in spacetime; this procedure then defines a particular kind of coordinate system that can be identified with measurable quantities by construction. A particularly simple example of such coordinates are *radar coordinates*:

This discussion is based on Ref. [145].

 $i \mid \triangleleft \text{Observer} \equiv \text{Clock following a time-like trajectory } \gamma(t) : [a, b] \rightarrow M$

i! The trajectory does not need to be a geodesic. The clock displays t along γ – but the parameter t is not required to be an affine parameter (in particular: proper time). If t does equal proper time Eq. (11.10) along γ (up to some offset), we call the clock an \leftarrow *ideal clock*.

 $\mathbf{ii} \mid \blacktriangleleft \text{Event } E \in M$

 \triangleleft Light signals emitted at $\gamma(t_1)$ & reflected at *E* & received at $\gamma(t_2)$:



 \rightarrow ** *Radar coordinates* (*T*, *R*, θ , φ):

$T := \frac{1}{2}(t_2 + t_1)$	** Radar time	(11.16a)

- $R := \frac{c}{2}(t_2 t_1) \qquad \stackrel{\text{\tiny \scalar}}{=} Radar \, distance \tag{11.16b}$
- $\theta := \langle \Psi Altitude \text{ of reflection at } \gamma(t_2) \rangle$ (11.16c)
- $\varphi := \langle \Psi Azimuth \text{ of reflection at } \gamma(t_2) \rangle$ (11.16d)

To define the altitude and azimuth, one must fix a smooth orthonormal \uparrow *tetrad* along γ .

Note that one can really measure (T, R, θ, φ) : Think of *E* as a point on the trajectory of a space probe flying away from Earth. You can periodically send directional radar pulses – that are reflected by the space probe – and use your Earth-bound clock to measure t_1 and t_2 (together with the angles θ and φ of the received reflection).

iii | Radar coordinates *cannot* cover all of spacetime in general!

The method can fail to assign coordinates to events E that are "shadowed" by other objects, or because γ and E are separated by an event horizon (e.g., a \uparrow *Rindler horizon*). It can also happen that the assignment is not unique if there are different null geodesics from $\gamma(t_1)$ to E and/or from E to $\gamma(t_2)$; this can happen due to spacetime curvature (\Rightarrow gravitational


lensing). However, one can show that there is always a finite "tube" around γ where these peculiarities can be excluded.

 \rightarrow Radar coordinates cover a "tube" around γ :



iv | Now that we can construct radar coordinates in the vicinity of a clock γ , we can perform the following experiment to check whether this clock is an \leftarrow *ideal clock* [i.e., the parameter t measures proper time Eq. (11.10)] [144]:



- **a** | Eject two free falling space probes along trajectories μ and $\overline{\mu}$ at t_0 .
- $\mathbf{b} \mid \text{ Track their trajectories with radar pulses} \rightarrow$

$$\mu = (R(T), T)$$
 and $\bar{\mu} = (\bar{R}(\bar{T}), \bar{T})$ (11.17)

(We omit the polar coordinates.)

c | γ is an ideal clock at t_0 iff $\|\dot{\gamma}(t_0)\|_{\gamma(t_0)} = c$ since [\leftarrow Eq. (11.10)]

$$d\tau = \frac{1}{c}ds = \frac{1}{c}\sqrt{g_{\mu\nu}dx^{\mu}dx^{\nu}} = \frac{1}{c}\sqrt{g_{\mu\nu}\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}}dt = \frac{1}{c}\underbrace{\|\dot{\gamma}\|_{\gamma}}_{c}dt = dt.$$
 (11.18)

One can show [144, 145] that this is the case if and only if

$$\left[\frac{R_T''}{1 - (R_T')^2}\right]_{T=t_0} = -\left[\frac{\bar{R}_{\bar{T}}''}{1 - (\bar{R}_{\bar{T}}')^2}\right]_{\bar{T}=t_0}$$
(11.19)

with the shorthand notation $R'_X \equiv \frac{dR}{dX}$.

This provides an operational procedure to check (in principle) that atomic clocks indeed measure the proper time Eq. (11.10) and are therefore ideal clocks.



5 | Simultaneity:

$\mathbf{i} \mid \triangleleft$ Two nearby clocks γ and $\tilde{\gamma}$:

We assume that they are within each others "tube" where radar coordinates can be defined.



We say:

<i>B</i> is <i>** Einstein synchronous</i> to <i>A</i>	:⇔	$T(\tilde{t}) = \frac{1}{2}(t_2 + t_1) = \tilde{t}$	(11.20a)
A is $*$ <i>Einstein synchronous</i> to B	:⇔	$\tilde{T}(t) = \frac{1}{2}(\tilde{t}_2 + \tilde{t}_1) = t$	(11.20b)

- Note that t
 i is measured by clock B while T(t
 i) is computed from t
 t t
- This is simply ← *Einstein synchronization* in SPECIAL RELATIVITY (← Section 1.1 and
 Problemset 1 last semester) generalized to arbitrary spacetimes. The synchronization constraint Eq. (11.20) is also known as ** Radar synchronization*.
- Recall that Einstein synchronization in SPECIAL RELATIVITY (i.e., on Minkowski space) could be proven to be *symmetric* and *transitive* for clocks at rest in the same inertial frame (
 Problemset 1 last semester). In the more general situation considered here, Einstein synchronization is neither symmetric nor transitive (note that, in general, there is no inertial frame that encompasses both clocks), see [145].
- One can show [145] that if the synchronization is *symmetric* (as any good synchronization should be), then the radar distances $R(\tilde{t})$ from A to B and $\tilde{R}(t)$ from B to A are necessarily constant and equal: $R(\tilde{t}) \equiv \tilde{R}(t) \equiv \text{const.}$
- ii | We now want to study possible obstructions to synchronizing clocks in curved spacetimes. For simplicity, we consider two *infinitesimally* separated clocks (right sketch).

This calculation follows LANDAU & LIFSHITZ [146] (§84, pp. 233–236).

 \triangleleft Two infinitesimally close clocks *A* and *B* separated by $d\vec{x} = \{dx^n\}$:

4 synchronous to
$$B \xrightarrow{\text{Eq. (11.20)}} x^0 + \underbrace{\delta x^0}_{\text{Offset}} = \frac{1}{2} (\tilde{x}_2^0 + \tilde{x}_1^0)$$
(11.21)

• We assume that the position of the clocks is labeled by x^m and their reading corresponds to the coordinate x^0 , i.e., these are *coordinate* clocks. They are *not* required to be ideal



clocks ticking off proper time; the following argument therefore applies to arbitrary coordinate systems with time-like coordinate x^0 and space-like coordinates $x^{1,2,3}$.

We say that a coordinate x^0 is time-like (at a given point $p \in M$) if the curve on M defined by varying x^0 and keeping x^m constant is a time-like curve in p (in a similar way we define x^m to be space-like). Mathematically, this means that $\partial_0 \equiv \frac{\partial}{\partial x^0} \in T_p M$ is a time-like vector:

$$\|\partial_0\|_p^2 = g_p(\partial_0, \partial_0) = g_{\mu\nu}(p) \underbrace{\mathrm{d}x^{\mu}(\partial_0)}_{\delta_0^{\mu}} \underbrace{\mathrm{d}x^{\nu}(\partial_0)}_{\delta_0^{\nu}} = g_{00} > 0.$$
(11.22)

Not every coordinate system has this property, but because of the Lorentzian signature of g, there are always (many) such coordinate systems. These coordinate systems are useful because their time-axis corresponds (at least locally) to possible trajectories of physical bodies (not necessarily free-falling ones). Thus one can think of the coordinate x^0 as *the time* (not necessarily proper time) measured by some (not necessarily free-falling) clock tracing out the time-axis through spacetime. Note that a coordinate can be time-like in one region of spacetime, become null at some point, and then space-like in another region. So the "type" of a coordinate is not fixed like that of a geodesic. Note also that not every coordinate system is guaranteed to have a time-like coordinate at all (this is possible for non-orthogonal coordinates which are rarely used).

• Note that the offset δx^0 could be absorbed into one of the clocks by shifting its reading (corresponding to a coordinate transformation). But we can also simply agree that two events at *A* and *B* are simultaneous iff their local clocks differ by $\delta x^0 (\rightarrow below)$. This is a generalization of the synchronization condition Eq. (11.20) with no downsides, at least for the comparison of two clocks.

Let w.l.o.g. $\tilde{x}_i^0 \equiv x^0 + dx_i^0 \rightarrow$

$$x^{0} + \delta x^{0} = x^{0} + \underbrace{\frac{1}{2}(\mathrm{d}x_{2}^{0} + \mathrm{d}x_{1}^{0})}_{\delta x^{0}}$$
(11.23)

iii | For the light signals used to synchronize the clocks we have:

$$ds^{2} = g_{00} (dx^{0})^{2} + 2g_{0m} dx^{0} dx^{m} + g_{mn} dx^{m} dx^{n} \stackrel{!}{=} 0$$
(11.24)

Here we separated the temporal from the spatial components n, m = 1, 2, 3.

Solve for $dx^0 \equiv dx_i^0 \xrightarrow{\circ} dx_i^0 x_i^0 x_i^$

$$dx_i^0 = \frac{1}{g_{00}} \left[-g_{0m} dx^m \mp \sqrt{(g_{0m}g_{0n} - g_{mn}g_{00}) dx^m dx^n} \right]$$
(11.25)

The minus sign corresponds to $dx_1^0 < 0$.

iv | Eq. (11.23) <u>Eq. (11.25)</u>

$$\delta x^{0} = -\frac{g_{0m}}{g_{00}} \mathrm{d} x^{m} \quad \Leftrightarrow \quad g_{00} \delta x^{0} + g_{0m} \mathrm{d} x^{m} = 0 \tag{11.26}$$

• *Interpretation:* δx^0 is the difference of the reading of two infinitesimally nearby clocks A at \vec{x} and B at $\vec{x} + d\vec{x}$ that indicates the time of two events happening *simultaneously* according to Einstein synchronization.



- You can think of δx⁰ as a ← connection relating nearby clocks (cf Eq. (10.34)): If A displays the time x⁰, we consider the time x⁰ + δx⁰ = ½(x̃₂⁰ + x̃₁⁰) displayed by B as "equal" (in the terminology of connections: "parallel", here better: simultaneous). If δx⁰ ≠ 0, the change in reading of nearby clocks is considered "fake"; this is not a problem in principle: If you have two clocks where the reading t of one always coincides with the reading t + Δt of another, you don't loose anything and can consider them as being synchronized (as long as you know what Δt is). However, there is a problem coming from δx⁰ ≠ 0 if you consider different paths in space to synchronize your clocks (→ next).
- $\mathbf{v} \mid \triangleleft$ Closed path in space:



 $\rightarrow g_{0m} \neq 0 \Rightarrow \delta x^0_{e=(ij)} \neq 0 \Rightarrow \Delta x^0 \equiv \sum_{e \in \text{Loop}} \delta x^0_e \neq 0$

Only in coordinates with $g_{0m} = 0 \Leftrightarrow \delta x^0 = 0$ the synchronization of clocks is *path-independent*. (Example: Minkowski space with clocks corresponding to inertial coordinates where $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.) If not, synchronizing clocks along a closed path can lead to the identification of different times x^0 and $x^0 + \Delta x^0$ of the same clock as simultaneous!

- In the vicinity of every space-like slice ("hypersurface") of an arbitrary spacetime it is possible to construct a coordinate system in which $g_{0m} = 0$. Because of Eq. (11.26), on such slices the synchronization of clocks is consistently possible (= path-independent). Furthermore, it is possible to tweak the coordinates such that $g_{00} = 1$ so that stationary coordinate clocks ($d\vec{x} = 0$) measure proper time (= are ideal clocks): $d\tau^2 = \frac{1}{c^2}ds^2 = g_{00}dt^2 + 0$; such coordinates are called \uparrow synchronous, see MISNER et al. [3] (§27.4, p. 717).
- The above argument shows that, in general, it is possible to fill space with ideal clocks and synchronize all of them. The question is whether they *stay* synchronized for all times (i.e., whether is it possible to synchronize clocks throughout *spacetime*).

The answer turns out to be negative because the synchronous coordinates, while being defined throughout *space* around a particular space-like hypersurface, cannot be extended to encompass all of *spacetime* (except for special cases like flat Minkowski space); they necessarily form "time singularities" [147]. This conclusion is reasonable if one





thinks of synchronous coordinates as being constructed by ejecting free-falling ideal clocks from the hypersurface (with arbitrary spatial coordinates (x^1, x^2, x^3)):

These clocks (the "time axes" of the synchronous coordinate system) follow geodesics. But we already know that, in generic spacetimes with curvature, geodesics tend to attract/repel each other and, eventually, *cross*. At this point the coordinate system becomes singular because the map between events (= points on the spacetime manifold) and coordinates is no longer unique.

- **6** | Spatial distances:
 - $i \mid \triangleleft Space-like$ curve $\gamma : [\lambda_a, \lambda_b] \rightarrow M$ on arbitrary spacetime (M, g):
 - ;! γ is not required to be a geodesic.
 - $\rightarrow **$ Proper distance:

$$L[\gamma] := \int_{\lambda_a}^{\lambda_b} \mathrm{d}\lambda \sqrt{-g_{\mu\nu}(\gamma(\lambda))\dot{\gamma}^{\mu}\dot{\gamma}^{\nu}} \equiv \int_{\gamma} \sqrt{-g_{\mu\nu}(x)\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}}$$
(11.27)

The minus is necessary because $\|\dot{\gamma}\|^2 < 0$ for a space-like curve.

i! While mathematically the proper distance is defined completely analogous to the proper time Eq. (11.10), its operational/physical role is very different: Whereas proper time can be immediately identified as the time displayed by an ideal clock that follows a time-like trajectory, there is nothing that "follows" a space-like trajectory; hence there is no immediate physical interpretation associated to the proper distance defined above.

ii | To obtain an operationally meaningful concept of distance, it is reasonable to use the \leftarrow radar distance R defined in Eq. (11.16) as a distance measure of spatially separated points.

To this end, consider again the two infinitesimally close clocks A and B from above:

 \rightarrow Coordinate time needed by radar pulse from B to A back to B:

$$dx_{\vec{\leftrightarrow}}^{0} := \tilde{x}_{2}^{0} - \tilde{x}_{1}^{0} = dx_{2}^{0} - dx_{1}^{0}$$

$$\stackrel{11.25}{=} \frac{2}{g_{00}} \sqrt{(g_{0m}g_{0n} - g_{mn}g_{00})} dx^{m} dx^{n}$$
(11.28)



 \rightarrow *Proper time* elapsed at position *B* during round trip:

$$d\tau_{\overrightarrow{\leftarrow}}^{2} = \frac{1}{c^{2}} ds_{\overrightarrow{\leftarrow}}^{2} = \frac{1}{c^{2}} g_{00} \left(dx_{\overrightarrow{\leftarrow}}^{0} \right)^{2}$$
$$= \frac{4}{c^{2}} \left(\frac{g_{0m} g_{0n}}{g_{00}} - g_{mn} \right) dx^{m} dx^{n}$$
(11.29)

Note that *B* is stationary in the considered coordinates so that $dx_{\not\leftarrow}^m = 0$ for m = 1, 2, 3.

 \rightarrow Infinitesimal *distance* between A and B:

$$dl^{2} := \left(\frac{c d\tau_{\overrightarrow{z}}}{2}\right)^{2} = \underbrace{\left(-g_{mn} + \frac{g_{0m}g_{0n}}{g_{00}}\right)}_{=:\tilde{g}_{mn}} dx^{m} dx^{n}$$
(11.30a)
$$\equiv \tilde{g}_{mn}(x) dx^{m} dx^{n}$$
(11.30b)

 \tilde{g}_{mn} : metric of three-dimensional space (in vicinity of B)

iii | Notes:

• It is straightforward to show that

$$-g^{lm}\tilde{g}_{mn} \stackrel{\circ}{=} \delta^l_n \tag{11.31}$$

so that the inverse spatial metric is the *negative* of the spatial part of the inverse metric:

$$\tilde{g}^{lm} = -g^{lm}$$
 with $\tilde{g}^{lm}\tilde{g}_{mn} = \delta_n^l$. (11.32)

• In general, it is operationally meaningless to integrate dl over a spatial curve $\eta: \int_{\eta} dl$. While every infinitesimal distance element dl does make sense for some observer (because we constructed it as the radar distance), this does not mean that *adding* different such elements along a curve η with constant coordinate time x^0 makes sense. This is only reasonable if one can establish an unambiguous notion of simultaneity along the spacetime curve defined by η and $x^0 = \text{const} - \text{which}$ is not always possible (as discussed above). Thus, in GENERAL RELATIVITY, there is no general concept of a "distance" between bodies that has objective and operational meaning.

(Note that we are not claiming that the length of a curve in space somehow depends on "how fast it is traversed": The spacetime curves we are considering are *space-like*, one cannot "traverse" them in any meaningful way! Only for the special cases where the metric $g_{\mu\nu}(x)$ is independent of time x^0 , the length of a spatial curve η can be defined by $\int_{\eta} dl$ and has a meaning that is independent of coordinates.)

7 | Speed of light:

Let us briefly comment on the role played by the speed of light in GENERAL RELATIVITY.

• Recall (Section 1.5):

In the global *inertial systems* of SPECIAL RELATIVITY, light always propagates with the same velocity $v_{\text{max}} = c$, and no signal can move faster.

• Problem:

"Velocity" is an observer-/coordinate-dependent quantity that depends on the choice of time and space coordinates. Objective statements about velocity therefore require the choice of a distinguished class of coordinate systems.



Note that this was also the case in SPECIAL RELATIVITY: The coordinate velocity is only constant c in *inertial coordinates* (where the *coordinate* velocity corresponds to a *physical* velocity because inertial coordinates are, by definition, Cartesian). By contrast, in the Rindler coordinates of an accelerated observer, the coordinate speed of light is only c *locally*, but can be less or more than c away from the observer.

 $\rightarrow \triangleleft$ Local inertial coordinates:

The **EEP** suggests: \rightarrow

The speed of light is constant (c) in locally inertial coordinates at any point of spacetime (= as measured in a freely falling laboratory).

• ¡! Remember that in SPECIAL RELATIVITY we went to great lengths to link the abstract notion of an inertial coordinate system to an operationally defined contraption of synchronized clocks and rods forming a Cartesian lattice. *Observing* (or *measuring*) events was then defined via this information-gathering contraption, and, as stressed previously, is different from *seeing* events (i.e., waiting for light signals to reach the camera of someone sitting in the origin of the rod lattice; recall the ← *Penrose-Terrell effect* mentioned in Section 2.1 and ④ Problemset 3 of last semester). Since, in SPECIAL RELATIVITY, inertial systems were assumed to be *global*, covering all of spacetime (i.e., the rod lattice was assumed to cover all of space and the clocks remained synchronized for all of times) we could *measure* events (times, distances, speeds) everywhere in spacetime, in particular: far away. Thus a statement like "the velocity of a light signal at Alpha Centauri was measured to be *c*" makes sense because we have a (magical) grid of synchronized clocks that reaches from Earth to Alpha Centauri (note that synchronized is short for "synchronized for all times", i.e., in particular, the clocks tick with the same rate).

In Section 8.2 we argued that global inertial systems do not survive the presence of gravity: they shrink to small, local patches on spacetime, namely free falling laboratories that are small enough to be not affected by tidal forces. But this means that we also *cannot* construct a universe-encompassing latticework of synchronized clocks, and, as a consequence, there is no longer a well-defined concept of *observing/measuring* distant events! In particular, there is no well-defined way for an observer located on Earth to *measure* the speed of a light signal at Alpha Centauri, we can only point a telescope into the direction and *watch* (= *see*). This is what astronomers do (it is all they *can* do) and they call it *observing* (they do it even in *observatories*); but keep in mind that this is not what we refereed to as *observing* in the context of SPECIAL RELATIVITY! We will adopt this new terminology henceforth.

Thus, in GENERAL RELATIVITY, we can only *measure* the speed of light *locally* (if one manages to setup a pair of synchronized clocks). The speed of *distant* light signals can only be *observed*, not *measured*. The constancy of the speed of light above refers only to *local measurements*, not to remote observations; the observed speed of light can be both smaller an larger than c!

(Whether the lab in which a local measurement of the speed of light is performed is inertial or accelerated actually doesn't matter: one always measures c. This is so because accelerated observers can describe physics locally by Rindler coordinates (\bigcirc Problemset 3), and in these the *local* (coordinate) speed of light is also c.)

• i! If you *calculate* the speed of light in *non-inertial* coordinates, the result is not necessarily c. For example, in the \rightarrow *Schwarzschild metric* of a spherically symmetric mass, the (coordinate) speed of light in Schwarzschild *coordinates* decreases when approaching the \rightarrow *Schwarzschild radius*. This corresponds to the well-known phenomenon that an observer far away from a black hole *sees* light freeze when approaching the event horizon.



Similarly, in Rindler coordinates (a useful coordinate system for observers with constant proper acceleration, \bigcirc Problemset 3), the (coordinate) velocity of distant light signals can be less or more than c, depending on their position.

Recall our discussion ← *above* on the cone field on Lorentzian manifolds that encodes the local causal structure of spacetime. The light cones are locally generated by null cones which, by definition, are spanned by the tangent vectors of light rays. This makes the statement that "the speed of light is constant in local inertial systems" a tautology in GENERAL RELATIVITY: There is no fixed background of distinguished coordinates with respect to which you could measure that *c* is constant. It is the *finiteness* of the speed of information propagation (e.g., by light) that guarantees a *local* causality structure on spacetime; this causality structure can be encoded by the cone field. One can then, without loss of generality, choose units of time and length such that, locally, the signals that span the null cones propagate with constant velocity *c*.

We already touched this topic in Section 1.4 where we derived the Lorentz transformation. There we realized that it is not so much the *constancy* of the speed of light that is important but its *finiteness* (the constancy follows from the finiteness, recall Eq. (1.73)). It is the finiteness of the speed of information propagation that induces a *local* causal structure of events.

8 | Implementing Einstein's Equivalence Principle EEP:

To implement the **EEP** into the physical models that are defined on the spacetime of GENERAL RELATIVITY, one can employ the following procedure:

§ Principle 3: Minimal-Coupling Principle MCP ("Comma-goes-to-Semicolon Rule")

i | Take a physical model (equation) in manifestly Lorentz covariant form.

The model is of course assumed to be valid in SPECIAL RELATIVITY (i.e., describe the laws of nature correctly in the globally inertial coordinates of flat Minkowski space).

ii Convert it into a generally covariant form by the following substitutions:

$$\partial_{\mu} \mapsto \nabla_{\mu} \quad (\text{or }, \mapsto;) \qquad \text{and} \quad \eta_{\mu\nu} \mapsto g_{\mu\nu}(x)$$
 (11.33)

If the physical model is given by a Lagrangian density (i.e., in an integral form), one must also ensure that the integrand transforms as a scalar by the substitution $d^4x \mapsto d^4x \sqrt{g}$ as discussed in Section 10.3.1.

iii | Assert the validity of this model in the curved spacetimes of GENERAL RELATIVITY.

Examples: \rightarrow Sections 11.2 and 11.3.

• The MCP has a similar status as \checkmark canonical quantization in quantum mechanics: It provides a (mathematically supported) guiding principle to "update" an "old" physical model to a "new" form that adapts the model to a more fundamental theory, while respecting some sort of correspondence principle (which is necessary because the outdated model works well in some domain).

Furthermore, the MCP ensures (at least in the cases relevant to us, \rightarrow *below*) that the constructed models respect the **EEP**, in that it asserts the *absence* of explicit (non-minimal) couplings to the curvature tensor (which, if present, would allow local experiments to detect the presence of a gravitational field, \rightarrow *below*).



(11.04)

• Just as canonical quantization works for most situations encountered in physics – but fails in certain edge cases (↑ *Weyl quantization*, ↑ *Groenewold's theorem*) –, the MCP works for most relativistic theories (in particular, all problems that we encounter in this course), but has some subtle ambiguities that prevent a unique outcome. The problem is that higher-order covariant derivatives do not commute, whereas higher-order partial derivatives do:

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Depending on which order of covariant derivatives you pick, you end up with theories that are equivalent in SPECIAL RELATIVITY (i.e., on flat Minkowski space), but differ by a curvature-dependent term in GENERAL RELATIVITY. Thus the MCP is only a unique recipe for first-order differential equations [148]. (Luckily, we are physicist and can experiments let decide which generally covariant model describes the laws of nature correctly.) For more details on this ordering ambiguity, see MISNER *et al.* [3] (pp. 388–389, §16.3 and pp. 390–391, Box 16.1).

• Here is an example to illustrate the MCP and contrast it to *non-minimal* coupling. The example also shows that non-minimal coupling typically leads to a violation of the EEP (put differently, the EEP lends credence to the MCP):

The real Klein-Gordon field is given by the Lorentz covariant action (in \leftarrow Section 7.1 we discussed the complex Klein-Gordon field)

$$S[\phi] = \int d^4x \left[\eta^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) - m^2 \phi^2 \right] = \int d^4x \left[\phi^{,\mu} \phi_{,\mu} - m^2 \phi^2 \right], \qquad (11.35)$$

the Euler-Lagrange equations of which are the Klein-Gordon equation

$$(\partial^2 + m^2)\phi(x) = 0.$$
(11.36)

If we want to study the Klein-Gordon field in the curved spacetime of GENERAL RELATIV-ITY, the MCP tells us to construct the generally covariant action (Problemset 4)

$$S_g[\phi] = \int \mathrm{d}^4 x \sqrt{g(x)} \left[g^{\mu\nu}(x) (\nabla_\mu \phi) (\nabla_\nu \phi) - m^2 \phi^2 \right]$$
(11.37a)

$$= \int d^4x \sqrt{g} \left[\phi^{;\mu} \phi_{;\mu} - m^2 \phi^2 \right], \qquad (11.37b)$$

which reduces to Eq. (11.35) on flat Minkowski space ($g = \eta$). The corresponding equation of motion is the generally covariant Klein-Gordon equation

$$(\Delta + m^2)\phi(x) = 0,$$
 (11.38)

where \triangle is the \leftarrow *Laplace-Beltrami operator* Eq. (10.97). In locally inertial coordinates, Eq. (11.38) reduces to Eq. (11.36), realizing the **EEP** (check this!).

Now let us couple the Klein-Gordon field in a *non-minimal* way to gravity by adding a scalar interaction with the Ricci scalar $R(x) \in Eq. (10.117)$ to the action (green):

$$\tilde{S}_{g}[\phi] = \int d^{4}x \sqrt{g} \left[\phi^{;\mu} \phi_{;\mu} - m^{2} \phi^{2} - \xi R(x) \phi^{2} \right], \qquad (11.39)$$

where $\xi \in \mathbb{R}$ is a coupling constant. The generally covariant equation of motion is clearly

$$[\Delta + m^2 + \xi R(x)]\phi(x) = 0, \qquad (11.40)$$



where R(x) depends on the metric $g_{\mu\nu}(x)$.

But Eq. (11.40) does *not* reduce to the Klein-Gordon Eq. (11.36) of SPECIAL RELATIVITY in local inertial frames! This is so because the term $\xi R(x)$ is a *scalar* that does not vanish on a curved spacetime in *any* coordinate system (in particular, locally inertial coordinates). Thus Eq. (11.40) explicitly violates the **EEP** because, using local measurements of the evolution of the Klein-Gordon field ϕ , a local observer can detect the presence of curvature (and thereby gravity).

• Be careful when making statements about *higher-order* covariant derivatives! For example, on a *flat* Minkowski space in *globally inertial coordinates*, all covariant derivatives in a generally covariant equation become partial derivatives:

$$T^{\dots}_{;\alpha;\beta} \xrightarrow{g_{\mu\nu} \to \eta_{\mu\nu}} T^{\dots}_{;\alpha,\beta} . \tag{11.41}$$

This is true because the Christoffel symbols are identically zero everywhere, so that all their derivatives vanish as well.

By contrast, on a *curved* spacetime in *locally inertial coordinates* [with metric as in Eq. (10.89)], this is *not* true:

$$T^{\dots}_{;\alpha;\beta} \xrightarrow{g_{\mu\nu} \to g_{\mu\nu}} T^{\dots}_{;\alpha;\beta}.$$
(11.42)

To see this, note that the left-hand side contains *derivatives* of the Christoffel symbols – which do not necessarily vanish in locally inertial coordinates (because coordinate transformations cannot make curvature go away).

This also follows from the Ricci identity Eq. (10.71):

$$T_{k;l;m} - T_{k;m;l} = R^{l}_{klm} T_{l} . (11.43)$$

On a curved space, the right-hand side does not vanish *in any coordinate system*, so that covariant derivatives do not commute; in particular, they cannot become partial derivatives.

This line of reasoning leads to a peculiar conclusion: Applying the MCP to higher-order differential equations can lead to generally covariant equations that contain *curvature terms*, and thereby violate the **EEP** (in its strictest form)! (Note that they do obey a correspondence princple in the sense that they reproduce the physics of SPECIAL RELATIVITY on flat Minkowski space.) This phenomenon is of course rooted in the fact that in locally inertial coordinates the metric is only Minkowskian *to first order*. For more details on this predicament (that most textbooks seem to be silent about) see Ref. [149]. CARROLL argues that curvature terms (from non-minimal or higher-order minimal coupling) may actually be present, but should be supressed by the Planck scale (\uparrow Ref. [4], §4.7, pp. 179–181).



↓Lecture 23 [04.06.24]

11.2. Classical mechanics in the gravitational field

We now apply the MCP to obtain a formulation of the classical mechanics of points on a given spacetime with Lorentzian metric $g_{\mu\nu}$.

- **1** | \triangleleft *Free* particle of mass *m*
 - \rightarrow EOM in local inertial system according to EEP and SPECIAL RELATIVITY (\leftarrow Eq. (5.46)):

$$m\frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} = 0 \tag{11.44}$$

 $u^{\mu} = \frac{dx^{\mu}}{d\tau}$: 4-velocity (in local inertial coordinates) τ : Proper time of particle

Note that you can multiply the mass m on the left-hand side, but it cancels anyway! This reflects the fact that, in SPECIAL RELATIVITY, the trajectory of a free particle is independent of its (inertial) mass.

2 | In local inertial coordinates it is $\Gamma^{\mu}_{\nu\rho} = 0 \rightarrow$

Eq. (11.44)
$$\stackrel{\Gamma=0}{\Leftrightarrow} \underbrace{m \frac{Du^{\mu}}{D\tau} \stackrel{10.37}{=} m \frac{d^2 x^{\mu}}{d\tau^2} + m \Gamma^{\mu}_{\nu\rho} \frac{dx^{\nu}}{d\tau} \frac{dx^{\rho}}{d\tau} = 0}_{\text{Generally covariant geodesic equation}}$$
(11.45)

 \rightarrow Eq. (11.45) valid in *arbitrary coordinates* (GRP) on *arbitrary spacetimes* (MCP / EEP)!

With this we mean that the MCP suggests that the correct equation of motion for a free particle on an arbitrary (potentially curved) spacetime of GENERAL RELATIVITY is Eq. (11.45), i.e., the \leftarrow geodesic equation!

 \rightarrow

In GENERAL RELATIVITY *free particles* follow *geodesics* in spacetime.

It is important to fully grasp what just happened (the procedure is deceptively simple but subtle):

i | We know from SPECIAL RELATIVITY that Eq. (11.44) describes the movement of free particles correctly (and globally) on flat Minkowski space. The Christoffel symbols of the Minkowski metric in (globally) inertial coordinates vanish *everywhere*, so that Eq. (11.44) is trivially equivalent to Eq. (11.45). But Eq. (11.45) is the (unique) generally covariant tensor equation that reduces to Eq. (11.44) in (globally) inertial coordinates. Eq. (11.45) still describes the physics of a free, relativistic particle on Minkowski space, but now in *arbitrary* coordinates. To reiterate: As long as you fix the metric of spacetime as the Minkowski metric (which therefore plays the role of a \leftarrow *background*), Eq. (11.45) is simply a more general (but equivalent) formulation of classical mechanics in SPECIAL RELATIVITY, i.e., there is no new physics contained in the equation!

What you witnessed is the transition from a *coordinate-specific* formulation of a physical model to a *generally covariant* formulation. This is the principle of general relativity **GRP** in action,



and, as we discussed in Section 9.2, it is physically vacuous (in particular, it is not specific to GENERAL RELATIVITY). Nonetheless we already gained something: Since Eq. (11.45) holds in arbitrary coordinates, you can use this equation to obtain the relativistic equations of motion in curvilinear coordinates (e.g., accelerated Rindler coordinates, \bigcirc Problemset 3). Just compute the Christoffel symbols of the Minkowski metric in these coordinates, and you are good to go!

ii | GENERAL RELATIVITY enters the stage in the next step (which is purely interpretational since Eq. (11.45) is already given and does not need to be modified):

The **EEP** claims that, even in the presence of gravity, the laws of SPECIAL RELATIVITY remain valid *locally*. Thus Eq. (11.44) must be valid in every *local* inertial frame of a potentially curved spacetime. Eq. (11.45) implements this demand because, for every point of spacetime and in *locally* inertial coordinates, the equation reduces to Eq. (11.44). That this happens is the motivation behind the **MCP**.

[Note that the local inertial coordinates x^{μ} are different from point to point! That is, astronauts in different space stations all can use Eq. (11.44) to describe free moving particles in their small labs, but their coordinate systems are not the same (and typically not even overlapping).]

The physically non-trivial claim, coming from **EEP** and built into **MCP**, is now that Eq. (11.45) describes the trajectories of free particles correctly on *all spacetimes* (and not only on flat Minkowski space). This claim is far from vacuous as it makes empirical statements about the motion of free particles in the presence of gravity (= curvature), something that SPECIAL RELATIVITY had nothing to say about. Whether Eq. (11.45) is correct in the presence of gravity is an empirical claim (as the validity of the **EEP** is) that needs to be tested experimentally.

Notes:

• We can rewrite Eq. (11.45) in the form

$$m \underbrace{\frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau}}_{\text{"4-acceleration"}} = \underbrace{-m\Gamma^{\mu}_{\nu\rho}u^{\nu}u^{\rho}}_{\text{"4-force"}}, \qquad (11.46)$$

which suggests the interpretation of the right-hand side as the "gravitational 4-force" acting on the test particle. The connection coefficients $\Gamma^{\mu}{}_{\nu\rho}$ then play the role of the "gravitational field strength" and (since $\Gamma \propto \partial g$) the metric $g_{\mu\nu}(x)$ can be identified as the "gravitational potential". In the Newtonian limit (\rightarrow *below*) this identification is indeed reproduced.

However, use these identifications with a grain of salt; the whole point of GENERAL REL-ATIVITY is to identify the effect of gravity as spacetime curvature (which we will finally do in Chapter 12), and not as a classical force (which can be present in addition to gravity, \rightarrow *below*). Note also that the "4-acceleration" in Eq. (11.46) is a *coordinate* acceleration and *not* a tensor, i.e., it cannot be identified with a physical acceleration [this is in contrast to the 4-acceleration in SPECIAL RELATIVITY, \leftarrow Eq. (4.49)].

The reason is that the coordinates x^{μ} in the definition of u^{μ} are *arbitrary*; in particular, they do *not* convey metric information on their own (recall our discussion of the role played by coordinates in Section 9.2). Hence the "4-force" on the right-hand side (which is also not a tensor!) does not correspond to a coordinate-independent, physical force; it is a fictitious "coordinate force", similar to the fictitious Coriolis force in classical mechanics (which is purely a consequence of a particular choice of coordinate system (corresponding to an inertial frame) where the Coriolis force vanishes everywhere. By contrast, the "4-force" in Eq. (11.46) can only be made vanish *locally* (in locally geodesic coordinates, that is) but not globally (if this is possible, the Christoffel symbols vanish everywhere and spacetime is flat).

• Please appreciate how elegant the implementation of the universality of free fall (= equivalence of gravitational and inertial mass, **WEP**) in this formalism is: In Eq. (11.45) there is only *one*



place to put a mass (in front of the absolute derivative). Only when we write the absolute derivative as sum of two terms, this single mass starts to play two (seemingly) different roles, namely that of *inertial* mass on the left-hand side of Eq. (11.46), and that of *gravitational* mass on the right-hand side. But the two are necessarily identical, a fact that Newtonian mechanics cannot explain! It is this natural emergence of the WEP that corroborates (and historically motivated) a *metric* theory of gravity.

Non-minimal coupling:

It is instructive to study what happens if we ignore the **MCP** and produce a non-minimally coupled, generally covariant equation. For example, we could postulate the following equation that (supposedly) describes the motion of a free particle in GENERAL RELATIVITY:

$$m\frac{\mathrm{D}u^{\mu}}{\mathrm{D}\tau} \stackrel{?}{=} \xi R_{;\sigma} u^{\sigma} u^{\mu} \tag{11.47}$$

with $\xi \in \mathbb{R}$ some coupling constant. This equation ...

- ... is generally covariant (\rightarrow implements **GRP**).
- ... reduces to Eq. (11.44) on flat Minkowski space (\rightarrow consistent with SPECIAL RELATIVITY).

The problem is that, on a generic curved spacetime, the curvature-related *tensor* $R_{;\sigma}$ does *not* vanish in locally inertial coordinates, so that Eq. (11.47) takes the locally inertial form

$$m\frac{\mathrm{d}\bar{u}^{\mu}}{\mathrm{d}\tau} = \xi \bar{R}_{;\sigma} \bar{u}^{\sigma} \bar{u}^{\mu} , \qquad (11.48)$$

which does *not* reproduce the physics of SPECIAL RELATIVITY [namely Eq. (11.44)], and therefore violates the **EEP**. Had we adhered to the **MCP**, we would have never added the curvature term in the first place, and this violation would not occur.

3 | <u>External forces:</u>

In SPECIAL RELATIVITY we not only discussed free particles but also ones that are acted upon by some external force [\leftarrow Eq. (5.6)]. Using the MCP we immediately obtain the generally covariant form of our relativistic equation of motion:

$$m\frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} = K^{\mu} \quad \xrightarrow{\mathrm{MCP}} \quad m\frac{\mathrm{D}u^{\mu}}{\mathrm{D}\tau} = m\left(\frac{\mathrm{d}^{2}x^{\mu}}{\mathrm{d}\tau^{2}} + \Gamma^{\mu}_{\nu\rho}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}\tau}\right) = K^{\mu} \quad (11.49)$$

Here K^{μ} is a placeholder for some force that transforms as a contravariant tensor and acts locally on the particle. We will find an explicit example \rightarrow *later* when we discuss the electromagnetic field.

 \rightarrow

Forces make trajectories *deviate* from geodesics.

(11.50)

For example, the force pushing you into your seat right now is the phenomenological consequence of *not* following a geodesic in spacetime. (A geodesic trajectory corresponds to free fall, but your seat is in the way!)

4 Some relations:

$$\mathbf{i} \mid u^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}$$
 is a tensor

 u^{μ} is not a tensor *field* as it is only defined along the trajectory $x^{\mu}(\tau)$, but $u^{\mu}\partial_{\mu} \in TM$ so that it transforms as a (1, 0) tensor.



 $\rightarrow ||u||^2 = g_{\mu\nu}u^{\mu}u^{\nu}$ is a scalar:

$$\|\dot{x}\|^{2} = g_{\mu\nu}u^{\mu}u^{\nu} = \left\{g_{\mu\nu}u^{\mu}u^{\nu}\right\}^{\mathrm{LI}} = \eta_{\mu\nu}\bar{u}^{\mu}\bar{u}^{\nu} \stackrel{4.48}{=} c^{2} > 0$$
(11.51)

(LI = Locally inertial coordinates)

 \rightarrow Physical trajectories $x^{\mu}(\tau)$ of massive particles must be *time-like*!

This is the generalization of Eq. (4.48).

ii | With this we find:

$$0 = \frac{dc^2}{d\tau} = \frac{D(g_{\mu\nu}u^{\mu}u^{\nu})}{D\tau} = 2g_{\mu\nu}u^{\nu}\frac{Du^{\mu}}{D\tau}$$
(11.52)

Here we used the metric-compatibility of the Levi-Civita connection Eq. (10.74) and the product rule for covariant/absolute derivatives. \rightarrow

$$\frac{\mathbf{D}u^{\mu}}{\mathbf{D}\tau}u_{\mu} = 0 \tag{11.53}$$

This is the generally covariant analog of Eq. (4.50).

iii | Eq. (11.49) $\xrightarrow{\text{Eq. (11.53)}}$

$$u_{\mu}K^{\mu} = 0 \tag{11.54}$$

This means that the 4-velocity u^{μ} of a physical trajectory is always orthogonal to the 4-acceleration $\frac{Du^{\mu}}{D\tau}$ and hence the 4-force K^{μ} .

iv | The \leftarrow 4-momentum is defined as previously:

$$p^{\mu} = m u^{\mu} \tag{11.55}$$

Eq. $(11.51) \rightarrow$

$$\|p\|^2 = g_{\mu\nu} p^{\mu} p^{\nu} = m^2 c^2$$
(11.56)

This is the generalization of Eq. (5.4).

5 | Variational principle (for a free particle):

As usual, the equation of motion can be found via a variational principle from an action. Since Eq. (11.45) is generally covariant, the Lagrangian must be a scalar. An application of the **MCP** to the action Eqs. (5.41) and (5.43) of a free particle in SPECIAL RELATIVITY immediately yields the correct result:

$$\underset{\text{Eq. (5.41)}}{\text{Eq. (5.43)}} \xrightarrow{\text{MCP}} S_g[x] = -mc \int_x ds = -mc \int_x \sqrt{g_{\mu\nu} dx^{\mu} dx^{\nu}}$$
(11.57)

with

$$\delta S_g[x] \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \text{Eq. (11.45)} \tag{11.58}$$

We do not need to prove this! This is exactly the variation that we used to *derive* the geodesic equation (which we now interpret as the equation of motion for a free particle!); \leftarrow Section 10.3.3.



6 | Newtonian approximation:

i | \triangleleft Non-relativistic particle in a *Newtonian* gravitational potential $\phi = -\frac{MG}{r}$:

$$L = -mc^2 + \frac{1}{2}mv^2 - m\phi$$
 (11.59)

 \rightarrow Non-relativistic (\approx) action:

$$S_g \approx \int \mathrm{d}t L = -mc \int \mathrm{d}t \left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right)$$
 (11.60)

To understand where Lagrangian & action come from, recall Eq. (5.42):

$$S_{\eta} = -mc^2 \int dt \sqrt{1 - v^2/c^2} \approx -mc^2 \int dt \left(1 - \frac{v^2}{2c^2}\right)$$
(11.61)

 \rightarrow Non-relativistic approximation of Lagrangian in SPECIAL RELATIVITY:

$$L = -mc^2 + \frac{1}{2}mv^2 \tag{11.62}$$

Above we simply added an additional Newtonian gravitational potential.

ii | Identify the line element in the fully relativistic action:

Eqs. (11.57) and (11.60)
$$\rightarrow ds \approx \left(c - \frac{v^2}{2c} + \frac{\phi}{c}\right) dt$$
 (11.63)

Use $d\vec{x} = \vec{v}dt$ and drop terms $\propto v^2/c^2$ (slow particle) and $\propto \phi^2/c^4$ (weak field) $\stackrel{\circ}{\rightarrow}$

$$g_{\mu\nu} \mathrm{d}x^{\mu} \mathrm{d}x^{\nu} = (\mathrm{d}s)^2 \approx \left(1 + \frac{2\phi}{c^2}\right) (c\,\mathrm{d}t)^2 - (\mathrm{d}\vec{x})^2$$
 (11.64)

This allows us to identify the Newtonian potential as 00-component of the metric tensor:

$$g_{00} \approx 1 + \frac{2\phi}{c^2}$$
 with $\phi = -\frac{MG}{r}$ (11.65)

- Note that this result is consistent with our previous interpretation of the metric as the analog of a gravitational potential in GENERAL RELATIVITY.
- This result demonstrates that the dominant effect of a weak gravitational field is the modification of the *time*-component of the metric, i.e., a modification of the tick-rate of clocks as a function of height (→ *gravitational time dilation*).

11.3. Electrodynamics in the gravitational field

Now we use the MCP to generalize \leftarrow classical electrodynamics to curved spacetimes:

- 1 | <u>Remember:</u> (Section 6.2)
 - Field strength tensor: $F_{\mu\nu} = A_{\nu,\mu} A_{\mu,\nu}$



• Homogeneous Maxwell equations: (Eqs. (6.42) and (6.50a))

$$\tilde{F}^{\mu\nu}_{,\nu} = 0 \quad \Leftrightarrow \quad F_{\langle\mu\nu,\lambda\rangle} = F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0 \qquad (11.66)$$

These equations are *identities* if $F_{\mu\nu}$ is expressed in terms of a gauge field A_{μ} .

• Inhomogeneous Maxwell equations: (Eq. (6.50b))

$$F^{\mu\nu}_{\ ,\nu} = -\frac{4\pi}{c} j^{\mu} \tag{11.67}$$

2 The field strength Lorentz tensor can be generalized to a proper tensor via the MCP:

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \xrightarrow{\text{MCP}} F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu} \stackrel{\circ}{=} A_{\nu,\mu} - A_{\mu,\nu}$$
(11.68)

This follows from the symmetry of the Christoffel symbols.

 \rightarrow No covariant derivatives needed!

Put differently: Our old field strength *Lorentz* tensor was a proper (0, 2) tensor all along!

3 | Homogeneous Maxwell equations (HME):

The homogeneous Maxwell equations follow directly with the MCP:

Eq. (11.66)
$$\xrightarrow{\text{MCP}} F_{\langle \mu\nu;\lambda\rangle} \stackrel{\circ}{=} F_{\langle \mu\nu,\lambda\rangle} = 0$$
 (11.69)

 \rightarrow The HME have the same form as in SPECIAL RELATIVITY

- If $F_{\mu\nu}$ is expressed in terms of the gauge field A_{μ} , this is again an *identity*, i.e., it is true for all gauge fields A_{μ} and hence does not impose constraints on A_{μ} . To see this without calculations, note that $F_{\mu\nu}$ does not contain connection coefficients due to Eq. (11.68). This means that in locally geodesic coordinates we have immediately $\{F_{\langle \mu\nu;\lambda\rangle}\}^{\text{LG}} = F_{\langle \mu\nu,\lambda\rangle} = 0$; since $F_{\langle \mu\nu;\lambda\rangle}$ is a tensor, it is $F_{\langle \mu\nu;\lambda\rangle} = 0$ in all coordinate systems. As this line of reasoning never imposes any constraint on A_{μ} , Eq. (11.69) is an identity.
- In coordinate-free notation, the homogeneous Maxwell equations read dF = 0, with the 2-form F = dA and the 1-form A (gauge connection); \leftarrow Eq. (6.70). That Eq. (11.69) is an identity simply follows from ddA = 0 since $d^2 = 0$ for the exterior derivative. The fact that all connection coefficients drop out, and the equations do not depend on the metric, is reflected by the fact that dF = 0 is a well-defined expression on any differentiable manifold neither connection nor metric required (e.g., in form of a Hodge dual).

Note that the equations are completely identical to \leftarrow Eq. (6.70), where we discussed the coordinate-free notation in the context of SPECIAL RELATIVITY. This emphasizes once more that general covariance is not a characteristic feature of GENERAL RELATIVITY.

4 Current:

Before we can discuss the inhomogeneous Maxwell equation, we must revisit the charge current:

i | Remember (Section 6.2): Charge $dq = \rho d^3x$ in volume $dV = d^3x$ is scalar quantity:





Recall that not d^4x but $\sqrt{g}d^4x$ transforms as a scalar [Eq. (10.101)]; in SPECIAL RELATIV-ITY, we only considered Lorentz transformations (which have g = 1) so that we didn't have to make this distinction.

This implies that the 4-current must be defined as follows to be a contravariant vector:

$$J^{\mu} := \frac{\rho}{\sqrt{g}} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}t} \stackrel{6.18}{=} \frac{j^{\mu}}{\sqrt{g}}$$
(11.71)

ii | Charge conservation is encoded by the covariant continuity equation:

$$J^{\mu}_{;\mu} \stackrel{\circ}{=} 0$$
 (Continuity equation) (11.72)

To show Eq. (11.72), use Eqs. (10.95) and (11.71) to rewrite the covariant divergence as

$$J^{\mu}_{;\mu} \stackrel{\text{10.95}}{=} \frac{1}{\sqrt{g}} \left(\sqrt{g} J^{\mu}\right)_{,\mu} \stackrel{\text{11.71}}{=} \frac{1}{\sqrt{g}} j^{\mu}_{,\mu} . \tag{11.73}$$

At every point we can transform into locally inertial coordinates where we know that

$$\{J^{\mu}_{;\mu}\}^{\mathrm{LI}} = j^{\mu}_{,\mu} \stackrel{6.24}{=} 0$$
 (11.74)

Because $J^{\mu}_{;\mu}$ is a scalar, Eq. (11.72) follows in all coordinate systems.

5 | Inhomogeneous Maxwell equations (IME):

We can now use the MCP to construct the IME valid on arbitrary spacetimes:

$$\stackrel{\text{Eq. (11.67)}}{\underset{\text{Eq. (11.71)}}{\longrightarrow}} \stackrel{\text{MCP}}{\longrightarrow} F^{\mu\nu}_{;\nu} \stackrel{10.96}{=} \frac{1}{\sqrt{g}} \left(\sqrt{g} F^{\mu\nu} \right)_{,\nu} = -\frac{4\pi}{c} J^{\mu}$$
(11.75)



- Using the form of the covariant divergence in the middle (which follows from Eq. (10.96) for an antisymmetric tensor), it is easy to verify that in locally inertial coordinates the special relativistic form Eq. (11.67) is recovered so that the **EEP** is satisfied. (To show this, use that in locally inertial coordinates first derivatives of the metric vanish.)
- In contrast to the HME in Eq. (11.69), the IME Eq. (11.75) are *not* identical to their Lorentz covariant counterparts Eq. (11.67) but true covariant extensions thereof. In particular, the metric makes an appearance in the equations. This means that, because of the IME, classical electrodynamics is not a *topological* but a *geometrical* field theory, in that its solutions depend on the geometry of spacetime. This is not surprising: One would expect the solutions for the electromagnetic field to be different if space were a sphere, for example. Put differently, the electromagnetic field reacts in a non-trivial way to curvature in spacetime. As we want a theory that reproduces the observed deflection of light in the vicinity of heavy masses (← Section 8.2), and we would like gravity to be completely encoded in the metric, this is certainly nice to see!
- That the current must satisfy the continuity equation Eq. (11.72) for Eq. (11.75) to have solutions is straightforward to show in a local inertial frame:

$$-\frac{4\pi}{c} \left\{ J^{\mu}_{;\mu} \right\}^{\text{LI 11.75}} = \left[\frac{1}{\sqrt{g}} \left(\sqrt{g} F^{\mu\nu} \right)_{,\nu} \right]_{,\mu} = \frac{1}{\sqrt{g}} \left(\sqrt{g} F^{\mu\nu} \right)_{,\nu,\mu} = 0.$$
(11.76)

Here we used that in locally inertial coordinates first derivatives of the metric vanish and *partial* derivatives commute, together with the antisymmetry of $F_{\mu\nu}$. (Note that you cannot – without additional input – conclude that $F^{\mu\nu}_{\ ;\nu;\mu} = 0$ since covariant derivatives in general do not commute! Here this is true because $F_{\mu\nu}$ is antisymmetric, as shown above.)

• In coordinate-free notation, also the IME looks the same as in SPECIAL RELATIVITY:

$$\star \mathbf{d}(\star F) \stackrel{*}{=} J \quad \Leftrightarrow \quad \text{Eq. (6.70b)} \tag{11.77}$$

with the 1-form $J = \frac{4\pi}{c} J_{\mu} dx^{\mu}$ and the 2-form $F = \frac{1}{2} F_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$. To derive this, one must use the definition of the Hodge star operator on arbitrary pseudo-Riemannian manifolds to show that $\star d(\star F) \stackrel{*}{=} F_{\mu : \nu} dx^{\mu}$.

The fact that the IME knows about the metric is reflected by the Hodge star operator in Eq. (11.77) (which is defined via the metric). That the equation looks the same as in SPECIAL RELATIVITY might be surprising at first, but this is the whole point of the MCP: the coupling to gravity is postulated to be *minimal* – and what is more minimal than not changing the equation at all? (Beware: That the equation looks the same does not mean that the EM field does not couple to the metric! What changed between SPECIAL RELATIVITY and GENERAL RELATIVITY is that, previously, the metric to define the Hodge star was *fixed* as the Minkowski metric, now it is a *dynamical* field with its own dynamics.)

$\mathbf{6} \mid \underline{\text{Action:}}$

The covariant action of electrodynamics follows via the MCP from the old Maxwell action, and by replacing the old current by the new contravariant one:

Eq. (6.56)
$$\xrightarrow{\text{MCP}} S_g[A] = \int d^4x \, \mathcal{L}(A, \partial A, g)$$
$$= \int d^4x \, \sqrt{g} \left[-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J^{\mu} A_{\mu} \right]$$
(11.78)



Note that the metric $g_{\mu\nu}$ is also hidden in the two contractions between the field strength tensors!

 $\stackrel{\circ}{\rightarrow}$ Euler-Lagrange Equations:

$$\frac{\partial \mathcal{L}}{\partial A_{\mu}} - \partial_{\nu} \frac{\partial \mathcal{L}}{\partial (\partial_{\nu} A_{\mu})} = 0 \quad \Leftrightarrow \quad F^{\mu\nu}_{;\nu} = -\frac{4\pi}{c} J^{\mu}$$
(11.79)

7 | Using the action it is possible to trace the continuity equation (= charge conservation) back to the invariance of the action under gauge transformations of the form $\tilde{A}_{\mu} = A_{\mu} + \partial_{\mu}\lambda$. To this end, consider the local gauge variation $\delta A_{\mu} = \partial_{\mu}\lambda$, generated by a compactly supported scalar $\lambda(x)$ (meaning: $\lambda(x)$ vanishes everywhere except for a finite region of spacetime), and compute the variation of the action:

$$\delta S_g = -\frac{1}{c} \int d^4 x \sqrt{g} J^\mu \partial_\mu \lambda = -\frac{1}{c} \underbrace{\int d^4 x \partial_\mu \left(\sqrt{g} J^\mu \lambda\right)}_{\substack{\text{Gauss} \\ g = 0}} + \frac{1}{c} \int d^4 x \lambda \partial_\mu \left(\sqrt{g} J^\mu\right) \,. \tag{11.80}$$

Here we used that $\delta F_{\mu\nu} = 0$ since $F_{\mu\nu}$ is gauge invariant (this is true whether or not A_{μ} extremizes the action). The first summand on the right vanishes because $\lambda(x)$ is compactly supported and vanishes on the surface the integration volume.

If A_{μ} solves the IME, and therefore extremizes the action, the variation vanishes: $\delta S_g = 0$. Since this must be true for arbitrary compactly supported $\lambda(x)$, the continuity equation follows:

$$\partial_{\mu}\left(\sqrt{g}J^{\mu}\right) = 0 \quad \Leftrightarrow \quad J^{\mu}_{;\mu} = 0.$$
 (11.81)

8 | Charged particle in an electromagnetic field:

It is now straightforward to write down a generally covariant equation that describes the motion o a charged particle in an electromagnetic field in an arbitrary gravitational field (= metric $g_{\mu\nu}$).

Recall Section 6.4 \rightarrow

Eq. (6.130)
$$\xrightarrow[Eq. (11.49)]{Eq. (11.49)}_{Eq. (11.68)} \frac{Dp^{\mu}}{D\tau} = \frac{dp^{\mu}}{d\tau} + m\Gamma^{\mu}_{\ \nu\rho}u^{\nu}u^{\rho} = \frac{q}{c}F^{\mu}_{\ \nu}u^{\nu}$$
(11.82)

Here we used the definition of the particle momentum Eq. (11.55).

From our discussions in Sections 11.2 and 11.3, it is clear that this covariant equation reduces to Eq. (6.130) in locally inertial coordinates, and thus obeys the **EEP**.

9 | Energy-momentum tensor:

The symmetric \leftarrow *Belinfante-Rosenfeld energy-momentum tensor* (BRT) of the general covariant theory Eq. (11.78) follows immediately:

Eq. (6.110)
$$\xrightarrow{\text{MCP}} T_{\text{em}}^{\mu\nu} = \frac{1}{4\pi} \left[g^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$$
 (11.83)

This tensor will describe the effect of energy and momentum carried by the electromagnetic field on the gravitational field (metric) of GENERAL RELATIVITY; i.e., Eq. (11.83) shows up on the right-hand side of the \rightarrow *Einstein field equations* as a source of gravity.



↓Lecture 24 [11.06.24]

11.4. The Hilbert Energy-Momentum Tensor

We have now discussed two covariant generalizations of classical theories that are valid on arbitrary spacetimes, specified by a given Riemannian metric $g_{\mu\nu}$. In this chapter, we study the implications of the general covariance of such theories to understand under which circumstances they feature conserved quantities. As a bonus, the central results of this chapter will be crucial for the derivation of the Einstein field equations in the next Chapter 12.

1 \triangleleft Generally covariant theory describing *matter fields* ϕ :

$$S_{g}[\phi] = \int \underbrace{\mathrm{d}^{4}x \sqrt{g}}_{\text{Scalar}} \underbrace{L(\phi, \partial\phi, g, \partial g)}_{\text{Scalar}} \equiv \int \mathrm{d}^{4}x \underbrace{\mathcal{L}(\phi, \partial\phi, g, \partial g)}_{\text{Scalar density}}$$
(11.84)

Recall Section 3.4 for the definition of \leftarrow tensor densities.

- The following is valid for arbitrary families of matter fields $\{\phi_k\}$; we omit the index k. In particular, we do not assume that the fields transform as scalars, they can be arbitrary tensor fields. The only important thing is that they are combined appropriately to a Lagrangian L that transforms as a scalar.
- "Matter" here refers to all degrees of freedom that are *not* the metric $g_{\mu\nu}$. So one example for ϕ would be the gauge field A^{μ} of classical electrodynamics, discussed in Section 11.3 with the action Eq. (11.78).
- We use the subscript g do indicate that the action depends on the metric (e.g., through covariant derivatives). If we consider these theories "stand alone", i.e., on a fixed spacetime background, the metric plays the role of a parameter (not a dynamical field), which motivates the subscript notation.
- The expression in Eq. (11.84) actually requires an additional prefactor $\frac{1}{c}$ for dimensional reasons because we measure time coordinates in units of length ($x^0 = ct$); we omit the prefactor because it is irrelevant in the following and would cancel anyway.

2 | Diffeomorphism invariance:

 \triangleleft Arbitrary coordinate transformation $\bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x}) \rightarrow$

$$\bar{\phi}(\bar{x}) := \mathcal{F}_{\phi}(\phi(x)) \quad \Leftrightarrow \quad \bar{\phi}(x) := \mathcal{F}_{\phi}(\phi(\varphi^{-1}(x))) \tag{11.85a}$$

$$\bar{g}(\bar{x}) := \mathcal{F}_g(g(x)) \quad \Leftrightarrow \quad \bar{g}(x) := \mathcal{F}_g(g(\varphi^{-1}(x)))$$
(11.85b)

 \mathcal{F} is shorthand for the transformation of the field components (e.g., $\mathcal{F} = 1$ for a scalar).

(Note that in x is a dummy variable in the right column; you can call it whatever you like.) For example, the metric tensor transforms as

$$\bar{g}(\bar{x}) = \mathcal{F}_g(g(x)) \quad \Leftrightarrow \quad \bar{g}_{\mu\nu}(\bar{x}) = \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} g_{\alpha\beta}(x).$$
(11.86)



By definition, S_g describes a generally covariant theory iff

$$L(\bar{\phi}(\bar{x}), \bar{\partial}\bar{\phi}(\bar{x}), \bar{g}(\bar{x}), \bar{\partial}g(\bar{x})) \stackrel{11.85}{=} L(\phi(x), \partial\phi(x), g(x), \partial g(x)).$$
(11.87)

This is a non-trivial constraint on the functional form of L that can be satisfied by constructing it from proper tensorial expressions to form a scalar.

For example, the Lagrangian of Maxwell theory (in vacuum) Eq. (11.78) satisfies

$$L(\bar{A}_{\mu}(\bar{x}), \bar{\partial}_{\nu}\bar{A}_{\mu}(\bar{x}), \bar{g}^{\mu\nu}(\bar{x})) = -\frac{1}{16\pi} \bar{g}^{\mu\alpha}(\bar{x}) \bar{g}^{\nu\beta}(\bar{x}) \bar{F}_{\mu\nu}(\bar{x}) \bar{F}_{\alpha\beta}(\bar{x})$$
(11.88a)

Use tensor transformation laws.

$$= -\frac{1}{16\pi} g^{\mu\alpha}(x) g^{\nu\beta}(x) F_{\mu\nu}(x) F_{\alpha\beta}(x)$$
(11.88b)

$$= L(A_{\mu}(x), \partial_{\nu}A_{\mu}(x), g^{\mu\nu}(x))$$
(11.88c)

with $\bar{F}_{\mu\nu}(\bar{x}) = \bar{\partial}_{\mu}\bar{A}_{\nu}(\bar{x}) - \bar{\partial}_{\nu}\bar{A}_{\mu}(\bar{x}).$ $\stackrel{\circ}{\rightarrow}$

$$S_{\bar{g}}[\bar{\phi}] \stackrel{\text{def}}{=} \int d^4x \sqrt{\bar{g}(x)} L(\bar{\phi}(x), \partial\bar{\phi}(x), \bar{g}(x), \partial\bar{g}(x))$$
(11.89a)

Rename dummy variables from x to \bar{x} (no substitution!).

$$= \int \mathrm{d}^{4}\bar{x}\sqrt{\bar{g}(\bar{x})}L(\bar{\phi}(\bar{x}),\bar{\partial}\bar{\phi}(\bar{x}),\bar{g}(\bar{x}),\bar{\partial}\bar{g}(\bar{x})) \tag{11.89b}$$

Variable substitution: $\bar{x} = \varphi(x)$

$$\stackrel{11.87}{=} \int d^4x \sqrt{g(x)} L(\phi(x), \partial \phi(x), g(x), \partial g(x))$$
(11.89c)

$$\stackrel{\text{def}}{=} S_g[\phi] \tag{11.89d}$$

$\rightarrow S_g$ is ** diffeomorphism invariant

- i! This means that any generally covariant theory has a symmetry in that the value of the action functional does not change under the substitution of fields (g, φ) → (ḡ, φ̄) defined in Eq. (11.85) for arbitrary diffeomorphisms φ. Note that we must replace *both* the metric g and the matter fields φ for this to work despite the fact that g is not (yet) a dynamical field.
- What happened above is similar to what we did in Section 1.2, where we rephrased the invariance under Galilei transformations as an active symmetry of a dynamical equation.

So far, we interpreted the map $\bar{x} = \varphi(x)$ as a *passive* coordinate transformation, i.e., both x and \bar{x} are thought to describe *the same point* on the manifold. In this reading, the fields $\phi(x)$ and $\bar{\phi}(\bar{x})$ describe the *same* physical state in different frames of reference. However, we can also interpret $\bar{x} = \varphi(x)$ as an *active* transformation in that φ actively moves the point corresponding to x to a *new* point corresponding to \bar{x} (in the same coordinates!). In this interpretation, one calls $\varphi a \uparrow diffeomorphism$ (here in a particular chart), and we interpret $\bar{\phi}$ as a *new* function, defined on the *same* coordinates, describing a *different* state of the system.

3 | Diffeomorphism invariance is a *continuous* symmetry:

⊲ Infinitesimal diffeomorphisms:

$$\bar{x}^{\mu}_{\varepsilon} = \varphi_{\varepsilon}(x^{\mu}) = x^{\mu} + \delta_{\varepsilon}x^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \quad \text{with} \quad |\varepsilon^{\mu}| \ll 1 \tag{11.90}$$



You should think of the vector field $\varepsilon^{\mu}\partial_{\mu} \in TM$ generating the infinitesimal diffeomorphism:

$$\bar{\phi}_{\varepsilon}(x) \stackrel{\text{11.85}}{=} \phi(\varphi_{\varepsilon}^{-1}(x)) \stackrel{\text{11.90}}{=} \phi(x-\varepsilon) \stackrel{\text{Taylor}}{=} \phi(x) - \varepsilon^{\mu}(x)\partial_{\mu}\phi(x) + \mathcal{O}(\varepsilon^2)$$
(11.91)

(here for a scalar field) so that

$$\delta_{\varepsilon}\phi(x) \equiv \phi_{\varepsilon}(x) - \phi(x) = -\varepsilon^{\mu}\partial_{\mu}\phi(x) \tag{11.92}$$

is the infinitesimal variation of the field (at the same point) due to the diffeomorphism; \leftarrow Eq. (6.79).

Remember \checkmark *Noether's (first) theorem*: (\leftarrow Eqs. (6.84) and (6.85))

Global continuous symmetry \rightarrow Conserved current

What are the consequences of diffeomorphism invariance?

Note that there are two peculiarities that prevent us from applying Noether's (first) theorem to diffeomorphism invariance:

• Diffeomorphism invariance is a *local* symmetry.

(Since the transformation $\varepsilon^{\mu}(x)$ can depend on the spacetime point x, one can consider transformations where $\varepsilon^{\mu}(x) = 0$ everywhere except for a compact subset of the manifold.)

- For the action to be invariant, we must also replace the metric g → ḡ (which is not a dynamical field but a parameter).
- 4 | Thus let us proceed carefully and step by step:
 - $i \mid \langle \bar{\phi}_{\varepsilon}(x) \rangle$ and $\bar{g}_{\varepsilon}(x)$ defined by Eqs. (11.85) and (11.90) \rightarrow

$$\delta_{\varepsilon}S := S_{\bar{g}_{\varepsilon}}[\bar{\phi}_{\varepsilon}] - S_g[\phi] \stackrel{11.89}{=} 0 \tag{11.93a}$$

$$= \int \mathrm{d}^4 x \left[\mathcal{L}(\bar{\phi}_{\varepsilon}, \partial \bar{\phi}_{\varepsilon}, \bar{g}_{\varepsilon}, \partial \bar{g}_{\varepsilon}) - \mathcal{L}(\phi, \partial \phi, g, \partial g) \right]$$
(11.93b)

ii | We can split the variation into two parts in lowest order of ε :

$$\delta_{\varepsilon}S = \underbrace{S_{\bar{g}_{\varepsilon}}[\phi] - S_{g}[\phi]}_{=:\delta_{g}S} + \underbrace{S_{g}[\bar{\phi}_{\varepsilon}] - S_{g}[\phi]}_{=:\delta_{\phi}S} + \mathcal{O}\left(\varepsilon^{2}\right) = 0 \quad (11.94)$$

iii $| \triangleleft Solutions \phi$ of the equations of motion \Leftrightarrow

$$\forall \varepsilon^{\mu}(x) : \quad \delta_{\phi} S = S_g[\bar{\phi}_{\varepsilon}] - S_g[\phi] = 0 \tag{11.95}$$

So that \rightarrow

$$0 = \delta_{\varepsilon} S \doteq S_{\bar{g}_{\varepsilon}}[\phi] - S_{g}[\phi]$$
(11.96a)

$$= \int \mathrm{d}^4 x \left[\mathcal{L}(\phi, \partial \phi, \bar{g}_{\varepsilon}, \partial \bar{g}_{\varepsilon}) - \mathcal{L}(\phi, \partial \phi, g, \partial g) \right]$$
(11.96b)

Here " \doteq " indicates an equality that is only valid "on shell", i.e., when the matter fields satisfy the matter equations of motion; for arbitrary fields, there are additional terms to be added. One can also say that "the equation is valid modulo EOMs".



iv | Let us write the variation of the metric as follows:

$$\bar{g}_{\varepsilon}(x) \equiv g(x) + \delta_{\varepsilon}g(x) \quad \text{with} \quad |\delta_{\varepsilon}g(x)| \in \mathcal{O}(\varepsilon)$$
 (11.97)

We derive an explicit expression for $\delta_{\varepsilon}g(x) \rightarrow below$.

Eq. (11.96b) \rightarrow

$$\mathcal{L}(\phi, \partial\phi, \bar{g}_{\varepsilon}, \partial\bar{g}_{\varepsilon}) - \mathcal{L}(\phi, \partial\phi, g, \partial g) \stackrel{\mathcal{O}(\varepsilon)}{=} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \delta_{\varepsilon} g^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}} \delta_{\varepsilon} g^{\mu\nu}_{,\lambda}$$
(11.98a)

$$= \left[\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \partial_{\lambda} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}}\right] \delta_{\varepsilon} g^{\mu\nu} + \partial_{\lambda} \left(\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}} \delta_{\varepsilon} g^{\mu\nu}\right)$$
(11.98b)

Here we are sloppy and write $g^{\mu\nu}_{,\lambda} \equiv g^{\mu\nu}_{,\lambda}$ to streamline the notation. In the second step we used $\partial_{\lambda}(\delta_{\varepsilon}g^{\mu\nu}) = \delta_{\varepsilon}g^{\mu\nu}_{,\lambda}$.

• \triangleleft *Compact* variation $\varepsilon^{\mu}(x) \rightarrow \delta_{\varepsilon} g^{\mu\nu} = 0$ on boundary of spacetime:

Eqs. (11.96b) and (11.98b)
$$\xrightarrow{\text{Gauss}} 0 = \delta_{\varepsilon}S \doteq \int d^4x \frac{\delta \mathscr{L}}{\delta g^{\mu\nu}} \delta_{\varepsilon} g^{\mu\nu}$$
 (11.99)

with **** variational derivative*

$$\frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} := \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - \partial_{\lambda} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}_{,\lambda}}.$$
 (11.100)

- vi | We now want to find an explicit expression for the variation $\delta_{\varepsilon}g^{\mu\nu}$ of the metric:
 - **a** | The metric transforms as a (2, 0) tensor:

$$\bar{g}_{\varepsilon}^{\mu\nu}(\bar{x}_{\varepsilon}) \stackrel{\text{ll.85b}}{=} \frac{\partial \bar{x}_{\varepsilon}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}_{\varepsilon}^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}(x)$$
(11.101a)

$$\stackrel{11.90}{=} g^{\mu\nu}(x) + \varepsilon^{\mu}{}_{,\alpha}g^{\alpha\nu}(x) + \varepsilon^{\nu}{}_{,\beta}g^{\mu\beta}(x) + \mathcal{O}\left(\varepsilon^{2}\right) \qquad (11.101b)$$

We dropped higher powers in the variation ε^{μ} and its derivatives. Note that we implicitly assume that derivatives of ε^{μ} are also infinitesimal; this is a restriction on reasonably smooth variations ε^{μ} (which we are free to impose).

b | On the other hand, we can also simply expand the new metric:

$$\bar{g}_{\varepsilon}^{\mu\nu}(\bar{x}_{\varepsilon}) = \bar{g}_{\varepsilon}^{\mu\nu}(x+\varepsilon) = \bar{g}_{\varepsilon}^{\mu\nu}(x) + g^{\mu\nu}_{,\lambda}\varepsilon^{\lambda} + \mathcal{O}\left(\varepsilon^{2}\right)$$
(11.102)

In the second term we replaced \bar{g} by g because their difference is of order ε which, together with the ε^{λ} , can be absorbed in $\mathcal{O}(\varepsilon^2)$.

 $c \mid$ Eqs. (11.101) and (11.102) \rightarrow

$$\delta_{\varepsilon}g^{\mu\nu}(x) = \bar{g}^{\mu\nu}_{\varepsilon}(x) - g^{\mu\nu}(x) = \varepsilon^{\mu}_{,\lambda}g^{\lambda\nu} + \varepsilon^{\nu}_{,\lambda}g^{\lambda\mu} - \varepsilon^{\lambda}g^{\mu\nu}_{,\lambda} \quad (11.103)$$

d | We can use covariant derivatives and metric-compatibility to simplify this expression: Eqs. (10.50), (10.56) and (10.74) \rightarrow

$$\delta_{\varepsilon}g^{\mu\nu} \stackrel{\circ}{=} \varepsilon^{\mu}_{\ ;\lambda}g^{\lambda\nu} + \varepsilon^{\nu}_{\ ;\lambda}g^{\lambda\mu} \tag{11.104}$$

This is the variation of the metric under the infinitesimal diffeomorphism φ_{ε} .



5 | Eq. (11.99) $\xrightarrow{\text{Eq. (11.104)}}$ (Use the symmetry of $g^{\mu\nu}$.)

$$0 \doteq \int d^4x \frac{\delta \mathscr{L}}{\delta g^{\mu\nu}} \left[\varepsilon^{\mu}_{;\lambda} g^{\lambda\nu} + \varepsilon^{\nu}_{;\lambda} g^{\lambda\mu} \right] = \int \underbrace{d^4x \sqrt{g}}_{\text{Scalar}} \underbrace{\frac{2}{\sqrt{g}} \frac{\delta \mathscr{L}}{\delta g^{\mu\nu}}}_{\rightarrow \text{Tensor}} \underbrace{\varepsilon^{\mu}_{;\lambda} g^{\lambda\nu}}_{\text{Tensor}}$$
(11.105)

This motivates the definition of the ... (recall the scalar Lagrangian $L \equiv \frac{x}{\sqrt{g}}$)

** (Hilbert) Energy-Momentum Tensor (HEMT):

$$T_{\mu\nu} := \frac{2}{\sqrt{g}} \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} = \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}L)}{\delta g^{\mu\nu}}$$
(11.106)

This tensor is always symmetric: $T_{\mu\nu} = T_{\nu\mu}$.

i! At this point it is unclear what $T_{\mu\nu}$ has to do with energy and momentum, and whether it relates to the EMT/BRT derived in Sections 6.3.1 and 6.3.2 from Noether's first theorem.

6 | With this new notation, we have:

$$0 \doteq \int d^4x \sqrt{g} T_{\mu}{}^{\lambda} \varepsilon^{\mu}{}_{;\lambda} = \int d^4x \sqrt{g} \Big[\underbrace{\left(\varepsilon^{\mu} T_{\mu}{}^{\lambda}\right)}_{\text{Surface term}} -\varepsilon^{\mu} T_{\mu}{}^{\lambda}{}_{;\lambda} \Big]$$
(11.107)

Here we used the Leibniz product rule for covariant derivatives in reverse.

Compact variation $\varepsilon^{\mu}(x) \xrightarrow{\text{Eq. (10.103)}}$

$$\int \mathrm{d}^4 x \sqrt{g} \,\varepsilon^\mu T_\mu^{\ \lambda}_{;\lambda} \doteq 0 \tag{11.108}$$

Valid for all *local* variations $\varepsilon^{\mu}(x) \to T_{\mu}^{\lambda}_{;\lambda} \doteq 0$

 \rightarrow (use that we can pull indices up under the covariant derivative)

$$T^{\mu\lambda}_{\ ;\lambda} \doteq 0 \tag{11.109}$$

Thus the covariant divergence of the HEMT vanishes for solutions ϕ of the matter EOMs and an arbitrary metric g.

- Eq. (11.109) is the generally covariant form of the current conservation Eq. (6.92) that follows from translation invariance for the (symmetric = Belinfante) energy-momentum tensor: $T^{\mu\lambda}_{,\lambda} = 0$. This suggests that the name "energy-momentum tensor" is warranted, though the exact relation between the previously defined BRT Eq. (6.106) is as of yet unclear. See also the $\rightarrow next$ two points.
- i! If you think of it, Eq. (11.109) looks like free lunch! We didn't specify any special properties of the matter theory $S_g[\phi]$ except for its general covariance, which, as argued in Section 9.2, is physically vacuous. This immediately implies that it cannot be possible to derive conserved quantities from Eq. (11.109) in general because these would be conserved in any reasonable theory of (fundamental) physics, independent of its symmetries!

We will study this in more detail in \rightarrow Section 11.5.



- Diffeomorphism invariance is a local symmetry in that its transformations can affect fields
 on compact regions of spacetime only; this makes it a *gauge symmetry* (→ *Hole argument*).
 Noether's *first* theorem applies to *global* symmetries and guarantees the existence of a
 conserved charge for each generator of the symmetry. By contrast, Noether's *second* theorem
 applies to *local* (gauge) symmetries (like diffeomorphism invariance) and implies the existence
 of *constraint equations* that reduce the number of degrees of freedom constrained by the EulerLagrange equations. From this angle, Eq. (11.109) is a product of Noether's *second* theorem;
 in particular, it does not come with conserved Noether charges (which come with Noether's *first* theorem).
- 7 | <u>Useful relations:</u> (Problemset 4)

The following relations are useful to compute the HEMT $T_{\mu\nu}$ for a specific Lagrangian L:

• With Eq. (10.91) it follows

$$\frac{\partial(\sqrt{g})}{\partial g_{\mu\nu}} = \frac{1}{2}\sqrt{g}g^{\mu\nu}.$$
(11.110)

Note that this is a derivative wrt. $g_{\mu\nu}$ and not $g^{\mu\nu}$!

• Because of $g^{\mu\rho}g_{\rho\nu} = \delta^{\mu}_{\nu}$ one finds for the derivative

$$\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}} \stackrel{\circ}{=} -g_{\alpha\mu}g_{\beta\nu} \tag{11.111}$$

and with this

$$\frac{\partial(\sqrt{g})}{\partial g^{\mu\nu}} = \frac{\partial(\sqrt{g})}{\partial g_{\alpha\beta}}\frac{\partial g_{\alpha\beta}}{\partial g^{\mu\nu}} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}$$
(11.112)

Note the additional minus!

8 | Example: Maxwell theory:

Details: → Problemset 4

i | Recall the scalar Lagrangian Eq. (6.56) of electrodynamics on curved spacetimes:

$$L(\partial A, g) = -\frac{1}{16\pi} F_{\alpha\beta} F_{\lambda\rho} g^{\lambda\alpha} g^{\rho\beta} \quad \text{with} \quad F_{\alpha\beta} = \partial_{\alpha} A_{\beta} - \partial_{\beta} A_{\alpha} \,. \tag{11.113}$$

ii | The variational derivative Eq. (11.100) simplifies because L is independent of derivatives of the metric:

$$\frac{\delta(\sqrt{g}L)}{\delta g^{\mu\nu}} \stackrel{11.100}{=} \frac{\partial(\sqrt{g}L)}{\partial g^{\mu\nu}} \stackrel{11.112}{=} -\frac{1}{2}\sqrt{g}g_{\mu\nu}L + \frac{1}{8\pi}\sqrt{g}F_{\mu\lambda}F^{\lambda}_{\ \nu}$$
(11.114)

iii | Eq. (11.106) $\stackrel{\circ}{\rightarrow}$

$$T_{\mu\nu} = \frac{1}{4\pi} \left[F_{\mu\lambda} F^{\lambda}_{\ \nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right]$$
(11.115)

 This is indeed the "old" Belinfante-Rosenfeld energy momentum tensor (BRT) of the electromagnetic field discussed in Eq. (6.110), only for an arbitrary metric g_{μν} in arbitrary coordinates instead of the Minkowski metric η_{μν} in inertial coordinates. This suggests that the Hilbert energy momentum tensor is the generally covariant generalization of the symmetric BRT to arbitrary spacetimes.



- One can show rigorously that the Hilbert energy momentum tensor (HEMT), defined above, and the symmetric Belinfante-Rosenfeld energy momentum tensor (BRT), defined in Section 6.3.2, lead to the same expressions [79]. So our result for electrodynamics is no coincidence.
- Therefore one can use Eq. (11.106) as an alternative to compute the symmetric energy momentum tensor for relativistic theories on Minkowski space (as an alternative to the symmetrization procedure discussed in Section 6.3.2). To do so, use the MCP to make a Lorentz invariant Lagrangian generally covariant, then assume that the coordinates are arbitrary for the computation, and specialize to inertial coordinates at the end (in which the metric takes the form $\eta_{\mu\nu}$).
- 9 | Example: Klein-Gordon field:

Details: September 4

i | \triangleleft Real Klein-Gordon field ϕ , Eq. (11.37):

$$L(\phi, \partial\phi, g) = \frac{1}{2}g^{\mu\nu}(\partial_{\mu}\phi)(\partial_{\nu}\phi) - \frac{m^2}{2}\phi^2$$
(11.116)

Note that this is a scalar Lagrangian; the covariant derivatives equal partial derivatives because ϕ is a scalar: $\partial_{\mu}\phi = \phi_{,\mu} = \phi_{;\mu} = \nabla_{\mu}\phi$.

ii | The variational derivative Eq. (11.100) simplifies because L is again independent of derivatives of the metric:

$$\frac{\delta(\sqrt{g}L)}{\delta g^{\mu\nu}} \stackrel{\text{l1.100}}{=} \frac{\partial(\sqrt{g}L)}{\partial g^{\mu\nu}} \stackrel{\text{l1.112}}{=} -\frac{1}{2}\sqrt{g}g_{\mu\nu}L + \frac{1}{2}\sqrt{g}(\partial_{\mu}\phi)(\partial_{\nu}\phi)$$
(11.117)

iii | Eq. (11.106) $\stackrel{\circ}{\rightarrow}$

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\left(\phi^{,\alpha}\phi_{,\alpha} - m^2\phi^2\right)$$
(11.118)

This is the HEMT of the real Klein-Gordon field.

iv | Let us verify that it satisfies Eq. (11.109):

$$T^{\mu\nu}_{;\nu} = (\phi^{,\mu}\phi^{,\nu})_{;\nu} - \frac{1}{2}g^{\mu\nu} \left(\phi^{,\alpha}\phi_{,\alpha} - m^2\phi^2\right)_{;\nu}$$
(11.119a)

$$= \phi^{,\mu} \phi^{,\nu}{}_{;\nu} \underbrace{+ \phi^{,\mu}{}_{;\nu} \phi^{,\nu} - g^{\mu\nu} \phi^{,\alpha} \phi_{,\alpha;\nu}}_{^{10,53} 0} + m^2 \phi^{,\mu} \phi \qquad (11.119b)$$

$$= \phi^{,\mu} \underbrace{\left(\phi^{,\nu}_{;\nu} + m^2 \phi \right)}_{^{11.38} \circ} \doteq 0.$$
(11.119c)

Here we used $g^{\mu\nu}{}_{;\rho} = 0$ since the connection is metric-compatible. Note that we had to invoke the equation of motion in the last step to show that the divergence vanishes.

11.5. ‡ Killing vector fields and conservation laws

As noted previously, Eq. (11.109) cannot be used to define conserved quantities in general. Here we discuss this in more detail and characterize the conditions under which conserved quantities actually do exist.

Killing (vector) fields are named after German mathematician \uparrow *Wilhelm Karl Joseph Killing* (1847–1923). Note that this is a situation were case sensitivity is of paramount importance: *Killing fields* and *the* \uparrow *Killing Fields* are not related whatsoever; luckily, we're only concerned with the former.



1 | <u>Problem:</u> (Compare this to Eq. (6.85) which implies Eq. (6.87).)

$$T^{\mu\nu}_{;\nu} \stackrel{10.57a}{=} T^{\mu\nu}_{,\nu} + \Gamma^{\nu}_{\rho\nu} T^{\mu\rho} + \Gamma^{\mu}_{\rho\nu} T^{\rho\nu} \stackrel{11.109}{\doteq} 0$$
(11.120a)

$$\stackrel{10.92}{\longleftrightarrow} \quad \frac{1}{\sqrt{g}} \left(\sqrt{g} T^{\mu\nu} \right)_{,\nu} + \underbrace{\Gamma^{\mu}_{\ \rho\nu} T^{\rho\nu}}_{\text{No surface term } \odot} \doteq 0 \tag{11.120b}$$

 \rightarrow Cannot be integrated by Gauss to define conserved charge!

- \rightarrow *Question:* Are there conditions under which conserved quantities can be defined?
- **2** | Let $\varepsilon^{\mu}(x) \equiv \varepsilon \xi^{\mu}(x)$ with $\varepsilon \ll 1$; $\lt \in$ Eq. (11.104) and demand

$$\delta_{\varepsilon}g^{\mu\nu} = \varepsilon \left(\xi^{\mu}_{;\lambda}g^{\lambda\nu} + \xi^{\nu}_{;\lambda}g^{\lambda\mu} \right) \stackrel{!}{=} 0 \tag{11.121}$$

For a given metric, this is a differential equation to be solved for vector fields $\xi^{\mu}(x)$. Pull indices down \rightarrow

$$\begin{aligned} \xi_{\mu;\nu} + \xi_{\nu;\mu} &= 0 & \text{** Killing equation} \\ \Leftrightarrow & \xi^{\mu}(x) & \text{** Killing (vector) field} \end{aligned}$$
(11.122)

- Whether the Killing Eq. (11.122) has solutions depends on the metric g_{μν}; it can have none / one / multiple solutions. One then says that the metric has no / one / multiple Killing fields.
- At this point it is unclear how Killing fields can help us to find conserved quantities. However, it is intuitively clear what Killing fields are: They generate a continuous group of diffeomorphisms that do not change the metric! Such special diffeomorphisms are called *↑ isometries* of the Riemannian manifold; i.e., the infinitesimal generators of the isometry group of a Riemannian manifold are the Killing fields. This group obviously depends on the geometry of the manifold, and thus its metric. Killing fields therefore characterize the *symmetries* of a spacetime manifold. A generic spacetime will have "bumps" and "twists", and therefore no symmetries/Killing fields.
- Example: Minkowski space:

On Minkowski space we can use global inertial coordinates, so that the covariant derivatives become partial derivatives everywhere:

$$\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0. \tag{11.123}$$

It is straightforward to check that the most general solution reads

$$\xi_{\mu}(x) = a_{\mu} + b_{\mu\nu}x^{\nu}$$
 with $b_{\mu\nu} = -b_{\nu\mu}$ (11.124)

with a_{μ} and $b_{\mu\nu}$ arbitrary integration constants. There are 4 linearly independent solutions parametrized by a^{μ} , and 6 solutions parametrized by the antisymmetric $b_{\mu\nu}$; so in total Minkowski space has 10 Killing vector fields. These correspond to 4 *translations* (the a_{μ} -solutions), 3 spatial rotations and 3 boosts (the $b_{\mu\nu}$ -solutions). The isometric diffeomorphisms of Minkowski space are the \leftarrow *Poincaré transformations*! Recall our discussion of the Lorentz group in Section 4.3.

One can show that a 3 + 1-dimensional spacetime can have at most 10 Killing fields. In general, a *D*-dimensional Riemannian manifold can have at most ½*D*(*D* + 1) Killing fields; such manifolds are called *↑* maximally symmetric spaces. Minkowski space is an example of a maximally symmetric space. For more details: *↑* CARROLL [4] (§3.8 & §3.9, pp. 133-144).

3 | Conserved quantities:

We show now that Killing fields can be used to construct conserved quantities:

i $| \triangleleft$ Spacetime with Killing vector field $\xi^{\mu} \rightarrow$ Define the 4-current

$$J^{\mu}_{\xi} := T^{\mu\nu} \xi_{\nu} \,. \tag{11.125}$$

Then it follows for the covariant divergence of this current:

$$\frac{1}{\sqrt{g}} \left(\sqrt{g} J_{\xi}^{\mu} \right)_{,\mu} \stackrel{10.95}{=} J_{\xi;\mu}^{\mu} = \left(T^{\mu\nu} \xi_{\nu} \right)_{;\mu} \stackrel{11.122}{=} \xi_{\nu} T^{\mu\nu}_{;\mu} \stackrel{\doteq}{=} 0 \tag{11.126}$$

Here we used that the contraction of a symmetric with an antisymmetric tensor vanishes.

ii | Then the charge

$$Q_{\xi} := \int d^3x \sqrt{g} J_{\xi}^0 \quad \text{is conserved:} \quad \frac{dQ_{\xi}}{dx^0} \doteq 0 \,. \tag{11.127}$$

This is true if $J_{\xi}^{i} = 0$ on the boundary of space; i.e., if one considers closed systems. *Proof.* From Eq. (11.126) we have

$$0 \doteq \int d^{3}x \left(\sqrt{g} J_{\xi}^{\mu}\right)_{,\mu} = \int d^{3}x \left(\sqrt{g} J_{\xi}^{0}\right)_{,0} + \int d^{3}x \left(\sqrt{g} J_{\xi}^{i}\right)_{,i}$$
(11.128)

and therefore

$$\frac{\mathrm{d}Q_{\xi}}{\mathrm{d}x^{0}} = \int \mathrm{d}^{3}x \left(\sqrt{g}J_{\xi}^{0}\right)_{,0} \doteq -\int \mathrm{d}^{3}x \left(\sqrt{g}J_{\xi}^{i}\right)_{,i} \stackrel{\mathrm{Gauss}}{=} -\int_{\partial} \mathrm{d}\sigma_{i}\sqrt{g}J_{\xi}^{i} = 0. \quad (11.129)$$

Here we used the assumption that the current vanishes on the spatial surface $\partial: J^i_{\xi} = 0$.

iii | For the special case of point mechanics, one finds a more explicit expression:

$$\frac{\mathrm{D}u^{\mu}}{\mathrm{D}\tau} = 0 \quad \text{with 4-velocity } u^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}.$$
 (11.130)

 \rightarrow With the Killing vector ξ^{μ} it follows:

$$\frac{\mathrm{d}(\xi_{\mu}u^{\mu})}{\mathrm{d}\tau} = \frac{\mathrm{D}(\xi_{\mu}u^{\mu})}{\mathrm{D}\tau} = \underbrace{u^{\mu}\frac{\mathrm{D}\xi_{\mu}}{\mathrm{D}\tau}}_{=0} + \xi_{\mu}\underbrace{\frac{\mathrm{D}u^{\mu}}{\mathrm{D}\tau}}_{=0} \stackrel{\circ}{=} 0 \tag{11.131}$$

The second summand vanishes because of Eq. (11.130) and the first one because of the Killing Eq. (11.122):

$$u^{\mu} \frac{\mathbf{D}\xi_{\mu}}{\mathbf{D}\tau} \stackrel{\mathbf{10.49}}{=} u^{\mu}\xi_{\mu;\nu} u^{\nu} = 0.$$
 (11.132)

Here we used that the Killing equation tells us that $\xi_{\mu;\nu}$ is an antisymmetric tensor and that the contraction of a symmetric with an antisymmetric tensor vanishes.

 \rightarrow Conserved quantity:

$$\xi_{\mu}u^{\mu} = \text{const} \tag{11.133}$$

This can be useful to integrate the geodesic equation on symmetric spacetimes.



4 | Stationary & Static spacetimes:

 \rightarrow

Using Killing fields, we can define two special classes of spacetimes with useful properties:

i | We can define the following special class of metrics g:

g is ** stationary : \Leftrightarrow g has (asymptotically) time-like Killing vector

"Asymptotically time-like" means that there is a Killing vector field that becomes time-like at infinity. It is possible that the Killing field becomes space-like in some finite region of space.

- ii | \triangleleft Time-like Killing field ξ^{μ} (i.e., $\xi^{\mu}\xi_{\mu} > 0$)
 - \rightarrow Choose *w.l.o.g.* coordinates where $\xi^{\mu} = (1, 0, 0, 0)$

This means that we choose the x^0 -axis such that it points along the Killing field.

Killing Eq. (11.122)
$$\stackrel{\text{Eq. (11.103)}}{\longleftrightarrow} \frac{\partial g^{\mu\nu}}{\partial x^0} = 0$$
 (11.134)

 $\rightarrow g_{\mu\nu}(x) = g_{\mu\nu}(\vec{x})$ independent of the time-like coordinate $x^0 \equiv t!$

 \rightarrow In these coordinates, a stationary metric has the general form:

$$ds^{2} = g_{00}(\vec{x})dt^{2} + g_{0i}(\vec{x})dtdx^{i} + g_{i0}(\vec{x})dx^{i}dt + g_{ij}(\vec{x})dx^{i}dx^{j}$$

(11.135)

- Interpretation: A stationary spacetime "does exactly the same thing at every time" [4].
- *Example:* The \uparrow *Kerr metric* of a rotating black hole is stationary.
- iii | There is an interesting subclass of stationary spacetimes:

 \triangleleft Stationary metric g. Iff there exists a coordinate system such that $g_{0i}(\vec{x}) = 0$,

$$ds^{2} = g_{00}(\vec{x})dt^{2} + g_{ij}(\vec{x})dx^{i}dx^{j}$$
(11.136)

the metric is called **** static*.

- Interpretation: A static spacetime "doesn't do anything at all" [4].
- *Example:* The (exterior) → *Schwarzschild metric* of a non-rotating black hole is static.
- The feature that distinguishes a stationary (non-static) metric Eq. (11.135) from a static metric Eq. (11.136) is that the latter is invariant under reversal t → -t of the time-like coordinate.

This makes sense if you compare the static Schwarzschild metric (non-rotating black hole) with the stationary Kerr metric (rotating black hole): If you invert time, a non-rotating black hole looks exactly the same, so also its metric should not change; hence it has the form Eq. (11.136) and is static. By contrast, the angular momentum of a rotating Kerr black hole changes sign under $t \mapsto -t$, so that it looks different in a time-reversed world; hence its metric should change as well, which necessitates the off-diagonal terms of a stationary (non-static) metric Eq. (11.135). (Note that the Kerr black hole still describes a *stationary* system in that its angular momentum doesn't change in time, and consequently the metric "does the same thing at every time".)



 $iv \mid$ In summary:



• *Counterexample:* The \uparrow *Kerr metric* is stationary but not static.

↓Lecture 25 [18.06.24]



12. The Einstein Field Equations

We are only one step away from completing the theoretical framework of GENERAL RELATIVITY.

In the previous Chapter 11 we studied how matter fields are affected by the metric of spacetime. What we are missing is the converse: How is the metric of spacetime determined in the first place? This is the question we will answer in this chapter, and it will lead us to the most important result of this course: The Einstein field equations.

12.1. Derivation of the Einstein field equations

In the following, we make the following general (and rather weak) assumptions:

§ Assumptions 2

3P1 Spacetime is a 3 + 1-dimensional Lorentzian manifold.

- **MTR** There exists a dynamical metric field g.
- **FLD** All other degrees of freedom ("matter") are described by fields ϕ .

Note that we write ϕ as placeholder for a family of (not necessarily scalar) fields.

VAR The classical dynamics of all fields can be described by a variational principle.

- LOC All actions are given by integrals over local Lagrangians.
- **COV** All theories are generally covariant (**GR**).

Follow the arguments below carefully; each step is quite simple, so that the derivation borders on magic:

1 | The action of Everything:

MTR + FLD + VAR $\rightarrow \triangleleft$ "Action of Everything":

$$S[g,\phi] = \underbrace{S[g]}_{\text{Only metric}} + \underbrace{S_g[\phi]}_{\text{Rest}}$$
(12.1)

Without loss of generality, we can divide the action into a purely metric part and a "rest", all terms of which contain at least one matter field.

Combining GENERAL RELATIVITY with the Standard Model of particle physics tells us what this action actually looks like (at least in the infrared limit), recall the \leftarrow *Core Theory* mentioned in Section 0.4. These details are not relevant for what follows, though.

2 | Equations of motion of Everything:



As usual, the physical solutions extremize the action (variational principle):

$$\operatorname{VAR} \to \delta S[g,\phi] \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \begin{cases} \delta_g S[g,\phi] = \delta_g S[g] + \delta_g S_g[\phi] \stackrel{!}{=} 0\\ \delta_\phi S[g,\phi] = \delta_\phi S_g[\phi] \stackrel{!}{=} 0 \end{cases}$$
(12.2)

To extremize the action, both equations on the right must be satisfied simultaneously:

• EOMs for *matter fields*:

$$\delta_{\phi} S_g[\phi] \stackrel{!}{=} 0 \tag{12.3}$$

These equations describe the dynamics of *matter fields* on a given "background" metric g.

 \rightarrow Already kown & understood! (\leftarrow Chapter 11)

An example is the Maxwell action (11.78), the variation of which leads to the Maxwell Eq. (11.79).

• EOMs for *metric field*:

$$\delta_g S[g] \stackrel{!}{=} -\delta_g S_g[\phi] \tag{12.4}$$

These equations describe the dynamics of the *metric* and its interaction with matter.

 \rightarrow New! What can we say about this equation of motion?

 $3 \mid LOC \rightarrow$

Because of locality, we can write both parts of the action as integrals of Lagrangian(densities):

$$S[g] = \int d^4x \sqrt{g} L_{\text{Metric}}(g, \partial g)$$
(12.5a)

$$S_g[\phi] = \int d^4x \sqrt{g} L_{\text{Matter}}(\phi, \partial\phi, g, \partial g)$$
(12.5b)

These expressions actually require an additional prefactor $\frac{1}{c}$ for dimensional reasons because we measure time coordinates in units of length ($x^0 = ct$); we omit these prefactors because they are irrelevant in the following and drop out in the next step anyway.

Eq. (12.4) is then equivalent to \rightarrow

$$\int \underbrace{\mathrm{d}^4 x \sqrt{g}}_{\mathrm{Scalar}} \underbrace{\frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g} L_{\mathrm{Metric}})}{\delta g^{\mu\nu}}}_{=: \Box_{\mu\nu} (\mathrm{unknown})} \underbrace{\delta g^{\mu\nu}}_{\mathrm{Variation}} \underbrace{\frac{1}{\sqrt{g}} \frac{\mathrm{d}^4 x \sqrt{g}}{\mathrm{Scalar}}}_{\mathrm{Scalar}} \underbrace{\frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g} L_{\mathrm{Matter}})}{\delta g^{\mu\nu}}}_{\underbrace{\delta g^{\mu\nu}}_{\mathrm{Variation}}} \underbrace{\delta g^{\mu\nu}}_{\mathrm{Variation}} (12.6)$$

Here we used the variational derivative Eq. (11.100), multiplied the equation by 2 and inserted \sqrt{g} to identify the Hilbert energy-momentum tensor Eq. (11.106) on the right-hand side.

Eq. (12.6) valid for all variations $\delta g^{\mu\nu}(x) \rightarrow$

$$\delta_g S[g,\phi] \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \Box_{\mu\nu} \stackrel{!}{=} -T_{\mu\nu} \tag{12.7}$$

- **4** | What do we know about $\Box_{\mu\nu}$?
 - □_{μν} depends on metric g and its derivatives ∂g, ∂²g,...
 Note that □_{μν} does not contain matter fields φ by construction.



• COV $\rightarrow \Box_{\mu\nu}$ is (0, 2)-tensor

This follows from the definition in Eq. (12.6) with the same arguments as for $T_{\mu\nu}$ in Section 11.4.

• $\Box_{\mu\nu}$ is symmetric

This follows from the definition in Eq. (12.6) with the same arguments as for $T_{\mu\nu}$ in Section 11.4.

• $\operatorname{COV} \to \Box_{\mu\nu}$ is identically divergence-free: $\Box^{\mu\nu}_{;\nu} \equiv 0$

An often heard argument for this condition is the following: Since $T_{\mu\nu}$ satisfies $T^{\mu\nu}_{;\nu} \doteq 0$ [recall Eq. (11.109)], Eq. (12.7) implies that $\Box^{\mu\nu}_{;\nu} = 0$. This argument is sloppy at best because of the little dot over the equal sign in $T^{\mu\nu}_{;\nu} \doteq 0$; recall that this indicates that the equation is only true for *special* matter fields, namely those that satisfy the equation of motion Eq. (12.3). Thus, from this line of argument, one can only conclude that $\Box^{\mu\nu}_{;\nu} \doteq 0$, i.e., $\Box_{\mu\nu}$ is divergence-free for solutions (g, ϕ) of the equations of motion.

But our claim – which is crucial for the next step – is much stronger: $\Box^{\mu\nu}_{;\nu} \equiv 0$ is an *identity* (that's why we use \equiv and not =), i.e., it is valid for *arbitrary* metric fields. That this must be true follows from our derivation in Section 11.4: Note that, because of Eq. (12.6), $\Box_{\mu\nu}$ plays formally (not physically, $\rightarrow later$) the role of $T_{\mu\nu}$ for a purely gravitational theory $S[g] = S_g[\bullet]$ without matter fields. One can then retrace our derivation in Section 11.4 with the simplification that $\delta_{\phi}S_g[\phi] \equiv 0$ is trivially satisfied (because there are no fields ϕ). Instead of $\Box^{\mu\nu}_{;\nu} \doteq 0$, one finds the identity $\Box^{\mu\nu}_{;\nu} \equiv 0$. This unconditional identity is therefore a consequence of the general covariance (= diffeomorphism invariance) of the gravitational action S[g] and the fact that it does not depend on any other fields. $\Box^{\mu\nu}_{;\nu} \equiv 0$ is an example of a so called \uparrow *Noether identity* that follows, via Noether's *second* theorem, from a group of local (gauge) symmetries (here: diffeomorphisms) [150].

These are all necessary properties of $\Box_{\mu\nu}$; no discussions!

5 We now make one (and the only) simplifying assumption, namely:

§ Assumptions 3

2ND The tensor $\Box_{\mu\nu}$ depends on g, ∂g , $\partial^2 g$ (but not on higher-order derivatives).

- This is the only simplicity assumption we use in our derivation. If you drop it, you can construct (more complicated) *modifications* of GENERAL RELATIVITY (→ *later*).
- Can you think of any equation of motion (classical or quantum, doesn't matter) that contains third- or even higher-order derivatives? No? Nothing? So our assumption isn't that outlandish after all ...
- **6** <u>Lovelock's theorem:</u>

We already know *two* tensors that satisfy all these properties:

Metric $g_{\mu\nu}$: $g^{\mu\nu}_{;\nu} \stackrel{10.74}{=} 0$ (Metric-compatibility) (12.8a) Einstein tensor $G_{\mu\nu}$: $G^{\mu\nu}_{;\nu} \stackrel{10.122}{=} 0$ (Bianchi identity) (12.8b)

Recall that the Einstein tensor depends linearly on the curvature tensor which, in turn, depends on second (and first) derivatives of the metric.



3P1 + **2ND** $\xrightarrow{\uparrow}$ Lovelock's theorem This list is exhaustive!

Lovelock's theorem states that, in D = 4 spacetime dimensions, the only divergence-free rank-2 tensors that can be constructed from the metric and its first and second derivatives are the Einstein tensor $G_{\mu\nu}$ and the metric $g_{\mu\nu}$ itself (for details see notes \rightarrow *below*).

- Since Lovelock's theorem is just a mathematical fact with a technical proof [135, 136], we take it at face value. Note that this does not open a conceptual gap in our derivation. We do not push any assumptions under the rug! See also MISNER *et al.* [3] (§17.1, pp. 407–408).
- Consider rank-2 tensors A^{ij} that are ...
 - (a) ... functions of the metric and its first two derivatives: $A^{ij} = A^{ij}(g, \partial g, \partial^2 g)$.
 - (b) ... divergence-free: $A^{ij}_{i} = 0$.
 - (c) ... symmetric: $A^{ij} = A^{ji}$.
 - (d) ... linear in $\partial^2 g$.

The statement of Lovelock's theorem is the following:

The only tensors with the properties (a)-(d) are G_{ij} and g_{ij} .

(This result is actually not due to LOVELOCK but CARTAN, WEYL and VERMEIL, see references in [136].)

Note that this statement is independent of the spacetime dimension D!

However, if one presumes that spacetime is D = 4-dimensional (which we did anyway starting from Chapter 11), LOVELOCK showed [136] that the assumptions of symmetry (c) and linearity in the second derivative (d) are *superfluous* and can be dropped!

Thus we are left with the only non-trivial assumption that the EOM of the metric field does not contain higher than second derivatives of the metric.

 \rightarrow Most general form of Eq. (12.7):

$$\Box_{\mu\nu} = \alpha G_{\mu\nu} + \beta g_{\mu\nu} \stackrel{!}{=} -T_{\mu\nu} \tag{12.9}$$

Note all conditions above are preserved by linear combinations.

7 | *** Einstein field equations (EFE)*:

Let us reshuffle and rename the unkown constants α and β a bit:

$$\underbrace{\frac{G_{\mu\nu}}{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}}_{\text{"Geometry"}} = -\underbrace{\kappa T_{\mu\nu}}_{\text{"Matter"}}$$
(12.10)

Two unknown parameters:

- ** Einstein gravitational constant κ
- ** Cosmological constant Λ

We will discuss these two parameters \rightarrow *below*.

Notes:



- The minus on the right-hand side of Eq. (12.10) depends on the convention; here we follow SCHRÖDER [2] (who follows the original convention by Einstein). There is a plethora of sign conventions in the literature, in some of which the minus in Eq. (12.10) is not present [↑ MISNER et al. [3] (first page)].
- If one removes the inconsistency of the linearized tensor gravity discussed in Section 8.2 (cf. Eq. (8.10), see also Problemset 1), one inevitably ends up with the Einstein field Eq. (12.10) [102]. Recall that we identified the *linearity* of Eq. (8.10) as the root cause for its inconsistency; in Section 8.2 we then argued on very general grounds that a relativistic theory of gravity must be *non-linear* in the gravitational field. Eq. (12.10) satisfies this: the Einstein tensor is *non-linear* in the metric (and its derivatives).

 \rightarrow

The superposition principle is not valid for the EFE!

This makes solving the EFE extremely hard in general.

- The above derivation of the EFEs used surprisingly few (and simple) assumptions. This makes the EFEs very "generic," and one shouldn't be surprised that there are many different routes to derive them. An overview over alternative derivations (or axiomatizations) of the Einstein field equations can be found in MISNER *et al.* [3] (pp. 417–428).
- The EFEs are the Euler-Lagrange equations that come from the variation of an action (which we don't know yet); i.e., we formulate GENERAL RELATIVITY in the \checkmark Lagrangian formalism. There is also a \checkmark Hamiltonian formulation of GENERAL RELATIVITY, the so called \uparrow ADM formalism [151], which plays an important role as a starting point for some theories of quantum gravity.
- The Einstein field equations are both very simple and very complicated:

They are simple in the sense that their derivation doesn't need much physical input; as we have seen above, under very general assumptions (like general covariance), the EFEs are *inevitable*. In that sense, GENERAL RELATIVITY is a very "cheap" theory (we don't have to "pay" with a lot of assumptions about reality).

On the other hand, because of their non-linearity, the EFEs are mathematically extremely complicated and hard to solve (\rightarrow *below*). This sounds bad, but is actually their greatest strength: because of their complexity, they predict and describe a plethora of non-trivial, unanticipated phenomena [black holes, gravitational waves, gravitational lenses, an expanding universe, ...; all of this is hidden in the innocuous-looking Eq. (12.10)].

Good physical theories have a high "compression ratio" of input vs. output: they describe a variety of phenomena with little input. This makes the EFEs (and thereby GENERAL RELATIVITY) one of the most successful physical theories of all time.

It is almost *too* good, at least as a starting point for a "theory of everything" (presumably a theory of quantum gravity). To find such a theory, we need *input*: features and phenomena of reality that we can use as starting points for an "inductive bootstrap" towards a more fundamental theory. The problem is that GENERAL RELATIVITY tells us that a big chunk of the crazy stuff happening in our world (black holes etc.) can be traced back to Eq. (12.10), which, as we have seen, is implied by rather generic assumptions about reality. Thus, while every viable theory of quantum gravity must necessarily lead to Eq. (12.10) in a classical regime, this might not be such a distinguishing feature as one might hope. Put differently: it might turn out to be hard to write down reasonable theories of quantum gravity that *do not* lead to Eq. (12.10).



• Some historical notes:

- A precursor to GENERAL RELATIVITY was developed by Einstein and his friend and colleague Marcel Grossmann (a mathematician who introduced Einstein to differential geometry) already in 1913, the so called *"Entwurftheorie"* [123]; it contained essentially all parts needed to formulate GENERAL RELATIVITY, but not yet the correct field Eq. (12.10).
- Einstein developed GENERAL RELATIVITY, culminating the EFEs Eq. (12.10), in a sequence of papers between October and November 1915 in the "Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin":
 - * On 4. November 1915, Einstein publishes "Zur allgemeinen Relativitätstheorie" [12] (extended by an addendum), where he proposed the (not yet quite correct) field equations $R_{\mu\nu} = -\kappa T_{\mu\nu}$. That is, he still missed the term $-\frac{1}{2}g_{\mu\nu}R$ that converts $R_{\mu\nu}$ into the Einstein tensor $G_{\mu\nu}$ (which satisfies the necessary condition $G^{\mu\nu}_{;\nu} \equiv 0$). (Beware: Einstein denoted the Ricci tensor by $G_{\mu\nu} © \odot$.)
 - * On 25. November 1915, Einstein published in "*Die Feldgleichungen der Gravitation*" [13] finally the correct field equations (without cosmological constant).

(Beware: Einstein's notation differs from the modern notation, so be careful when comparing Ref. [13] with Eq. (12.10); ⇒ Problemset 4.)

- * On 8. Februar 1917, Einstein introduces the cosmological term $\Lambda g_{\mu\nu}$ in "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie" [15] [Eq. (13a) on p. 151].
- The German mathematician David Hilbert arrived at the Einstein field equations almost at the same time as Einstein (↑ p. 8 in Ref. [152]). Hilbert introduced the HEMT Eq. (11.106) and obtained Eq. (12.10) (without cosmological constant) directly via the variation of an action (the *→ Einstein-Hilbert action*), derived from a Lagrangian (which Hilbert called "Weltfunktion", essentially our "Action of Everything").
- A first comprehensive account of GENERAL RELATIVITY, summarizing all his previous results that had appeared in many different papers, was provided by Einstein in "*Die Grundlage der allgemeinen Relativitätstheorie*" in 1916 [21].
- Details on the historical genesis of the Einstein field equations can be found in Ref. [153].

8 | <u>Trace-inverted form:</u>

Let $T := T^{\mu}_{\mu}$ be the trace of the energy-momentum tensor $\stackrel{\circ}{\rightarrow}$

Eq. (12.10)
$$\Leftrightarrow R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) + \Lambda g_{\mu\nu}$$
 (12.11)

This is the (completely equivalent) trace-inverted form of the Einstein field equations.

Proof. Taking the trace on both sides of Eq. (12.10) yields

$$R^{\mu}_{\ \mu} - \frac{1}{2}R\delta^{\mu}_{\mu} + \Lambda\delta^{\mu}_{\mu} = -\kappa T^{\mu}_{\ \mu} \quad \Leftrightarrow \quad R = \kappa T + 4\Lambda \tag{12.12}$$

where we used $\delta^{\mu}_{\mu} = 4$. We can now apply Eq. (12.12) to replace R in Eq. (12.10),

$$R_{\mu\nu} - \frac{1}{2}(\kappa T + 4\Lambda)g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}, \qquad (12.13)$$

which can be reshuffled to Eq. (12.11).


9 | Vacuum field equations:

The cosmological constant has the interpretation of a vacuum energy (\rightarrow *below*). If we assume this contribution to be absent, "vacuum" means "no energy & momentum":

Eq. (12.11)
$$\xrightarrow{\Lambda=0, T_{\mu\nu}=0} R_{\mu\nu} = 0$$
 (12.14)

→ Vacuum solutions = ** *Ricci-flat* spacetime manifolds

- i! Note that $R_{\mu\nu\rho\sigma} = 0$ implies $R_{\mu\nu} = 0$ but not the other way around! Ricci-flat spacetimes are therefore not necessarily flat (= Minkowskian).
- Note that $R_{\mu\nu} = 0$ is equivalent to $G_{\mu\nu} = 0$.
- Simplest solution: Minkowski space $g_{\mu\nu} = \eta_{\mu\nu}$

There are also more complicated, non-trivial solutions; e.g., the \rightarrow *Schwarzschild solution*, which describes the exterior geometry of a spherically symmetric mass, or gravitational wave solutions (note that these waves propagate through vacuum: $T_{\mu\nu} = 0$).

• Eq. (12.14) looks simple, right? Well, not so much:

$$0 \stackrel{!}{=} R_{\mu\nu} \tag{12.15a}$$

$$\frac{1}{2}g^{\lambda\pi} \left(g_{\lambda\pi,\mu,\nu} + g_{\mu\nu,\lambda,\pi} - g_{\lambda\nu,\mu,\pi} - g_{\mu\pi,\lambda,\nu}\right)$$

$$+ g^{\lambda\pi} \left(\Gamma_{\rho\mu\nu}\Gamma^{\rho}_{\ \lambda\pi} - \Gamma_{\rho\mu\pi}\Gamma^{\rho}_{\ \lambda\nu}\right)$$

$$(12.15b)$$

$$\stackrel{10.79}{=} \frac{1}{2} g^{\lambda \pi} \left(g_{\lambda \pi, \mu, \nu} + g_{\mu \nu, \lambda, \pi} - g_{\lambda \nu, \mu, \pi} - g_{\mu \pi, \lambda, \nu} \right)$$
(12.15c)
$$+ \frac{1}{4} g^{\lambda \pi} g^{\rho \eta} \left(g_{\rho \mu, \nu} + g_{\nu \rho, \mu} - g_{\mu \nu, \rho} \right) \left(g_{\eta \lambda, \pi} + g_{\pi \eta, \lambda} - g_{\lambda \pi, \eta} \right)$$

$$= \frac{1}{2}g^{\lambda\pi} \left(g_{\lambda\pi,\mu,\nu} + g_{\mu\nu,\lambda,\pi} - g_{\lambda\nu,\mu,\pi} - g_{\mu\pi,\lambda,\nu}\right)$$
(12.15d)
$$= \frac{1}{2}g^{\lambda\pi} \left(g_{\lambda\pi,\mu,\nu} + g_{\mu\nu,\lambda,\pi} - g_{\lambda\nu,\mu,\pi} - g_{\mu\pi,\lambda,\nu}\right)$$
(12.15d)
$$+ \frac{1}{4}g^{\lambda\pi}g^{\rho\eta} \left[\begin{array}{c} g_{\rho\mu,\nu}g_{\eta\lambda,\pi} + g_{\nu\rho,\mu}g_{\eta\lambda,\pi} - g_{\mu\nu,\rho}g_{\eta\lambda,\pi} \\ + g_{\rho\mu,\nu}g_{\pi\eta,\lambda} + g_{\nu\rho,\mu}g_{\pi\eta,\lambda} - g_{\mu\nu,\rho}g_{\pi\eta,\lambda} \\ - g_{\rho\mu,\nu}g_{\lambda\pi,\eta} - g_{\nu\rho,\mu}g_{\lambda\pi,\eta} + g_{\mu\nu,\rho}g_{\lambda\pi,\eta} \\ - g_{\rho\mu,\pi}g_{\eta\lambda,\nu} - g_{\pi\rho,\mu}g_{\eta\lambda,\nu} + g_{\mu\pi,\rho}g_{\eta\lambda,\nu} \\ - g_{\rho\mu,\pi}g_{\nu\eta,\lambda} - g_{\pi\rho,\mu}g_{\nu\eta,\lambda} + g_{\mu\pi,\rho}g_{\nu\eta,\lambda} \end{array} \right]$$

 $[+g_{\rho\mu,\pi}g_{\lambda\nu,\eta} + g_{\pi\rho,\mu}g_{\lambda\nu,\eta} - g_{\mu\pi,\rho}g_{\lambda\nu,\eta}]$

Happy solving! $\bigcirc \bigcirc \rightarrow$

Even the vacuum EFEs are extremely complicated and only a few exact solutions are known.

If one allows for a finite cosmological constant Λ ≠ 0, and considers an otherwise empty universe (T_{µν} = 0), one finds the more general vacuum EFE

$$R_{\mu\nu} = \Lambda g_{\mu\nu} \,. \tag{12.16}$$

Solutions of this equation are called ** *Einstein manifolds*.

Note that flat Minkowski space does *not* solve this equation for $\Lambda \neq 0$; since interstellar space (= vacuum) is very close to flat Minkowski space (SPECIAL RELATIVITY is valid to good approximation), this already tells us that the cosmological constant, if nonzero, cannot be very large in our universe. This is why the cosmological constant Λ is often set to zero for non-cosmological calculations (e.g., for tests in the solar system).

10 | Properties:



• How many *independent* EFEs are there?

The Einstein field Eq. (12.10) in vacuum (without cosmological constant)

$$G_{\mu\nu} = 0 \tag{12.17}$$

is a set of *second-order* partial differential equations (PDEs) that determine the evolution of the metric tensor field $g_{\mu\nu}(x)$. Thus, for a *three*-dimensional spatial slice at time coordinate x_*^0 , you can provide initial data $g_* \equiv g_{\mu\nu}(x_*^0, \vec{x})$ and $\dot{g}_* \equiv g_{\mu\nu,0}(x_*^0, \vec{x})$, and the EFE should provide you with a solution $g_{\mu\nu}(x)$ defined on the full spacetime (ignoring issues with singularities). Since $G_{\mu\nu}$ is symmetric, the EFEs correspond to 10 PDEs, which matches the 10 independent components of the metric $g_{\mu\nu}$ (which is also symmetric).

There is a catch, though: The four Bianchi *identities* $G^{\mu\nu}_{;\nu} \equiv 0$ ($\mu = 0, \dots, 3$) tell us that not all of these 10 PDEs are independent. Due to these constraints, we actually loose four of the 10 equations, which makes the EFEs *underconstrained*. That is, we should expect that solutions $g_{\mu\nu}$ of the EFEs retain four unconstrained degrees of freedom that can be changed arbitrarily. This reflects of course our freedom to change coordinates! Viewed as an active transformation, this freedom corresponds to the diffeomorphism invariance of the Einstein-Hilbert action, which must be interpreted as a *gauge symmetry* (with four generators): different solutions $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ that are related by a coordinate transformation / diffeomorphism describe the *same* physics! The Bianchi identities $G^{\mu\nu}_{;\nu} \equiv 0$ can then be interpreted as \uparrow *Noether identities*, following from Noether's *second* theorem.

• Degrees of freedom:

So how many *physical* degrees of freedom do the EFEs then actually describe? Subtracting the 4 gauge DOF from the 10 DOF of the metric yields 6 DOF; but, as we will later see in our discussion of \rightarrow *gravitational waves*, this cannot be the end of the story because gravitational waves have only *two* polarizations and not 6 (just like photons)! What is going on?

To solve this puzzle, we must first recognize that all dynamical degrees of freedom of a *deterministic* theory, described by a second-order PDE, are encoded in the initial data (g_*, \dot{g}_*) . Since the EFEs describe a gauge theory, they are only deterministic if we throw all gauge-equivalent solutions into a common "gauge equivalence class"; thus let $[g_*, \dot{g}_*]$ denote the class of field configurations on the spatial slice at x^0_* that are equivalent modulo coordinate transformations (diffeomorphisms). The physical degrees of freedom are then the DOF that parametrize these gauge classes; and – according to our argument above – there should be 6 such degrees of freedom (in configuration space, not in phase space).

The problem is that not all initial field configurations $[g_*, \dot{g}_*]$ are allowed (= yield solutions) because the initial data must satisfy *four* constraint equations:

$$G^{\mu 0} = f(g, \dot{g}, \partial_i g, \partial_i^2 g) \stackrel{!}{=} 0 \text{ for } \nu = 0, \dots, 3.$$
 (12.18)

These equations are just part of the EFEs Eq. (12.17); the point is that the $G^{0\nu}$ are functions of only *first* time derivatives of the metric. Hence they are not evolution equations at all – they are *constraint* equations that must be satisfied by the initial data $[g_*, \dot{g}_*]$. Put differently: You cannot hand in an arbitrary initial configuration $[g_*, \dot{g}_*]$ and expect the EFEs to spit out a solution. Only the special subclass of initial configurations that satisfy Eq. (12.18) yield solutions. As Eq. (12.18) provides four constraints, this cuts down the physical DOF by another 4. So in summary there are only 10 - 4 - 4 = 2 physical DOF described by the EFEs per point of *space*, which matches the two polarizations of gravitational waves.

How to see that Eq. (12.18) is correct? We want to avoid an expansion of the Einstein tensor in terms of the metric (because it is ugly). To this end, expand the Bianchi identity:

$$G^{\mu\nu}_{;\nu} \stackrel{10.5/a}{=} G^{\mu0}_{,0} + G^{\mu i}_{,i} + \ldots \equiv 0$$
(12.19)



where the \ldots part does not contain derivatives of G. So we have

$$G^{\mu 0}_{,0} \equiv -G^{\mu i}_{,i} - \dots$$
 (12.20)

But the right-hand side contains at most *second* time derivatives of the metric. Since this is an identity, $G^{\mu 0}$ can only contain at most *first* time derivatives of the metric. This is the statement of Eq. (12.18).

Quite surprisingly, the equations of motion for the *matter* fields Eq. (12.3) are already contained in the integrability constraint T^{μν}_{;ν} = 0 that follows from the identity G^{μν}_{;ν} ≡ 0 (↑ Ref. [154]). Put differently:

The Einstein field Eq. (12.10) are not only the differential equations that determine the geometry of spacetime in response to the energy and momentum of the matter fields, but, at the same time, determine the evolution of the matter fields themselves!

This is possible because the EFEs are non-linear [154].

To understand how strange this is, recall our theory in Section 6.4 that described the joint evolution of an electromagnetic field coupled to charged, massive particles. There we derived *two* equations of motion: the "matter EOM" Eq. (6.130) describes the motion of particles in response to the EM field, and the "field EOM" Eq. (6.125) describes the evolution of the EM field in response to the current produced by the charged particles. To describe the evolution of the full system, one needs *both* EOMs – one cannot derive the Lorentz force law Eq. (6.132) from the inhomogeneous Maxwell Eq. (6.125) (at least not without assuming conservation of total energy and momentum).

Naïvely, the Einstein field Eq. (12.10) parallel the inhomogeneous Maxwell Eq. (6.125) in that they describe the response of a field (the metric) to a source (energy & momentum). The difference is that the EFEs are so restrictive (due to their non-linearity), that they already contain (local) conservation of energy and momentum, and thereby the matter EOMs! Thus, in GENERAL RELATIVITY, the geometry of spacetime and the evolution of matter are so tightly interwoven, that one can only solve them together (which makes solving the EFEs in general extremely hard, if not impossible).

As an example, consider \uparrow *Einstein-Maxwell theory* that describes a universe filled with Maxwell's EM field but nothing else (no charges). The source of the gravitational field is then given by the HEMT Eq. (11.115) of Maxwell theory,

$$T_{\mu\nu} = \frac{1}{4\pi} \left[F_{\mu\lambda} F^{\lambda}_{\ \nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right], \qquad (12.21)$$

and the equations of motion of the coupled system read

Eq. (12.3) $\Leftrightarrow F^{\mu\nu}_{;\nu} = 0$ (Inhomogeneous Maxwell eqs.), (12.22a)

Eq. (12.4)
$$\Leftrightarrow G_{\mu\nu} = -\kappa T_{\mu\nu}$$
 (Einstein field eqs.). (12.22b)

We assume that $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ with the gauge field A_{μ} , so that the homogeneous Maxwell equations Eq. (11.69) are identically satisfied.

If we combine Eq. (12.21) with Eq. (12.22b), we obtain the ** Einstein-Maxwell equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{\kappa}{4\pi} \left[F_{\mu\lambda} F^{\lambda}_{\ \nu} + \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right].$$
(12.23)

The crucial (and surprising) insight is that Eq. (12.23) [equivalently: Eq. (12.22b) and Eq. (12.21)] already *contains the inhomogeneous Maxwell* Eq. (12.22a):

$$G_{\mu\nu} = -\kappa T_{\mu\nu} \quad \stackrel{*}{\Rightarrow} \quad F^{\mu\nu}_{\ ;\nu} = 0. \tag{12.24}$$



So writing down the Einstein field equations – with an explicit expression of the energymomentum tensor in terms of the matter fields on the right – is tantamount to writing down *all* equations of motion!

For more details [and a proof of Eq. (12.24)] see MISNER et al. [3] (§20.6, pp. 471–483).

For a fully "geometrized" formulation of Einstein-Maxwell theory see Ref. [155].

12.1.1. Newtonian limit

We now want to study the relation between the EFEs and Newtonian mechanics to determine the Einstein gravitational constant κ via a correspondence principle.

11 | <u>Non-relativistic limit:</u>

• Slowly varying, weak gravitational fields \rightarrow Metric almost Minkowskian:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$$
 with small perturbation $|h_{\mu\nu}(x)| \ll 1$ (12.25)

In the following, we keep only the lowest order terms in $h_{\mu\nu}$.

• Slow bodies $(v \ll c) \rightarrow$ Source of gravity = Mass density $\rho(x)$ (= rest energy)

$$T_{\mu\nu} = \begin{cases} \rho c^2 & \mu\nu = 00\\ 0 & \text{otherwise} \end{cases} \Rightarrow T = T^{\mu}_{\ \mu} = \rho c^2 \qquad (12.26)$$

The energy-momentum tensor of a \uparrow *perfect fluid* of mass-energy density $\rho(x)$, pressure p(x), and 4-velocity field $u^{\mu}(x)$ is given by

$$T^{\mu\nu}(x) = \left(\rho + \frac{p}{c^2}\right) u^{\mu} u^{\nu} - p g^{\mu\nu} .$$
 (12.27)

In a comoving frame (where the fluid is at rest), it is $u^{\mu} = (c, 0, 0, 0)$ and $g^{\mu\nu} = \eta^{\mu\nu}$ so that

$$T^{\mu\nu} = \operatorname{diag}\left(\rho c^{2}, p, p, p\right) \stackrel{p\approx0}{\approx} \operatorname{diag}\left(\rho c^{2}, 0, 0, 0\right)$$
(12.28)

if we assume the pressure to be negligible wrt. the rest energy.

12 | $\triangleleft \mu \nu = 00$ in Eq. (12.11) $\xrightarrow{\Lambda = 0}$

We are interested in the $\mu\nu = 00$ component because in Eq. (11.65) we related this component of the metric with the Newtonian gravitational potential in the non-relativistic limit.

$$R_{00} = -\frac{\kappa}{2}\rho c^2 + \mathcal{O}(\hbar)$$
(12.29)

We drop $h_{\mu\nu}(x)$ on the right-hand side because this is a higher-order perturbation that can be neglected for $|h_{\mu\nu}| \ll 1$.

i | Because of $h_{\mu\nu}(x)$ there is *no* global inertial coordinate system; but there is a coordinate system where $\mathcal{O}(\Gamma) = \mathcal{O}(h)$ with connection coefficients $\Gamma \sim |\Gamma^{\rho}_{\mu\nu}|$.

Eqs. (10.104) and (10.106) \rightarrow

$$R_{00} = \partial_0 \Gamma^{\mu}_{\ 0\mu} - \partial_\mu \Gamma^{\mu}_{\ 00} + \mathcal{O}(\hbar^2)$$
(12.30)



$$R_{00} = \underline{\partial_0} \Gamma^{\mu}_{0\mu} - \underline{\partial_0} \Gamma^{0}_{00} - \overline{\partial_i} \Gamma^{i}_{00} \approx -\overline{\partial_i} \Gamma^{i}_{00}$$
(12.31)

with Christoffel symbols Eq. (10.79)

$$\Gamma^{i}_{00} \stackrel{12.25}{\approx} -\frac{1}{2} \partial^{i} h_{00} . \qquad (12.32)$$

Here we also dropped time derivatives.

 \rightarrow

$$R_{00} \approx \frac{1}{2} \partial_i \partial^i h_{00} = -\frac{1}{2} \Delta h_{00}$$
(12.33)

Here we used $\eta_{\mu\nu}$ for pulling indices up/down since the modification by $h_{\mu\nu}$ yields higherorder terms $\mathcal{O}(h^2)$.

iii | Eq. (12.29) $\xrightarrow{\text{Eq. (12.33)}}$

$$\Delta h_{00} \approx \kappa \rho c^2 \tag{12.34}$$

13 | Einstein gravitational constant:

Recall that we can identify the 00-component of the metric with the Newtonian gravitational potential in the non-relativistic limit:

Eq. (12.34)
$$\xrightarrow{\text{Eqs. (11.65) and (12.25)}} (h_{00} \approx 2\phi/c^2)$$

$$\Delta \phi = \frac{1}{2} \kappa c^4 \rho \quad cf. \text{ Newtonian gravity Eq. (8.4):} \quad \Delta \phi = 4\pi G \rho \qquad (12.35)$$

The validity of Newtonian gravity in the non-relativistic limit requires the identification:

$$\kappa = \frac{8\pi G}{c^4} \approx 2.076\,65 \times 10^{-43}\,\mathrm{N}^{-1} \tag{12.36}$$

- κ plays the role of a *coupling constant* in Eq. (12.10): It describes the coupling between metric/geometry (= gravitational field) and matter. If you set κ to zero, matter and energy no longer curve spacetime and gravitational systems (like our solar system) can no longer exist.
- The fact that κ is extremely small (in units of everyday life) tells us that the coupling of matter to the spacetime geometry is extremely weak. Note that this weakness is due to the smallness of Newton's gravitational constant G and the largeness of the speed of light c.

This explains why it took us so long the figure out that masses curve spacetime: Since κ is so small, spacetime is *extremely* "stiff" (much stiffer than steel or glass), so that masses of everyday life have no perceivable effect on it. This is also why space around us is essentially Euclidean, despite the presence of Earth. In an imaginary world where $\kappa \sim 1 \text{ N}^{-1}$, space(time) would "wobble" like jelly when you move; you could see this because of the \rightarrow *deflection of light* and \rightarrow *gravitational lensing*. For example, you could tell whether an opaque bottle is full or empty from the way it distorts what you see in its vicinity.

14 | Newtonian dynamics:



 $i \mid \triangleleft$ Geodesic equation Eq. (11.45) for a test particle:

$$\frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} = -\Gamma^{\mu}_{\ \alpha\beta} \frac{\mathrm{d}x^{\alpha}}{\mathrm{d}\tau} \frac{\mathrm{d}x^{\beta}}{\mathrm{d}\tau} \tag{12.37}$$

In the non-relativistic limit, this should lead to Newton's equation in a gravitational field.

ii | \triangleleft Non-relativistic particle: $\tau \approx t = x^0/c$ and $u^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} \approx (c, 0, 0, 0)$

Eq. (12.37)
$$\rightarrow \frac{d^2 x^i}{dt^2} = -\Gamma^i_{\ 00} c^2 \stackrel{12.32}{=} \frac{1}{2} c^2 \partial^i h_{00} = -\frac{1}{2} c^2 \partial_i h_{00}$$
 (12.38)

iii | Comparison with the Newtonian equation of motion Eq. (8.5):

$$\ddot{\vec{x}} = -\nabla\phi \,. \tag{12.39}$$

Identification: $\phi = \frac{c^2}{2}h_{00} \rightarrow$

$$g_{00} \stackrel{12.25}{=} \eta_{00} + h_{00} = 1 + \frac{2\phi}{c^2} \checkmark$$
(12.40)

This is consistent with Eq. (11.65).

12.1.2. The cosmological constant

15 | Cosmological constant:

To find an interpretation for the cosmological constant Λ , we study its effects on non-relativistic Newtonian physics:

i | Retrace our steps to derive the non-relativistic limit of the EFEs above, but now including the cosmological constant:

Add $\Lambda g_{00} \approx \Lambda \eta_{00}$ on right-hand side of Eq. (12.29) $\xrightarrow{\circ}$

$$\Delta \phi = 4\pi G \rho - \Lambda c^2 \equiv 4\pi G (\rho + \rho_{\Lambda})$$
(12.41)

with additional "mass density" $\rho_{\Lambda} := -\Lambda c^2/(4\pi G)$

ii | $\triangleleft \rho = 0$ (vacuum) & $\Lambda \neq 0$:

$$\Delta \phi = -\Lambda c^2 = \text{const} \quad \Rightarrow \quad \phi(\vec{r}) \stackrel{\circ}{=} -\frac{\Lambda c^2}{6}r^2 \tag{12.42}$$

The solution follows with the boundary condition $\phi(\vec{0}) = 0$.

 $\stackrel{\circ}{\rightarrow}$ Gravitational acceleration:

$$\vec{g}(\vec{r}) = -\nabla\phi = \frac{\Lambda c^2}{3}\vec{r} = \begin{cases} \text{repulsive} \quad \Lambda > 0\\ \text{attractive} \quad \Lambda < 0 \end{cases}$$
 (12.43)

• Note that our choice to set the gravitational potential to zero in the origin $\vec{r} = 0$ is arbitrary: Consider two test bodies at positions \vec{r}_A and \vec{r}_B . Because of the universality of



free fall their masses don't matter, and their relative acceleration due to the gravitational potential is

$$\vec{a}_{AB} = \vec{g}(\vec{r}_A) - \vec{g}(\vec{r}_B) = \frac{\Lambda c^2}{3}(\vec{r}_A - \vec{r}_B) = \frac{\Lambda c^2}{3}\Delta \vec{r}_{AB}$$
 (12.44)

This demonstrates how strange the effect of the cosmological constant is: All bodies accelerate away or towards one another, and the acceleration only depends on and is proportional to their relative distance vector. The effect is therefore completely homogeneous and space behaves like a dough that rises or collapses, with massive bodies being dragged along like raisins.

- Thus $\Lambda > 0$ acts like "antigravity" and blows up the universe, $\Lambda < 0$ does the opposite.
- For large Λ > 0, the universe would blow up so fast that neither stars nor galaxies could form. Conversely, for large Λ < 0 the universe would have already collapsed. Thus we can exclude both large positive and large negative values for Λ (→ *below*).
- iii | We can conclude:

The cosmological constant makes the non-relativistic limit of GENERAL RELATIVITY deviate from Newtonian mechanics: It predicts a homogeneous long-range repulsion ($\Lambda > 0$) or attraction ($\Lambda < 0$) that *increases* with the distance. Thus, if it Λ is non-zero, it must be very small to be consistent with our observations and can only be relevant on cosmological scales.

Einstein introduced the cosmological constant in 1917 in "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie" [15] [Eq. (13a) on p. 151, Einstein denoted our Λ by λ]. Its purpose was to allow for cosmological solutions of the EFEs that describe a static and finite universe (at this time, it was widely believed that the universe was static).

When Edwin Hubble showed 1929 that the universe is actually *expanding* (and therefore non-stationary) [156], the cosmological constant lost its purpose and was abandoned by Einstein and contemporaries (though Einstein was quite stubborn and hesitant to acknowledge non-stationary solutions as mathematically sound and physically reasonable [157–159]). Einstein later referred to the introduction of the cosmological constant as *"his biggest blunder"* [160].

In hindsight, Einstein's "biggest blunder" was not the introduction of the cosmological term in the first place (given the state of knowledge in 1917, it was a reasonable approach), but his later refusal and hesitant acceptance of non-static solutions, supporting evidence notwithstanding.

• How small is small? First, note that

$$[\Lambda] \stackrel{12.12}{=} [R] \stackrel{10.117}{=} [R_{\mu\nu}] \stackrel{10.114}{=} [R_{\mu\nu\rho\sigma}] \stackrel{10.105}{=} L^{-2}$$
(12.45)

so that $\Lambda^{-1/2}$ is a length scale. Since the Newtonian limit has been successfully tested in our solar system (without any evidence for strange long-range acceleration effects), modifications due to Λ , if present, must be much larger than this length scale; this yields an upper bound

$$|\Lambda| \lesssim (\text{Size of the solar system})^{-2}$$
 (12.46)

for the cosmological constant. For more details, see Refs. [161, 162].



- Today we know that the universe is not only steadily expanding: the expansion is *accelerating*. In a strange turn of events, these observations led to a revival of the cosmological constant, because it can be used to model such accelerated expansions (→ ΛCDM). By now, there is striking evidence that Λ > 0 in our universe [163,164]. The physical mechanism behind a non-zero cosmological constant is unknown (↑ *dark energy*).
- We can bring the cosmological term in Eq. (12.10) to the other side,

$$G_{\mu\nu} = -\kappa \left(T_{\mu\nu} + \frac{\Lambda}{\kappa} g_{\mu\nu} \right) \equiv -\kappa \left(T_{\mu\nu} + T_{\mu\nu}^{\rm vac} \right) , \qquad (12.47)$$

which suggests the definition of a "vaccuum contribution" to the total energy-momentum tensor:

$$T_{\mu\nu}^{\rm vac} = \frac{\Lambda}{\kappa} g_{\mu\nu} \,. \tag{12.48}$$

In this reading, even "empty" space $(T_{\mu\nu} = 0)$ contains a homogeneously distributed form of energy $(T_{\mu\nu}^{vac} \neq 0)$ that acts as a source of gravity and is responsible for blowing up or collapsing spacetime.

While this may sound exotic, it is actually what one would expect from \uparrow quantum field theory and the \uparrow Standard Model of particle physics: In quantum mechanics, you learn that even the ground state (= lowest energy state) of a harmonic oscillator has a finite \downarrow ground state energy of $\frac{\hbar\omega}{2}$. The same is true for the ground state (= vacuum) of the quantum fields that permeate space and describe all the fundamental particles (leptons, quarks, gauge bosons). That is, quantum field theory predicts that even the vacuum has a finite "vacuum energy density", and it is reasonable to conjecture that this might translate into the cosmological constant of GENERAL RELATIVITY in the classical limit.

But there is a problem: We argued above that $\Lambda \neq 0$ can only be *small*. But quantum field theory tells us that the vacuum energy should be *large*; more precisely: the cosmological constant predicted by quantum field theory is by a factor of $10^{50} - 10^{120}$ larger than the observed one (the factor depends on how exactly one evaluates quantum field theory)! This is of course ridiculous and has been dubbed "the worst prediction in the history of physics." It is at present unknown how to solve this conundrum, see Refs. [161,162,165–167] for more details on the \uparrow cosmological constant problem.

• To understand why a contribution to the HEMT of the form Eq. (12.48) can be interpreted as the energy of the vacuum, we can use the (classical) Klein-Gordon field theory Eq. (11.116):

$$L(\phi, \partial\phi, g) = \underbrace{\frac{1}{2}(\partial^{\mu}\phi)(\partial_{\mu}\phi)}_{\text{Kinetic energy}} - \underbrace{\frac{m^{2}}{2}\phi^{2}}_{\text{Potential energy}}.$$
 (12.49)

It's Hilbert energy-momentum tensor (Problemset 4) reads [Eq. (11.118)]:

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\left(\phi^{,\alpha}\phi_{,\alpha} - m^2\phi^2\right).$$
(12.50)

The "vacuum" is the lowest-energy state ϕ_0 of the field. (In particle physics, the quantum fields that describe fundamental particles permeate space; they cannot "go away". "Vacuum" then means "no particles", which translates to "no excitation of the field".) Classically, the field of the lowest-energy state tries to minimize the kinetic energy; and it can do so by being constant: $\phi_0 = \text{const.}$ The HEMT in the vacuum state then reads



which has exactly the form of Eq. (12.48) with the identification $\Lambda \kappa = m^2 \phi_0^2/2$. This explains the hypothesis that a non-zero cosmological constant could be due to the vacuum energy of the (quantum) fields that describe the fundamental particles of the Standard Model (or some other yet unknown field).

Remark: You may complain that the classical ground state of the Klein-Gordon field is $\phi_0 = \text{const} = 0$, since the field also minimizes the *potential energy* (which is a harmonic potential ϕ^2), so that $T_{\mu\nu}^{\text{vac}} = 0$. This is of course correct. But first note that this is a feature of the particular potential chosen and does not affect the form $T_{\mu\nu}^{\text{vac}} \propto g_{\mu\nu}$, which is crucial for our argument. Furthermore, remember that we are actually dealing with *quantum* fields in the classical limit. So actually one should use *expectation values* to compute the classical HEMT: $T_{\mu\nu}^{\text{vac}} = \frac{m^2}{2} \langle \phi^2 \rangle_0 g_{\mu\nu}$. And just like $\langle x^2 \rangle_0 > 0$ for a quantum harmonic oscillator in its ground state (recall that it is a \checkmark *coherent state*), one also finds $\langle \phi^2 \rangle_0 > 0$ due to the *quantum fluctuations* of the Klein-Gordon field.



↓Lecture 26 [25.06.24]

12.2. The Einstein-Hilbert action

In our derivation of the Einstein field equations in Section 12.1 we assumed that there is an action with a local Lagrangian that gives rise to the dynamics of the metric field. By exploiting Lovelock's theorem, we managed to derive the equation of motion without ever specifying the Lagrangian explicitly. Since the EFEs are conceptually simple and "inevitable", we should expect the action that gives rise to these equations to be simple and "inevitable" as well:

1 $| \triangleleft$ Generally covariant action for metric field $g_{\mu\nu}$:

(We omit prefactors for the sake of clarity.)

$$S[g] = \int \underbrace{d^4x \sqrt{g}}_{\sim \partial^2 g} \left[\underbrace{1 + R + R^2 + R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\pi}R^{\mu\nu\sigma\pi} + \dots}_{\sim (\partial^2 g)^2}\right]$$
(12.52)

<u>Goal</u>: We want a *second*-order differential equation for $g_{\mu\nu}$ (\leftarrow 2ND in Section 12.1) Facts:

- If the Lagrangian only depends on *first* derivatives, the Euler-Lagrange equations include at most *second* derivatives. ⁽²⁾
- Problem: There is no scalar including only first derivatives of the metric! ③

This is easy to see: At every point we can choose locally geodesic coordinates where all first derivatives of the metric vanish. Since a scalar is independent of coordinates, this leaves only the trivial possibility that the scalar does not depend on the first derivatives at all, and is therefore constant.

• The next best (and only!) scalar *linear* in second derivatives is the Ricci scalar.

This is our last hope to obtain a (non-trivial) second-order differential equation, since the *linearity* in $\partial^2 g$ makes all terms of potentially third and higher derivatives vanish. \odot

To understand why, remember (well, probably you don't remember because this is rarely covered in basic courses on classical mechanics) that for a Lagrangian L(t, q, q', q'') that depends on *second* derivatives q'', the Euler-Lagrange equation reads

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial q'} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} \frac{\partial L}{\partial q''} = 0. \qquad (12.53)$$

If $L(t, q, q', q'') = f(t)q''(t) + \tilde{L}(t, q, q')$ only depends linearly on the second derivative, $\frac{\partial L}{\partial q''}$ does not contain the function q(t), so that there are no derivatives beyond q'' that can show up in the Euler-Lagrange equation.

That the Ricci scalar R is the *only* (non-trivial) scalar that can be constructed from the metric and its first and second derivatives, and is linear in the latter, has been shown by HERMANN VERMEIL in 1917 [168]; this statement is known as \uparrow *Vermeil's theorem*.



2 | <u>The ** Einstein-Hilbert action:</u>

These arguments lead us to propose following the simple action:

$$S[g] := \underbrace{\frac{c^3}{16\pi G}}_{(2\kappa c)^{-1}} \int d^4x \sqrt{g} \left(R - 2\Lambda\right)$$
(12.54)

- The factor 2 in front of the cosmological constant is chosen such that the Euler-Lagrange equations take the conventional form of the Einstein field equations. The global prefactor $\frac{1}{2\kappa c}$ is irrelevant for our current purpose because, first, we are interested in pure gravity (= no matter action $S_g[\phi]$), and second, we are only interested in *classical* equations of motion, i.e., we don't use the action to define a \checkmark *path integral* (which would be needed for a theory of *quantum* gravity). The strange additional c in the prefactor $\frac{1}{2\kappa c}$ is necessary for dimensional reasons (\rightarrow *below*) and due to our choice to measure time coordinates in units of length: $x^0 = ct$.
- If you want to use an action for a path integral, it must have the *dimension* of an action $E \cdot T = M \cdot L^2 \cdot T^{-1}$, so that the exponent of $\exp\left[\frac{i}{\hbar}S\right] = \exp\left[2\pi i \frac{S}{\hbar}\right]$ is a dimensionless number [\hbar is the reduced Planck's constant, the "quantum of action" (*Wirkungsquantum*) that quantifies the strength of quantum fluctuations].

A bit of dimensional analysis yields for Eq. (12.54)

$$[S] = [\kappa]^{-1} \underbrace{[c^{-1}d^4x]}_{L^{3} \cdot T} \underbrace{[g]^{\frac{1}{2}}}_{1} \underbrace{[R]}_{L^{-2}} \stackrel{!}{=} M \cdot L^2 \cdot T^{-1} \Rightarrow [\kappa] = T^2 \cdot M^{-1} \cdot L^{-1} \quad (12.55)$$

which is conveniently the dimension of $\kappa = \frac{8\pi G}{c^4}$.

• The Einstein-Hilbert action was introduced by mathematician DAVID HILBERT in 1915 in Ref. [152] – where he independently found the Einstein field equations (essentially at the same time as Einstein, give or take a few days). For a historical account on the race between Einstein and Hilbert see Ref. [153].

3 | Euler-Lagrange equations:

One can now check (\rightarrow *below*) that the stationary solutions satisfy the EFE in vacuum Eq. (12.10):

$$\delta S[g] \stackrel{!}{=} 0 \quad \stackrel{*}{\Leftrightarrow} \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \stackrel{!}{=} 0 \tag{12.56}$$

i! Please appreciate this almost magical result: You start by writing down the *simplest non-trivial covariant action* that can be constructed from the metric, *and the Einstein field equations follow*.

This underpins our previous statement that the EFEs are "inevitable" under quite general assumptions. It also illustrates in which sense GENERAL RELATIVITY is *simple*: Without cosmological constant (and dropping the unnecessary prefactor), the action of GENERAL RELATIVITY in vacuum is

$$S[g] = \int \mathrm{d}^4 x \sqrt{g} R \,. \tag{12.57}$$

If this isn't simple and elegant, what is?

The proof of Eq. (12.56) is straightforward but a bit tedious (Problemset 5):



i | With Eq. (11.112) it follows

$$\delta(\sqrt{g}) = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}$$
(12.58)

which is the variation of the cosmological term in Eq. (12.56).

ii | The variation of the more complicated first term in Eq. (12.56) yields

$$\delta(\sqrt{g}R) = \delta(\sqrt{g}R_{\mu\nu}g^{\mu\nu}) = \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} + \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{g} \qquad (12.59)$$

where we again made use of Eq. (11.112); the variation $\delta R_{\mu\nu}$ remains to be evaluated.

iii | To do so, we proceed in *locally geodesic coordinates* where the Christoffel symbols vanish and we can use Eq. (10.104) so that

$$R_{\mu\nu} = R^{\lambda}_{\ \mu\nu\lambda} = \Gamma^{\lambda}_{\ \mu\lambda,\nu} - \Gamma^{\lambda}_{\ \mu\nu,\lambda} , \qquad (12.60)$$

and therefore

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu} \left[\left(\delta \Gamma^{\lambda}_{\ \mu\lambda} \right)_{,\nu} - \left(\delta \Gamma^{\lambda}_{\ \mu\nu} \right)_{,\lambda} \right]$$
(12.61a)

$$= \left(g^{\mu\nu}\delta\Gamma^{\lambda}_{\ \mu\lambda} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\ \mu\lambda}\right)_{,\nu} \tag{12.61b}$$

$$\equiv C^{\nu}_{,\nu} . \tag{12.61c}$$

Here we used $\delta(\Gamma, \nu) = (\delta\Gamma)_{,\nu}$ and that $g^{\mu\nu}_{,\sigma} = 0$ in locally geodesic coordinates; we also renamed the indices $\lambda \leftrightarrow \nu$ in the second term.

iv | In locally geodesic coordinates, Eq. (12.61) is equivalent to $g^{\mu\nu}\delta R_{\mu\nu} = C^{\nu}_{;\nu}$. If C^{ν} is a *vector field* (\rightarrow *below*), then this equation is actually valid in *arbitrary* coordinates, and we can apply Gauss's theorem:

$$\int d^4x \sqrt{g} g^{\mu\nu} R_{\mu\nu} = \int d^4x \sqrt{g} C^{\nu}_{;\nu} \stackrel{10.95}{=} \int d^4x \left(\sqrt{g} C^{\nu}\right)_{,\nu} = \oint d\sigma_{\nu} \sqrt{g} C^{\nu} = 0.$$
(12.62)

The surface integral vanishes because $C \propto \delta \Gamma \propto \delta g^{\mu\nu} = 0$ on the surface, which is true if we consider local variations of the metric. We have thereby shown that the first summand in Eq. (12.59) does not contribute to the variation of the Einstein-Hilbert action and can therefore be dropped.

So why is C^{ν} , defined in Eq. (12.61), a vector field? This is not so obvious because it is defined in terms of connection coefficients $\Gamma^{\mu}_{\nu\rho}$ – which are not tensors! The crucial point is that the coordinate transformation law Eq. (10.39) of connection coefficients is tensorial up to a non-tensorial contribution that depends only on the coordinate transformation (but not the connection itself): Let $\tilde{g} = g + \delta g$ be an infinitesimal variation of the metric. Then the variation of the coefficients of the Levi-Civita connection is

$$\delta\Gamma = \Gamma(\tilde{g}) - \Gamma(g), \qquad (12.63)$$

where we omit indices for clarity and the dependence on the metric is given by Eq. (10.79). Under an arbitrary coordinate transformation, this difference transforms like a tensor because the problematic term in Eq. (10.39) is independent of the metric and therefore cancels in the difference.

v | We can now combine our results:

The variation of the Einstein-Hilbert action Eq. (12.54) evaluates to

$$\delta S[g] \stackrel{\substack{12.58\\12.59}}{=} \frac{1}{2\kappa c} \int d^4 x \sqrt{g} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right] \delta g^{\mu\nu} \stackrel{!}{=} 0, \qquad (12.64)$$



since this variation must vanish for all $\delta g^{\mu\nu}$, this is equivalent to the Einstein field Eq. (12.10) in vacuum $(T_{\mu\nu} = 0)$.

4 Action with matter:

We can now insert the Einstein-Hilbert action into the "Action of Everything" introduced in Section 12.1:

Eq. (12.1) Eqs. (12.5b) and (12.54)

$$S[g,\phi] = \frac{1}{c} \int d^4x \sqrt{g} \left[\frac{1}{2\kappa} \left(R - 2\Lambda \right) + L_{\text{Matter}} \right]$$
(12.65)

i! Now the prefactor with the Einstein gravitational constant κ is important: it determines the coupling strength between gravity and matter.

The additional prefactor $\frac{1}{c}$ is only needed for dimensional reasons because we measure time in units of length: $x^0 = ct$; it does not affect the equations of motion.

$$\delta_g S[g,\phi] \stackrel{!}{=} 0 \quad \stackrel{12.5a}{\longleftrightarrow} \quad R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}$$
(12.66)

- 5 | What is the energy-momentum tensor of the gravitational field?
 - i | Since the metric is now our dynamical "gravitational potential", it should be able to carry energy (and momentum) in some form. And surely it does: The first (indirect) observation of → gravitational waves was based on the observation of a neutron star circling a pulsar (the ↑ Hulse-Taylor pulsar, also known as PSR B1913+16). Over time their orbital period changes, indicating a decay of the orbit [169–171]. But this means that energy must be radiated away, and the only possible carrier is gravitational waves! (By the way, the observations match precisely the quantitative predictions of GENERAL RELATIVITY.) So clearly gravitational waves which are excitations of the metric field carry energy.
 - ii | It is therefore reasonable to expect that there is an energy-momentum tensor associated to the gravitational field, just as there is for any other field that carries energy and momentum.

Recall that the Hilbert energy-momentum tensor was defined in Eq. (11.106) as

$$T_{\mu\nu}^{\text{Matter}} = \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}L_{\text{Matter}})}{\delta g^{\mu\nu}}, \qquad (12.67)$$

and by comparison with previous results (e.g., electrodynamics) we verified that this quantity indeed captures the concepts of energy (currents) and momentum (currents) correctly [recall Eq. (6.110)].

iii | But in GENERAL RELATIVITY, the metric field is "just another dynamical field," described by an action (the Einstein-Hilbert action), which is given by the Lagrangian

$$L_{\text{Metric}}(g, \partial g, \partial^2 g) \stackrel{12.54}{=} \frac{1}{2\kappa} (R - 2\Lambda).$$
 (12.68)

It is therefore reasonable to expect that the energy-momentum tensor of the gravitational field is given by

$$T_{\mu\nu}^{\text{Metric}} \stackrel{?}{=} \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}L_{\text{Metric}})}{\delta g^{\mu\nu}} \stackrel{12.6}{\stackrel{12.64}{=}} \kappa^{-1}(G_{\mu\nu} + \Lambda g_{\mu\nu}).$$
(12.69)

This already looks strange: This is the left-hand side of the Einstein field equations!



iv | Let us assume that there is no matter present, so that the EFEs read $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$. Since the propagation of gravitational waves does not rely on the presence of matter, they should still be able to carry energy. However:

Eq. (12.69)
$$\xrightarrow{12.10} T_{\mu\nu}^{\text{Metric}} \doteq 0$$
. (12.70)

So the HEMT $T_{\mu\nu}^{\text{Metric}}$ of pure gravity vanishes for all solutions of the field equations. This tells us that $T_{\mu\nu}^{\text{Metric}}$ is *not* a reasonable choice for the EMT of the gravitational field.

[Side note: What happened here is a consequence of the diffeomorphism invariance of the Einstein-Hilbert action (which is a gauge symmetry). A global continuous symmetry yields a conserved current via Noether's *first* theorem. A *local* continuous symmetry yields Noether identities via Noether's *second* theorem. The latter necessarily make the conserved quantity associated to "global gauge transformations" vanish on-shell [77]. Here this means $T_{\mu\nu}^{\text{Metric}} \doteq 0$. This is similar to the vanishing of the Hamiltonian of the reparametrization invariant theory in Section 5.4.]

• But there is something even weirder going on: If this tensor would describe the energy density of the gravitational field, then (local) energy conservation means

$$(T^{\text{Metric}})^{\mu\nu}_{;\nu} = \kappa^{-1} (G^{\mu\nu}_{;\nu} + \Lambda g^{\mu\nu}_{;\nu}) \equiv 0.$$
(12.71)

This follows from the Bianchi identity Eq. (10.122) and metric compatibility Eq. (10.74).

Compare this to the "normal" (local) energy-momentum conservation Eq. (11.109) of proper matter fields, which is only true for fields that satisfy the equations of motion (\doteq), i.e., it provides a *constraint* on field evolutions that can be realized in nature (one often employs such constraints to solve complicated equations of motion). But since Eq. (12.71) is an *identity*, it does not constrain the field evolution $g_{\mu\nu}$ whatsoever! This makes the constrain of "energymomentum conservation" rather vacuous, and the "energy" defined by Eq. (12.69) a quite useless quantity (independent of the fact that it vanishes on-shell for pure gravity).

At this point it should be clear that Eq. (12.69) is *not* a reasonable candidate for the energymomentum tensor of the gravitational field.

But we can escalate the situation further by asking ...

vi | What is the total energy-momentum tensor of *Everything*?

Well, the "Action of Everything" is Eq. (12.1), so that the "HEMT of Everything" should be

$$T_{\mu\nu}^{\text{Everything}} \stackrel{?}{=} \frac{2}{\sqrt{g}} \frac{\delta(\sqrt{g}[L_{\text{Metric}} + L_{\text{Matter}}])}{\delta g^{\mu\nu}}$$
$$= T_{\mu\nu}^{\text{Metric}} + T_{\mu\nu}^{\text{Matter}} \stackrel{12.69}{=} \frac{1}{\kappa} (G_{\mu\nu} + \Lambda g_{\mu\nu}) + T_{\mu\nu}^{\text{Matter}}.$$
(12.72)

That's the pinnacle ob absurdity! This is simply what one obtains from the Einstein field Eq. (12.10) if one collects all terms on one side. Since every realizable configuration (g, ϕ) of all fields must satisfy the EFE, the "energy-momentum tensor of Everything" again vanishes on-shell:

$$T_{\mu\nu}^{\text{Everything}} \doteq 0. \tag{12.73}$$

We could boldly conclude that "the total energy of the universe is zero," but this misses the point because Eq. (12.73) has no operational meaning – it is simply the Einstein field equations in disguise.

[If you recall the general structure of the theory considered in Section 12.1 and take Eq. (11.106) as the definition of the energy-momentum tensor, we have just shown that *the energy-momentum tensor of any diffeomorphism invariant theory vanishes on-shell*. A theory is diffeomorphism invariant if (1) it is background independent (= g is a dynamical field) and (2) the action is generally covariant.]



vii | If $T_{\mu\nu}^{\text{Metric}}$ is not a reasonable choice for the energy-momentum tensor of the gravitational field, what is? The answer may be surprising:

There is no energy-momentum tensor of the gravitational field. Gravitational energy is necessarily *non-local*.

Important: This does *not* mean that the gravitational field carries no energy. It only means that this energy cannot be associated to a *local energy density* in any reasonable way; gravitational energy is necessarily *delocalized*.

viii | Here is another (hand-waving) argument to reason why there cannot be a energy-momentum tensor for gravity (the argument is flawed [172], but demonstrates at least that the gravitational field is "different"):

The energy-momentum tensors of all field theories relevant to fundamental physics are quadratic in *first* derivatives of the field [Examples: \leftarrow Eq. (11.115) for electrodynamics and \leftarrow Eq. (11.118) for the Klein-Gordon field]. Since $g_{\mu\nu}$ is the field of GENERAL RELATIVITY, a conventional EMT should then be quadratic in $g_{\mu\nu,\rho}$. But we already argued previously that one cannot construct a tensor from first derivatives of the metric alone (because one can make these derivatives vanish in locally geodesic coordinates). Hence such a "conventional" EMT cannot exist for the gravitational field.

The flaw of this argument, pointed out in Ref. [172], is of course that we already know that gravity is very different from all other fundamental field theories (recall Section 8.2). Thus it is quite a leap of faith to exclude an energy-momentum tensor that depends on higher-than-first derivatives, based solely on our experience that other field theories behave differently.

ix | The problem concerning the energy of the gravitational field has a long-standing history; for more details see Ref. [172] and references therein. See also MISNER et al. [3] (§20.2, pp. 466-468). For a discussion of the so called ↑ energy-momentum pseudotensor that can be used to study the energy of gravitational waves see CARROLL [4] (§7.6, pp. 307-315).

12.3. **‡** Modifications of GENERAL RELATIVITY

- Our approach to come up with a covariant action already suggests modifications of GENERAL RELATIVITY by adding higher-order curvature terms, recall Eq. (12.52). This begs the question in which ways GENERAL RELATIVITY can be *modifield* to obtain other relativistic theories of gravity.
- i! So far, GENERAL RELATIVITY has passed the test of time with flying colors:
 - → *Applications* in Chapter 13

Despite some unexplained phenomena (\leftarrow *below*), there is currently no widely accepted evidence that GENERAL RELATIVITY *needs* to be modified (at least not on the classical level, i.e., in the "infrared limit").

- 1 | Two classes of modifications:
 - <u>Modifications in the UV-limit</u> (= at high energies, small distances ...)

Motivation: GENERAL RELATIVITY is not a quantum theory \rightarrow Quantum gravity?

This is quite *uncontroversial*: It is widely believed that GENERAL RELATIVITY is the classical limit of a more fundamental theory that is most likely governed by the laws of quantum



mechanics (that is, a theory of \uparrow quantum gravity). The modifications due to quantum effects will become important on the \downarrow *Planck scale* at the latest; on a semi-classical level, this might manifest as additional curvature terms showing up in the action / field equations, which modify the predictions of GENERAL RELATIVITY on very small (high) length (energy) scales. Note that such modifications are not relevant for large-scale physics (such as the motion of galaxies or the expansion of the universe).

Not everyone agrees that gravity must be quantized; Roger Penrose, for example, advocates that "quantum mechanics must be gravitized." He denies that gravity has a quantum nature at all, and that the collapse of the quantum wavefunction is an objective dynamical process, induced by gravity, that makes a unique, classical, macroscopic world emerge out of a microscopic quantum world [173]. This is in direct contradiction to most other interpretations of quantum mechanics (collapse theories are not interpretations but *modifications* of quantum mechanics) like \uparrow *Everett's many-worlds interpretation* or \uparrow *decoherence theory*.

- \rightarrow *Excursions* in Part III
- <u>Modifications in the IR-limit</u> (= at low energies, large distances ...)

Motivation: Unexplained gravitational phenomena on large scales.

This is *controversial* and a stance not shared by many physicists: Modifications of GENERAL RELATIVITY in the IR-limit typically means messing with well-established classical observations such as Newtonian gravity and/or the equivalence principle (in its various incarnations, \leftarrow Section 9.1). While it is of course possible that these classical laws and principles are violated by some as of yet undiscovered physical process (which then would require a revision of GENERAL RELATIVITY as taught in this course), there is currently no hard evidence for such (regarding the problem of *dark matter*: $\rightarrow next$).

- \rightarrow *Below* we briefly discuss potential IR-modifications of GENERAL RELATIVITY.
- 2 | Dark matter:
 - i | Arguments for IR-modifications must be based on *classical*, *large-scale* observations that don not match the predictions of GENERAL RELATIVITY and/or its non-relativistic limit: Newtonian gravity.
 - ii | The most prominent and widely accepted discrepancy of this kind is based on the *rotation* curves of galaxies: The gravitational pull experienced by stars in the outskirts of galaxies is much larger than computed from the mass of gas and stars one observes (using GENERAL RELATIVITY in the non-relativistic limit, i.e., Newtonian mechanics). That this discrepancy is real is confirmed beyond any doubt.

If one plots the orbital velocities of stars in spiral galaxies over their distance from the galaxy center, one obtains curves that flatten out for large distances (Fig. 12.1 a). This is true for almost all galaxies (there are a few exceptions though). If one uses telescopes to infer the mass-energy distribution of these galaxies (i.e., gas, dust and stars), and then computes how the velocity profile according to Newtonian mechanics should look like (by equating centrifugal and gravitational force), one finds that the rotation curves should *drop off* for large distances (Fig. 12.1 b). In a nutshell: the stars in the outskirts of galaxies are *too fast*, if they were only pulled by the gravitational force (but they are not!).

- iii | Potential solutions to the puzzle fall into two categories:
 - GENERAL RELATIVITY is correct



FIGURE 12.1. • Why dark matter? (a) That the rotation curves of galaxies do not match the visible matter distribution was first noticed in 1970 [174] and repeatedly confirmed over the years [175, 176]. For a review on the rotation curves of spiral galaxies see Ref. [177]. (Plot from Ref. [175].) (b) Without a modification of GENERAL RELATIVITY (\uparrow MOND, \uparrow TeVeS, ...), there must be invisible matter ("dark matter") responsible for this phenomenon. The distribution of this hypothetical matter can then be mapped out by studying rotation curves [178]. (Plot from Ref. [178].) (c) Evidence for dark matter: Shown is a superimposed image of the galaxy cluster 1E 0657-56 (↑ Bullet cluster). Pink: X-ray (hot gas, baryonic matter); White/Orange: Optical (stars, baryonic matter); Blue: Lensing map (baryonic and dark matter). This direct observation of a spatial dislocation of baryonic and gravitating matter is believed to be a strong evidence for the existence of dark matter [179]. These measurements can even be used to constrain the properties of dark matter [180]. Photo: https://chandra.si.edu/photo/2006/1e0657/. (d) By now there are observations of other galaxy clusters with similar features, such as MACS J0025.4-1222 [181]. The color map is the same as for c). Photo: https://www.chandra.harvard.edu/photo/2008/macs/. (e) The observation of the ultra-diffuse galaxy NGC 1052-DF2 revealed a rotation curve consistent with the absence of dark matter [182, 183]. A similar galaxy without dark matter was found guickly after [184]. Note that the absence of dark matter in specific galaxies can be interpreted as evidence for the existence of dark matter. Photo: https://esahubble.org/images/heic1806a/.

 \rightarrow There must be matter/energy in galaxies that we cannot detect ("see").

$\rightarrow ** Dark matter?$

Note that "dark matter" is a placeholder term. It is simply matter that we cannot detect for whatever reason. There is nothing magical about it $(\rightarrow below)$.

• GENERAL RELATIVITY is not correct

(on very large length-scales or at very low accelerations)

\rightarrow Modifications of GENERAL RELATIVITY?

Because stars in the outskirts of galaxies are non-relativistic (low velocities, weak gravitational fields), they must be described by the non-relativistic limit of the correct relativistic theory of gravity; for GENERAL RELATIVITY, this is good old Newtonian gravity. So a modification of GENERAL RELATIVITY that makes sense of the flat rotation curves of galaxies *without* postulating additional "dark" matter must necessarily modify Newton's law of universal gravitation. But this law works perfectly well in our



solar system (up to corrections that GENERAL RELATIVITY can explain). Thus any reasonable IR-modification of GENERAL RELATIVITY must ensure that its hampering with Newton's law only affects extremely large distances (and therefore extremely low gravitational accelerations). The most prominent theory of this kind is called \uparrow *Modified Newtonian Dynamics (MOND)*; in it original formulation by MILGROM [185–187] it is simply a non-relativistic, non-covariant modification of Newtonian gravity. Since (local) Lorentz covariance and the principle of general covariance are rather well-established cornerstones of physics that one shouldn't carelessly mess with, it is desirable to derive the modifications proposed by MOND from a generally covariant modification of GENERAL RELATIVITY; such a modification was proposed by BEKENSTEIN and dubbed \uparrow *Tensor-Vector-Scalar gravity (TeVeS)* [188]. It is a rather contrived theory that is significantly more complex than GENERAL RELATIVITY.

Recent studies (using new data from space-borne observatories that piled up over the last few years) have shown that the modifications proposed by MOND-like theories do not match observations [189–192]. In short: the future for MOND-like modifications of GENERAL RELATIVITY looks bleak. (There is nothing wrong with this; that's how science works: there is a unexplained phenomenon; you propose a solution and derive its implications; as more observations pour in, you check whether they are compatible with your theory; if not, you modify or, if this doesn't help, abandon the theory.)

DEPARTMENT OF ASTROPHYSICS MOTTO: YES EVERYISODY HAS ALREADY HAD THE IDEA, "MAYBE THERE'S NO DARK MATTER-GRAVITY JUST WORKS DIFFERENTLY ON LARGE SOALES!" IT SOUNDS GOOD BUT DOESN'T REALLY FIT THE DATA.

Here is the corresponding XKCD comic that sums up the situation quite well:

Source: https://xkcd.com/1758/

iv | The case for dark matter:

To be very clear: While the rotation curve/dark matter discrepancy has been the strongest case for potential IR-modifications of GENERAL RELATIVITY, this route has always been pursued only by a minority of physicists.

To laymen and students of physics alike, the alternative "solution" to postulate "dark matter" to patch up the discrepancy between observations and GENERAL RELATIVITY often looks like a cheap cop-out (Fig. 12.1 b). However, there are reasons why the majority of physicists believe that this is the most promising route to solve the puzzle:

1. Postulating unseen particles has been successful in the past. For example, Wolfang Pauli postulated 1930 the neutrino to explain missing momentum in radioactive beta decay (the neutrino was found in 1956). Peter Higgs (and others) postulated 1964 the Higgs boson to explain the mass of the weak gauge bosons (which was then discovered at the LHC in 2012). In 1973 the third generation of quarks (later called *top* and *bottom*) was predicted to explain CP violations in the decay of kaons (the bottom quark was discovered in 1977, the top quark in 1995).

[To be fair: Postulating the existence of things that have not yet been observed has



also failed in the past. For example, to explain the anomalous perihelion precession of Mercury, a new planet named \uparrow *Vulcan* was postulated (orbiting between Sun and Mercury). The planet does not exist, and today we know that corrections due to GENERAL RELATIVITY are responsible for the precession.]

- 2. The Standard Model of particle physics, our best theory of the very small, describes the properties of fundamental particles. While the model is restricted by symmetries (one of them being Lorentz invariance), it still can be modified and extended in many ways; for example, it is quite natural to add right-handed ↑ *sterile neutrinos* without breaking the math. Thus, from the viewpoint of a particle physicist, it is general practice to extend theories by new particles (= fields) and study the consequences. "Dark matter" could just be one or more fields the excitations of which evaded our detectors so far (sterile neutrinos are such a candidate for dark matter).
- 3. By now, there is strong *indirect* evidence for dark matter (whatever it is made of) from astronomical observations (Fig. 12.1):
 - Fig. 12.1 c and Fig. 12.1 d show images of the galaxy clusters 1E 0657-56 (↑ Bullet cluster) [179, 180] and MACS J0025.4-1222 [181], respectively. The images superimpose data from different instruments: Pink denotes X-rays that indicate where the hot interstellar gas is located. The white/orange structures on the black background are the optical signatures of galaxies coming from stars. The most interesting is the blue cloud: it encodes the distribution of gravitating matter inferred from a so called ↑ lensing map. The idea is to use → gravitational lenses to map out the mass distribution of a region of space. Essentially you look how the light coming from the stars in the background is disturbed by masses in the foreground. In these pictures, only the blue density map is sensitive to dark matter (because dark matter does not emit light, but distorts light from the background stars).

The situation in both Fig. 12.1 c and Fig. 12.1 d is similar: we see the aftermath of two clusters of galaxies that collided. This sounds more exciting than it actually is, because galaxies (and even more so clusters of galaxies) consist mostly of empty space with a bit of dust and gas. This means that in such collisions there are almost no collisions of *stars*; they all miss each other! By contrast, the low-density gas between the stars behaves like a fluid; the two "blobs" of interstellar gas hit each other and slow down. This is what the two pink clouds in Fig. 12.1 c show: the X-ray emitting gas is lagging behind the actual stars (mostly in the blue region) that missed each other and are flying to the left and right.

So far, there is no hint of dark matter: the blue (gravitating) mass is on top of the stars and the gas is lagging behind. The twist is that almost all of the (visible) mass of a galaxy (cluster) comes from the gas between the stars - and not the stars themselves! This might sound strange, but there is a lot of space between stars, and even if this space is *almost* vacuum, the total mass still outweighs the stars significantly. But now we have a problem: If most of the *visible* mass is gas (pink), why is most of the *gravitating* mass where the stars are (blue)? Well, because the blue cloud is mostly caused by the dark matter halo of the two galaxies, and not by the stars! And this fits exactly the properties expected for dark matter: The particles making up dark matter cannot interact in any significant way, otherwise we would have already detected them. But this means that a cloud of dark matter does not behave like "normal gas" would; in particular, two colliding clouds of dark matter cannot slow each other down. Thus it is perfectly consistent that the two dark matter clouds (blue) passed each other, just as the visible stars did (but for very different reasons). It is this observable separation between visible mass (pink) and gravitating mass (blue) that makes a strong case for dark matter.

• Another recent observation supporting the existence of dark matter is, quite surprisingly, the observation of the ultra-diffuse galaxy NGC 1052-DF2 (Fig. 12.1 e) with a rotation curve that is consistent with the *absence* of dark matter [182, 183]



(by now a second of these rare galaxies has been found [184]). The argument is quite simple:

If you want to avoid dark matter, you must mess with GENERAL RELATIVITY (like MOND and TeVeS do) and thereby Newton's law of universal gravitation. But now that we have examples of galaxies *where Newtonian gravity works without postulating dark matter*, you have a problem: why is your modification not valid for these galaxies? You cannot go around and modify the laws of physics from place to place! But if dark matter exists (and is responsible for the flat rotation curves), it is at least plausible that a few galaxies with an extravagant history somehow got their cloud of dark matter stripped away (perhaps by the gravitational interaction with another galaxy), and therefore have rotation curves that fall off, without the need for additional mass.

In summary, it seems likely that dark matter exists and is responsible for the rotation curve problem. Conversely, it seems more and more unlikely that GENERAL RELATIVITY must be modified anytime soon. But until we identify and measure what dark matter actually is, we don't know for sure.

3 | Potential modifications:

Arguments for or against modifications of GENERAL RELATIVITY aside, which possibilities do we have to construct alternatives to GENERAL RELATIVITY?

For more details on alternative theories of gravity see Ref. [111] and CARROLL [4] (§4.8, pp. 181–190).

 \leftarrow Lovelock's theorem [135, 136, 172]:

 $\left.\begin{array}{c} \text{Only a metric field} \\ \text{Second-order field equations} \\ \text{Four-dimensional spacetime} \\ \text{Local action} \end{array}\right\} \quad \Rightarrow$

GENERAL RELATIVITY

(12.74)

- \rightarrow Options for modifications of GENERAL RELATIVITY:
 - \triangleleft Other fields in addition to (or replacing) the metric
 - \triangleleft Scalar fields

Theories that augment the metric tensor field $g_{\mu\nu}$ by an additional scalar field ϕ are known as \uparrow *scalar-tensor theories* of gravity. You may wonder how ϕ differs from any other matter field? The reason why ϕ cannot be simply identified as another matter field is that its coupling to the metric is *non-minimal*. Note that this suggests a definition which fields describe *matter* and which describe *gravity*: Matter fields are minimally coupled to the metric, additional gravitational fields are non-minimally coupled. Since non-minimally coupled fields tend to violate the equivalence principle, such theories often violate its *strong* version **SEP**.

Example: One of the first and most famous scalar-tensor theories is \uparrow *Brans-Dicke theory* [193]. It is defined by the gravitational action (here for c = 1)

$$S_{\rm BD}[g,\phi] = \frac{1}{16\pi} \int d^4x \sqrt{g} \left[\phi R - \frac{\omega}{\phi} g^{\mu\nu} (\partial_\mu \phi) (\partial_\nu \phi) \right]$$
(12.75)

with massless scalar field $\phi(x)$ and $\stackrel{*}{*}$ Dicke coupling constant ω . For $\omega \to \infty$ one recovers GENERAL RELATIVITY. In theories of this kind, the scalar field $\phi(x)$ can be



interpreted as a position and time dependent replacement of the gravitational coupling constant $1/\kappa \propto 1/G$.

- ⊲ Connections *other than* the Levi-Civita connection

As discussed in Section 9.4, the concepts of *connection* and *metric* are independent in principle. Only when demanding a *metric-compatible* and *torsion-free* connection does one obtain the unique Levi-Civita connection and everything is determined by the metric alone.

Example: One could start with the Einstein-Hilbert action, but treat connection Γ and metric g as independent fields:

$$S[g,\Gamma] := \frac{1}{2\kappa c} \int \mathrm{d}^4 x \sqrt{g} g^{\mu\nu} R_{\mu\nu}(\Gamma) \,. \tag{12.76}$$

Here, the curvature is directly computed from the connection via Eq. (10.70); this is known as the \uparrow *Palatini action* (\bigcirc Problemset 5).

Quite surprisingly, if one starts from Eq. (12.76) and assumes either ...

- * Γ is metric-compatible, or ...
- * Γ is torsion-free,

the variation $\delta_{\Gamma}S$ of the action wrt. the connection coefficients $\Gamma^{\mu}_{\nu\rho}$ vanishes only for the Levi-Civita connection (which brings us back to GENERAL RELATIVITY). Only if one drops all restrictions, and allows for arbitrary connections, does one find a modification of GENERAL RELATIVITY.

However, note that the *difference* of two arbitrary connections is a *tensor* [this follows from Eq. (10.39)]. But this means that *w.l.o.g.* you can write any connection in the form

$$\Gamma^{\mu}_{\nu\rho} = \underbrace{\begin{cases} \mu \\ \nu\rho \end{cases}}_{\text{Levi-Civita}} + \underbrace{T^{\mu}_{\nu\rho}}_{\text{Tensor}}, \qquad (12.77)$$

so that the decoupling of metric and connection boils down to the extension of GENERAL RELATIVITY by some additional tensor field $T^{\mu}_{\nu\rho}$.

For example, see Refs. [132, 133] for potential extensions of GENERAL RELATIVITY by allowing connections with torsion.

- ...

There is of course no limit to your imagination. One can consider any combination of arbitrary-rank tensor fields to augment the metric. One example is the previously mentioned \uparrow *Tensor-Vector-Scalar gravity (TeVeS)* by Bekenstein [188] which, as the name suggests, comprises a metric tensor field, a vector field, and a scalar field.

• < Higher than second derivatives of the metric in the field equations

Example: $\uparrow f(R)$ -gravity theories [194] are defined by the generalized Einstein-Hilbert action

$$S_f[g] := \frac{1}{2\kappa c} \int \mathrm{d}^4 x \sqrt{g} f(R) \tag{12.78}$$

with some differentiable function $f : \mathbb{R} \to \mathbb{R}$ that specifies the theory. For f(R) = R one recovers GENERAL RELATIVITY (without cosmological constant), but for $f(R) \neq R$ the field equations differ from the EFEs (and typically contain higher than second derivatives of the metric).



• \triangleleft Spacetime dimensions $D \neq 4$

It is straightforward to generalize GENERAL RELATIVITY (e.g., using the Einstein-Hilbert action) to arbitrary spacetime dimensions D. For an example, in \bigcirc Problemset 4 we study D = 2+1-dimensional GENERAL RELATIVITY. These theories can behave very differently from D = 3 + 1-dimensional GENERAL RELATIVITY. However, they obviously do not describe reality correctly as our spacetime has undeniable D = 3 + 1 dimensions. The only viable route is then to postulate additional spatial dimensions and "curl them up" (\uparrow compactification) so that one cannot see them on the large scales accessible to us. Such theories are known as \uparrow Kaluza-Klein theories because THEODOR KALUZA introduced a D = 4 + 1-dimensional version in 1921 [195], which was later extended by OSKAR KLEIN in 1926 [89, 196].

Example: A simple metric of a D = 4 + d-dimensional spacetime could have the form

$$ds^{2} = G_{ab} dX^{a} dX^{b} \equiv \underbrace{g_{\mu\nu}(x) dx^{\mu} dx^{\nu}}_{\text{Observable 4D spacetime}} + \underbrace{b^{2}(x)\gamma_{ij}(y) dy^{i} dy^{j}}_{\text{Compact extra dimensions}}$$
(12.79)

with a, b = 0, ..., d+3 for $d > 0, \mu, \nu = 0, 1, 2, 3$, and i, j = 1, ..., d are the coordinates of the compactified additional d dimensions.

As action one could postulate the Einstein-Hilbert action,

$$S[G] := \frac{1}{16\pi G_{4+d}} \int d^{4+d} X \sqrt{G} R[G], \qquad (12.80)$$

where G_{4+d} denotes the "gravitational constant" of this hypothetical 4 + d-dimensional theory and R[G] is the Ricci scalar computed from the 4 + d-dimensional metric G_{ab} . Note that this is not equivalent to GENERAL RELATIVITY in 4 + d dimensions because (1) we constrain the form of the metric to Eq. (12.79), and (2) the additional d space dimensions are compact and not extended.

Remarkably, if one integrates out these compact extra dimensions (\uparrow dimensional reduction), one finds theories equivalent to GENERAL RELATIVITY with extra fields [under additional constraints on γ_{ij} , the simple metric (12.79) yields a \leftarrow scalar-tensor theory, see CARROLL [4] (§4.8, pp. 186–189) for details]. The intuition behind this is that the geometric degrees of freedom of the curled-up dimensions manifest in the extended 4D spacetime as additional fields that can couple non-minimally to the metric.

In a nutshell: Extending spacetime by compact extra dimensions is often equivalent to adding new fields on a 4D spacetime (without extra dimensions).

• < Field equations that *cannot* be derived from the metric variation of an action.

The field equations of such theories are not necessarily rank-2 tensor equations; and even if so, they are not necessarily symmetric in the indices and/or divergence-free (recall our derivation in Section 12.1 of these properties starting from a covariant action).

• *⊲ Non-local* theories

Physicists don't like non-local theories very much. Whenever you work with a continuum theory that can be described by (a set of) differential equations on some manifold, the theory is local. Non-local theories therefore must be described by other equations (for example: \uparrow *integro-differential equations*). Such theories and equations are often hard to work with. Fortunately, nature seems to be rather local, which explains the prevalence of local theories in physics (though this might be an illusion of sorts).



12.4. ‡ Diffeomorphism invariance and the Hole argument

Now that GENERAL RELATIVITY has been fully developed as a relativistic theory of gravity, there are a few conceptual issues that need to be clarified.

1 | The Hole argument:

This discussion is based on Ref. [197]. For reviews of the hole argument see Refs. [198, 199].

 $i \mid \triangleleft Fields (g, \phi) \& Action of Everything (AoE) S[g, \phi] = S[g] + S_g[\phi]$

Recall Eq. (12.1) in Section 12.1.

 \triangleleft Diffeomorphism $\varphi \in \text{Diff}(M) \rightarrow \text{Transformed fields } (\bar{g}, \bar{\phi})$

Recall Eq. (11.85) in Section 11.4.

i! Remember that we interpret $(\bar{g}, \bar{\phi})$ as new/different fields (in the same coordinates).

AoE is generally covariant $\xrightarrow{\text{Eq. (11.89)}}$ AoE is diffeomorphism invariant:

$$S[\bar{g},\bar{\phi}] = S[g,\phi] \tag{12.81}$$

This implies for the EOMs \rightarrow

$$\delta S[\bar{g}, \bar{\phi}] = 0 \quad \Leftrightarrow \quad \delta S[g, \phi] = 0 \tag{12.82}$$

In words: If (g, ϕ) is a solution of the equations of motion (the Einstein field equations and the matter EOMs), then the new fields $(\bar{g}, \bar{\phi})$ obtained by any diffeomorphism φ are another solution. The group of diffeomorphisms Diff(M) on the spacetime manifold is therefore a *symmetry/invariance group* of GENERAL RELATIVITY.

Note that this hinges on the fact that both the metric g and the matter fields ϕ are transformed by the same diffeomorphism.

ii | At this point, it is unclear why the fact that Diff(M) is a symmetry group of GENERAL RELATIVITY poses a problem. To understand the issue, recall our current interpretation:

Spacetime =
$$\left(\underbrace{\text{Manifold } M}_{\text{Coincidence classes}}, \underbrace{\text{Metric } g}_{\text{Gravitational}}\right)$$
 (12.83)

Diffeomorphism invariance then implies that the following two field configurations (sketched here on a 2D spacetime manifold for simplicity) both satisfy the EOMs if one of them does:



The crucial point is that diffeomorphisms φ can act *locally* on compact regions of spacetime (here the "hole"), leaving the fields everywhere else unchanged.

Here the conincidence classes of events E that make up the spacetime manifold M are denoted by gray dots, the fields (both metric g and matter ϕ) are indicated by contour lines. Note that the two diffeomorphic field configurations (g, ϕ) and $(\bar{g}, \bar{\phi})$ differ only in a compact region denotes as "hole".

\rightarrow Problem:

The problem that arises from such a construction can be phrased in various ways:

• Assume that time runs upwards in the 2D patch of spacetime above. The two field configurations (g, ϕ) and $(\bar{g}, \bar{\phi})$ are then identical in the "past" (= lower boundary of the patch), but differ in the "hole". But this is a problem for *determinism*: A useful physical model should make unambiguous predictions for the future evolution of a system, based on a set of initial data. The diffeomorphism invariance of GENERAL RELATIVITY thwarts this, for it cannot distinguish between the two field evolutions above that coincide in the past.

 \rightarrow Is general relativity indeterministic?

- In our current reading, the points of the spacetime manifold are (coincidence classes of) events E. The fields (metric and matter alike) are functions on M. If we interpret the red contour line in the sketch as the trajectory of a particle (given by the excitation of some field), diffeomorphisms can be used to deform this trajectory arbitrarily. But the statement "the red particle passes through event E" is not invariant under such transformations. This seems to be problematic because it is a statement about \leftarrow conicidences, and as such should be objective (as argued in Section 1.1). Put differently: The EOMs of GENERAL RELATIVITY cannot decide wether the particle meets the event E or not!
 - \rightarrow What is the relation between *fields* and (coincidence classes of) *events*?
- Einstein originally considered a spacetime filled with matter except for a "hole" that was assumed to be free of matter (that's were the term "hole" comes from). He then asked whether the metric in the hole was determined by the distribution of matter (and the metric) outside the hole. Diffeomorphism invariance said *no*. This implies that knowledge of the distribution of matter *outside* the hole, together with the initial geometry of spacetime, is not enough to predict the metric *inside* the hole. Einstein called this a "violation of the law of causlity" which is essentially the problem of *indeterminism* identified above.

Einstein introduced the argument in late 1913 to rationalize his failure to find a generally covariant field equations that were consistent with Newtonian gravity in the non-relativistic limit. He used the hole argument to convince himself that a generally covariant theory of gravity was *impossible* (\leftarrow *last point above*). The argument then coaxed him into a (misguided) search for *non-covariant* field equations that, in hindsight, delayed the genesis of GENERAL RELATIVITY by two years. Einstein found the flaw in his argument in late 1915 (\rightarrow *next*); freed from this conceptual roadblock, he published the correct field Eq. (12.10) shortly after.

iii | Solution:

To solve the problems above, we have no choice but to concede the following:

• If we want GENERAL RELATIVITY to be a deterministic (= predictive) theory, we must *identify* diffeomorphic solutions as *physically indistinguishable*.

[Similar to gauge fields A_{μ} and \tilde{A}_{μ} that are related by $\tilde{A}_{\mu} = A_{\mu} + \partial_{\mu}\lambda$ are physically indistinguishable in electrodanymics.]



• We cannot interpret the points $E \in M$ of the manifold as observable entities that exist (in some physical sense) independent of the fields.

 \rightarrow ** Leibniz equivalence:

- Diffeomorphic solutions $(g, \phi) \stackrel{\varphi}{\sim} (\bar{g}, \bar{\phi})$ describe *the same* physics.
- GENERAL RELATIVITY is a gauge theory; Diff(M) is its gauge group.
- The spacetime manifold *M* itself does *not* exist as physical entity.
- The *fields* (g, ϕ) on the spacetime manifold M exist as physical entities.
- The points $E \in M$ of the spacetime manifold cannot exist in the same way the fields on the manifold do. The stance that M exists as a physical entity is known as \uparrow manifold substantivalism; the hole argument is therefore an argument against this philosophical reading of GENERAL RELATIVITY. (See also my perspective \rightarrow below.) Note that this does not affect the independent existence of the metric field, which is responsible for the elevation of "spacetime" from a static background to a dynamical participant in the evolution of the universe. This view is known as \uparrow substantivalism (without the prefix "manifold") and remains unaffected by the hole argument.
- *Disclaimer:* No matter what I claim here, you will always find a paper by a philosopher of science who disagrees. That's fine; the whole purpose of philosophy is to disagree about stuff that we cannot (yet) pin down by experiments.
- <u>Historical note:</u>

Einstein finally discarded the hole argument and embraced Leibniz equivalence (which led him to his field equations in November 1915). On January 3. 1916, Einstein writes in a letter to his friend Michele Besso [200]:

An der Lochbetrachtung war alles richtig bis auf den letzten Schluss. Es hat keinen physikalischen Inhalt, wenn inbezug auf dasselbe Koordinatensystem K zwei verschiedene Lösungen G(x) und G'(x) existieren. Gleichzeitig zwei Lösungen in dieselbe Mannigfaltigkeit hineinzudenken, hat keinen Sinn und das System K hat ja keine physikalische Realität. Anstelle der Lochbatrachtung tritt folgende Überlegung. Real ist physikalisch nichts als die Gesamtheit der raumzeitlichen Punktkoinzidenzen. Wäre z. B. das physikalische Geschehen aufzubauen aus Bewegungen materieller Punkte allein, so wären die Begegnungen der Punkte, d. h. die Schnittpunkte ihrer Weltlinien das einzig Reale, d. h. prinzipiell beobachtbare. Diese Schnittpunkte bleiben natürlich bei allen Transformationen erhalten (und es kommen keine neuen hinzu), wenn nur gewisse Eindeutigkeitsbedingungen gewahrt bleiben. Es ist also das natürlichste, von den Gesetzen zu verlangen, dass sie nicht mehr bestimmen als die Gesamtheit der zeiträumlichen Koinzidenzen. Dies wird nach dem Gesagten bereits durch allgemein kovariante Gleichungen erreicht.

For your entertainment, the letter also contains the following (unrelated) statement:

Das Studium von Minkowski würde Dir nichts helfen. Seine Arbeiten sind unnütz kompliziert.

iv | ‡ Another perspective:

What follows is my own take on what diffeomorphism invariance might tell us about reality. The purpose of the following arguments is to demonstrate that the "hole issue", diffeomorphism invariance, and general covariance, are all "symptoms" of mathematical surplus structure that – while being useful for our description of GENERAL RELATIVITY- cannot (and should not) be identified with real physical entities.



a | Let me start by pointing out an intrinsic flaw of our previous interpretation of the mathematical objects we are working with. So far, our reading was as follows:

Spacetime manifold M = Set of *coincidence classes* $E = \{e_1, e_2, ...\}$ of *events* e_i

Gravitational field g = Tensor field on $M: g: M \to T^*M \otimes T^*M$

Electromagnetic field A = Vector field on $M: A: M \rightarrow TM$

Klein-Gordon field ϕ = Scalar field on $M: \phi: M \to \mathbb{R}$

Now think of a coincidence class including the events "particle here" and "photon here". We can think of this as a combined event where the photon is absorbed or emitted by the particle. But particle physics tells us that there are no particles (funny, I know), just fields. In the modern reading of quantum field theory, "particles" are simply localized (and quantized) excitations of fields. For simplicity, let us say that the event "photon here" simply means $A(E) \neq 0$ for some point $E \in M$, and the coincidental presence of the particle is similarly described by $\phi(E) \neq 0$ (since ϕ is a scalar, this would have to be a scalar particle like the Higgs boson [which is not electrically charged]; but this is not important here).

This shows that elementary events of the type "particle of type X is here" are associated with specific *values* of fields of type X – not with their *arguments* (= points of the manifold)! But if these points are supposed to be coincidence classes of events, we arrive at a strange circular construction where fields are defined on points that contain (and are characterized by) the value of the field at that very point.

In a nutshell:

All observable features of physical systems are determined by the values of fields, their coincidences and causal relations.

This suggests that the points E of the spacetime manifold M cannot be events themselves (nor can they be classes of events). However, we were also not far off identifying spacetime points with coincidence classes of events. Let me explain:

b | Let us put forward the following <u>Postulate:</u>

Only events (e), related by coincidences (\sim) and a causal partial order (\prec) exist.

Recall our discussion in SPECIAL RELATIVITY of events and their relations in Section 1.6.

That we identify these events with realizations of values of fields it not important.

 \rightarrow Reality is a causal network of events, grouped by coincidences:





This looks rather messy! It is certainly hard to formulate a workable model (= theory) for this reality without putting in a bit more effort into the layout of the causal graph:

- c | Unexplained fact: The causality graph of our universe is 4D-embeddable.
 - "4D-embeddable" means that you can lay out the graph on a 4D manifold such that the edges (= causal relations) only connect "nearby" nodes (= events) of the graph ("nearby" being defined by the ↑ *topology* of the manifold).
 - If you randomly construct a causality graph, there is absolutely no reason to expect that it has this property. Hence this is a feature of reality that must be explained (*→ below*). Note that the embeddability is a *local* feature; we do not claim that the graph can be embedded in a topologically trivial manifold like ℝ⁴.

This suggests the following procedure to lay out the causality graph:

- Construct a 4D manifold using *empty boxes* (these are the points of the manifold) by arranging them in a 4D hypercubic lattice (for simplicity). This is our new spacetime manifold *M*. Note that it doesn't contain any events yet; its a completely artificial structure without physical existence.
- (2) Place events of the causality graph into the boxes of the manifold such that ...
 - ... events that coincide (\sim) are placed in the *same* box.
 - ... events connected by an edge (\prec) are placed in *nearby* boxes.

This procedure succeeds because the graph is 4D-embeddable.

(Since empty boxes don't exist, you can think of them as not being there at all.)

 \rightarrow For example:



There you have it. This is the structure we called "spacetime manifold". You can even see the light/null-cone structure of a Lorentzian manifold emerge from the causal relations (recall Section 11.1).

What changed is our interpretation: The points $E \in M$ are the *boxes themselves* (not the sets of events collected in them). Nothing of this construction has to do with reality (we don't change the causality graph); this *layout* is merely a convenient way to *represent* reality.



d | The diffeomorphism invariance of GENERAL RELATIVITY (and thus the hole argument) are now trivial consequences of the fact that the above description to lay out the causality graph on a 4D grid of boxes *is not unique*: It is obvious that there is a quite lot of freedom in placing the causally related events in nearby boxes.

For example, an alternative layout that differs from the previous one only in the orange "hole" is the following:



This is the discrete version of Einstein's "hole diffeomorphisms". From this perspective, it is trivial that such a transformation must be *gauge* because all physics is encoded in the causality graph of events (which remains the same). Note how the emerging light/null-cone texture is "warped", as expected from an (active) diffeomorphism that affects the metric.

e | We now understand diffeomorphism invariance and the hole argument. But what about *general covariance*? Up to now we didn't even mention *coordinates*!

Coordinates are simply *labels* that we print onto the boxes to refer to them in our equations. This is what we mean by a chart that assigns the coordinates x^{μ} to a point (= box) $E \in M$. It is also convenient to assign the label of a box to all events placed *in* the box; this is what we mean by expressions like $A^{\mu}(x)$ if $A(E) = A^{\mu}(x)\partial_{\mu}$ is the value of the field at $E \in M$.





For example, here is a systematic way to label the boxes and 5 exemplary events (e.g., the values of the EM field):

Now that we have boxes (that don't exist) with labels (that don't exist either), the duality of (active) diffeomorphisms and (passive) coordinate transformations becomes evident:

(A) Diffeomorphism (active view):

Keep the labels of the boxes, but move the events around (thereby assigning new labels to the events).

(B) Coordinate transformation (passive view):

Keep the events in their boxes, but change the labels of all boxes (thereby assigning new labels to the events).

Note that both transformation can lead to the same labeling of events (if the coordinate transformation is chosen "inverse" to the diffeomorphism)!

- From this perspective, the statement that GENERAL RELATIVITY is gauged by (active) diffeomorphisms is dual to the statement that its equations are generally covariant, i.e., form invariant under (passive) coordinate transformations.
- This equivalence hinges on the fact that *all physical content is encoded in the causality graph* which implies that *the boxes are not physical entities*; this is ← *background independence*. By contrast, a physical theory that is background *dependent* assigns physical reality to the boxes themselves (but not the labels) by associating them with some events (= field) that are not moved around with the other events. With such "static" events in place, the duality between diffeomorphisms and coordinate transformations is lost!

This is why SPECIAL RELATIVITY can be formulated generally covariant without being diffeomorphism invariant. In this case, the "static" background structure is the Minkowski metric and the boxes make up Minkowski space.

v | Comment on \uparrow *Scientific realism*:



When we deny the manifold M existence (and relegate it to a useful auxiliary structure of "empty boxes"), we must find an answer to the following question (otherwise the 4D embeddability of the causal network of events is a "miracle"):

Why does the causal graph have the topology of a 4D manifold?

- I have no answer to this question \odot (and there is certainly no consensus among scientists, let alone philosophers). However, it seems that any reasonable theory beyond GENERAL RELATIVITY (quantum gravity ...) must answer this question.
- A potential solution to the question is the line of arguments discussed in Section 4.4.
- Scientific realism is the epistemological stance that *there exists physical entities out there* that we descibe by our theories independent of whether (and how) we observe them. In philosophy, scientific realism is an attempt to explain "why science works."

[For example: To understand the effectiveness of Maxwell's equation in describing electromagnetic phenomena, it is certainly useful to assume that the electromagnetic field $F_{\mu\nu}$ (or, to some extent, the gauge field A_{μ}) really exists – despite the fact that nobody has ever directly observed these fields.]

2 Where is SPECIAL RELATIVITY?

- i | Here is a riddle:
 - 1. In SPECIAL RELATIVITY we were proud of our discovery that Maxwell's equations were forminvariant under Lorentz transformations (Lorentz covariant) but *not* under Galilei transformations.
 - 2. In GENERAL RELATIVITY we were proud of our discovery that coordinate systems don't exist, and all fundamental physical theories must be expressible in a generally covariant form. We achieved this for Maxwell's equations.
 - 3. But then these generally covariant Maxwell equations must be forminvariant under both Lorentz and Galilei transformations (among others). The distinguished status of Lorentz transformations seems to be lost.

What is going on?

ii | We use the massless Klein-Gordon field Eq. (11.36) for its simplicity to resolve the puzzle:

You can of course use the (more complicated) Maxwell equations to make the same points.

a | The Klein-Gordon field theory is defined in SPECIAL RELATIVITY as follows:

$$\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi(x) = 0 \tag{12.84}$$

- Black: Equation (= definition of the theory/model)
- Red: Solution (= possible evolution)
- **b** | \triangleleft Arbitrary diffeomorphism $\bar{x} = \varphi(x)$

$$\rightarrow$$
 Define new field $\bar{\phi}(\bar{x}) := \phi(\varphi^{-1}(\bar{x})) = \phi(x)$

 $\rightarrow \varphi$ is *symmetry* of Eq. (12.84) iff

$$\begin{cases} \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi(x) = 0\\ \eta^{\alpha\beta}\bar{\partial}_{\alpha}\bar{\partial}_{\beta}\phi(\bar{x}) = 0 \end{cases} \qquad \stackrel{\varphi}{\underset{\text{sym.}}{\leftarrow}} \qquad \begin{cases} \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\bar{\phi}(x) = 0\\ \eta^{\alpha\beta}\bar{\partial}_{\alpha}\bar{\partial}_{\beta}\bar{\phi}(\bar{x}) = 0 \end{cases}$$
(12.85)

 \Leftrightarrow



The differential equations in braces are trivially equivalent because x and \bar{x} are dummy variables (α and β are dummy indices) and the equations are assumed to be satisfied for all coordinates x / points on the manifold.

Let us check under which conditions on φ we can get from the left-hand side of Eq. (12.85) to the righ-hand side (and vice versa):

$$\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi(x) = 0 \qquad (12.86a)$$

$$\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi(\bar{x}) = 0 \qquad (12.86b)$$

$$\Leftrightarrow \qquad \eta^{\mu\nu}\partial_{\mu}\left[\left(\bar{\partial}_{\beta}\bar{\phi}(\bar{x})\right)\frac{\partial\bar{x}^{\beta}}{\partial x^{\nu}}\right] = 0 \qquad (12.86c)$$

$$\Leftrightarrow \quad \eta^{\mu\nu} \left[\left(\bar{\partial}_{\alpha} \bar{\partial}_{\beta} \bar{\phi}(\bar{x}) \right) \frac{\partial \bar{x}^{\beta}}{\partial x^{\nu}} \frac{\partial \bar{x}^{\alpha}}{\partial x^{\mu}} + \left(\bar{\partial}_{\beta} \bar{\phi}(\bar{x}) \right) \frac{\partial^2 \bar{x}^{\beta}}{\partial x^{\mu} \partial x^{\nu}} \right] = 0 \quad (12.86d)$$

When is this expression equivalent to the right-hand side of Eq. (12.85)? First, the linear order term must vanish. This implies

$$\frac{\partial^2 \bar{x}^{\beta}}{\partial x^{\mu} \partial x^{\nu}} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \bar{x}^{\beta} = M^{\beta}_{\ \alpha} x^{\alpha} + b^{\beta} , \qquad (12.87)$$

which means that the diffeomorphism φ must be an *affine map*. With this constraint, Eq. (12.86d) simplifies to

$$\left(M^{\alpha}_{\ \mu}\eta^{\mu\nu}M^{\beta}_{\ \nu}\right)\bar{\partial}_{\alpha}\bar{\partial}_{\beta}\bar{\phi}(\bar{x})=0.$$
(12.88)

This is equivalent to the right-hand side of Eq. (12.85) if

$$\left(M^{\alpha}_{\ \mu}\eta^{\mu\nu}M^{\beta}_{\ \nu}\right) \stackrel{!}{=} \eta^{\alpha\beta} . \tag{12.89}$$

But this is the defining relation for *isometries* of Minkowski space [\leftarrow Eq. (4.21)], and we already know that this defines the Lorentz group O(1, 3) (recall Section 4.2). Hence we can conclude:

 $\stackrel{\circ}{\rightarrow}$

The symmetries (\leftarrow *invariance group*, Section 1.2) of the Klein-Gordon equation include \leftarrow *Poincaré transformations*.

i! This is a statement about *active* transformations of fields: Poincaré transformations are a "machine" to construct new solutions of the Klein-Gordon equation. First, this is a useful mathematical tool, and second, it is physically significant as it implies that if the field evolution ϕ can be observed, then so can $\overline{\phi}$. Nothing of this has to do with coordinates!

This suggests the following definition:

A theory is *relativistic* (in the sense of SPECIAL RELATIVITY) if its invariance group contains the *Poincaré group*.

Note that this definition makes no reference to coordinate transformations and how equations transform under such!



c | But in SPECIAL RELATIVITY we always talked about "Lorentz covariant equations" that do not change under Lorentz/Poincaré transformations, now interpreted as *coordinate transformations*.

To understand how this relate to the previous discussion, let us once again focus on the Klein-Gordon equation, but now we perform a

 \triangleleft Arbitrary coordinate transformation $\bar{x} = \varphi(x)$

It is convenient to interpret Eq. (12.84) as a generally covariant equation:

$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi = 0 \tag{12.90}$$

with $g^{\mu\nu}(x) = \eta^{\mu\nu}, \nabla_{\nu}\phi = \partial_{\nu}\phi$, and

$$\nabla_{\mu}\Box_{\nu} = \partial_{\mu}\Box_{\nu} - \Gamma^{\alpha}_{\ \mu\nu}\Box_{\alpha} \quad \text{with} \quad \Gamma^{\alpha}_{\ \mu\nu} = 0.$$
 (12.91)

Under $\bar{x} = \varphi(x)$ the equation remains *forminvariant* in the sense that:

$$g^{\alpha\beta}(x)\nabla_{\alpha}\nabla_{\beta}\phi(x) = 0 \quad \stackrel{\bar{x}=\varphi(x)}{\longleftrightarrow} \quad \bar{g}^{\mu\nu}(\bar{x})\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\bar{\phi}(\bar{x}) = 0 \quad (12.92)$$

with

$$\bar{g}^{\mu\nu}(\bar{x}) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}(x) = \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \eta^{\alpha\beta}$$
(12.93)

and

$$\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\bar{\phi}(\bar{x}) \stackrel{\text{def}}{=} \bar{\partial}_{\mu}\bar{\partial}_{\nu}\bar{\phi}(\bar{x}) - \bar{\Gamma}^{\alpha}_{\ \mu\nu}\bar{\partial}_{\alpha}\bar{\phi}(\bar{x})$$
(12.94a)

$$\stackrel{10.39}{=} \left(\partial_{\alpha}\partial_{\beta}\phi(x)\right)\frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}}\frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} + \left(\partial_{\beta}\phi(x)\right)\frac{\partial^{2}x^{\beta}}{\partial \bar{x}^{\mu}\partial \bar{x}^{\nu}} \qquad (12.94b)$$
$$- \left(\partial \bar{x}^{\alpha} \quad \partial^{2}x^{\rho}\right) + \left(\partial \bar{x}^{\beta}\phi(x)\right)\frac{\partial x^{\beta}}{\partial \bar{x}^{\mu}\partial \bar{x}^{\nu}} = \left(\partial \bar{x}^{\alpha}\phi(x)\right)^{2} + \left(\partial \bar{x}^{\alpha}\phi(x)\right)^{2$$

$$= \frac{1}{\sqrt{\partial x^{\mu}} \partial \bar{x}^{\mu}} \frac{\partial \bar{x}^{\nu}}{\partial \bar{x}^{\mu}} \int \left(\partial_{\beta} \phi(x) \right)^{\mu} \frac{\partial \bar{x}^{\alpha}}{\partial \bar{x}^{\alpha}}$$

$$= \frac{12.91}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \nabla_{\alpha} \nabla_{\beta} \phi(x) \qquad (12.94c)$$

Of course you don't have to do this step-by-step calculation; the whole point of introducing covariant derivatives was that the object transforms like a tensor!

- **d** | Now comes the punchline:
 - General covariance:

The property Eq. (12.92) is what we call *general covariance*; it is valid for *arbitrary* coordinate transformations φ , including Lorentz *and* Galilei transformations:

$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\phi(x) = 0 \quad \stackrel{\varphi = \begin{cases} \text{Lorentz} \\ \text{Galilei} \\ \dots \\ \hline \end{array}}{\not{g}^{\mu\nu}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}\bar{\phi}(\bar{x})} = 0 \quad (12.95)$$

But this does not imply that φ is a symmetry of the equation because the transformed equation on the right is not functionally equivalent to the equation on the left! This means: If you relabel the dummy variable $\bar{x} \mapsto x$ in the right equation, you don't end up with original equation on the left because in general:

$$g^{\mu\nu}(x) \neq \bar{g}^{\mu\nu}(x) \text{ and } \nabla_{\mu} \neq \bar{\nabla}_{\mu}$$
 (12.96)

 \rightarrow The transformed solution $\overline{\phi}(x)$ solves a *functionally different* equation!



- Let use show explicitly that the two equations are not functionally identical. To this end, introduce the explicit notation $\Gamma^{\alpha}_{\mu\nu}[g](x)$, which tells us to use the metric $g_{\mu\nu}$ to compute the Christoffel symbols from their definition Eq. (10.79), and interpret the result as a function of the spacetime coordinates x.

With this notation, the left equation of (12.95) reads explicitly

$$g^{\mu\nu}(x)\left[\partial_{\mu}\partial_{\nu}\phi(x) - \Gamma^{\alpha}_{\ \mu\nu}[g](x)\partial_{\alpha}\phi(x)\right] = 0, \qquad (12.97)$$

whereas the right equation reads

$$\bar{g}^{\mu\nu}(\bar{x})\left[\bar{\partial}_{\mu}\bar{\partial}_{\nu}\bar{\phi}(\bar{x}) - \Gamma^{\alpha}_{\ \mu\nu}[\bar{g}](\bar{x})\bar{\partial}_{\alpha}\bar{\phi}(\bar{x})\right] = 0.$$
(12.98)

These two equations have the same form – they are *forminvariant*; and Eq. (12.98) is equivalent to Eq. (12.97) if both $g_{\mu\nu}$ and ϕ transform as usual for a tensor and a scalar. But the variable \bar{x} in Eq. (12.98) is a dummy variable (ignoring potential domain issues); thus let us rename it $\bar{x} \mapsto x$ so that the differential equation reads

$$\bar{g}^{\mu\nu}(x)\left[\partial_{\mu}\partial_{\nu}\bar{\phi}(x) - \Gamma^{\alpha}_{\ \mu\nu}[\bar{g}](x)\partial_{\alpha}\bar{\phi}(x)\right] = 0.$$
(12.99)

But this differential equation is not the same as Eq. (12.97) because $\bar{g}^{\mu\nu} \neq g^{\mu\nu}$ for arbitrary transformations φ . This is why the new function $\bar{\phi}(x)$ solves a *different* equation in general – and not the old one. But then φ does not automatically lead to a symmetry that can be used to construct new solutions from old ones.

- Some might complain: Wait, wasn't the point of general covariance that equations are *forminvariant* under arbitrary coordinate transformations? Well, yes, but with "forminvariance" one means exactly the above transformation; and not that the equation remains functionally *identical*.

Recall (Chapter 3) that the whole point of introducing tensors and "generally covariant equations" (= tensor equations) was to characterize coordinate-*dependent* (!) equations that encode coordinate-*independent* equations (= relations between geometric objects). The transformation Eq. (12.95) guarantees that the equation can be written coordinate-*free* as (\rightarrow *below*)

$$g^{ab}\nabla_a\nabla_b\phi = 0, \qquad (12.100)$$

and that's the whole point.

• Symmetry:

For a scalar field, a transformation φ is a symmetry if, for a solution $\phi(x)$, the new function $\overline{\phi}(x) := \phi(\varphi^{-1}(x))$ is another solution of the *old* equation:

$$g^{\alpha\beta}\nabla_{\alpha}\nabla_{\beta}\phi(x) = 0 \qquad \xleftarrow{\varphi = \begin{cases} \text{Lorentz ?} \\ \text{Galilei ?} \\ \dots ? \\ & & \\$$

This is clearly not the same equivalence as in Eq. (12.95).

Eq. (12.101) & Eqs. (12.97) and (12.99) \rightarrow

$$\Gamma^{\alpha}_{\ \mu\nu}[\bar{g}] \stackrel{!}{=} \Gamma^{\alpha}_{\ \mu\nu}[g] = 0 \quad \text{and} \quad \bar{g}^{\mu\nu} \stackrel{!}{=} g^{\mu\nu} = \eta^{\mu\nu} \tag{12.102}$$

Using the transformation of connection coefficients Eq. (10.39), one immediately derives Eq. (12.87) from the first condition; this again implies an affine



form Eq. (12.87) for the transformation φ . The second condition is equivalent to Eq. (12.89) and restricts φ to the *isometry group* of Minkowski space, that is: \leftarrow *Poincaré transformations*.

 $\stackrel{\circ}{\rightarrow}$ Symmetries of the Klein-Gordon equation on Minkowski space:

 $g^{\alpha\beta} \nabla_{\alpha} \nabla_{\beta} \phi(x) = 0 \quad \stackrel{\varphi = \{ \text{Poincaré}}{\longleftrightarrow} \quad g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \bar{\phi}(x) = 0 \quad (12.103)$

Brief round-up:

- We started by writing the (special) relativistic Klein-Gordon equation in tensorial form. The equation then becomes generally covariant, i.e., forminvariant under arbitrary coordinate transformations (in particular: Galilei transformations). But these do not translate to (active) symmetries: The transformations of fields that map solutions onto new solutions are still only Poincaré transformations. The takeaway is that Galilei transformations (or any other non-Poincaré transformations) *are not isometries of Minkowski space*, and this spoils their use for constructing new solutions from old ones.
- A nice benefit of the generally covariant form Eq. (12.90) is that it can be used to define the Klein-Gordon field on arbitrary curved spacetimes, not only on Minkowski space. If $g_{\mu\nu}$ is the metric of some generic spacetime, the equation remains of course forminvariant under coordinate transformations. But which of these passive transformations can be reinterpreted as active *symmetries*? Our argument above still goes trough and we are tasked with finding the *isometries* of the new spacetime. But, as mentioned in Section 11.5, a generic spacetime doesn't have any Killing fields, and therefore also no (continuous group of) symmetries. Thus, on a generic spacetime, the Klein-Gordon equation does not have the Poincaré group as (part of) its symmetry group, because the spacetime on which it is formulate doesn't have this symmetry either.

iii | Question:

Is it possible to construct a generally covariant theory for which every (passive) coordinate transformation can be interpreted as an (active) symmetry?

Compare Eq. (12.97) and Eq. (12.99):

Solve for
$$\phi$$
:
$$\begin{cases} g^{\mu\nu}(x) \left[\partial_{\mu} \partial_{\nu} \phi(x) - \Gamma^{\alpha}_{\mu\nu} [g](x) \partial_{\alpha} \phi(x) \right] = 0\\ \bar{g}^{\mu\nu}(x) \left[\partial_{\mu} \partial_{\nu} \bar{\phi}(x) - \Gamma^{\alpha}_{\mu\nu} [\bar{g}](x) \partial_{\alpha} \bar{\phi}(x) \right] = 0 \end{cases}$$
(12.104)

Problem: $\phi(x)$ and $\overline{\phi}(x)$ solve *different* equations (compare the black equations).

Idea: Interpret the metric as *solution* and not as background (= part of the equation).

$$\rightarrow$$

Solve for
$$(g,\phi)$$
:
$$\begin{cases} g^{\mu\nu}(x) \left[\partial_{\mu}\partial_{\nu}\phi(x) - \Gamma^{\alpha}_{\ \mu\nu}[g](x)\partial_{\alpha}\phi(x) \right] = 0\\ \bar{g}^{\mu\nu}(x) \left[\partial_{\mu}\partial_{\nu}\bar{\phi}(x) - \Gamma^{\alpha}_{\ \mu\nu}[\bar{g}](x)\partial_{\alpha}\bar{\phi}(x) \right] = 0 \end{cases}$$
(12.105)

 (g, ϕ) and $(\bar{g}, \bar{\phi})$ solve *the same* equation \odot .

 \rightarrow We just prepared the theory for \leftarrow background independence.



Whether the theory really is background independent depends on the presence and type of additional equations of motion that constrain the metric field (\rightarrow *below*). However, what we can say is that the theory has no longer any *absolute objects* (= non-dynamical tensor fields).

We conclude:



- As argued above, diffeomorphism invariance must be interpreted as a *gauge symmetry* that relates physically equivalent solutions. This means that our "trick" to declare the metric as a dynamical field to lift all coordinate transformations to (active) symmetries (now also Galilei transformations are symmetries!) didn't really work out as intended. It is as if we wanted "too much": Now that *every* diffeomorphism is a symmetry, *none* of them is *physical* anymore all of them are *gauge*! But the good old (physical) Poincaré symmetry can of course be resurrected for metric *solutions* that have the appropriate Killing fields.
- Interpreting Eq. (12.90) as an equation for (g, φ) makes the theory diffeomorphism invariant, i.e., every coordinate transformation can be interpreted as an active symmetry transformation. Without restricting the new dynamical field g_{μν} by an additional equation of motion [e.g., the Einstein field equation Eq. (12.10) (→ *below*)], this is a rather useless construction because the theory has solutions for *every* metric you ask for. Hence it cannot *predict* anything about the metric, only about the evolution of the Klein-Gordon field in relation to the metric.

iv | Conclusion:

We can sum up our findings as follows:

- Global Lorentz/Poincaré transformations $\phi \stackrel{\Lambda}{\mapsto} \bar{\phi}$ of matter fields are *not* symmetries of GENERAL RELATIVITY, because the metric typically lacks the necessary Killing fields.
- Global Lorentz/Poincaré transformations $(g, \phi) \stackrel{\Lambda}{\mapsto} (\bar{g}, \bar{\phi})$ of *both* matter fields and metric are *gauge* symmetries of GENERAL RELATIVITY; they have *no* physical significance.
- So is SPECIAL RELATIVITY gone? Well, yes, if we identify the theory with "global Lorentz symmetry" the answer must be affirmative: GENERAL RELATIVITY does *not* contain SPECIAL RELATIVITY in its pure form because spacetime is a dynamical field and solving for it usually does *not* produce flat Minkowski space. Only in the situations where it does, GENERAL RELATIVITY reduces to SPECIAL RELATIVITY (which is approximately true in interstellar space far away from matter, and with appropriate boundary conditions).
- *Is this a problem?* The answer is of course *no*, but it is instructive understand why:

Minkowski space is the defining entity of SPECIAL RELATIVITY and has two characteristic features: it is *flat* and it is *Lorentzian* [it has metric signature (1, 3)]. The crucial insight is that its *flatness* is not a characteristic feature of reality, it is a simplicity assumption that makes SPECIAL RELATIVITY "unnaturally symmetric" (10 Killing fields!). Hence the global Lorentz symmetry of relativistic theories – which is imprinted by the symmetry of spacetime due to general covariance – is "unnatural" *mutatis mutandis*. The core feature of reality, that SPECIAL RELATIVITY actually brought to the table, is the *locality of causality*, which is ensured by the *Lorentz signature* (1, 3) alone. The realization that this core feature is not tied to the flatness of Minkowski space leads directly to GENERAL RELATIVITY.

 \rightarrow *Essence* of SPECIAL RELATIVITY that survives in GENERAL RELATIVITY:

$$\begin{cases} \text{General covariance} \\ \text{Lorentzian metric} \end{cases} \Rightarrow \begin{cases} \text{Local Lorentz symmetry} \\ \text{Locality of causality} \\ \text{Local constancy of the speed of light} \end{cases}$$

- Recall the "spoiler" in Section 0.6.
- *Aside:* You know now three theories to describe classical mechanics: Good old Newtonian mechanics, SPECIAL RELATIVITY, and GENERAL RELATIVITY. It is established practice to teach these subjects in this very order:

$$\underbrace{\underbrace{Newtonian mechanics}_{2nd \ term}}_{fit} \xrightarrow{then} \underbrace{\underbrace{SPECIAL \ RELATIVITY}_{5th \ term}}_{fit} \xrightarrow{then} \underbrace{\underbrace{GENERAL \ RELATIVITY}_{6th \ term}}_{fit}$$

It seems to be consensus that the reason for this is how hard or easy the subjects are. While there certainly is some (pedagogic) truth to this assessment, I would like to point out that the "complexity" of a theory can be gauged in (at least) two different ways. For lack of better terms I will refer to them as *operational complexity* and *conceptual complexity*.

Operational complexity captures how hard it is to work out actual problems in the respective theory. This leads to the following grading:

$$\vec{F} = m\vec{a} \xrightarrow{\text{simpler}} K^{\mu} = m \frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} \xrightarrow{\text{simpler}} K^{\mu} = m \frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} + m \Gamma^{\mu}_{\ \alpha\beta} u^{\alpha} u^{\beta}$$

Since students must solve problems to internalize a theory, it is this order, from the operationally simple Newtonian mechanics to the operationally hard GENERAL RELA-TIVITY, that supports the conventional approach to teach these subjects.

By contrast, *conceptual complexity* captures how much "conceptual scaffolding" is needed to formulate the theory precisely. Based on the discussions above, my claim is that the order is exactly opposite:

GENERAL RELATIVITY
$$\xrightarrow{\text{simpler}}_{\text{than}}$$
 SPECIAL RELATIVITY $\xrightarrow{\text{simpler}}_{\text{than}}$ Newton

The argument is simple: While more symmetries (or structures) make *solving* problems easier (thereby *lowering* the *operational* complexity), they tend to clutter the conceptual framework of a theory (thereby *increasing* the *conceptual* complexity). In addition, they often obfuscate the actually important structures of the theory (recall the discussion of the flatness of Minkowski space above).

One might counter that surely the conceptual framework of Newtonian mechanics is not harder than that of SPECIAL RELATIVITY! I beg to disagree: If one carves out the mathematics of Newtonian spacetime *properly*, one has to deal with \uparrow *affine*


manifolds, \uparrow *fiber bundles*, etc...[for a proper definition of Newtonian spacetime, see STRAUMANN [9] (pp. 10–16)]. Put bluntly: Newtonian mechanics looks conceptually "simple" because it is usually not done rigorously! (At least compared to how rigorously we do GENERAL RELATIVITY.)

3 Interlude: *** Abstract index notation*:

It is often convenient to write equations in a coordinate-free form, without loosing information about the types of tensors involved, and how they act on each other. This is achieved by abstract index notation:

Scalar:	$\phi:=\phi$	$\in \mathbb{R}$	(12.106a)
Vector:	$A^a := A^{\mu} \partial_{\mu}$	$\in TM$	(12.106b)

Vector: $A^a := A^{\mu} \partial_{\mu} \quad \in TM$ Covector: $B_a := B_{\mu} dx^{\mu} \quad \in T^*M$ (12.106c)

Mixed tensor: $T^a_{\ \ \nu} := T^{\mu}_{\ \ \nu} \partial_{\mu} \otimes dx^{\nu} \in TM \otimes T^*M$ (12.106d)

i! The roman indices a, b, ... are not numerical indices, they are labels that indicate "slots" of tensors and how they are applied to each other. For example:

$$B_a A^a := B_a (A^a) = B_\mu dx^\mu (A^\nu \partial_\nu) = B_\mu A^\nu dx^\mu (\partial_\nu) = B_\mu A^\nu \delta^\mu_\nu = B_\mu A^\mu .$$
(12.107)

Example: The Klein-Gordon equation in coordinate-free notation reads:

$$\eta^{ab} \nabla_a \nabla_b \phi = 0 \tag{12.108}$$

where $\eta_{ab} = g_{\mu\nu}(x) dx^{\mu} dx^{\nu}$ denotes the Minkowski metric. [This is not the *matrix* $\eta_{\mu\nu} =$ diag (1, -1, -1, -1)! Furthermore, the *components* $g_{\mu\nu}(x)$ only equal $\eta_{\mu\nu}$ in inertial coordinates.]

4 | Background independence:

How does the concept of \leftarrow *background independence* (Section 9.2) mesh with these concepts? This discussion is based on Ref. [119].

i We have found the following implication:

$$\begin{array}{c} \text{Background} \\ \text{independence} \end{array} \Rightarrow \begin{cases} \text{No absolute objects} \\ \text{General covariance} \end{cases} \Rightarrow \begin{array}{c} \text{Diffeomorphism} \\ \text{invariance} \end{array}$$
(12.109)

This suggests the following identification:

(?) A theory is background independent *iff* it is diffeomorphism invariant.

ii | Problem:

 \triangleleft The following theory (in abstract index notation):

KG-SRT:
$$\begin{cases} g^{ab} \nabla_a \nabla_b \phi = 0 & \text{(Matter EOM)} \\ R^a_{bcd} = 0 & \text{(Metric EOM)} \end{cases}$$
(12.110)

This theory ...

- ... is coordinate-free (in components: generally covariant).
- ... has no absolute objects [solutions are tuples (g_{ab}, ϕ)].



• ... is equivalent to the Klein-Gordon field in SPECIAL RELATIVITY.

(The only flat metric on $M \simeq \mathbb{R}^4$ is the Minkowski metric $g_{ab} = \eta_{ab}$.)

 \rightarrow KG-SRT is a *diffeomorphism invariant* formulation of SPECIAL RELATIVITY!

But clearly SPECIAL RELATIVITY should count as a background-*dependent* theory, for it is defined on non-dynamical Minkowski space! How can we extract this characteristic feature from artificially diffeomorphism invariant formulations like KG-SRT?

iii | To this end, let us compare KG-SRT with the Klein-Gordon field in GENERAL RELATIVITY:

$$\begin{array}{l} \text{KG-GRT:} \begin{cases} g^{ab} \nabla_a \nabla_b \phi = 0 & (\text{Matter EOM}) \\ R_{ab} - \frac{1}{2} R g_{ab} = -\kappa T_{ab} & (\text{Metric EOM}) \end{cases} \tag{12.111} \end{array}$$

Here T_{ab} depends on the KG-field with components given in Eq. (11.118) for m = 0.

 \rightarrow Compare solutions:

KG-0

KG-SRT
$$\Rightarrow$$
 $(\eta_{ab}, \phi^1), (\eta_{ab}, \phi^2), (\eta_{ab}, \phi^3), \dots$ (12.112)

$$\exists \mathbf{RT} \quad \Rightarrow \quad (g_{ab}^1, \phi^1), \, (g_{ab}^2, \phi^2), \, (g_{ab}^3, \phi^3), \dots \tag{12.113}$$

- \rightarrow KG-SRT has a "hidden" *absolute object* (η_{ab}) shared by all solutions!
- iv | The fact that all solutions of a theory like SPECIAL RELATIVITY share some invariant objects allows for a cascade of "specializations" of the formulation of the theory:

Formulation	Solve for	Diff. inv.	Coord. free	Gen. cov.		
$g^{ab} \nabla_a \nabla_b \phi = 0$ $R^a_{\ bcd} = 0$	g_{ab}, ϕ	√	~	1		
Fix shared metric $g_{ab} = \eta_{ab} = \tilde{\eta}_{\mu\nu}(x) dx^{\mu} dx^{\nu}$ as absolute element \rightarrow						
$\eta^{ab} \nabla_a \nabla_b \phi = 0$	ϕ	×	✓	~		
Write in components wrt. some coordinates \rightarrow						
$\tilde{\eta}^{\mu\nu}(x)\nabla_{\mu}\nabla_{\nu}\phi=0$	φ	×	×	1		
Choose inertial coordinates to exploit symmetry of Minkowski space \rightarrow						
$\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi=0$	ϕ	×	×	×		

- All formulations above are equivalent in the sense that they describe the same physics.
- Thus, the very fact that we could formulate SPECIAL RELATIVITY in a *non*-diffeomorphism invariant form (and even a *non*-generally covariant form) characterizes it as a background-*dependent* theory.
- For GENERAL RELATIVITY, the first step (were one fixes the metric as an absolute element of the theory) fails and the cascade cannot take off. This prevents nondiffeomorphism invariant formulations of the theory. Since there is no distinguished metric, there cannot be a distinguished coordinate system, and thereby no non-generally covariant formulation either.



v | This leads us to the following refined definition of background independence:

Background independent theories (like GENERAL RELATIVITY) are characterized by their *lack* of a formulation that is *not* diffeomorphism invariant.

This explains why we didn't encounter a formulation of GENERAL RELATIVITY that is *not* generally covariant, while we did use such formulations when discussing SPECIAL RELATIVITY.

↓Lecture 27 [02.07.24]



13. Applications & Predictions

Now that the framework of GENERAL RELATIVITY is fully developed, we can start using it. As already mentioned, solving the Einstein field equations is hard, and in most realistic scenarios impossible. This is why we focus on the simplest and most symmetric settings – which still does not save us from mathematical complexity. Thus, instead of struggling with *conceptual* subtleties, we will mostly fight *technical* (mathematical) issues in this chapter.

i! Studying applications and predictions of GENERAL RELATIVITY is a vast topic, deserving its own course. This chapter only scratches the surface of this multifaceted (and active) field of research.

A comprehensive review of experimental tests of GENERAL RELATIVITY can be found in Ref. [201].

13.1. The gravitational field of a spherical mass

The first (and most important) exact solution of the Einstein field equations was obtained by German physicist KARL SCHWARZSCHILD at the end of 1915, only a few weeks after Einstein published his field equations; the so called \rightarrow *Schwarzschild metric* was published in January 1916 [202]. Schwarzschild found his solution while serving in the German army (during World War I); he died only a few months later in May 1916 (due to a disease he developed at the Russian front).

Here is how Schwarzschild sells his solution in Ref. [202] (§2):

Hr. EINSTEIN hat gezeigt, daß dies Problem [der sphärisch symmetrischen Massenverteilung] in erster Näherung auf das Newtonsche Gesetz führt [\leftarrow Section 12.1.1] und daß die zweite Näherung die bekannte Anomalie in der Bewegung des Merkurperihels richtig wiedergibt [\rightarrow Section 13.2.1]. Die folgende Rechnung liefert die strenge Lösung des Problems. Es ist immer angenehm, über strenge Lösungen einfacher Form zu verfügen [recht hat er \odot]. Wichtiger ist, daß die Rechnung zugleich die eindeutige Bestimmtheit der Lösung ergibt, über die Hrn. EINSTEINS Behandlung noch Zweifel ließ, und die nach der Art, wie sie sich unten einstellt, wohl auch nur schwer durch ein solches Annäherungsverfahren erwiesen werden könnte. Die folgenden Zeilen führen also dazu, Hrn. EINSTEINS Resultat in vermehrter Reinheit erstrahlen zu lassen.

Einstein was surprised that Schwarzschild succeeded so quickly in deriving an exact solution for his field equations. He writes on 29. December 1915 [203] in a letter to Schwarzschild:

Ihre Rechnung, die den Eindeutigkeitsbeweis für das Problem liefert, ist höchst interessant. Hoffentlich veröffentlichen Sie dieselbe bald! Ich hätte nicht gedacht, dass die strenge Behandlung des Punktproblems so einfach wäre.

Approximately spherically symmetric masses are ubiquitous in our universe: think of planets, stars, and black holes. It is thus a reasonable first step, after setting up the Einstein field equations, to ask what the metric induced by spherically symmetric bodies looks like (in vacuum, outside of the mass itself), and which modifications of the dynamics of test particles moving in such gravitational fields GENERAL RELATIVITY predicts. In this section, we discuss this scenario in detail.

Here we consider only spherically symmetric mass distributions that are *non-rotating* and *uncharged*. In particular the first assumption is often not satisfied by objects in space (celestial bodies typically rotate); taking into account the *angular momentum* of masses leads to a (more complicated) cousin of the Schwarzschild metric, the so called \uparrow *Kerr metric* (which we will not discuss here due to its complexity).



13.1.1. Spherically symmetric spacetimes

1 | Recall: Minkowski metric in spherical coordinates:

$$ds^{2} = c^{2}dt^{2} - \underbrace{(dr^{2} + r^{2}d\Omega^{2})}_{d\bar{x}^{2}} \quad \text{with} \quad d\Omega^{2} := d\theta^{2} + \sin^{2}\theta \, d\varphi^{2}$$
(13.1)

 \triangleleft Most general spherically symmetric metric (= invariant under spatial rotations $\vec{x}' = R\vec{x}$):

$$ds^{2} = A(r,t) dt^{2} - B(r,t) dr^{2} + 2C(r,t) dt dr - \underbrace{D(r,t)}_{=r^{2} (wlog)} d\Omega^{2}$$
(13.2)

A, B, C, D: Undetermined functions

• A metric that is spherically symmetric should allow for coordinates that reflect this symmetry. This means that the metric "looks the same" in all directions, i.e., no coefficient $g_{\mu\nu}$ can depend on θ or φ (above A, B, C, D). Furthermore, the metric should not contain any off-diagonals that mix angles $d\theta$ and $d\varphi$ with either time dt or the radial part dr. For fixed time t and radius r, a spherically symmetric metric must describe, well, a sphere, so that the only allowed length element is $d\Omega^2$, possibly scaled by a constant.

These statement are sloppy. What we really want is a metric that has *three* (linearly independent) *space-like* \leftarrow *Killing vector fields* that satisfy the Lie algebra $\mathfrak{so}(3)$ (the algebra of angular momentum operators in quantum mechanics) – and therefore represent spatial rotations; such spacetimes are called \uparrow *spherically symmetric* because their isometries (generated by Killing vectors) include the rotation group SO(3). One can then show that for such spherically symmetric spacetimes there exist coordinates in which the metric has the form (13.2). This is similar to Section 11.5 were we studied how the existence of a time-like Killing vector restricts the components of the metric in appropriately chosen coordinates.

- Note that we do not assume that the metric is *← stationary* or even *← static*, nor do we restrict its asymptotic behavior.
- That one can always choose coordinates where $D(r, t) = r^2$ is easy to see: Simply define new coordinates $\bar{r} := \sqrt{D(r, t)}$ and $\bar{t} := t$, and use the transformation $d\bar{r} = \partial_r \sqrt{D(r, t)} dr + \partial_t \sqrt{D(r, t)} dt$ and $d\bar{t} = dt$ to rewrite ds^2 . This modifies the prefactors $A \to \bar{A}$, $B \to \bar{B}$ and $C \to \bar{C}$, but does not introduce additional terms beyond $d\bar{t}^2$, $d\bar{r}^2$ and $d\bar{t}d\bar{r}$ in the metric. Finally, rename $\bar{r} \mapsto r, \bar{t} \mapsto t, \bar{A} \mapsto A$ etc.

2 Define new time coordinate $\bar{t} = \bar{t}(t, r)$ and a suitable function $\omega = \omega(t, r)$ such that

$$\mathrm{d}\bar{t} = \omega(\mathrm{A}\mathrm{d}t + C\,\mathrm{d}r) \tag{13.3}$$

That this is always possible is straightforward to see: First, note that the expression Adt + Cdr is not necessarily an \uparrow *exact* differential form, i.e., it is not guaranteed for $\bar{t}(t, r)$ to exist. This is why we need the additional function $\omega(t, r)$. On a suitable domain (it must be \uparrow *contractible*), the \uparrow *Poincaré lemma* tells us that every \uparrow *closed form* is exact. This means that if we can choose $\omega(t, r)$ such that $\omega(Adt + Cdr)$ becomes *closed*, we know that $\bar{t}(t, r)$ exists such that Eq. (13.3) holds.

A differential form is closed if its exterior derivative vanishes:

$$0 \stackrel{!}{=} \mathbf{d}[\omega(A\mathbf{d}t + C\,\mathbf{d}r)] = [\partial_r(\omega A) - \partial_t(\omega C)]\,\mathbf{d}r \wedge \mathbf{d}t\,. \tag{13.4}$$

This condition is equivalent to a first-order partial differential equation for ω ,

$$\partial_r(\omega A) = \partial_t(\omega C) \quad \Leftrightarrow \quad (\partial_r \omega)A + (\partial_r A)\omega = \partial_t(\omega)C + \partial_t(C)\omega, \tag{13.5}$$



for which you need to find only one non-zero solution ω , given the functions A and C. Such a solution is called \uparrow *integrating factor*.

$$A dt^{2} + 2C dt dr \stackrel{\circ}{=} \frac{d\overline{t}^{2}}{A\omega^{2}} - \frac{C^{2} dr^{2}}{A}$$
(13.6)

This trick eliminates the mixed term dt dr (we drop again all bars and rename prefactors):

$$ds^{2} = A(r,t) dt^{2} - B(r,t) dr^{2} - r^{2} d\Omega^{2}$$
(13.7)

3 | Lorentz signature of $ds^2 \to A > 0$ and $B > 0 \to$ Define $A \equiv e^{\nu}c^2$ and $B \equiv e^{\lambda} \to$ Here $\nu = \nu(r, t)$ and $\lambda = \lambda(r, t)$ are undetermined functions:

$$ds^{2} = e^{\nu} d(ct)^{2} - e^{\lambda} dr^{2} - r^{2} \left(d\theta^{2} + \sin^{2} \theta d\varphi^{2} \right) \equiv g_{\mu\nu} dx^{\mu} dx^{\nu}$$
(13.8)

Note that our coordinates are $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi)$.

In these coordinates the only non-zero components of the metric tensor are:

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & & \\ & g_{11} & \\ & & g_{22} & \\ & & & g_{33} \end{pmatrix}_{\mu\nu} = \begin{pmatrix} e^{\nu} & & & \\ & -e^{\lambda} & & \\ & & -r^2 & & \\ & & & -r^2 \sin^2 \theta \end{pmatrix}_{\mu\nu}$$
(13.9)

4 Our final goal is to solve the Einstein field equations for a point mass using the rotation symmetric ansatz Eq. (13.8). To this end, we need the curvature tensor, and therefore the ...

Christoffel symbols:

A straightforward but tedious calculation yields the non-zero components:

Eq. (13.9) $\xrightarrow{\text{Eq. (10.79)}}$

$$\Gamma^{0}_{\ 00} = \frac{\dot{\nu}}{2}, \qquad \Gamma^{0}_{\ 01} = \frac{\nu'}{2}, \qquad \Gamma^{0}_{\ 11} = \frac{\lambda}{2}e^{\lambda-\nu} \qquad (13.10a)$$

$$\Gamma^{1}_{00} = \frac{\nu'}{2} e^{\nu - \lambda}, \quad \Gamma^{1}_{01} = \frac{\lambda}{2}, \qquad \Gamma^{1}_{11} = \frac{\lambda'}{2}$$
(13.10b)
$$\Gamma^{1}_{22} = -r e^{-\lambda}, \quad \Gamma^{1}_{33} = -r e^{-\lambda} \sin^{2} \theta$$

$$\Gamma^{1}_{22} = -re^{-\kappa}, \quad \Gamma^{1}_{33} = -re^{-\kappa}\sin^{2}\theta$$

$$\Gamma^{2}_{12} = \frac{1}{r}, \qquad \Gamma^{2}_{33} = -\sin\theta\cos\theta \qquad (13.10c)$$

$$\Gamma^{3}_{13} = \frac{1}{r}, \qquad \Gamma^{3}_{23} = \cot\theta$$
 (13.10d)

with abbreviations $\dot{\Box} \equiv \frac{\partial \Box}{\partial (ct)} = \frac{\partial \Box}{\partial x^0}$ and $\Box' \equiv \frac{\partial \Box}{\partial r} = \frac{\partial \Box}{\partial x^1}$. We extend this convention to higher derivatives in the obvious way.

All not listed components either vanish or are given by the symmetry of the Christoffel symbols.



5 | Einstein field equations:

Another straightforward (but even more tedious) calculation yields the non-zero components of the Einstein tensor. With these, the Einstein field equation reads:

$$G_0^{\ 0} = e^{-\lambda} \left(\frac{1}{r^2} - \frac{\lambda'}{r} \right) - \frac{1}{r^2} = -\kappa T_0^{\ 0}$$
(13.11a)

$$G_0^{\ 1} = e^{-\lambda} \frac{\lambda}{r} = -\kappa T_0^{\ 1}$$
(13.11b)

$$G_{1}^{-1} = e^{-\kappa} \left(\frac{1}{r^{2}} + \frac{\nu}{r} \right) - \frac{1}{r^{2}} = -\kappa T_{1}^{-1}$$
(13.11c)
$$\left(-\frac{1}{r^{2}} - \frac{\nu}{r} \right)^{2} = \nu^{2} + \nu^{2} +$$

$$G_2^{\ 2} = \begin{cases} \frac{1}{2}e^{-\kappa} \left(\nu'' + \frac{\nu}{2} + \frac{\nu-\kappa}{r} - \frac{\nu}{2} \right) \\ -\frac{1}{4}e^{-\nu} \left(2\ddot{\lambda} + \dot{\lambda}^2 - \dot{\lambda}\dot{\nu} \right) \end{cases} = -\kappa T_2^{\ 2}$$
(13.11d)

$$G_3^{\ 3} = G_2^{\ 2} = -\kappa T_3^{\ 3}$$
 (13.11e)

All other components of the Einstein tensor vanish.

Our rotation symmetric ansatz Eq. (13.9) for the metric of course imposes restrictions on the form of the energy-momentum tensor for which solutions exists. Note that the Einstein tensor contains *second*-order derivatives for it derives from the curvature tensor.

13.1.2. Birkhoff's theorem

6 $| \triangleleft$ Spherically symmetric solutions *in vacuum*: $T_{\mu\nu} = 0$



This means that we are interested in the metric *outside* of spherically symmetric bodies (like planets and stars). As this is exactly were we would like to test GENERAL RELATIVITY (e.g., by following test particles on their geodesics), this simplifications is actually well motivated.

7 | \triangleleft First three equations of Eq. (13.11):

Eq. (13.11a)
$$\Leftrightarrow e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0$$
 (13.12a)

Eq. (13.11b)
$$\Leftrightarrow$$
 $\dot{\lambda} = 0$ (13.12b)

Eq. (13.11c)
$$\Leftrightarrow e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0$$
 (13.12c)

Eq. (13.12b) $\rightarrow \lambda = \lambda(r)$



$$\lambda'(r) + \nu'(r,t) = 0 \quad \Leftrightarrow \quad \nu'(t,r) = -\lambda'(r) \tag{13.13}$$

Integration yields the form

$$v(t,r) = v(r) + f(t)$$
. (13.14)

The fact that v(r) relates to $\lambda(r)$ is not important right now. The crucial point is that the space and time dependency of v(t, r) separated into two summands:

 \triangleleft Coordinate transformation $\bar{t} = \bar{t}(t)$ with $d\bar{t} = e^{f/2} dt$:

$$e^{\nu(t,r)} \mathsf{d}(ct)^2 \stackrel{13.14}{=} e^{\nu(r)} e^{f(t)} \mathsf{d}(ct)^2 = e^{\nu(r)} \, \mathsf{d}(c\bar{t})^2 \tag{13.15}$$

That such a coordinate transformation always exists is easy to see: The integral $\bar{t}(t) := \int_{t_0}^t e^{f(s)/2} ds$ does the job by construction.

Eq. (13.8) $\xrightarrow{\bar{t} \mapsto t}$ Most general spherically symmetric solution in vacuum:

$$ds^{2} = e^{\nu(r)} d(ct)^{2} - e^{\lambda(r)} dr^{2} - r^{2} d\Omega^{2}$$
(13.16)

Our new insight is that $\lambda = \lambda(r)$ and $\nu = \nu(r)$ do not depend on the time coordinate:

 \rightarrow Metric is \leftarrow *static*.

9 | Eq. (13.13) is still valid: [combine Eqs. (13.13) and (13.14)]

$$\lambda'(r) + \nu'(r) = 0 \quad \Rightarrow \quad \lambda(r) + \nu(r) = 0 \tag{13.17}$$

There is of course also an integration constant. But this constant can be absorbed in the term $e^{\nu(r)} d(ct)^2$ by another coordinate transformation (rescaling) of the time coordinate.

10 | Let us once again go back to the Einstein field equations:

Eq. (13.12a)
$$\Leftrightarrow e^{-\lambda(r)} (1 - \lambda' r) = 1$$
 (13.18)

Use substitution $\alpha(r) := e^{-\lambda(r)} \rightarrow$

$$\alpha + \alpha' r = 1 \tag{13.19}$$

 \rightarrow Solution:

$$\alpha = 1 + \frac{a}{r} = e^{-\lambda} \stackrel{13.17}{=} e^{\nu} \quad \text{with integration constant } a. \tag{13.20}$$

 \triangleleft Spatial infinity $r \rightarrow \infty$: [Use $\lim_{r \rightarrow \infty} e^{\lambda} = 1 = \lim_{r \rightarrow \infty} e^{\nu}$.]

$$\lim_{r \to \infty} \mathrm{d}s^2 \stackrel{\mathrm{13.16}}{=} \mathrm{d}(ct)^2 - \mathrm{d}r^2 - r^2 \,\mathrm{d}\Omega^2 = \langle \mathrm{Minkowski \ space} \rangle \tag{13.21}$$

 \rightarrow Metric is *asymptotically flat*.

11 | Check that ...

- Eqs. (13.11a) and (13.17) \rightarrow Eq. (13.11c) solved \checkmark
- Eq. (13.20) $\xrightarrow{\circ}$ Eqs. (13.11d) and (13.11e) solved \checkmark



12 *** *Birkhoff's theorem*:

We can summarize our results as follows:

Every *spherically symmetric* solution of the Einstein field equations *in vacuum* is *static* and *asymptotically flat*.

- The theorem was proven by American mathematician GEORGE DAVID BIRKHOFF in 1923 [204]. However, the same result was obtained already in 1921 by Norwegian physicist JØRG TOFTE JEBSEN [205]. Birkhoff's theorem is therefore a typical example for ↑ *Stigler's law* [206] according to which no scientific discovery is named after its original discoverer.
- If you think about it, this result is quite surprising as we didn't exploit any properties of the energy-momentum tensor that produces the gravitational field except for its spherical symmetry. This means that our result holds also for *time-dependent* distributions of mass/energy as long as the time dependence does not break the spherical symmetry. For example, consider a pulsating (non-rotating) star:



Birkhoff's theorem demands that the metric outside of this star is nonetheless *static* and asymptotically flat. This implies in particular that such a time-dependent object *cannot emit* gravitational waves!

[A similar situation occurs when a dying star explodes in a supernova: If the explosion is spherically symmetric, such an event cannot emit gravitational waves.]

13.1.3. The Schwarzschild metric

13 | The derivation above yields the most general solution of the vacuum EFEs that are spherically symmetric:

Eqs. (13.16) and (13.20) \rightarrow

$$ds^{2} = \left(1 + \frac{a}{r}\right) d(ct)^{2} - \left(1 + \frac{a}{r}\right)^{-1} dr^{2} - r^{2} d\Omega^{2}$$
(13.22)

 \rightarrow The parameter *a* must be determined by the mass of the object that generates this metric.

14 $| \triangleleft$ Correspondence principle:

In the non-relativistic weak-field limit, we must recover the Newtonian gravitational potential:

$$1 + \frac{a}{r} \stackrel{13.22}{=} g_{00} \stackrel{11.65}{\approx} 1 + \frac{2\phi}{c^2} = 1 - \frac{2GM/c^2}{r} \quad \text{with} \quad \phi = -\frac{GM}{r}$$
(13.23)

Here M is the mass of the central spherically symmetric body.

 $\rightarrow a = -\frac{2GM}{c^2} \equiv -r_s$ with the ...

$$r_s = \frac{2GM}{c^2}$$
 ** Schwarzschild radius (13.24)



Thus we finally find the *** Schwarzschild metric

$$ds^{2} = \left(1 - \frac{r_{s}}{r}\right) d(ct)^{2} - \left(1 - \frac{r_{s}}{r}\right)^{-1} dr^{2} - r^{2} d\Omega^{2}$$
(13.25)

expressed in ** Schwarzschild coordinates (ct, r, θ, φ) with $d\Omega^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2$.

15 Comments:

- i! Do not make the mistake to interpret t and r as measurable times and distances, respectively. These are just *coordinates* and one must compute coordinate independent proper times and distances to check if, how, and where they relate to observable quantities (\rightarrow *below*). (Recall the remarks in Section 9.2 about the role played by coordinates in GENERAL RELATIVITY.) Note also that for $r < r_s$ the coordinate t is actually *space-like* whereas r is time-like.
- In Schwarzschild coordinates, the metric Eq. (13.25) has two singularities:
 - 1. On the sphere with $r = r_s$ (\rightarrow event horizon) the prefactor of dr^2 diverges and the prefactor of dt^2 vanishes. You show on \bigcirc Problemset 6 that this singularity is an artifact of the Schwarzschild *coordinates*, and that it can be remedied by choosing better coordinates (e.g. \land *Kruskal–Szekeres coordinates*). The *metric* can then be smoothly extended beyond the horizon without anything fancy happening on the horizon itself.
 - 2. At r = 0 the prefactor of dt^2 blows up and the prefactor of dr^2 vanishes. In contrast to the coordinate singularity at $r = r_s$, the singularity at r = 0 of the interior solution is "physical" in the sense that there coordinate-*independent* quantities (scalars built from the curvature tensor) diverge. However, keep in mind that for "normal" bodies like planets and stars, the Schwarzschild metric is not valid in the interior anyway, so that this singularity has no physical relevance in these scenarios. Only for black holes this singularity is relevant as it heralds the breakdown of GENERAL RELATIVITY.
- Inspection of Eq. (13.25) shows that the ratio r_s/r quantifies the deviations from flat Minkowski space. Because the Schwarzschild solution Eq. (13.25) is only valid *outside* of the mass, relativistic effects become important if one can approach the body to $r \sim r_s$, i.e., the (coordinate) radius *R* of the body must be of the order of the Schwarzschild radius. Conversely, for bodies with $R \gg r_s$ it is necessarily $r \gg r_s$ such that the dominant effect of the Schwarzschild metric is described by Newtonian gravity:



The situation of a \rightarrow *black hole* where $r < r_s$ is possible will be discussed in Section 13.3; in the following we assume $r \ge R > r_s$ so that neither the coordinate singularity at $r = r_s$ nor the physical singularity at r = 0 are relevant.



• i! Do not forget that the Schwarzschild solution Eq. (13.25) is only valid in vacuo, i.e., *outside* the gravitating mass. The metric in the *interior* is different from Eq. (13.25) (in particular: non-singular). This means that if the Schwarzschild radius $r_s < R$ is "buried" in the body, *it has no physical significance*. There is no event horizon close to the center of Earth!

For the exact solution of the interior of a star see ↑ WEINBERG [121] (§11.1, pp. 299–304).

- Because of the coordinate singularity at $r = r_s$, the Schwarzschild solution in Eq. (13.25) actually separates into *two* independent solutions for the EFEs in vacuum: The extended *outer* solution for $r > r_s$ (with time-like coordinate t) and the bounded *inner* solution for $0 < r < r_s$ (with time-like coordinate r). Since the metric is undefined at $r = r_s$, it is a priori unclear whether (and if, how) these two "patches" can be glued together to form a single, contiguous spacetime that solves the vacuum EFEs. That (and how) this is possible can be seen for example in \uparrow *Kruskal-Szekeres coordinates* (\bigcirc Problemset 6).
- If one plugs in the numbers, the Schwarzschild radius of a spherical mass M is roughly

$$r_s \sim 3 \times \left(\frac{M}{M_{\odot}}\right) \,\mathrm{km}$$
 (13.26)

where M_{\odot} is the mass of the Sun. One finds for example:

	<i>r</i> _s [m]	r_s/R
Erde	9×10^{-3}	10 ⁻⁹
Sonne	3×10^3	10 ⁻⁶
White dwarf	3×10^3	3×10^{-4}
Neutron star	3×10^3	0.3

This explains why the Newtonian approximation has been so successful in our Solar System.

It is clear that we should compare r_s to the *coordinate* radius R of the spherical body, i.e., the radial Schwarzschild coordinate r = R where the surface of the body is located (since the terms r_s/r compare r_s to the *coordinate* r). Remembering our warning above (that coordinates cannot directly be identified with physical quantities), you might object that equating R with the *measured* radius of (e.g.) Earth is not justified. This is indeed a valid objection; however, \rightarrow *below* we will see that the coordinate radius has a straightforward physical meaning – which justifies the numbers above (although the interpretation is not the one you might expect).

- According to Birkhoff's theorem, the Schwarzschild metric is the *unique* solution of the vacuum field equations outside of a *spherically symmetric, non-rotating, uncharged mass.* That the mass is non-rotating is important, because a finite angular momentum breaks the rotation symmetry of the problem. That the mass is uncharged is important, because otherwise the electromagnetic field *outside* the mass would be non-zero and our assumption $T_{\mu\nu} = 0$ would be invalid.
- One can loosen these restrictions and solve the EFEs for more general scenarios:

Rotating?	Charged?	Metric	Ref.	Found
×	×	← Schwarzschild	[202]	1916
×	1	↑ Reisser-Nordström	[207-210]	1916
\checkmark	×	↑ Kerr	[211]	1963
✓	1	↑ Kerr-Newman	[212, 213]	1965

Because most celestial bodies rotate, these generalizations (in particular the Kerr metric) are often more useful to describe real phenomena than the Schwarzschild metric (like black



holes). However, for slowly rotating bodies the Schwarzschild metric often provides good approximations to explain a variety of phenomena (\rightarrow Section 13.2; but not always, \rightarrow *Lense-Thirring effect* in \bigcirc Problemset 6).

• In his derivation, Schwarzschild used both *time-independence* and *asymptotic flatness* as independent assumptions [202]. The contribution by Birkhoff and Jebsen was to show that both assumptions are superfluous [204, 205]: That the solution must be static and asymptotically flat is already implied by its rotational symmetry.

16 | Proper time:

Let us now study how the Schwarzschild coordinates relate to measurable proper time:

- $i \mid \triangleleft$ Ideal clock at rest in Schwarzschild coordinates:
 - \rightarrow Proper time:

$$\mathrm{d}\tau \stackrel{11.10}{=} \frac{1}{c} \mathrm{d}s \stackrel{13.25}{=} \sqrt{1 - \frac{r_s}{r}} \mathrm{d}t \tag{13.27}$$

- $\rightarrow \Delta \tau < \Delta t$ for $r_s < r < \infty$
- ii | \triangleleft Asymptotic observer at $r \rightarrow \infty$:

$$\lim_{r \to \infty} \mathrm{d}\tau = \mathrm{d}t \tag{13.28}$$

(13.29)

 \rightarrow We can conclude:

Schwarzschild time t = Proper time of observer at spatial infinity

iii | In summary, the clocks of stationary observers at finite distance to the mass run always *slower* than the clocks at spatial infinity. The closer the clock to the Schwarzschild radius, the slower it ticks. We can illustrate this as follows:



• To draw the null cones in a Schwarzschild rt-diagram, note that $ds^2 \stackrel{!}{=} 0$ implies

$$\frac{\mathrm{d}(ct)}{\mathrm{d}r} = \pm \left(1 - \frac{r_s}{r}\right)^{-1} \tag{13.30}$$

for constant θ and φ . So for $r \to \infty$ the cones open with 90°, as in flat Minkowski space; for $r \to r_s$ the cones close up and become degenerate at the Schwarzschild radius.



17 | Proper distance:

How do the Schwarzschild coordinates relate to proper distances?

- $\mathbf{i} \mid \triangleleft \text{Time slice } t = \text{const} (\mathrm{d}t = 0)$
 - \rightarrow Spatial metric: [For a formal definition see Eq. (11.30) or Eq. (11.27).]

$$dl^{2} = \left(1 - \frac{r_{s}}{r}\right)^{-1} dr^{2} + r^{2} \left(d\theta^{2} + \sin^{2}\theta \, d\varphi^{2}\right)$$
(13.31)

ii | \triangleleft Circumference of a great circle \mathcal{C} of *coordinate* radius r ($\theta = \frac{\pi}{2}$):

$$L[\mathcal{C}] := \underbrace{\int_{\mathcal{C}} dl}_{\substack{\text{Coordinate} \\ \text{independent}}} = \underbrace{r \int_{0}^{2\pi} d\varphi}_{\substack{\text{Coordinate} \\ \text{dependent}}} = 2\pi r$$
(13.32)

Similarly, one finds $A[\mathscr{S}] = 4\pi r^2$ for the *surface* of a sphere \mathscr{S} with coordinate radius r. \rightarrow The coordinate r directly relates to lengths of circles (and areas of spheres). Note that both $L[\mathscr{C}]$ and $A[\mathscr{S}]$ are geometric (= coordinate independent) quantities.

iii | But what about *radial* proper distances?

 \triangleleft Radial segment \mathcal{L} from r_1 to r_2 ($\theta = \text{const}$ and $\varphi = \text{const}$):

$$L[\mathcal{L}] := \int_{\mathcal{L}} \mathrm{d}l = \int_{r_1}^{r_2} \frac{\mathrm{d}r}{\sqrt{1 - \frac{r_s}{r}}} =: \Delta R(r_1, r_2) > r_2 - r_1 \qquad (13.33)$$

Note that we cannot compute distances from the center r = 0 because, first, we would integrate over the coordinate singularity (and start at the singularity at r = 0), and second, for $r < r_s$ the coordinate becomes time-like and the integral actually measures a *time* and not a length! This is why we consider distances between two points with radial coordinates $r_2, r_1 > r_s$.

We conclude:

The radial proper distance is *larger* than the coordinate distance.

(13.34)

iv $| \triangleleft$ Two great circles C_i with radii $r_2 > r_1 \rightarrow$

$$\frac{\delta U}{\delta R} := \frac{L[\mathcal{C}_2] - L[\mathcal{C}_1]}{\Delta R(r_1, r_2)} = \frac{2\pi(r_2 - r_1)}{\Delta R(r_1, r_2)} \stackrel{13.33}{<} \frac{2\pi(r_2 - r_1)}{r_2 - r_1} = 2\pi$$
(13.35)

This means that the circumference varies "less than usual": $\delta U < 2\pi \delta R$.

Compare this to Euclidean geometry:

$$\frac{\delta U}{\delta R} := \frac{2\pi r_2 - 2\pi r_1}{r_2 - r_1} = 2\pi \quad \Rightarrow \quad \delta U = 2\pi \delta R \tag{13.36}$$

Note that the ratio defined in Eq. (13.35) makes use of geometric properties of the space(time) only; i.e., both $L[C_i]$ and Δd are (in principle) measurable quantities that do not depend on coordinates.



 \rightarrow Space is *non-Euclidean*!

The fact that this ratio is *smaller* than 2π tells us that the spatial curvature is *positive*. For example, a two-dimensional sphere has positive curvature and the same feature:



(This is an extreme example where the ratio is zero.)

• | Let us approximate the measure Eq. (13.35) and apply it to the Solar System to get a feeling for how non-Euclidean space actually is in our neighborhood:

We assume $r_s \ll r$ and $r_2 - r_1 \gg r_s$ (which is satisfied for all situations in the Solar System). Eq. (13.33) \rightarrow

$$\Delta R(r_1, r_2) \approx \int_{r_1}^{r_2} \mathrm{d}r \left(1 + \frac{1}{2} \frac{r_s}{r} \right) = r_2 - r_1 + \frac{r_s}{2} \ln \frac{r_2}{r_1}$$
(13.37)

With this and Eq. (13.35) we find:

$$\frac{2\pi(r_2 - r_1)}{\Delta R(r_1, r_2)} \stackrel{\circ}{\approx} 2\pi \left[1 - \underbrace{\frac{1}{2} \left(\frac{r_s}{r_2 - r_1} \right) \ln \frac{r_2}{r_1}}_{\text{Non-Euclid. correction } \epsilon} \right].$$
(13.38)

For example, let $r_1 = 7 \times 10^8$ m be the radius of the Sun and $r_2 = 5.8 \times 10^{10}$ m the semi-major axis of Mercury. With the Schwarzschild radius $r_s = 3 \times 10^3$ m (of the Sun) one finds the non-Euclidean correction $\epsilon \approx 10^{-7}$.

 \rightarrow The deviations from Euclidean geometry in the Solar System are miniscule.

This explains why the Euclidean space used in Newtonian mechanics is such a good approximation to describe the Solar System!

18 Alternative coordinates:

There is a zoo of different coordinate systems adapted to the Schwarzschild metric, all with distinct advantages and disadvantages. Here we introduce one alternative coordinate system to demonstrate that the singularity at $r = r_s$ is an artifact of Schwarzschild coordinates:

For a motivation of the widely used ↑ *Kruskal–Szekeres coordinates*: ⊖ Problemset 6.

i | \triangleleft Coordinate transformation $\bar{r} = \bar{r}(r)$ with

$$r \equiv \left(1 + \frac{r_s}{4\bar{r}}\right)^2 \bar{r} \tag{13.39}$$

and $r \geq r_s \Leftrightarrow \bar{r} \geq r_s/4$.

Eq. (13.25)
$$\stackrel{\circ}{\rightarrow}$$

$$ds^{2} = \left(\frac{1 - \frac{r_{s}}{4\bar{r}}}{1 + \frac{r_{s}}{4\bar{r}}}\right)^{2} d(ct)^{2} - \left(1 + \frac{r_{s}}{4\bar{r}}\right)^{4} \underbrace{\left(d\bar{r}^{2} + \bar{r}^{2}d\Omega^{2}\right)}_{\equiv d\bar{x}^{2} + d\bar{y}^{2} + d\bar{z}^{2}}$$
(13.40)



with ** isotropic coordinates $(c\bar{t}, \bar{x}, \bar{y}, \bar{z})$ or $(c\bar{t}, \bar{r}, \theta, \varphi)$.

Note that the \bar{r} -dependent scaling now affects all spatial coordinates equally; hence *isotropic* coordinates [cf. Eq. (13.25)].

ii | *Important:* No divergence/singularity for $\bar{r} \rightarrow r_s/4$ in Eq. (13.40)!

Although the *divergence* (singularity) at the event horizon is gone, the metric is still *degenerate* at $\bar{r} = r_s/4$ since the component $\bar{g}_{00} = 0$ vanishes [\leftarrow Eq. (3.46)]. The \uparrow *Kruskal–Szekeres* coordinates you study in \bigcirc Problemset 6 do not have this problem and are non-degenerate and non-singular on the event horizon.

iii | \triangleleft Weak field limit $\bar{r} \gg r_s \stackrel{\circ}{\rightarrow}$ (expand linearly in $\frac{r_s}{\bar{r}}$)

$$\mathrm{d}s^2 \approx \left(1 - \frac{r_s}{\bar{r}}\right)\mathrm{d}(ct)^2 - \left(1 + \frac{r_s}{\bar{r}}\right)\left(\mathrm{d}\bar{r}^2 + \bar{r}^2\mathrm{d}\Omega^2\right) \tag{13.41}$$

19 | Cosmological constant:

Retracing the solution in Section 13.1.2 – but now including the cosmological constant in the EFEs – yields the ** Schwarzschild de Sitter metric*

$$ds^{2} = \left(1 - \frac{r_{s}}{r} - \frac{\Lambda r^{2}}{3}\right) d(ct)^{2} - \left(1 - \frac{r_{s}}{r} - \frac{\Lambda r^{2}}{3}\right)^{-1} dr^{2} - r^{2} d\Omega^{2}.$$
 (13.42)

[\uparrow de Sitter space is the maximally symmetric (= 10 Killing vectors) spacetime with constant positive scalar curvature (R > 0); you can think of it as the generalization of spheres in Euclidean space. De Sitter space is the maximally symmetric vacuum solution of the EFEs with positive cosmological constant – analog to Minkowski space for the case of vanishing cosmological constant.]

Due to the additional terms in Eq. (13.42), the asymptotic metric for $r \to \infty$ is no longer flat Minkowski space but positively curved de Sitter space. In the non-relativistic limit, the gravitational potential can be identified via Eq. (11.65) as

$$\phi = -\frac{GM}{r} - \frac{c^2 \Lambda}{6} r^2 \,. \tag{13.43}$$

This is a modification of Newtonian gravity and consistent with our previous result Eq. (12.42).



↓Lecture 28 [09.07.24]

13.2. Tests of GENERAL RELATIVITY in the Solar System

With the Schwarzschild metric at hand, we can finally derive predictions of GENERAL RELATIVITY that can be used to distinguish the theory from its non-relativistic predecessor, Newtonian dynamics. Here we focus on tests and predictions that are applicable to scales within our Solar System. Hence we omit the cosmological constant ($\Lambda = 0$) and can also safely assume $r \gg r_s$, so that the singularities of the Schwarzschild metric can be ignored:



Einstein introduced and studied 1916 in Ref. [21] (§22, pp. 818–822) what are today known as the "*Three classical tests of* GENERAL RELATIVITY". He summarized and popularized them 1919 in an article written for the London Times [214, 215]:

- The perihelion precession of Mercury (\Rightarrow Section 13.2.1)
- The deflection of light by the Sun (\Rightarrow Section 13.2.2)
- The gravitational redshift of light (\Rightarrow Section 13.2.4)

In Einstein's words [214] (p. 209):

Die neue Theorie der Gravitation weicht in prinzipieller Hinsicht von der Theorie Newtons bedeutend ab. Aber ihre praktischen Ergebnisse stimmen mit denen der Newton'schen Theorie so nahe überein, dass es schwer fällt, Unterscheidungs-Kriterien zu finden, die der Erfahrung zugänglich sind. Solche haben sich bis jetzt gefunden

1) in der Drehung der Ellipsen der Planetenbahnen um die Sonne (beim Merkur bestätigt).

2) in der Krümmung der Lichtstrahlen durch die Gravitationsfelder (durch die englischen Sonnenfinsternis-Aufnahmen bestätigt).

3) in einer Verschiebung der Spektrallinien nach dem roten Spektralende hin des von Sternen bedeutender Masse zu uns gesandten Lichtes (bisher nicht bestätigt).

Der Hauptreiz der Theorie liegt in ihrer logischen Geschlossenheit. Wenn eine einzige aus ihr gezogene Konsequenz sich als unzutreffend erweist, muss sie verlassen werden; eine blosse Modifikation erscheint ohne Zerstörung des ganzen Gebäudes unmöglich.



13.2.1. Apsidal precession

The first and most famous application of GENERAL RELATIVITY was and is the explanation of the anomalous apsidal precession of Mercury's orbit:



Problem:

Taking into account all known gravitational perturbations (mostly due to other planets) explains Mercury's apsidal precession up to a deviation of [216]

$$\Delta \varphi_{?} \approx (42.56 \pm 0.94)^{\prime\prime} \text{ per century}$$
(13.44)

which remains mysterious in Newton's theory ©.

Solution: GENERAL RELATIVITY ©

The fact that GENERAL RELATIVITY can be used to compute $\Delta \varphi_{?}$ precisely was a triumph for Einstein, and paved the way for a quick adoption of the theory. Einstein derived $\Delta \varphi_{?}$ in his famous paper "*Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie*" [14], published on 18. November 1915.

If you look sharply at the publication date, you might wonder how Einstein was able to pull off this feat if his foundational paper "Die Feldgleichungen der Gravitation" [13] (in which he published the Einstein field equations) appeared later, namely on 25. November 1915. The reason is that he used the "wrong" equations $R_{\mu\nu} = -\kappa T_{\mu\nu}$ [which he introduced in Ref. [12] on 11. November 1915 (with a correction added on 18. November), see Eq. (16b) on p. 800 (remember that Einstein's notation differs from ours)] to do the Mercury calculation. Because this calculation rests on the *vacuum* field equations only, and $G_{\mu\nu} = 0 \Leftrightarrow R_{\mu\nu} = 0$, these results remained unaffected by his later modification of the field equations. Einstein writes in Ref. [13]:

Die Feldgleichungen für das Vakuum, auf welche ich die Erklärung der Perihelbewegung des Merkur gegründet habe, bleiben von dieser Modifikation [the addition of the term $\frac{1}{2}g_{\mu\nu}T$ in the trace-inverted form Eq. (12.11)] unberührt.

‡ Reminder: The Kepler problem in Newtonian mechanics

Let us first revisit the two-body problem in Newtonian mechanics so that we can compare it to the modifications due to the Schwarzschild geometry later:

1 | System: \triangleleft Test mass *m* in gravitational field of heavy mass $M \gg m$:

We use spherical coordinates (r, θ, φ) on Euclidean space to exploit the rotational symmetry.

Rotational symmetry \rightarrow Conservation of angular momentum \rightarrow *w.l.o.g.* $\theta = \frac{\pi}{2}$



 \rightarrow Lagrangian of test mass:

$$L = \underbrace{\frac{1}{2}m\left(\dot{r}^{2} + r^{2}\dot{\varphi}^{2}\right)}_{\text{Kinetic energy}} + \underbrace{\frac{GmM}{r}}_{\substack{Gravitational energy}}$$
(13.45)

2 | Integration:

Integrating the equations of motion of this system is simplified by exploiting its symmetries:

 $\mathbf{i} \mid \varphi$ cyclic (the Lagrangian does not depend on $\varphi) \rightarrow$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad l := m \underbrace{r^2 \dot{\phi}}_{=:h} = \mathrm{const}$$
(13.46)

 \rightarrow Angular momentum *l* is conserved

ii | Eq. (13.45) translation-symmetric in time $t \rightarrow$

$$E := H = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - \frac{GmM}{r} = \text{const}$$
(13.47)

 \rightarrow Energy *E* is conserved

iii | Use $h = r^2 \dot{\varphi} = \text{const}$ and assume $r = r(\varphi) \rightarrow \dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\varphi} \dot{\varphi}$

The assumption $r = r(\varphi)$ restricts the set of solutions to the ones we are interested in. There are of course also radial solutions with $\varphi = \text{const}$ and r = r(t), but these are not important for our application to describe planets in the Solar System.

Eq. (13.47)
$$\rightarrow E = \frac{1}{2}m\left[\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2\frac{h^2}{r^4} + \frac{h^2}{r^2}\right] - \frac{GmM}{r}$$
 (13.48)

iv | \triangleleft New radial coordinate $u := \frac{1}{r} \rightarrow u' := \frac{\mathrm{d}u}{\mathrm{d}\varphi} = -\frac{1}{r^2} \frac{\mathrm{d}r}{\mathrm{d}\varphi}$

Eq. (13.48)
$$\rightarrow E = \frac{1}{2}mh^2 \left[(u')^2 + u^2 \right] - GmMu$$
 (13.49)

v | Assume $u' \neq 0$ and derive Eq. (13.49) wrt. φ :

Solutions with u' = 0 imply r = const and correspond to circular orbits.

Eq. (13.49)
$$\xrightarrow{\frac{d}{d\varphi}} u'' + u \stackrel{\circ}{=} A$$
 with $A := \frac{GM}{h^2}$ (13.50)

We will find a similar (but modified) equation of this form in the Schwarzschild geometry.

3 Solution:

Adding homogeneous solutions to the particular solution A of Eq. (13.50) yields the general solution:

$$u = \frac{1}{r} = A \left[1 + e \cos(\varphi - \varphi_0) \right]$$
(13.51)

$\rightarrow \downarrow$ Conic sections

For 0 < e < 1 the orbit $r = r(\varphi)$ describes an *ellipse* with perihelion at $\varphi = \varphi_0$ (*w.l.o.g.* $\varphi_0 = 0$) and *eccentricity* e. For e = 0 one obtains the circular solution with radius A^{-1} .

That Eq. (13.51) describes ellipses for 0 < e < 1 with eccentricity *e* is not obvious because this equation is the \checkmark *polar form* of the ellipse equation with φ measured wrt. one of the \checkmark *foci* of the ellipse. (Remember that we put the heavy mass in the origin r = 0 of our coordinate system.)



The Kepler problem in Schwarzschild spacetime

We can now tackle the same problem (that is, the motion of a test mass in the gravitational field of a much heavier body) in GENERAL RELATIVITY by using ...

- ... the Schwarzschild metric produced by the Sun (Section 13.1.3).
- ... that the test mass follows geodesics in this metric (Section 11.2).
- 4 | System:

The geodesic equation follows from the Lagrangian: [\leftarrow Eq. (10.126) ff. in Section 10.3.3]

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \quad \text{with Schwarzschild metric } g_{\mu\nu} \tag{13.52}$$

 \triangleleft Schwarzschild coordinates $(x^0, x^1, x^2, x^3) = (ct, r, \theta, \varphi) \xrightarrow{\text{Eq. (13.25)}}$

$$L = \frac{1}{2} \left[\left(1 - \frac{r_s}{r} \right) c^2 \dot{t}^2 - \left(1 - \frac{r_s}{r} \right)^{-1} \dot{r}^2 - r^2 \left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta \right) \right]$$
(13.53)

with $\dot{\Box} := \frac{d\Box}{d\tau}$ and proper time τ .

- We can parametrize the geodesic with proper time because $m \neq 0$ for the test mass.
- We could also plug the Christoffel symbols Eq. (13.10) [together with Eq. (13.20)] into the geodesic equation Eq. (10.131) and solve it. Here we follow a more pedestrian (and less technical) approach to work out the differences to the Newtonian case above.
- **5** | Equation of motion:
 - i \triangleleft Euler-Lagrange equation for $x^2 = \theta$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad \Leftrightarrow \quad \ddot{\theta} + \frac{2}{r} \dot{r} \dot{\theta} - \dot{\varphi}^2 \sin \theta \cos \theta = 0 \tag{13.54}$$

Solved by $\theta = \frac{\pi}{2} = \text{const}$ (without imposing restrictions on the other coordinates!)

As before, this restricts our solutions to the plane with $\theta = \frac{\pi}{2}$; because of the spherical symmetry of the problem this no actual restriction.

ii | This choice simplifies the Lagrangian:

Eq. (13.53)
$$\xrightarrow{\theta = \frac{\pi}{2}} L_1 = \frac{1}{2} \left[\left(1 - \frac{r_s}{r} \right) c^2 \dot{t}^2 - \left(1 - \frac{r_s}{r} \right)^{-1} \dot{r}^2 - r^2 \dot{\varphi}^2 \right]$$
 (13.55)

The subscript reminds us that this Lagrangian only describes motions in the $\theta = \frac{\pi}{2}$ plane. iii | \triangleleft Euler-Lagrange equation for $x^1 = r \stackrel{\circ}{\rightarrow}$

$$\left(1 - \frac{r_s}{r}\right)^{-1}\ddot{r} + \frac{1}{2}\frac{r_s}{r^2}c^2\dot{r}^2 - \frac{1}{2}\left(1 - \frac{r_s}{r}\right)^{-2}\frac{r_s}{r^2}\dot{r}^2 - r\dot{\varphi}^2 = 0$$
(13.56)

This complicated EOM is not needed because we exploit enough integrals of motion (\rightarrow below).

iv | Cyclic coordinates $x^0 = ct$ and $x^3 = \varphi \rightarrow$ Integrals of motion:

$$\frac{\partial L_1}{\partial (c\dot{t})} = \left(1 - \frac{r_s}{r}\right)c\dot{t} =: k = \text{const} \quad \text{and} \quad \frac{\partial L_1}{\partial \dot{\phi}} = r^2\dot{\phi} =: h = \text{const} \quad (13.57)$$



 $\mathbf{v} \mid \ m \neq 0 \xrightarrow{\text{Eq. (11.51)}} \|\dot{x}\|^2 = g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} = c^2 > 0$

With this we restrict our derivation to *time-like* solutions (as needed for a massive test particle). The fact that $\|\dot{x}\|^2 = \text{const}$ was proven in Eq. (11.3) and is a consequence of $x^{\mu}(\tau)$ describing a geodesic *and* τ being an \leftarrow *affine parameter* [which is true for all solutions of the geodesic equation Eq. (10.131)]. That the constant equals c^2 selects a specific affine parameter, namely the *proper time* τ .

$$g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = c^2 \xrightarrow[\theta=\frac{\pi}{2}]{} \left(1 - \frac{r_s}{r}\right)c^2\dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2 = c^2 \quad (13.58)$$

vi | Eqs. (13.57) and (13.58) $\xrightarrow{\circ}$

$$\frac{1}{r^4} \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2} \left(1 - \frac{r_s}{r}\right) \left(1 + \frac{c^2 r^2}{h^2}\right) - \frac{k^2}{h^2} = 0 \tag{13.59}$$

Here we assumed $\dot{\varphi} \neq 0$ (thereby excluding radial motions) and used the chain rule,

$$\frac{\dot{r}^2}{\dot{\varphi}^2} = \left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 \,,\tag{13.60}$$

once again imposing a restriction to solutions of the form $r = r(\varphi)$.

vii | \triangleleft New radial coordinate $u := \frac{1}{r}$

Eq. (13.59)
$$\xrightarrow{\circ}$$
 $(u')^2 + u^2 = \frac{k^2 - c^2}{h^2} + \frac{c^2 r_s}{h^2} u + r_s u^3$ (13.61)

Here we used again $u' := \frac{\mathrm{d}u}{\mathrm{d}\varphi} = -\frac{1}{r^2} \frac{\mathrm{d}r}{\mathrm{d}\varphi}$.

viii | Assume again $u' \neq 0$ and derive Eq. (13.61) wrt. φ :

Solutions with u' = 0 imply r = const and correspond to circular orbits.

Eq. (13.61)
$$\xrightarrow{\frac{d}{d\varphi}} u'' + u = A \underbrace{+ \frac{3}{2} r_s u^2}_{\text{cf. (13.50)}}$$
 (13.62)

with $A = \frac{GM}{h^2}$ and $r_s = \frac{2GM}{c^2}$.

Note that the new term couples with the length scale r_s of GENERAL RELATIVITY and makes the differential equation *non-linear*.

Both the Newtonian source A and the relativistic correction $\propto r_s$ on the right-hand side include Newtons gravitational constant G. However, only r_s contains the speed of light c, which marks this correction as *relativistic*.

Perihelion precession in GENERAL RELATIVITY

6 | Approximate solution:

Up to this point our derivation is exact. However, the differential equation Eq. (13.62) is no longer linear and hard to solve exactly. Thus we apply some (well justified) approximations to solve it:



$$u'' + \left(1 - \frac{3}{2}\frac{r_s}{r}\right)u = A \tag{13.63}$$

so that the deviations of the Newtonian case Eq. (13.50) are controlled by $\frac{r_s}{r}$.

In the Solar System this ratio is very small:

$$\left(\frac{r_s}{r}\right)_{\text{Mercury}} \sim 7 \times 10^{-8}$$
 (13.64)

Here we used the Schwarzschild radius $r_s=3\times10^3\,{\rm m}$ of the Sun and the perihelion $r\sim4.6\times10^{10}\,{\rm m}$ of Mercury.

 $\rightarrow \triangleleft r_s u^2$ a *perturbation* for the Newtonian solution (13.51),

$$u_0 := A \left[1 + e \cos \varphi \right] \,. \tag{13.65}$$

 \rightarrow First-order perturbation:

$$u'' + u \approx A + \frac{3}{2}r_s A^2 \underbrace{\left[1 + 2e\cos\varphi + e^2\cos^2\varphi\right]}_{u_0^2}$$
(13.66)

Here we inserted the unperturbed solution u_0 into the perturbation of the EOM. Solving this equation yields a first-order correction. One could then reinsert this solution into the equation and repeat the procedure until one converges to a fixed point – and thereby a solution of the non-linear equation. For our purpose, the first-order correction is already sufficient.

ii | The eccentricity of most planets is very small (= their orbits are almost circular). For example: $e_{\text{Mercury}} \approx 0.2 \rightarrow \text{drop } \mathcal{O}(e^2) \text{ terms} \rightarrow$

$$u'' + u - A \approx 3r_s A^2 e \cos\varphi \tag{13.67}$$

Here we also dropped the constant $\frac{3}{2}r_sA^2$ on the right-hand side because it can be absorbed into a (small) shift of the constant A on the left-hand side (which will not affect our conclusions below).

iii | Eq. (13.67) is linear \rightarrow Use unperturbed solution u_0 for ansatz:

$$u \approx u_0 + u_1 \implies \underbrace{\left(u_0'' + u_0 - A\right)}_{\stackrel{13.50}{=} 0} + u_1'' + u_1 = 3r_s A^2 e \cos \varphi$$
 (13.68)

 $\stackrel{\circ}{\rightarrow}$ Particular solution:

$$u_1 = \frac{3}{2} r_s A^2 e \,\varphi \,\sin\varphi \tag{13.69}$$

 \rightarrow Solution:

$$u \approx u_0 + u_1 \stackrel{13.65}{=} A \left[1 + e \cos \varphi + \underbrace{\frac{3}{2} r_s A \varphi}_{=: \Delta(\varphi)} e \sin \varphi \right]$$
(13.70)

oreti



$$\Delta(\varphi) = \frac{3}{2}r_s A\varphi = 3\left(\frac{GM}{ch}\right)^2 \varphi \ll 1$$
(13.71)

Recall that $h = r^2 \dot{\varphi}$ [\leftarrow Eq. (13.57)]. Since the motion of planets is non-relativistic, we can estimate $h \sim (1 \text{ AU})^2 (2\pi/1 \text{ yr}) \sim 10^{15} \text{ m}^2/\text{s}$. Plugging in the other constants and the mass of the Sun yields $\frac{3}{2}r_s A \sim 10^{-7}$, which, together with $\varphi \in [0, 2\pi) \sim 1$, justifies the following approximation:

Eq. (13.70)
$$\xrightarrow{o}$$
 $u \approx A \{1 + e \cos[\varphi - \Delta(\varphi)]\}$ (13.72a)

$$= A \left\{ 1 + e \cos\left[\left(1 - \frac{3}{2} r_s A \right) \varphi \right] \right\}$$
(13.72b)

Here we used $\cos \Delta(\varphi) \approx 1$ and $\sin \Delta(\varphi) \approx \Delta(\varphi)$, together with the trigonometric identity $\cos[\varphi - \Delta(\varphi)] = \cos \varphi \cos \Delta(\varphi) + \sin \varphi \sin \Delta(\varphi)$.

 \rightarrow Rosetta orbit (\leftarrow *Sketch above*)

To see that this equation describes a Rosetta orbit note that $\Delta(\varphi)$ plays the role of an *angle-dependent* phase shift. Hence the object follows ellipses that slowly rotate themselves with φ about the focus point in which the central mass resides.

7 | Perihelion precession:

To evolve from one perihelion to the next, the argument of the cosine in Eq. (13.72b) must advance by 2π (because then the radial distance $u = \frac{1}{r}$ is again the same). \rightarrow

$$(1 - \frac{3}{2}r_s A)\varphi_1 \equiv 2\pi \quad \Leftrightarrow \quad \varphi_1 = \frac{2\pi}{1 - \frac{3}{2}r_s A} \stackrel{\text{Taylor}}{\approx} 2\pi + \underbrace{3\pi r_s A}_{=:\Delta\varphi_1}$$
 (13.73)

 $\Delta \varphi_1$: Angle by which the perihelion advances after one revolution.

Here we used again that $\frac{3}{2}r_s A \ll 1$ is very small.

8 | It is convention to express $\Delta \varphi_1$ in terms of the parameters of the (Newtonian) elliptical orbit (13.51):

$$r = \frac{\ell}{1 + e \cos \varphi} \quad \text{with} \quad \ell := \frac{1}{A} \tag{13.74}$$

with ...

- \downarrow eccentricity e,
- \uparrow semi-latus rectum ℓ ,
- \checkmark perihelion distance $r_{\min} := \frac{\ell}{1+e} = a(1-e)$, and
- ↓ semi-major axis a.

Eq. (13.73) →

$$\Delta \varphi_1 = 3\pi \frac{r_s}{\ell} = \frac{3\pi}{1+e} \frac{r_s}{r_{\min}} = \frac{6\pi GM}{c^2 r_{\min}(1+e)} = \frac{3\pi r_s}{a(1-e^2)}$$
(13.75)

oreti



• If a(b) is the semi-major (semi-minor) axis of an ellipse, and the focal points are at $|c| := \sqrt{a^2 - b^2}$ on the x-axis, the eccentricity is given by $e := \frac{c}{a} = \sqrt{1 - b^2/a^2}$. In our context, the perihelion distance is then $r_{\min} := a - c = a(1 - e)$. The parameter ℓ in the polar representation is given by $\ell := r_{\min}(1 + e) = a(1 - e^2) = \frac{b^2}{a}$ and called \uparrow semi-latus rectum:



• For Mercury we have $r_{\min} \approx 4.6 \times 10^{10}$ m and $e \approx 0.206$, and the Schwarzschild radius of the Sun is $r_s \approx 2952$ m. Plugging in the numbers in Eq. (13.75) yields $\Delta \varphi_1 \approx 0.103''$, which is extremely small and not measurable. However, for each revolution around the Sun this shift accumulates, so that the effect amplifies over time. This is why the perihelion precession is typically measured in angular advance per 100 years. The orbital period of Mercury is $T \approx 0.241$ yr which leads to $N \approx 415$ revolutions per century. The prediction of GENERAL RELATIVITY for the perihelion advance of Mercury is then:

$$(\Delta \varphi)_{\text{Theory}} \approx 43.0'' \text{ per century}$$
 (13.76)

Subtracting all known Newtonian effects (mostly due to planetary perturbations, this contribution is $\sim 532''$ per century, i.e., much larger than the relativistic effect) from the observed precession results in an unexplained difference of [216]

$$(\Delta \varphi)_{\text{Observation}} = (42.56 \pm 0.94)'' \text{ per century}$$
(13.77)

(a more recent analysis can be found in [217]). To comment this result in Einstein's words [14]:

Die Rechnung liefert für den Planeten Merkur ein Vorschreiten des Perihels um 43" in hundert Jahren, während die Astronomen 45" \pm 5" als unerklärten Rest zwischen Beobachtungen und Newtonscher Theorie angeben.

Dies bedeutet volle Übereinstimmung.

- Since the general relativistic perihelion precession scales with $\frac{r_s}{r_{\min}}$ and accumulates with each revolution around the Sun, it is understandable that the effect was first observed for Mercury, the planet with the smallest perihelion distance ($\sim 0.31 \text{ AU}$) and the shortest orbital period ($\sim 0.24 \text{ yr}$). Nowadays one can measure the (considerably smaller) effect also for Venus and Earth.
- Compare this to your results of ⇒ Problemset 1 where you studied relativistic, non-metric, linear theories of gravity:

For the scalar theory you found

$$(\Delta \varphi_1)_{\text{Scalar}} = -\frac{2\pi GM}{c^2 r_{\min}(1+e)}, \qquad (13.78)$$

which has both a wrong sign and a wrong prefactor compared to Eq. (13.75).

For the linear tensor theory you found

$$(\Delta \varphi_1)_{\text{Tensor}} = \frac{8\pi GM}{c^2 r_{\min}(1+e)},$$
(13.79)

which has now the correct sign but still a wrong prefactor compared to Eq. (13.75).

These comparisons explain why we claimed that these theories make wrong predictions.

• Compare Eq. (13.75) with Einstein's result in Ref. [14] [Eq. (13) on p. 838]. Note that Einstein did not (and could not) know about the Schwarzschild solution at the time; he therefore employed approximate techniques to construct an appropriate metric. Since we also made approximations in the same order, the results coincide.

13.2.2. Deflection of light

We now study the second of Einstein's classical tests of GENERAL RELATIVITY: the deflection of light in the gravitational field of heavy bodies.

1 | Light rays follow *null* geodesics [\leftarrow Eq. (11.5)]

We could now plug the Christoffel symbols of the Schwarzschild spacetime Eq. (13.10) [together with Eq. (13.20)] into the geodesic equation Eq. (10.131) and solve it for *light-like/null* trajectories (\leftarrow Eq. (11.5)).

2 | There is a simpler method, though:

We can exploit our results for *massive* test particles in Section 13.2.1 to directly obtain the differential equation that describes null geodesics in the $\theta = \frac{\pi}{2}$ plane. The trick is that the geodesics of these particles must continuously morph into the null geodesics of light rays in the limit $m \to 0$ (for constant momentum).

 \triangleleft Eqs. (13.46) and (13.57): $l = hm = mr^2\dot{\varphi} \rightarrow \text{Orbital angular momentum}$

Light (photons) has momentum $(p = \hbar k)$ but no mass (m = 0).

$$\rightarrow \lim_{m \to 0} l = \lim_{m \to 0} hm \stackrel{!}{>} 0 \rightarrow \lim_{m \to 0} h = \infty \rightarrow \lim_{m \to 0} A = \lim_{m \to 0} \frac{GM}{h^2} = 0$$

With this limit we find:

Eq. (13.62)
$$\xrightarrow{m \to 0} u'' + u = \frac{3}{2}r_s u^2$$
 with $u = \frac{1}{r}$ and $u'' = \frac{d^2 u}{d\varphi^2}$ (13.80)

3 | <u>Solution</u>:

We solve Eq. (13.80) perturbatively along the same lines as in Section 13.2.1:

i | \triangleleft Homogeneous/linear part of Eq. (13.80):

$$u_0'' + u_0 = 0 \quad \Rightarrow \quad u_0 = \frac{1}{r} = \frac{1}{b}\sin(\varphi - \varphi_0)$$
 (13.81)

Here, b and φ_0 parametrize the initial state.

We set *w.l.o.g.* $\varphi_0 = 0$ in the following. (Because of rotation symmetry.)

 $\triangleleft u_0$ in Cartesian coordinates: (Note that $b = r \sin \varphi = \text{const}$ for the solution u_0 .)

$$\vec{x}(\varphi) \equiv \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} r \cos \varphi \\ r \sin \varphi \end{pmatrix} \stackrel{13.81}{=} \begin{pmatrix} b \cot \varphi \\ b \end{pmatrix}$$
(13.82)

 \rightarrow "Straight line" with ** impact parameter b



The solution describes a horizontal line parallel to the x-axis that goes from $x = +\infty$ for $\varphi = 0$ to $x = -\infty$ for $\varphi = \pi$ and passes by the origin (where the Sun would be) at distance b.

i! The trajectory u_0 does *not* solve the geodesic equation Eq. (13.80) of the Schwarzschild spacetime; it is therefore not a "straight line" (= autoparallel curve) in this metric. Only in the special case where $M = 0 \Rightarrow r_s = 0$ (i.e., when the Sun is gone) does u_0 describe the trajectory of light rays correctly. This is consistent as in this situation spacetime is flat and we would expect light to follow straight lines in Cartesian coordinates (which one can choose globally on a flat spacetime).

ii | Perturbative equation:

We plug in the unperturbed solution u_0 for the non-linear perturbation in Eq. (13.80):

$$u'' + u \approx 3 \frac{r_s}{2b^2} \sin^2 \varphi \tag{13.83}$$

 $\stackrel{\circ}{\rightarrow}$ Particular solution:

$$u_1 = \frac{r_s}{2b^2} \left(1 + \cos^2 \varphi \right)$$
(13.84)

iii $| \rightarrow$ First-order solution:

$$u = \frac{1}{r} \approx u_0 + u_1 = \frac{1}{b}\sin\varphi + \frac{r_s}{2b^2} \left(1 + \cos^2\varphi\right)$$
(13.85)

4 | We can again introduce spatial "Cartesian" coordinates: \rightarrow

$$y = r \sin \varphi \stackrel{13.85}{=} b - \frac{r_s}{2b} r \left(2\cos^2 \varphi + \sin^2 \varphi \right) \stackrel{13.82}{=} b - \underbrace{\frac{r_s}{2b} \frac{2x^2 + y^2}{\sqrt{x^2 + y^2}}}_{\text{New! Cf. Eq. (13.82)}}$$
(13.86)

- \rightarrow No longer "straight lines" in "Cartesian" coordinates!
- **5** $| \triangleleft$ Asymptotic behavior for $x \rightarrow \pm \infty$:

$$y \approx b - \frac{r_s}{b} |x| \tag{13.87}$$

Do not forget the absolute value when evaluating $x^2/\sqrt{x^2}$!

This result an be illustrated as follows:





 \rightarrow Deflection angle: (use the small angle approximation tan $\alpha \approx \alpha$)

$$\delta = 2\alpha \stackrel{\alpha \ll 1}{\approx} 2\frac{r_s}{b} = \frac{4GM}{c^2 b} \stackrel{\text{Sun}}{\approx} \frac{1.75''}{b/R_{\odot}}$$
(13.88)

Here R_{\odot} denotes the radius of the Sun.

• i! We established above that for M = 0 we can use Cartesian coordinates to describe the trajectory of the light ray on flat Minkowski space. Once we switch on gravity $(M \neq 0)$, we can of course still use the coordinates (x, y), defined via Schwarzschild coordinates (r, φ) in the usual way. However, we cannot simply assume that they continue to have *metric meaning* (i.e., are *Cartesian*)! (Recall that r has no direct metric meaning in the Schwarzschild geometry either [\in Eq. (13.33)].) Thus the fact that Eq. (13.86) no longer describes a "straight line" in (x, y) coordinate space has no physical interpretation a priori.

Luckily, we usually observe celestial bodies from *far away* (and the light deflected by them reaches them from far away). Since we know that the Schwarzschild metric induced by these objects is asymptotically flat, we can study the light long before and after it entered the gravitational field of these objects. In these regions, spacetime is approximately Minkowskian, and the coordinates (x, y) are approximately Cartesian (they are spatial components of an inertial coordinate system which, as discussed in Section 1.1, carries metric information). Since the angle Eq. (13.88) is defined between two straight lines in this region of space, it has physical meaning and observable effects (\rightarrow *below* and Section 13.2.3).

• The predicted deflection of light that passes close by the Sun ($b \sim R_{\odot} \Rightarrow \delta \approx 1.75''$) was first measured by ARTHUR EDDINGTON and collaborators during their famous expedition to West Africa (Príncipe) and Brazil (Sobral) [218], where they exploited the solar eclipse on 29. May 1919 to observe stars that are visible close to the solar disk only when it is covered by the moon ($\uparrow Eddington experiment$). In their paper, they distinguish three possible outcomes: (1) light is not deflected by gravity, (2) light is deflected by the Newtonian angle $\delta/2 \approx 0.87''$ ($\rightarrow below$), or (3) light is deflected by the angle $\delta \approx 1.75''$ predicted by GENERAL RELATIVITY. They summarize their meticulous analysis as follows (p. 332):

Thus the results of the expeditions to Sobral and Principe can leave little doubt that a deflection of light takes place in the neighbourhood of the Sun and that it is of the amount demanded by EINSTEIN'S generalised theory of relativity, as attributable to the Sun's gravitational field.

This result mad headlines all over the world, contributed to the wide acceptance of GENERAL RELATIVITY, and catapulted Einstein to fame.

For a review of various experimental results (up to 1960) regarding the deflection of light see Ref. [219]. The precision of the Eddington experiment was rather low (and its significance later debated, see Ref. [220] for a review). However, later variations of the experiment that used radio waves instead of light verified the predictions of GENERAL RELATIVITY to very high precision. Ref. [221], for example, reports only a deviation of $\delta_{\text{measured}}/\delta_{\text{predicted}} = 1.007 \pm 0.009$ from the predictions of GENERAL RELATIVITY.

• In the aftermath of establishing SPECIAL RELATIVITY, Einstein studied uniformly accelerated frames of reference and already proposed the equivalence principle, equating uniform acceleration with uniform gravitational fields. This led him 1907 to the prediction that light must be deflected by gravity. He states in Ref. [96] (p. 212):

Es folgt hieraus, daß die Lichtstrahlen, [...], durch das Gravitationsfeld gekrümmt werden; [...]



Later, in 1911, Einstein elaborated on this idea in his paper "Über den Einfluß der Schwerkraft auf die Ausbreitung des Lichtes" [103] and predicted the deflection angle

$$\delta_{\text{Newton}} = \frac{2GM}{c^2 b} = \frac{\delta}{2} \,, \tag{13.89}$$

which is exactly *half* the prediction (13.88) of GENERAL RELATIVITY. He evaluates it for a light ray that skims the Sun and concludes (p. 908):

Ein an der Sonne vorbeigehender Lichtstrahl erlitte demnach eine Ablenkung vom Betrage $4 \cdot 10^{-6} = 0.83$ Bogensekunden.

The result (13.89) can be obtained by postulating that Newtonian gravity also affects light rays because, according to SPECIAL RELATIVITY, photons have a "dynamical mass" $m = E/c^2 = h\nu/c^2$. Due of the universality of free fall, the trajectory of a particle that shoots by the Sun (and is on an unbound trajectory) only depends on its initial velocity and position (and not its mass). It is then reasonable to postulate that the same trajectories are followed by photons with c as initial velocity. This purely Newtonian calculation (with appropriate approximations) yields the deflection angle Eq. (13.89).

In the course of completing GENERAL RELATIVITY in 1915, Einstein realized that the actual deflection angle predicted by GENERAL RELATIVITY is *twice* his original prediction of 1911. He presented his results in a meeting of the Prussian Academy of Science on 25. March 1915 [222] and published the calculation of the correct deflection angle (13.88) in his famous 1916 paper [21] (which sums up the results accumulated during 1915).

Fun fact: While the deflection angle 1.7" is correctly stated in Ref. [21], the corresponding equation (74) on page 822 is actually off by the important factor of 2 due to a printing error; it *should* read $B = \frac{2\alpha}{\Delta} = \frac{\kappa M}{2\pi\Delta}$ with $\kappa = \frac{8\pi K}{c^2}$, where K denotes Newton's gravitational constant, B is the deflection angle, and Δ the impact parameter [223].

The difference between Newtonian and generally relativistic predictions of the deflection angle can be traced back to the curvature of *space* that is missing in the former (Newtonian space is Euclidean) and included in the latter [due to the factor $(1 - r_s/r)^{-1}$ for dr in the Schwarzschild metric (13.25), recall Eq. (13.33) ff]. Note that because of Eqs. (11.64) and (11.65), the prefactor $(1 - r_s/r)$ of dt in (13.25) is responsible for reproducing Newtonian physics; it is then the additional prefactor $(1 - r_s/r)^{-1}$ of dr that is responsible for doubling the deflection angle in the Schwarzschild metric.

13.2.3. Gravitational lensing

A direct consequence of the deflection of light is that large masses can act as "lenses" for distant observers:

6 | Example: Here we consider the most symmetric (and rarest) scenario:

 \triangleleft *Collinear* constellation with ...

- light source S (e.g., a galaxy),
- heavy mass/lens L (e.g., another galaxy),
- observer O (a telescope on Earth).



Axial symmetry \rightarrow Point source appears ring-shaped \rightarrow ** Einstein ring

 \rightarrow Straightforward linearized trigonometry leads to the angular size of the Einstein ring:

$$\theta_E \stackrel{\circ}{=} \sqrt{\frac{4GM}{c^2} \cdot \frac{d_{LS}}{d_L d_S}} \quad ** Einstein angle$$
(13.90)

For details: ↑ CARROLL [4] (§8.6, p. 349 ff.).

• i! Because of the 1/b dependence of the deflection angle (13.88), gravitational lenses do not have a focal point but a focal *line* (along the optical axis, behind the lens). Thus, strictly speaking, gravitational lenses are no lenses:



- If source and/or observer are slightly off-axis, the Einstein ring typically breaks into two copies of the imaged object. If the alignment is almost perfect, the ring can morph into a "horseshoe Einstein ring", as shown in Fig. 13.1 (a). When the lens breaks the rotational symmetry (think of an elongated galaxy), the image can consist of four copies of the same object, called an *↑ Einstein cross* [Fig. 13.1 (b)]. Typical constellations are even less symmetric and produce a warped mess, as shown in Fig. 13.1 (c).
- That massive bodies can act as "gravitational lenses" was discussed by Einstein in 1936 [224]. Since Einstein considered *stars* as lenses, he came to the conclusion that the effect was way too small to be observable:

Therefore, there is no great chance of observing this phenomenon, even if dazzling by the light of the much nearer star B is disregarded.

However, one year later, FRITZ ZWICKY suggested that *galaxies* might be massive enough to cause observable lensing effects [225]. The first gravitational lens (indeed caused by a galaxy) was then observed in 1979 [226] (the \uparrow *Twin Quasar*, a single quasar that appears



twice due to a gravitational lens). The first complete Einstein ring was observed later (in 1997) by the Hubble telescope in the infrared [227].

Nowadays, a plethora of gravitational lenses have been identified (\Rightarrow Fig. 13.1).

7 Observations:

Here a few examples of observed gravitational lenses:



FIGURE 13.1. • **Gravitational lenses: (a)** A horseshoe Einstein ring photographed by Hubble in 2011: "The gravity of a luminous red galaxy in the foreground has gravitationally distorted the light from a much more distant blue galaxy." [228] (b) An \uparrow Einstein cross photographed by Hubble in 2012: "The foreground galaxy's gravity acts as a lens that bends and amplifies the light from a quasar behind *it, producing four images of the distant object.*" [229] (c) A large gravitational lens photographed by the James Webb Space Telescope in 2023: "A galaxy cluster in the foreground has magnified distant galaxies, warping their shapes and creating the bright smears of light spread throughout this image." [230]

8 | Applications:

Nowadays, gravitational lensing is used as a *tool* in astronomy:

For more details: \uparrow Ref. [231].

- ↑ *Microlensing:* Gravitational lensing of background light sources by small, mostly invisible objects (like exoplanets, neutron stars, black holes ...) can be used to detect and study them. These objects are too light to cause observable distortions of the image; however, lensing also changes the apparent *brightness* of the background object and changes in brightness over time can be detected even if the lensing itself cannot be resolved.
- ↑ Weak lensing: The lensing that produces Einstein rings and multiple images of the same object is called ↑ strong lensing (and is quite rare). In most directions of space, there are no observable strong lensing phenomena. By contrast, weak lensing describes the slight and ubiquitous "warping" of background sources by the foreground mass distribution. This warping can be used statistically to gain information about the (often invisible) mass distribution in the foreground [recall the blue ↑ lensing map used to study the ← Bullet cluster in Fig. 12.1 (c)].
- Since strong gravitational lenses can magnify extremely distant objects, it has been hypothesized to use the Sun as a "telescope" [232]. The problem with this proposal is that the nearest point on the half-infinite focal line of the Sun is about 550 AU (astronomical units = Sun-Earth distances) away and this is where a space-borne observatory would have to be in order to use the Sun as a lens. For comparison, Voyager 1 is with only ~ 164 AU the most distant spacecraft we managed to deploy.



13.2.4. Gravitational redshift

Based on the conservation of energy, and the possibility to create and annihilate particles from and into photons (reflecting the equivalence of mass an energy), we already concluded in Section 8.3 that the wavelength of light that escapes a gravitational potential must increase, i.e., the light must be *redshifted*. Here we finally confirm this prediction within the full framework of GENERAL RELATIVITY:

1 | \triangleleft Stationary emitter E at \vec{x}_E and receiver R at \vec{x}_R in a static metric:

Definition of a static metric: \leftarrow Eq. (11.136) in Section 11.5.



The following derivation does not rely on the Schwarzschild metric; we will specialize to this particular metric later.

2 | \triangleleft Light signal emitted by *E* at t_E and received by *R* at t_R :

The light follows a *light-like* trajectory. \rightarrow ds² = 0 $\xrightarrow{\text{Eq. (11.136)}}$

$$ct_R - ct_E = c \int_{t_E}^{t_R} dt = \int_{\mathcal{P}_{ER}} \underbrace{\sqrt{\frac{-g_{ij}}{g_{00}}} dx^i dx^j}_{\text{Independent of } t}$$
(13.91)

 \mathcal{P}_{ER} is the spatial path followed by the light signal from \vec{x}_E to \vec{x}_R .

3 \triangleleft *Second* light signal from *E* to *R*:

Metric static \rightarrow Signal follows the same path $\mathcal{P}_{ER} \xrightarrow{\text{Eq. (13.91)}}$

$$\underbrace{ct_R - ct_E}_{\text{first signal}} = \underbrace{ct'_R - ct'_E}_{\text{second signal}} \quad \Leftrightarrow \quad \Delta t_R := t'_R - t_R = t'_E - t_E =: \Delta t_E \quad (13.92)$$

This means that the *coordinate* time differences between the first and the second signal are the same for both emitter and receiver!

Assume that at the first signal a laser is switched *on*, and on the second signal it is switched *off*. Let there be *n* oscillations of the electromagnetic field emitted by *E* and received by $R \rightarrow$

$$\frac{n}{\Delta t_R} \stackrel{13.92}{=} \frac{n}{\Delta t_E}$$
(13.93)

i! This is the *coordinate* frequency of the light at *R* and *E*, not the measured frequency!



$$\Delta \tau_{E/R} \stackrel{11.136}{=} \sqrt{g_{00}(\vec{x}_{E/R})} \,\Delta t_{E/R} \tag{13.94}$$

Recall that we assume E and R to be stationary in the chosen coordinates $x^{\mu} = (ct, \vec{x})$.

 \rightarrow With this we find for the *measured* frequencies of the emitted and received light:

$$\frac{\nu_R}{\nu_E} = \frac{n/\Delta\tau_R}{n/\Delta\tau_E} \stackrel{13.94}{=} \sqrt{\frac{g_{00}(\vec{x}_E)}{g_{00}(\vec{x}_R)}} \cdot \frac{n/\Delta t_R}{n/\Delta t_E} \stackrel{13.93}{=} \sqrt{\frac{g_{00}(\vec{x}_E)}{g_{00}(\vec{x}_R)}}$$
(13.95)

5 So far we only used that the metric is static; now we specialize to the metric of a spherical mass:
 ⊲ Schwarzschild metric Eq. (13.25) →

$$\frac{\nu_R}{\nu_E} = \sqrt{\frac{1 - r_s/r_E}{1 - r_s/r_R}} \quad \text{or} \quad 1 + z := \frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1 - r_s/r_R}{1 - r_s/r_E}}$$
with ** *Redshift parameter z*
(13.96)

For $r_R > r_E$ it follows $\lambda_R > \lambda_E \Leftrightarrow z > 0 \rightarrow *$ Gravitational redshift

- The gravitational redshift was first experimentally probed and verified by ROBERT POUND and GLEN REBKA in 1960 with their famous ↑ *Pound-Rebka experiment* [104, 105]. The experiment was conducted in a laboratory on Earth and exploited the extremely high spectral resolution provided by the ↑ *Mößbauer effect*.
- The gravitational redshift also affects photons emitted by the Sun and received on Earth. This particular probe of the redshift has been successful as well [233], but is complicated by the motion of the emitting atoms on the Sun (which causes random Doppler shifts).

6 | Approximations:

• In many situations it is $\frac{r_s}{r} \ll 1 \rightarrow$ Newtonian approximation:

$$1 + z \stackrel{\circ}{\approx} 1 + \frac{r_s}{2} \left(\frac{1}{r_E} - \frac{1}{r_R} \right) \tag{13.97}$$

To show this, expand Eq. (13.96) in first order of r_s/r_E and r_s/r_R .

Let $r_R = r_E + \Delta h$ with height difference $\Delta h \ll r_E \rightarrow$

$$1 + z \stackrel{\text{Taylor}}{\approx} 1 + \frac{r_s}{2} \cdot \frac{\Delta h}{r_E^2} = 1 + \underbrace{\frac{GM}{r_E^2}}_{=g} \cdot \frac{\Delta h}{c^2} = 1 + \frac{g\Delta h}{c^2}$$
(13.98)

Here g is the gravitational acceleration at the emitter (and receiver, since $\Delta h \ll r_E$).

 \rightarrow Same result as Eq. (8.14) in Section 8.3 \odot

We therefore confirmed our previous derivation in the Newtonian limit. The fact that light is affected by gravity (and redshifted if it leaves a gravitational potential) is therefore not a consequence of the particular structure of the Einstein field equations (we didn't now about



them in Section 8.3), but follows from the principles of SPECIAL RELATIVITY, together with the **EEP** of GENERAL RELATIVITY.

This explains how Einstein could predict the gravitational redshift in 1907 (p. 209 of Ref. [96]) without knowing about curved spacetime. However, this approach is only applicable to *homogeneous* gravitational fields (which is often justified on Earth). The exact value of the redshift for light that traverses large distances (and thereby probes the non-homogeneity of gravitational fields), and/or comes close to the Schwarzschild radius, can only be computed with the machinery of GENERAL RELATIVITY as applied above (including the EFEs).

In astronomical scenarios, the emitters are often excited atoms close to the surface of a star, so that r_E = R_{*} with R_{*} the radius of the star. Telescopes on Earth are the receivers, so that usually r_R → ∞ is a good approximation. One then finds for the < redshift parameter z:

$$1 + z \stackrel{13.96}{\approx} \left(1 - \frac{r_s}{R_*} \right)^{-\frac{1}{2}}$$
(13.99)

Such redshifts can be measured by spectroscopy since we know the optical transitions of the elements that serve as emitters (e.g., hydrogen). Spectroscopic analysis of the light emitted by a star then reveals the redshift by comparison with the wavelength one would measure for the same elements in a laboratory on Earth.

Beware: The situation is significantly complicated by various other phenomena that can change the wavelength of light. For example, relative motion leads to the \downarrow *Doppler effect*. Furthermore, the metric of our universe is *not* a static Schwarzschild metric but describes an expanding spacetime. This leads to an additional \uparrow *cosmological redshift* that depends on the time the light requires to reach us.

13.2.5. Gravitational time dilation

The gravitational time dilation Eq. (13.94) causes the gravitational redshift discussed in Section 13.2.4. We also covered it in our discussion of the role played by the Schwarzschild time coordinate [\leftarrow Eq. (13.27)]. So all important mathematical results have already been stated before:

7 $| \triangleleft$ Two stationary clocks A/B located at $\vec{x}_i = (r_i, \theta_i, \varphi_i)$ (i = A, B):

The clocks are stationary in Schwarzschild coordinates.

Measure proper time between the same *coordinate* time slices t and $t + \Delta t \xrightarrow{\text{Eq. (13.94)}}$

$$\frac{\Delta \tau_A}{\Delta \tau_B} \stackrel{13.94}{=} \sqrt{\frac{g_{00}(\vec{x}_A)}{g_{00}(\vec{x}_B)}} \stackrel{13.25}{=} \sqrt{\frac{1 - r_s/r_A}{1 - r_s/r_B}}$$
(13.100)

 $r_B > r_A \implies \Delta \tau_B > \Delta \tau_A \rightarrow * Gravitational time dilation$

To understand why and how *B* "sees" *A* tick slower: \rightarrow *below*.

• Like the gravitational redshift, gravitational time dilation is not a probe for the validity of the Einstein field equations (at least if applied in the weak field limit) but of the EEP. That is, the effect can be derived from assuming the validity of SPECIAL RELATIVITY together with the equivalence principle.

This explains how Einstein could predict the gravitational time dilation already in 1907 (p. 208-209 of Ref. [96]).



- Recall that special relativistic time dilation (\leftarrow Section 2.2) is *symmetric* in that two (inertial) observers both measure the clock of the other tick slower. This symmetric effect is somewhat artificial because it relies on the comparison of *different clocks* which makes apparent the relativity of simultaneity. To compare two clocks that travel different paths *twice*, at least one had to be accelerated (in Minkowski space!), recall our discussion of the twin "paradox" in Section 2.4. In this scenario, the effect was no longer symmetric and both observers agreed on their relative time delay. Gravitational time dilation is a generalization of this phenomenon to curved spacetime. It is also asymmetric in that an Earth-bound observer sees the clock of an asymptotically distant observer run *faster*, whereas this observer sees the Earth-bound clock run *slower*.
- The slowdown of time becomes extreme if one approaches the Schwarzschild radius. As already discussed, this is impossible for "normal" objects like planets and stars, which is why the scenario is irrelevant for physics in the solar system. However, if we *could* approach the event horizon of a black hole (we don't have to reach it, being nearby $r \gtrsim r_s$ is enough), the effect of gravitational time dilation can become arbitrarily large.

Fun fact: This effect is one of the main plot points of the 2014 movie INTERSTELLAR. In the movie, the protagonists land on "Miller's planet" – a planet that orbits a supermassive black hole – where *one hour* proper time corresponds to *seven years* proper time at $r \rightarrow \infty$ (e.g., on Earth). If you stay too long (say one day) and fly back to Earth, everyone you knew will be long dead \odot .

8 | To understand how stationary observers at different locations (in Schwarzschild coordinates) "see" clocks tick, consider the following setup:

(This is the sketch from Section 13.1.3, reprinted for your convenience.)



Let A be a stationary clock in the gravitational field at r_A and B a clock at spatial infinity ("far away"): $r_B \rightarrow r_{\infty} = \infty$. Assume A sends the reading of its clock at coordinate times t_A^1 and t_A^2 with radio signals to B. Because the Schwarzschild metric is static, the spacetime trajectories of the two signals are congruent, so that the coordinate time differences Δt between the two signals are the same for A and B [see sketch, mathematically this follows from Eq. (13.92)].

But the two clock readings sent by A differ by

$$\Delta \tau_A = \sqrt{1 - \frac{r_s}{r_A}} \,\Delta t = \sqrt{1 - \frac{r_s}{r_A}} \,\Delta \tau_\infty \ < \ \Delta \tau_\infty \ , \tag{13.101}$$

which is less than the time $\Delta \tau_{\infty}$ elapsed for *B* between the two messages.



 \rightarrow *B* concludes that the clock *A* runs slower!

9 | Weak-field approximation:

 \triangleleft Weak-field limit $\frac{r_s}{r} \ll 1$: Eq. (13.100) $\xrightarrow{\text{Eq. (13.97)}}$

$$\frac{\Delta \tau_A}{\Delta \tau_B} - 1 \approx \frac{r_s}{2} \left(\frac{1}{r_B} - \frac{1}{r_A} \right) \tag{13.102}$$

 \rightarrow Relative tick rate:

$$\frac{\Delta T}{T} \equiv \frac{\Delta \tau_A - \Delta \tau_B}{\Delta \tau_B} \stackrel{13.102}{\approx} \frac{r_s}{2} \left(\frac{1}{r_B} - \frac{1}{r_A}\right) \stackrel{13.98}{\approx} \frac{g\Delta h}{c^2}$$
(13.103)

with gravitational acceleration $g = \frac{GM}{r_{P}^{2}}$.

Here we used $r_A = r_B + \Delta h$ with height difference $\Delta h \ll r_B$ as in Eq. (13.98).

If we use $g = 9.81 \text{ m}^2/\text{s}$ and the height of Mount Everest $\Delta h = 8848 \text{ m}$, we find $\Delta T/T \sim 10^{-12}$ for the relative frequency difference between a clock on sea level and one on the summit. This variation is small but well within the precision of modern atomic clocks (\rightarrow *below*).

10 | Experiments:

- The space-born ↑ *Gravity Probe A* experiment (1976) was one of the first to directly measure the effect of gravitational time dilation [234]. It confirmed the prediction of GENERAL RELATIVITY to high precision.
- For the ← *Hafele-Keating experiment* (1971), the gravitational time dilation due to the difference in height between the airplanes and the ground-based reference clock had to be taken into account to match observation and theory [47, 48]; recall Problemset 5.
- Modern ↑ *optical atomic clocks* are precise enough to directly measure the gravitational time dilation simply by lifting them a few centimeters [235]:



FIGURE 13.2. • **Gravitational time dilation measured by optical clocks:** In 2010, the precision of optical clocks (a modern variety of atomic clocks) reached levels that allowed for the direct verification of the gravitational time dilation by elevating one clock by 33 cm (between measurement numbers 13 and 14 in panel B) [235]. The frequency (tick rate) of the clock clearly increases, as predicted by GENERAL RELATIVITY.

• The recent development of smaller and more robust optical clocks gave birth to the new field of ↑ *relativistic geodesy*, i.e., the mapping of Earth by using optical clocks to measure *heights* by proxy of gravitational time dilation (see Ref. [236] and references therein). Note that gravitational time dilation does not actually measure heights (with respect to whatever)



but the *gravitational potential*. This means that the tick rate of your clock also changes when you are above a geological anomaly with higher/lower average mass density (which is also valuable information):



FIGURE 13.3. • **Relativistic Geodesy:** By now, optical clocks have become small enough so that one can use them *to measure hights* [236]; this establishes the field of \uparrow *relativistic geodesy*: the measurement of differences in the gravitational potential by exploiting generally relativistic effects and our technological ability to measure times extremely precisely.

Famously, both special relativistic (← Section 2.2) and gravitational time dilation are relevant effects that must be taken into account for the ↑ *Global Positioning System (GPS)* to work. The system is based on a fleet of satellites equipped with atomic clocks that broadcast their time (plus additional data) to Earth; these timestamps can be used by Earth-bound receivers to calculate their position relative to (at least) three satellites. Special relativistic time dilation makes the clocks of the satellites run *slower* with respect to stationary clocks on Earth, whereas gravitational time dilation makes them run *faster*. For the orbit of GPS satellites, the gravitational time dilation dominates, so that their clocks run *faster* than Earth-bound clocks. This effect (among others) must be taken into account for the system to function; see Ref. [237] for details.



13.2.6. Shapiro time delay

Besides the three classical tests of GENERAL RELATIVITY (perihelion precession, deflection of light / lensing, gravitational redshift / time dilation), there is is a *fourth* test proposed 1964 by IRWIN SHAPIRO [238]: Light that travels through the gravitational field of a heavy mass takes a bit longer than it would without the mass:

 $1 \mid \langle Radar signal bounced between Earth and satellite (or another planet):$

For a strong effect, the satellite must be in (approximate) \uparrow *superior conjunction* with Sun, such that the signal passes close to the Sun and experiences a strong gravitational field.



Question: How much time elapses on Earth during a round trip of the signal?

To simplify calculations, we make the following (justified) approximations:

- The deflection of the signal is small and can be neglected.
- The radar signal is fast so that we can consider all bodies as stationary.
- The time elapsed on Earth is assumed to be approximately Schwarzschild coordinate time.
- $\mathbf{2} \mid \triangleleft \theta = \frac{\pi}{2} \text{ plane} \xrightarrow{\text{Eq. (13.25)}}$

$$ds^{2} = \left(1 - \frac{r_{s}}{r}\right) d(ct)^{2} - \left(1 - \frac{r_{s}}{r}\right)^{-1} dr^{2} - r^{2} d\varphi^{2} \stackrel{\text{Light}}{=} 0$$
(13.104)

Light/radar ray follows (approximately) straight line $y = r \sin \varphi = b = \text{const} \stackrel{\circ}{\rightarrow}$

$$d(ct)^{2} = \left[\left(1 - \frac{r_{s}}{r} \right)^{-2} + \left(1 - \frac{r_{s}}{r} \right)^{-1} \frac{b^{2}}{(r^{2} - b^{2})} \right] dr^{2}$$
(13.105)

To show this use $0 = db = \sin \varphi \, dr + r \cos \varphi \, d\varphi$ and $\tan^2 \varphi = b^2/(r^2 - b^2)$.

3 | Take root & expand in linear order of $\frac{r_s}{r} \rightarrow \frac{\circ}{r}$

$$d(ct) \approx \frac{dr}{\sqrt{1 - \frac{b^2}{r^2}}} \left(1 + \frac{r_s}{r} - \frac{1}{2} \frac{r_s b^2}{r^3} \right)$$
(13.106)


$$c\tilde{T} = \underbrace{\sum_{\substack{P+x_E\\\text{Euclidean}\\\text{distance}}}^{\equiv cT} + \underbrace{r_s \ln \frac{(r_P + x_P)(r_E + x_E)}{b^2} - \frac{r_s}{2} \left(\frac{x_P}{r_P} + \frac{x_E}{r_E}\right)}_{\text{Additional time delay due to gravity} \to \text{Shapiro delay}}$$
(13.107)

Since *r* is not single-valued on the trajectory from Earth to the satellite, one has to add up the integrals from r_E to r = b (segment x_E) and from r = b to r_P (segment x_P). To derive the result, use $x_i = \sqrt{r_i^2 - b^2}$ for i = E, P.

 $2\tilde{T}$: (Coordinate) time for round trip of signal with Sun ($r_s > 0$)

2T: (Coordinate) time for round trip of signal *without Sun* ($r_s = 0$)

Ignoring the gravitational time dilation on Earth (due to the gravitational field of the Sun), $2\tilde{T}$ is approximately the time measured by a clock on Earth for a round trip of the signal.

5 Let $\Delta T := \tilde{T} - T$ and assume $r_E, r_P \gg b$ so that $x_E \approx r_E$ and $x_P \approx r_P \rightarrow b$

$$\Delta T \approx \frac{r_s}{c} \left(\ln \frac{4x_P x_E}{b^2} - 1 \right) > 0 \qquad \text{** Shapiro time delay} \tag{13.108}$$

 \rightarrow Light travels *slower* in a gravitational field than in flat Minkowski space!

i! This slowdown is "seen" by an observer at infinity (it is a slowdown in the *Schwarzschild coordinate velocity*); in every *local inertial frame* the speed of light remains constant (namely c, \leftarrow Section 11.1).

- The effect is more pronounced for smaller impact parameters *b*; this explains why approximate superior conjunction is needed to measure the effect.
- To be fully correct, one has to translate the *coordinate* time delay Eq. (13.108) via Eq. (13.94) into the *proper* time delay measured by clocks on Earth (using the Schwarzschild metric of the Sun). We omit this correction here because it is irrelevant for understanding the Shapiro effect qualitatively.
- To get a feeling for the magnitude of the effect, let us assume we bounce radar signals off Mercury ($x_P \approx 5.8 \times 10^{10}$ m) and receive them on Earth ($x_E \approx 14.9 \times 10^{10}$ m). The smallest impact parameter possible is the radius of the Sun: $b \approx 6.96 \times 10^8$ m. With the Schwarzschild radius $r_s \approx 3 \times 10^3$ m (of the Sun) one finds a one-way delay of $\Delta T \approx 102 \,\mu$ s, which is of course well within the capabilities of modern clocks.
- The relativistic time delay was proposed by IRWIN SHAPIRO as a fourth test of GENERAL RELATIVITY in Ref. [238]. First results were obtained by reflecting radar signals off Venus and Mercury [239, 240] and confirmed the predictions of GENERAL RELATIVITY. The most precise measurement thus far was obtained by monitoring the radio link of the Cassini spacecraft; these measurements confirmed the predictions of GENERAL RELATIVITY with a relative deviation of only $\sim 2 \times 10^{-5}$ [241].

oreti



↓Lecture 29 [16.07.24]

13.3. Black holes

Black holes are one of the most surprising and fascinating predictions of GENERAL RELATIVITY. The following discussion focuses on the non-rotating, uncharged *Schwarzschild black hole (SBH)* and merely scratches the surface of this vast (and active) area of research.

For more details: ↑ CARROLL [4] (§5.6–§5.8 and §6.1–§6.7) and ↑ MISNER *et al.* [3] (§31–§34).

1 | <u>Preface</u>: \triangleleft Euclidean plane \mathbb{R}^2 with Cartesian coordinates (x, y)Define New coordinates:

$$\eta := x \text{ and } \xi := y - \frac{1}{x}$$
 (13.109)



Consider a trajectory that follows the x-axis (red). In (η, ξ) -coordinates, this trajectory escapes to $+\infty$ and comes back from $-\infty$. The y-axis does not show up in the (η, ξ) at all (which is why the crossing of x- and y-axis cannot be represented). So if, by chance, you end up with (η, ξ) coordinates to describe the red trajectory along the x-axis, you might expect some strange physical behavior at $\eta = x = 0$. But the same trajectory in "good" Cartesian coordinates (x, y) reveals that nothing is happening at all; the divergence is an artifact of the chosen coordinates – it does not correlate to any physical phenomenon of the trajectory.

 \rightarrow Lesson:

"Bad coordinates" can produce weird behavior that is not rooted in physics!

We show now that the singularity of the Schwarzschild metric at the event horizon $r = r_s$ is of this type, i.e., Schwarzschild coordinates are "bad coordinates":

2 | Remember: Schwarzschild metric (13.25) in Schwarzschild coordinates (ct, r, θ, φ) :

$$ds^{2} = \left(1 - \frac{r_{s}}{r}\right) d(ct)^{2} - \left(1 - \frac{r_{s}}{r}\right)^{-1} dr^{2} - r^{2} d\Omega^{2}$$
(13.110)

with singularities at r = 0 and $r = r_s$:





- In this (r, t)-diagram, the Schwarzschild metric is encoded by the texture of local null cones; \leftarrow Sections 11.1 and 13.1.3, in particular Eq. (13.30). In Schwarzschild coordinates (r, t), these close up at $r = r_s$ and open again in the interior $(r < r_s)$, but are rotated there: time flows to the left and ends at the r = 0 singularity.
- Trajectories of light rays (null geodesics) are drawn both outside and inside the black hole
 (● Problemset 7). Light falling towards the black hole seems to shoot off to infinity and
 apparently never reaches the horizon at r = r_s. It is unclear if and how these connect to the
 interior geodesics that terminate at the singularity at r = 0. It should be obvious that this
 "problem" is very similar to the strange behavior of the x-axis in our ← Euclidean example.
- In the exterior, there are both light rays that escape to infinity and rays that approach the horizon (and presumably cross it somehow). By contrast, in the interior all future-directed null geodesics terminate at the singularity. This demonstrates that when you enter a SBH, *nothing* can prevent you from hitting the singularity: Once you traverse the horizon, you enter a region of spacetime *where time inevitably ends*.

Claim: Singularity at $r = r_s$ is due to coordinates (the singularity at r = 0 is not!)

- On → Problemset 6 you showed that a free falling observer reaches (and crosses) the event horizon at r = r_s in *finite* proper time. This already suggests that the divergence of the metric components at r = r_s (and the time-like and light-like geodesics) do not herald a break-down of the metric itself (and thereby local physics), but instead indicates how an infinitely distant observer *sees* the probe fall towards the black hole (which are two very different things).
- To support this claim, we must provide a coordinate transformation [in the spirit of the inverse of Eq. (13.109)] such that the Schwarzschild metric becomes regular on the horizon at $r = r_s$:

3 | *<* <u>Kruskal-Szekeres coordinates</u>:

Define the new coordinates (T, R) via ...

$$T := \begin{cases} \sqrt{\frac{r}{r_s} - 1} e^{\frac{r}{2r_s}} \sinh\left(\frac{ct}{2r_s}\right) & \text{for } r \ge r_s \\ \sqrt{1 - \frac{r}{r_s}} e^{\frac{r}{2r_s}} \cosh\left(\frac{ct}{2r_s}\right) & \text{for } r < r_s \end{cases}$$
(13.111a)

$$R := \begin{cases} \sqrt{\frac{r}{r_s} - 1} e^{\frac{r}{2r_s}} \cosh\left(\frac{ct}{2r_s}\right) & \text{for } r \ge r_s \\ \sqrt{1 - \frac{r}{r_s}} e^{\frac{r}{2r_s}} \sinh\left(\frac{ct}{2r_s}\right) & \text{for } r < r_s \end{cases}$$
(13.111b)

For a motivation of these transformations:
Problemset 7

The angular coordinates θ and φ remain unmodified.

 $\stackrel{\circ}{\rightarrow}$ Schwarzschild metric in Kruskal-Szekeres coordinates: (Details: \bigcirc Problemset 7)

$$ds^{2} = (2r_{s})^{2} \frac{e^{-r/r_{s}}}{r/r_{s}} \left(dT^{2} - dR^{2} \right) - r^{2} d\Omega^{2}$$
(13.112)

where r = r(T, R) is implicitly determined via

$$T^2 - R^2 \stackrel{13.111}{=} \left(1 - \frac{r}{r_s}\right) e^{\frac{r}{r_s}}.$$
 (13.113)

Use $\cosh^2 x - \sinh^2 x = 1$ to show this.

i! Neither the metric coefficients Eq. (13.112) nor the coordinate functions Eq. (13.111) have singularities at $r = r_s$. [Only at the central singularity r = 0 the metric Eq. (13.112) is still singular.] This shows that the singularity at $r = r_s$ is a *coordinate singularity* specific to Schwarzschild coordinates:

 \rightarrow No singular behavior on the event horizon $(r = r_s)!$

- 4 Discussion:
 - $\xrightarrow{\circ}$ All \leftarrow null cones open with $\pm 45^{\circ}$.
 - \rightarrow All light rays are straight lines with slope ± 1 , like in a Minkowski diagram!

Note that null cones in the *RT*-diagram are determined by $d\Omega = 0$ and ds = 0, which implies $dT = \pm dR$

• $\stackrel{\circ}{\rightarrow}$ Metric Eq. (13.112) is defined for

$$|T| < \sqrt{1 + R^2}$$
 with $R \in [-\infty, \infty]$, (13.114)

bounded by singularities at $T = \pm \sqrt{1 + R^2}$.

It is easy to show that

$$T^2 - R^2 \stackrel{\text{13.113}}{=} \left(1 - \frac{r}{r_s}\right) e^{\frac{r}{r_s}} \le 1$$
 (13.115)

since the right side has the global maximum 1 at r = 0 [where Eq. (13.112) is singular].



• Event horizon(s):

$$r = r_s \quad \xrightarrow{13.113} \quad T = \pm R \tag{13.116}$$

- i! The event horizon of the Schwarzschild black hole corresponds to the segment T = R for R > 0. The other three segments do not belong to the original Schwarzschild solution (\rightarrow *below*).
- Since all null cones have slope ±1, this shows that the event horizon is a *null hypersurface*: it is tangential to local null cones and acts as a "causal membrane" through which timeand light-like trajectories can traverse only in one direction.
- If you plug $r = r_s$ into the coordinate transformation Eq. (13.111), you find (T, R) = (0, 0) for *all* Schwarzschild times $-\infty < t < +\infty$. This means that the *complete* "horizon" at $r = r_s$ in an rt-diagram is identified as a *single* event in Kruskal coordinates.

To understand what is going on, you should *start* with Kruskal coordinates (which are the "good" coordinates that faithfully represent the Schwarzschild spacetime). The transition to Schwarzschild coordinates then "stretches" the single event at (T, R) = (0, 0) in time direction to become the vertical line at $r = r_s$ in a Schwarzschild diagram. As a side effect, this transformation "pushes" all events on the event horizon (T = R, R > 0) to $t = +\infty$ in the Schwarzschild diagram. Since trajectories that enter the black hole cross the horizon via such events, these trajectories now vanish to and emerge from $t = +\infty$ in the *rt*-diagram. As no trajectory that originates at $t > -\infty$ can cross the horizon at (T, R) = (0, 0), no trajectory in the *rt*-diagram can cross the vertical line at $r = r_s$.

This is similar to the coordinate singularity introduced by Eq. (13.109) in the Euclidean plane which maps the point (x, y) = (0, 0) (actually the complete y-axis) to infinity, so that all trajectories that cross this point do so "outside" of the $\eta - \xi$ -coordinate plane.

• Let us check in which regions of the RT-plane the coordinate transformation Eq. (13.111) maps the exterior $(r > r_s)$ and interior $(r < r_s)$ of the Schwarzschild solution, respectively:

Exterior:
$$\begin{cases} r_s < r < +\infty \\ -\infty < t < +\infty \end{cases} \xrightarrow{13.111} \begin{cases} 0 < R < +\infty \\ -R < T < R \end{cases}$$
(13.117a)

Interior:
$$\begin{cases} 0 < r < r_s \\ -\infty < t < +\infty \end{cases} \xrightarrow{13.111} \begin{cases} -\infty < R < +\infty \\ |R| < T < \sqrt{1 + R^2} \end{cases}$$
(13.117b)

→ Only half of the *RT*-plane (→ Region I and II) corresponds to the Schwarzschild solution!
Draw lines of constant Schwarzschild coordinates *r* and *t* into the *RT*-plane:

$$r = \text{const} \xrightarrow{13.113} T = \pm \sqrt{\text{const} + R^2} \rightarrow \text{Hyperbolas}$$
 (13.118a)
 $t = \text{const} \xrightarrow{13.111} T = \text{const} \times R \rightarrow \text{Straight lines}$ (13.118b)

$$t = \text{const} \xrightarrow{\text{result}} T = \text{const} \times R \longrightarrow \text{Straight lines} (13.1 (through origin))$$

In particular:

$$T = \pm \sqrt{1 + R^2} \Leftrightarrow \underbrace{r = 0}_{\text{Singularity}} \text{ and } T = \pm R \Leftrightarrow \underbrace{r = r_s}_{\text{Horizon}}$$
 (13.119a)
 $T = +R \Leftrightarrow t = +\infty \text{ and } T = -R \Leftrightarrow t = -\infty$ (13.119b)



All of this is straightforward to see if one uses Eq. (13.111) to show that

$$\frac{T}{R} = \begin{cases} \tanh\left(\frac{ct}{2r_s}\right) & \text{for } r \ge r_s \\ \coth\left(\frac{ct}{2r_s}\right) & \text{for } r < r_s \end{cases}$$
(13.120)

and remembers that $\lim_{x \to \pm \infty} \tanh(x) = \pm 1$ and $\lim_{x \to \pm \infty} \coth(x) = \pm 1$.

- Since the metric is regular everywhere except for the singularity at $T = \pm \sqrt{1 + R^2}$ (which corresponds to the physical singularity at r = 0) both time- and light-like trajectories can now cross continuously the event horizon at R = T (R > 0). The sketch below demonstrates that trajectories that do so necessarily terminate at the singularity, whereas trajectories that avoid the horizon can escape this fate.
- $\rightarrow * Kruskal diagram:$



Notes:

- For now, focus only on the (uncolored) half plane lower-bounded by the diagonal T = -R (Region I and II). These two regions correspond to the exterior and interior solution of the Schwarzschild metric that are separated by the coordinate singularity at r = r_s in Schwarzschild coordinates. To two additional colored Regions III and IV are briefly discussed → *below*. [Note that the coordinate transformation Eq. (13.111) does not map events that can be labeled by the original Schwarzschild coordinates into these regions.]
- Do not forget that the radial coordinates θ and φ still exist! This means that every point in the Kruskal diagram corresponds to a "sphere of events". For example, the line R = T(R > 0) describes the evolution of the *spherical* event horizon of the SBH through time. This means that a light signal emitted *on* the horizon with geodesics *along* the diagonal R = Tcorresponds to a photon that is emitted radially outward on some point of the black hole horizon (determined by the angles θ and φ that are not shown). The Kruskal diagram shows that such a photon remains trapped on the horizon forever. Figuratively speaking: On the



event horizon space "flows" towards the singularity "with the speed of light", such that only radially "fleeing" light can "stand still" and evade being sucked into the singularity.

• Here is how to "see" the transformation from the Kruskal diagram to the Schwarzschild diagram above:

First, delete the colored Regions III and IV. Next, fold up the T = -R diagonal about the origin (T, R) = (0, 0), merging the two semi-infinite segments for R < 0 and R > 0 (these segments correspond to $t = -\infty$). This transformation straightens the hyperbolic r = const lines (all vertical, if you rotate the whole diagram appropriately by 45°). But the time lines t = const are still star-like, with all lines emanating from the origin (R, T) = (0, 0). Thus, in the last step, "stretch" this point to an infinite line along the T = R axis, thereby pushing the semi-infinite segments to $t = \pm\infty$. This transformation makes the t = const lines parallel and perpendicular to the r = const lines. The result of these transformations is the Schwarzschild diagram $\leftarrow above$. Due to the "stretching", all trajectories that cross the event horizon are pushed to $t = +\infty$ and seem to leave the rt-plane. This is the coordinate singularity in Schwarzschild coordinates.

5 | Maximally extended Schwarzschild solution:

Observation: Metric Eq. (13.112) solves EFEs also for T < -R.

So by introducing Kruskal coordinates, we not only got rid of the singular behavior at the event horizon; we also found two new "patches" of spacetime that solve the EFEs and can be "glued" to the Schwarzschild spacetime:

\rightarrow Spacetime *extended* by Region III and IV:

Region III:
 White hole (= time-reversed black hole)

In contrast to black holes, signals can only *leave* the white hole via its event horizon, but no signal can ever *enter*. As the Kruskal diagram shows, mathematically, white holes are the time-reversed solutions of black holes.

Region IV: "Mirror universe"

(= another asymptotically flat region; cannot be reached from Region I)

This region is another asymptotically flat region of spacetime that looks like the exterior Region I of the SBH that we occupy. Note that it is impossible to exchange signals between Region I and Region IV, neither via the black hole nor via the white hole (though an observer *within* the SBH could receive signals from both universes). The two regions are connected by a \uparrow *wormhole*, a so called \uparrow *Einstein-Rosen bridge* [242] that, unfortunately, closes too quickly to be traversable by time-like observers; \uparrow CARROLL [4] (§5.7, pp.227–228).

[The Einstein-Rosen bridge is nothing magical: Consider the Kruskal diagram above and remember that it continues for $R \to \pm \infty$ and $T \to \pm \infty$. Now "zoom out"; the spacetime region depicted above becomes a small "throat" that forms a bridge between the large, asymptotically flat Region I on the right and Region IV on the left. This is the Einstein-Rosen bridge. That it cannot be traversed follows directly from the Kruskal diagram. This non-traversability can be traced back to the short lifetime of the wormhole: it opens at T = -1 and closes shortly after at T = +1.]

Notes:

• i! Neither Region III nor IV exist for solutions of the EFEs that describe black holes that form *dynamically* (as it happens in our universe via gravitational collapse). By contrast, the Schwarzschild solution describes a highly artificial scenario: a black hole in an empty universe that exists forever (for an asymptotic observer). There is currently no evidence for white



holes or the mirror universe; there are also no physical processes known that could lead to the formation of a white hole.

- It is impossible to represent all four regions I-IV at the same time in Schwarzschild coordinates. However, since Regions III and IV are mirrored versions of Regions II and I, one can map them to *another* set of Schwarzschild coordinates that describe these new regions of spacetime (again with a coordinate singularity).
- The difference between our original Schwarzschild spacetime (Region I and II) and the new one (Regions I-IV) is that the latter is ↑ *geodesically complete*, whereas the former is not:

A spacetime is *geodesically complete* if you can shoot off geodesics from any point in any direction (of spacetime), and these geodesics either go on forever (i.e., they are defined for all \leftarrow affine parameters $\lambda \in \mathbb{R}$), or terminate for finite λ at a singularity of the metric. Inspection of the Kruskal diagram shows that the union of Regions I-IV indeed has this property. However, removing a single one of the four regions immediately creates a (non-singular) boundary that can be reached by some geodesics for *finite* λ . This means in particular that the original Schwarzschild solution is *not* geodesically complete (due of the non-singular boundary T = -R). This explains why the Kruskal solution (comprised of Regions I-IV) is called *maximally extended Schwarzschild solution*.

You may now wonder whether this means that we can somehow reach "the end of spacetime" if Regions III and IV were absent (which they most likely are in reality). The answer is *no*, as the Kruskal diagram shows: To reach the boundary T = -R, you would have to follow a time-like trajectory (not necessarily a geodesic) *into the past*; no future-directed time-like trajectory can hit this boundary! This also answers how signals emitted from the *white* hole would reveal themselves: For an asymptotic observer, these are signals that appeared on the event horizon in the distant past ($t = -\infty$) and slowly make their way up towards asymptotic Observer when a light source falls into the black hole (where it "freezes" for $t \to +\infty$).

6 | Penrose diagrams:

Goal: Represent causal structure of complete spacetime in finite domain.

Idea: Coordinate transformations with *arctangent* can be used to "make infinities finite":

arctan :
$$\mathbb{R} \to \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$$
 with $\lim_{x \to \pm \infty} \arctan(x) = \pm \frac{\pi}{2}$ (13.121)

Since coordinates obtained in such a way are defined on the compact interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, such transformations yield \uparrow *compactifications* of spacetimes (they include "infinities" as *points*).

 \rightarrow Solution (sketch):

$$\underbrace{\overbrace{(T,R)}^{\text{Time/Space}}}_{\text{Kruskal}} \xrightarrow[Vertarrow by 45^{\circ}]{\text{Null/Null}} \xrightarrow[Vertarrow by 45^{\circ}]{\text{Compactify}} \xrightarrow[Vertarrow by -45^{\circ}]{\text{Null/Null}} \xrightarrow[Vertarrow by -45^{\circ}]{\text{Time/Space}} \underbrace{\overbrace{(\tilde{T},\tilde{R})}^{\text{Time/Space}}}_{\text{Penrose}} (13.122)$$

The additional rotations to and from null coordinates are needed to ensure that all null cones keep their constant 90° opening angle after the transformation (a convenient feature of Kruskal coordinates that we want to inherit).





Notes:

- ** Conformal infinity: (= boundary of conformally compactified spacetime)
 - $i^{\pm} \equiv$ future/past time-like infinity This is where time-like trajectories come from $(t = -\infty)$ and go to $(t = +\infty)$.
 - $i^0 \equiv$ spatial infinity This is where space-like trajectories go to for $r \to \infty$.
 - $\mathcal{J}^{\pm} \equiv \text{future/past null infinity}$

All null geodesics headed into the asymptotically flat region of spacetime end at \mathcal{J}^+ . Conversely, all null geodesics that come from this region start at \mathcal{J}^- .

i! These symbols label the *type* of a boundary (point) of a conformally compactified spacetime region. The fact that both Region I and IV have the same labels on their boundary does *not* mean that these are the same points (= are topologically identified).

- i! The points i^{\pm} of future/past time-like infinity are distinct from the adjacent singularities. This means that future-directed, time-like trajectories in Region I can hit i^+ without hitting the singularity.
- The transformation from Kruskal to Penrose coordinates produces a metric that is related to the Minkowski metric by a *conformal* transformation (at least in *RT*-slices). Penrose diagrams are therefore also called *conformal diagrams*. You can think of conformal transformations as a local, angle-preserving "stretching and squeezing" of the metric: g̃_{µν}(x) = Ω(x)g_{µν}(x). This explains why the null cone texture look like that of Minkowski space. But of course the maximally extended Schwarzschild metric is *not* the Minkowski metric, it only has a comparable *causality structure* (but different geodesics). The light cone field does not convey information of how fast clocks tick locally, since this information is hidden in the scaling Ω(x) of the metric (and is lost if one considers conformal equivalence classes; ← Section 11.1).
- The use of conformal compactifications to describe asymptotically flat solutions of the Einstein field equations was spearheaded independently by PENROSE [243] and CARTER [244]. Diagrams of this type are therefore also called *Penrose-Carter diagrams*.



7 | Horizon \rightarrow Interior \rightarrow Singularity:

Now that we smoothly connected the exterior to the interior of a SBH, let us comment on a few important points:

• Question: What happens the moment one crosses the horizon?

Answer: The **EEP** asserts that spacetime is locally flat (= Minkowskian). We just showed that the metric is regular on the horizon, so the **EEP** is valid there, too. This means that if you are in a spaceship (and the black hole is large enough, so that tidal forces can be neglected on the horizon), there is no experiment that you can perform inside your ship to detect when you cross the horizon. In particular, you continue to *see* the universe outside of the black hole (\rightarrow Simulation below).

Regarding the (non-)observability of event horizons: ↑ Ref. [245].

• Question: What is the spacetime inside the black hole like?

Answer: The spacetime inside a SBH is an infinitely long 3D cylinder $\mathbb{R} \times S^2$ that gets longer and thinner with time. After a finite time, it shrinks & stretches to a line; this is the singularity. This means in particular that you should not think of the singularity at r = 0 as a *point* in *space*; it is rather a one-dimensional, space-like manifold of *events* in *time* (these events demarcate quite literally the *end* of time). Since you cannot stop time from passing, you cannot avoid the singularity from happening.

To understand the interior geometry, start from the Schwarzschild metric Eq. (13.25) and note that for $r < r_s$ the coordinate $r \in (0, r_s)$ is *time*-like, whereas the coordinate $ct \in (-\infty, \infty)$ is *space*-like. To make the metric look more natural, let us rename $r \mapsto ct$ and $ct \mapsto \rho$; the interior of a SBH is then described by the metric

$$ds^{2} = \left(1 - \frac{r_{s}}{ct}\right) d\rho^{2} - \left(1 - \frac{r_{s}}{ct}\right)^{-1} d(ct)^{2} - (ct)^{2} d\Omega^{2}, \qquad (13.123)$$

where time flows from $ct = r_s$ to ct = 0 (where the singularity happens). Note that this is not a static (or stationary) metric! Spacetime *itself* shrinks to the singularity, which explains why neither light nor material objects can avoid it.

A fixed time-slice t = const is then described by the spatial metric

$$dl^{2} = \left(\frac{r_{s}}{ct} - 1\right) d\rho^{2} + (ct)^{2} d\Omega^{2}, \qquad (13.124)$$

with $0 < ct < r_s$ and $-\infty < \rho < +\infty$. This metric describes an infinitely long 3D cylinder $\mathbb{R} \times S^2$ of radius ct. With time running from r_s to 0, the radius of the cylinder shrinks to zero while being stretched along the ρ direction (which is experienced as \uparrow *Spaghettification* by poor observers in this spacetime). For $ct \to 0$, time ends and space becomes an infinite one-dimensional line; this is the singularity.

• *Question:* Does the physical singularity really exist?

Almost certainly not (this is a majority view among physicists). The singularity is a mathematical artifact that indicates a breakdown of GENERAL RELATIVITY. It is expected that *before* one reaches the singularity (when the length scale associated to the curvature is of the order of the \checkmark *Planck length* $l_p \sim 1.6 \times 10^{-35}$ m), quantum effects become important and classical GENERAL RELATIVITY no longer describes spacetime correctly (\rightarrow quantum gravity in Part III).



8 Characteristics of a Schwarzschild black hole:

The following concepts are important to characterize the vicinity of a Schwarzschild black hole:

• (Event) Horizon at $r_s = \frac{2MG}{c^2}$:

The event horizon at r_s is a "null hypersurface" in spacetime (= its normal 4-vector is null everywhere, \leftarrow *above*) and therefore serves as a "causal membrane": time-like (and null) signals can only traverse the hypersurface in one direction. But this is not what makes it special (light cones have this property too, \leftarrow Section 11.1). What makes the event horizon special is that all time-like (and null) trajectories that cross it *eventually terminate at the singularity*. Thus the horizon separates the static exterior Schwarzschild solution from the non-stationary interior solution that ends at the singularity.

There are some deep conceptual (and operational) subtleties concerning event horizons that we sweep under the rug: \uparrow Ref. [245].

• Photon sphere at $r_{\rm ph} = \frac{3}{2}r_s$:

This is the sphere, traced out by circular null geodesics, on which light can orbit the black hole. When you hover at $r_{\rm ph}$ over a SBH, you can see the back of your head! These photon orbits are unstable: If light hits the photon sphere inclining inwards, it necessarily crosses the event horizon later. Conversely, light traversing the photon sphere outwards escapes and is projected (and magnified) onto the \rightarrow *shadow* (in some direction). Below the photon radius $r_{\rm ph}$, there are no circular geodesics (neither time-like nor null, \rightarrow *below*), so that everything that dips beneath $r_{\rm ph}$ and falls freely (!) eventually ends up in the black hole. (With a powerful enough rocket you can still escape from this region, though.)

The radius of the photon sphere can be easily derived from Eq. (13.80) which describes null geodesics in the $\theta = \frac{\pi}{2}$ plane. Setting $u = r_{\rm ph}^{-1} = \text{const}$ to find possible circular photon orbits yields the unique photon radius

$$r_{\rm ph}^{-1} = \frac{3}{2} r_s r_{\rm ph}^{-2} \quad \Rightarrow \quad r_{\rm ph} = \frac{3}{2} r_s \,.$$
 (13.125)

• <u>Shadow</u> with impact parameter $r_{\rm sh} = \frac{\sqrt{27}}{2} r_s \approx 2.6 r_s$:

The shadow is a disk of apparent radius $r_{\rm sh}$ that appears black if the SBH is isotropically illuminated from all sides. It is an image of both the photon sphere and the event horizon. The center of the shadow contains the near side of the event horizon, followed by ring of the *backside*, followed by another (thinner) ring of the near side again, ... add infinitum. The shadow is therefore not only significantly larger than the event horizon, it contains infinitely many copies of images of the *complete* event horizon.

Note that the shadow does not need to be black: If you throw a light source towards the SBH, some of its light can escape (increasingly redshifted) – even from within the photon sphere – until the source reaches the event horizon. This light will show up in the shadow, where you can track (multiple copies) of the light source fading away (asymptotically being "frozen" due to gravitational time dilation). The shadow typically appears black because there is not much stuff below the $\rightarrow ISCO$ that can serve as a light source emitting into the shadow.

To calculate the asymptotic shadow radius, we must do a bit of hand waving: $r_{\rm sh}$ is the asymptotic impact parameter of parallel light rays that separate *scattered* rays from *absorbed* rays. It is evident that such a critical parameter must exist, because rays that pass the black hole far away are not absorbed, whereas rays that hit the event horizon head on clearly are. The rays that separate these two cases must be the ones that "touch" the photon sphere



tangentially (because rays that intersect the photon sphere inevitably hit the horizon, and rays that do not touch the photon sphere at all cannot be absorbed and must be scattered).

To solve the problem, we start from Eq. (13.61) (valid for massive particles) and take again the limit $m \to 0$. As argued in Section 13.2.2, this implies $h \to \infty$; but we must be careful: The quotient

$$\frac{k}{h} \stackrel{\textbf{13.57}}{=} \left(1 - \frac{r_s}{r}\right) \frac{c\dot{t}}{r^2\dot{\varphi}} = \left(1 - \frac{r_s}{r}\right) \frac{c}{r^2} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^{-1} =: \beta < \infty$$
(13.126)

remains clearly finite, even for light-like geodesics. The analog of Eq. (13.61) for such geodesics reads then

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \beta^2 + r_s u^3 - u^2 \tag{13.127}$$

for some constant of motion β . A geodesic that touches the photon sphere at $r = r_{\text{ph}} = \frac{3}{2}r_s$ must therefore satisfy

$$0 \stackrel{!}{=} \left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \beta^2 + \frac{r_s}{r_{\mathrm{ph}}^3} - \frac{1}{r_{\mathrm{ph}}^2} \quad \Rightarrow \quad \beta \stackrel{\circ}{=} \frac{2}{\sqrt{27}} r_s^{-1} \tag{13.128}$$

for some angle φ (remember that u = 1/r and a trajectory is tangential to a circle if its radius does not change in first order of the angle). Now we must only relate the constant of motion β to the impact parameter of the ray. Asymptotically $(r \to \infty \Rightarrow u \to 0)$, the ray satisfies

Eq. (13.127)
$$\xrightarrow{r \to \infty} \left| \frac{\mathrm{d}u}{\mathrm{d}\varphi} \right| = \beta$$
. (13.129)

In the same limit, the angle is given by $\varphi \approx \sin \varphi = \frac{r_{\rm sh}}{r} = r_{\rm sh}u$, where $r_{\rm sh}$ is the impact parameter of the incident ray; this relation implies

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{1}{r_{\mathrm{sh}}} \,. \tag{13.130}$$

Comparing Eqs. (13.129) and (13.130) yields the apparent shadow radius $r_{\rm sh} = \sqrt{27}/2 r_s$.

Innermost stable circular orbit (ISCO) at $r_{isco} = 3r_s$:

In Newtonian mechanics, the Kepler problem allows for *stable* orbits at any distance from the central mass. (If you poke a mass on such an orbit slightly, its orbit might get deformed to an ellipse, but it remains *bounded*.) Not so in the Schwarzschild metric: Solving the geodesic equation for massive particles yields only stable circular orbits down to a minimal radius, the so called *** innermost stable circular orbit (ISCO)*. Matter that orbits below this radius is eventually absorbed by the black hole or knocked out of this region to higher orbits (or to infinity). This means that an *accretion disk* of hot matter can only be stable down to the ISCO, so that r_{isco} marks the inner edge of the accretion disk (if there is one).

To calculate the ISCO, we start from Eq. (13.62) and plug in a constant radius r = R = const:

$$\frac{1}{R} = \frac{GM}{h^2} + \frac{3}{2}r_s\frac{1}{R^2}.$$
(13.131)

This yields for circular geodesics the possible radii

$$R \stackrel{\circ}{=} \frac{h^2 \pm \sqrt{h^4 - 6GMr_sh^2}}{2GM} \,. \tag{13.132}$$



Remember [Eq. (13.57)] that h is the angular momentum (per unit mass) of the orbiting test particle. In the limit of large angular momenta $(h \to \infty)$, we find two possible orbits for each angular momentum:

$$R \stackrel{\text{Taylor}}{\approx} \frac{h^2 \pm h^2 (1 - 3GM r_s / h^2)}{2GM} \approx \begin{cases} \frac{h^2}{GM} & (+) \\ \frac{3}{2} r_s & (-) \end{cases}.$$
 (13.133)

The first solution $R = \frac{h^2}{GM}$ is the Kepler orbit of Newtonian mechanics (which we know is stable). The second solutions approach the photon sphere and do not have a counterpart in Newtonian mechanics; one can show that these circular orbits are *unstable* [\uparrow CARROLL [4] (§5.4, pp. 205–212)]. The smallest stable (pseudo-Newtonian) orbit follows then from Eq. (13.132) when the two solutions coincide, i.e.,

$$h_{\rm isco}^4 - 6GMr_s h_{\rm isco}^2 \stackrel{!}{=} 0 \quad \Rightarrow \quad h_{\rm isco}^2 = 3r_s^2 c^2 \,, \tag{13.134}$$

for which Eq. (13.132) yields the ISCO radius $R = r_{isco} = 3r_s$.

Note that these values are specific to non-rotating Schwarzschild black holes and are generally different (and more complicated) for more realistic (= rotating) \uparrow *Kerr black holes*.

 \rightarrow Graphical summary:





\rightarrow Front view: (from slightly above the accretion disk [eye in sketch above])

The raytraced renderings are taken from https://rantonels.github.io/starless. The renderer is available on Github: https://github.com/rantonels/starless.



- This demonstrates that no matter from which perspective you view a black hole (which is illuminated by a thin and luminous accretion disk), you always see the *entire* disk and the *entire* event horizon (actually multiple times).
- A nice visualization of the appearance of a black hole to a distant observer can be found on Derek Muller's YouTube channel *Veritasium*:

SouTube Video: How to Understand What Black Holes Look Like

 The popular renderings (like the one above, or the one in the movie INTERSTELLAR, → *below*) use *thin* and *luminous* accretion disks as light sources because they help to visualize the warped spacetime structure in the vicinity of a black hole.

By contrast, real black holes (like Messier M87* depicted by the EHT collaboration, \rightarrow *below*) are more likely surrounded by *thick* and *dim* accretion *regions* instead [246]. The image of such a black hole isn't as pretty and "clean", but still characterized by a central shadow bounded by a pronounced bright ring from light that grazes the photon sphere [247]. The EHT collaboration explains in Ref. [246]:

For accreting black holes embedded in a geometrically thick, optically thin emission region, [...], the combination of an event horizon and light bending leads to the appearance of a dark "shadow" together with a bright emission ring that should be detectable through very long baseline interferometery [...] experiments.

• For more details on the depiction of black holes under various lighting conditions (and the roles played by the photon orbit $r_{\rm ph}$ and the shadow $r_{\rm sh}$), see Ref. [248].

9 | History & Theory milestones:

• For a long time, the role played by the singularity at r_s of the Schwarzschild metric was unclear. In 1939, Einstein eventually tried to show that objects that are smaller than their Schwarzschild radius (= black holes) cannot form dynamically [249]. He concluded:

The essential result of this investigation is a clear understanding as to why the "Schwarzschild singularities" do not exist in physical reality. [...] The "Schwarzschild singularity" does not appear for the reason that matter cannot be concentrated arbitrarily. And this is due to the fact that otherwise the constituting particles would reach the velocity of light.



So Einstein neither liked nor believed in the existence of black holes. At the same time, OPPENHEIMER and SNYDER (tacitly ignoring Einstein's results) showed that – and by which mechanism – a collapse into a black hole *is* possible [250]; the limiting mass above which this collapse occurs is known as the \uparrow *Tolman-Oppenheimer-Volkoff limit* [251,252]. See Ref. [253] for a historical account on how black holes evolved from a mathematical curiosity into a physical possibility.

- That the event horizon at the Schwarzschild radius is only a *coordinate* singularity that acts as a "causal membrane" was finally worked out by FINKELSTEIN in 1958 [254]. Shortly after, KRUSKAL found his convenient coordinate system that allowed him to construct the maximal extension of the Schwarzschild solution [255].
- In the early seventies, work by several physicists established what is today known as the
 No-hair theorem [256–258] (due to the required mathematical assumptions this is more a
 conjecture than a theorem): All possible black hole solutions of the EFEs are characterized by
 only three numbers: their *mass, angular momentum*, and *charge*. (These are exactly the three
 numbers that parametrize the ↑ *Kerr-Newman metric*, i.e., the metric of a rotating, charged
 mass.) In contrast to "normal" bodies like planets or stars which can be distinguished by
 additional features (like their surface structure [these are the "hair"]) two black holes with
 the same mass, angular momentum, and charge, look *exactly* the same.
- In contrast to the (potential) coordinate singularities at the event horizon, the singularities in the *center* of black hole solutions are true (physical) singularities (were the curvature diverges). These singularities are "outside" of space and time and not within the predictive domain of GENERAL RELATIVITY (→ *quantum gravity*). For a long time, it was believed that these "holes" in GENERAL RELATIVITY are mere artifacts of unnaturally symmetric solutions. It was shown by PENROSE and HAWKING in the late sixties that this is not so [259, 260]: Spacetime singularities are *generic* features of many solutions of the EFEs They are part of GENERAL RELATIVITY, whether we like it or not.
- In 1975, HAWKING performed quantum field theory calculations on the static background metric of a black hole (this is not quantum gravity!) and showed that it would emit thermal black body radiation [261]. This ↑ *Hawking radiation* does not escape from withing the event horizon but still causes the black hole to loose mass. If the mass lost by this process is not compensated by absorbed matter, the black hole slowly "evaporates" (which leads, inter alia, to the famous ↑ *Black hole information paradox*). Both Hawking radiation and the evaporation of black holes are so far *hypotheses* without experimental evidence.

10 Observations:

- The first identified black hole was Cygnus X-1, which is accompanied by a star in close proximity (this binary system belongs to the Milky Way). By observing the Doppler shift of the spectrum of its companion, one can extract information about the star's orbit, and from this derive the presence (and mass) of the black hole [262, 263]. Today, the mass of Cygnus X-1 is estimated as 21 M_☉ (solar masses).
- Modern astronomy provides a variety of (indirect) methods that have been successfully employed for the identification and characterization of black holes. For example, the supermassive black hole Sagittarius A* in the center of our Milky Way has been characterized very precisely by observing the motion of stars and gas in its vicinity (↑ R. Genzel's Nobel Lecture [264] and references therein). Another example is the detection of an isolated stellar-mass black hole in 2022 by means of *← microlensing* [265].
- The first detection GW150914 of a gravitational wave in 2016 by the LIGO and VIRGO



collaboration was well described by the merger of two black holes with masses $\sim 35 M_{\odot}$ and $\sim 30 M_{\odot}$, respectively (Fig. 13.8 and \rightarrow Section 13.4); the product of the merger is predicted to be a Kerr black hole with mass $62 M_{\odot}$. The detection of gravitational waves therefore serves as independent (and indirect) evidence for the existence of black holes. These events also allow for the test of GENERAL RELATIVITY in the very strong-field regime.

The first direct depiction of the accretion disk and the shadow of a black hole was published 2019 by the ↑ *Event Horizon Telescope (EHT) collaboration*, a global network of radio telescopes. The first black hole studied in this way was the supermassive black hole Messier 87* (~ 6 × 10⁹ M_☉) in the center of the galaxy Messier 87. In 2022, the collaboration published a similar analysis for Sagittarius A* (~ 4 × 10⁶ M_☉), the supermassive black hole in the center of our Milky Way:



FIGURE 13.4. • **Messier 87* (2019):** Messier 87 (M87) is a galaxy 55 million light years away from Earth; M87* is the supermassive black hole believed to lie at the center of this galaxy. The analysis that leads to and contextualizes this picture is detailed in a sequence of papers by the Event Horizon Telescope Collaboration [246, 266–273] (the picture is from Ref. [246]).



FIGURE 13.5. • **Sagittarius A* (2022):** Sagittarius A* is the supermassive black hole at the center of our Milky Way. The analysis that leads to and contextualizes this picture is detailed in a sequence of papers by the Event Horizon Telescope Collaboration [274–279] (the picture is from Ref. [274]).



11 | Visualizing black holes:

By now you have all the tools necessary to write computer code that simulates what one would see close to a black hole: The rays emitted by light sources in the vicinity of the black hole (like background galaxies and glowing gas in its accretion disk) follow light-like trajectories determined by the geodesic Eq. (10.131), and the Christoffel symbols needed for this computation derive from the metric of the black hole as a solution of the Einstein field Eq. (12.10) [the Schwarzschild metric Eq. (13.25), if the black hole does not rotate and is uncharged]. If you want to be sophisticated, you not only trace rays, but emit signals with fixed frequency in the proper time of each light source and compute the frequency measured at the location of your virtual camera, thereby faithfully reproducing gravitational and Doppler shifts in the color of light (Section 13.2.4).

• Luckily, you don't have to write your own code. Here is a Python based raytracer for the Schwarzschild metric to play with:

➡ github.com/rantonels/starless

Example renderings with explanations can be found on

rantonels.github.io/starless

• If you don't want to run code, NASA has you covered [280].

Here is a visually stunning simulation of what you would see during a plunge into a supermassive ($4.3 \times 10^6 M_{\odot}$) Schwarzschild black hole:

Plunge into a Black Hole Explained (Video)

Compare the features in the video with our discussion above.

• *Fun fact:* In the 2014 science fiction film INTERSTELLAR, the protagonists land on a planet orbiting the (fictional) supermassive black hole "Gargantua". The VFX shots that show the black hole were computed from the ↑ *Kerr metric* (i.e., the black hole rotates). Up to artistic modifications (like purposefully omitting the Doppler effect and gravitational redshift, ↑ Figure 15 of Ref. [281]), the depicted scenery is in accordance with GENERAL RELATIVITY:



FIGURE 13.6. • The fictional black hole "Gargantua" in the movie Interstellar (2014): The company Double Negative Ltd. wrote dedicated rendering software – based on mathematical input from (now Nobel laureate) Kip Thorne – to simulate the ray propagation in the vicinity of a spinning (Kerr) black hole [281–283]. The software was used to render the fictional black hole "Gargantua" (a) in various scenes of the movie Interstellar (b) by Christopher Nolan; the rendering in (a) is taken from [281]. Notably, the VFX artists won the 87th Academy Awards for best visual effects; in that regard the author lists of Refs. [281,282] are quite remarkable: In 2015, Paul Franklin won an Oscar for visualizing phenomena of GENERAL RELATIVITY, and Kip Thorne was awarded the 2017 Nobel Prize in Physics for describing phenomena of GENERAL RELATIVITY (gravitational waves, not black holes).



↓Lecture 30 [00.00.00]

13.4. Gravitational waves

So far we only studied (asymptotically) static solutions of the Einstein field equations that are caused locally by a source of gravity (a heavy mass) and become asymptotically flat. But the metric has degrees of freedom that are independent of matter because Ricci flatness ($R_{\mu\nu} = 0$) is a weaker constraint than Riemann flatness ($R_{\mu\nu\alpha\beta} = 0$). Ricci-flat excitations of the metric field can traverse a spacetime without matter and are called *gravitational waves*; they demonstrate the "non-Machianity" of GENERAL RELATIVITY (\leftarrow Section 9.3): Spacetime (= the metric field) exists as an independent dynamical entity!

Here we study gravitational fields in the weak-field limit of linearized gravity:

Details: September 6

- **1** | Linearized gravity:
 - i | Rationale: The deformations of spacetime due to gravitational waves (GW) are tiny.

At least here on Earth this must be true because otherwise we would have experienced the dynamical nature of spacetime long ago. This approximation is of course not justified in the strong-field regime (e.g., close to merging neutron stars or black holes).

 $\rightarrow \triangleleft$ Small deviations $|h_{\mu\nu}(x)| \ll 1$ from Minkowski space $\eta_{\mu\nu}$:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) + \mathcal{O}(h^2)$$
(13.135)

Goal: Linearize GENERAL RELATIVITY in the perturbation $h_{\mu\nu}$.

 \rightarrow *Rules*:

- Drop all non-linear terms in $h_{\mu\nu}$ and its derivatives $h_{\mu\nu,\rho}$...
- Raise and lower indices with $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$.
- The inverse metric in linear order is (check this!)

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad \text{with} \quad h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta} . \tag{13.136}$$

The idea is to approximate GENERAL RELATIVITY as a linear, Lorentz covariant field theory of the tensor field $h_{\mu\nu}(x)$ on a static Minkowski background $\eta_{\mu\nu}$. Since the decomposition Eq. (13.135) of $g_{\mu\nu}$ is not unique (in linear order), this will be a *gauge theory* with gauge field $h_{\mu\nu}(x)$ (\Rightarrow below; recall also \bigcirc Problemset 1). For details see \uparrow CARROLL [4] (§7.1, pp. 274–278).

ii | We can now express all relevant quantities in linear order of the perturbation $h_{\mu\nu}$:

Christoffel symbols
$$\xrightarrow{10.79}$$
 $\hat{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2} \eta^{\lambda\rho} \left(h_{\rho\nu,\mu} + h_{\rho\mu,\nu} - h_{\mu\nu,\rho} \right)$ (13.137a)
Riemann tensor $\xrightarrow{10.106}$ $\hat{R}_{\alpha\mu\nu\beta} = \frac{1}{2} \left(h_{\mu\nu,\beta,\alpha} - h_{\alpha\nu,\mu,\beta} - h_{\mu\beta,\nu,\alpha} + h_{\alpha\beta,\mu,\nu} \right)$ (13.137b)
Ricci tensor $\xrightarrow{10.114}$ $\hat{R}_{\mu\nu} = \frac{1}{2} \left(h_{\mu\nu,\alpha}{}^{,\alpha} - h^{\alpha}{}_{\nu,\mu,\alpha} - h^{\alpha}{}_{\mu,\nu,\alpha} + h^{\alpha}{}_{\alpha,\mu,\nu} \right)$ (13.137c)

We mark linearized quantities with a hat ^.



$$\Phi_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \quad \text{with} \quad h := h^{\alpha}{}_{\alpha} \,. \tag{13.138}$$

Eq. (13.137c) $\xrightarrow{\circ}$

$$\hat{R}_{\mu\nu} = \frac{1}{2} h_{\mu\nu,\alpha}{}^{,\alpha} - \frac{1}{2} \left(\Phi_{\mu}{}^{\alpha}{}_{,\alpha,\nu} + \Phi_{\nu}{}^{\beta}{}_{,\beta,\mu} \right)$$
(13.139)

iv | Gauge symmetry:

< Arbitrary small vector field $\xi^{\mu}(x) \sim |h_{\mu\nu}| \ll 1$; define the transformation

$$h_{\mu\nu}(x) := h_{\mu\nu}(x) - \xi_{\mu,\nu}(x) - \xi_{\nu,\mu}(x)$$
(13.140)

Since ξ^{μ} is assumed to be of the same order as $h_{\mu\nu}$, the new perturbation $h_{\mu\nu}$ is also of the same order. Note that Eq. (13.140) is *not* a coordinate transformation!

Eq. (13.137b) $\xrightarrow{\circ}$

$$\hat{R}_{\alpha\mu\nu\beta}\left[\bar{h}_{\mu\nu}\right] = \hat{R}_{\alpha\mu\nu\beta}\left[h_{\mu\nu}\right]$$
(13.141)

\rightarrow Eq. (13.140) is a *gauge symmetry* of linearized GENERAL RELATIVITY

This means that both $h_{\mu\nu}(x)$ and $\bar{h}_{\mu\nu}(x)$ describe the same physical situation in linearized gravity, and we can use the gauge freedom Eq. (13.140) to simplify equations.

Remember that the Einstein field equations follow from the Einstein-Hilbert action – which is defined in terms of the curvature. The linearized field equations then follow from the corresponding action with linearized curvature $\hat{R}_{\alpha\mu\nu\beta}$. Because of Eq. (13.141), this action is invariant under the transformation Eq. (13.140) (this is analogous to the invariance of the electromagnetic field strength tensor $F_{\mu\nu}$ and thereby the Maxwell action under gauge transformations). Consequently, the linearized Einstein field equations must have the symmetry Eq. (13.140). Since this is a *local* symmetry (choose $\xi^{\mu}(x)$ non-zero on a compact region of spacetime), it must be a *gauge* symmetry, i.e., it relates physically equivalent field configurations (\leftarrow Section 6.2).

 $\mathbf{v} \mid \text{Eq.} (13.140) \rightarrow$

$$\bar{h} = h - 2\xi^{\mu}_{\ \mu} \tag{13.142}$$

With this we find the gauge transformation of the auxiliary field Φ : Eq. (13.138) \rightarrow

$$\bar{\Phi}_{\mu\nu} = \Phi_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu} \xi^{\alpha}_{,\alpha}$$
(13.143)

In particular:

$$\bar{\Phi}^{\alpha}_{\mu}{}_{,\alpha} = \Phi^{\alpha}_{\mu}{}_{,\alpha} - \underbrace{\xi^{\alpha}_{\mu}{}_{,\alpha}{}_{,\alpha}}_{\equiv \Box \xi_{\mu}}$$
(13.144)

Here it is $\Box = \eta^{\alpha\beta} \partial_{\alpha} \partial_{\beta}$.

vi | For given $\Phi_{\mu}{}^{\alpha}$, solve differential equation $\Box \xi_{\mu} = \Phi_{\mu}{}^{\alpha}{}_{,\alpha}$ for ξ_{μ} \rightarrow Choose gauge *w.l.o.g.*

$$\bar{\Phi}^{\ \alpha}_{\mu\ ,\alpha} = 0 \tag{13.145}$$

This gauge is called * *Hilbert gauge* (or *Lorenz gauge* or *harmonic gauge*); it is analog to the Lorenz gauge $A^{\alpha}_{,\alpha} = 0$ in electrodynamics.

In the following we fix this gauge and drop the bars: $\overline{\Phi} \mapsto \Phi$.

vii | Eq. (13.139) $\xrightarrow{\text{Eq. (13.145)}}$

$$\hat{R}_{\mu\nu} = \frac{1}{2} \Box h_{\mu\nu} \quad \Rightarrow \quad \hat{R} = \frac{1}{2} \Box h \tag{13.146}$$

With this we find for the linearized Einstein tensor:

$$\hat{G}_{\mu\nu} := \hat{R}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \hat{R} \stackrel{13.138}{\stackrel{13.146}{=}} \frac{1}{2} \Box \Phi_{\mu\nu}$$
(13.147)

Eq. $(12.10) \rightarrow$ Linearized Einstein field equations:

$$\Box \Phi_{\mu\nu} = -2\kappa \hat{T}_{\mu\nu} \tag{13.148}$$

The hat on $\hat{T}_{\mu\nu}$ indicates that the HEMT is computed with $\eta_{\mu\nu}$ and not with $g_{\mu\nu}$ (i.e., in zeroth order of $h_{\mu\nu}$). Eq. (13.148) is then a linear differential equation in $h_{\mu\nu}$.

To get a small excitation $h_{\mu\nu}$, the source $T_{\mu\nu}$ must already be of the same order in *zeroth* order of $h_{\mu\nu}$; otherwise the assumption that $h_{\mu\nu} \ll 1$ is inconsistent.

Eq. (12.11) \rightarrow *Trace inverted* field equations:

$$\Box h_{\mu\nu} = -2\kappa \left(\hat{T}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \hat{T} \right)$$
(13.149)

 \rightarrow Linearized *vacuum* field equations ($\hat{T}_{\mu\nu} = 0$):

$$\Box \Phi_{\mu\nu} = 0 \quad \Leftrightarrow \quad \Box h_{\mu\nu} = 0 \tag{13.150}$$

viii | Solutions must satisfy the Hilbert gauge Eq. (13.145):

Eqs. (13.145) and (13.148) \rightarrow

$$\partial_{\nu}\hat{T}^{\mu\nu} = 0 \quad \Leftrightarrow \quad \frac{1}{c}\partial_{t}\hat{T}^{\mu0} = -\partial_{k}\hat{T}^{\mu k}$$
 (13.151)

 \rightarrow Energy-momentum conservation

This means that for solutions of Eq. (13.148) to exist, the source $\hat{T}^{\mu\nu}$ must satisfy Eq. (13.151).

2 | Particular solution:

All solutions of Eq. (13.148) can be constructed as sum of a particular solution and the homogeneous solutions. The latter correspond to *plane waves* and will be discussed \rightarrow *below*. Here we focus on the perturbations of the metric sourced by the HEMT $\hat{T}_{\mu\nu}$:

i | Inhomogeneous solution of Eq. (13.148):

$$\Phi_{\mu\nu}(t,\vec{x}) \stackrel{*}{=} -\frac{\kappa}{2\pi} \int_{V} d^{3}y \frac{\hat{T}_{\mu\nu}(t_{\text{ret}},\vec{y})}{|\vec{x}-\vec{y}|} \quad \text{with} \quad t_{\text{ret}} = t - \frac{1}{c} |\vec{x}-\vec{y}| \qquad (13.152)$$

This follows with the \checkmark *retarded Green's function* of the wave operator \Box that you already know from your electrodynamics course.

 $ii \mid \triangleleft$ Fourier transform (in time):

$$\tilde{\phi}(\omega, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int dt \, e^{-i\omega t} \phi(t, \vec{x}) \tag{13.153a}$$

$$\phi(t, \vec{x}) = \frac{1}{\sqrt{2\pi}} \int d\omega \, e^{i\omega t} \tilde{\phi}(\omega, \vec{x})$$
(13.153b)

Here, $\phi(t, \vec{x})$ is any function of spacetime, e.g., $\Phi_{\mu\nu}$ or $\hat{T}_{\mu\nu}$. Eq. (13.152) \rightarrow

$$\tilde{\Phi}_{\mu\nu}(\omega,\vec{x}) \stackrel{\circ}{=} -\frac{\kappa}{2\pi} \int_{V} \mathrm{d}^{3} y e^{-i\frac{\omega}{c}|\vec{x}-\vec{y}|} \frac{\hat{T}_{\mu\nu}(\omega,\vec{y})}{|\vec{x}-\vec{y}|}$$
(13.154)

iii $| \triangleleft$ Localized, distant, slowly moving source:



Approximations:

- Source localized & distant $\rightarrow |\delta \vec{r}| \ll \vec{r}$
- Source slowly moving $\rightarrow \omega \ll \frac{c}{|\delta \vec{r}|}$

With $\vec{y} - \vec{x} \equiv \vec{r} + \delta \vec{r} \xrightarrow{\text{Eq. (13.154)}}$

$$\tilde{\Phi}_{\mu\nu}(\omega,\vec{x}) \approx -\frac{\kappa}{2\pi} \frac{-e^{i\frac{\omega}{c}r}}{r} \int_{V} \mathrm{d}^{3}y \,\tilde{\tilde{T}}_{\mu\nu}(\omega,\vec{y}) \tag{13.155}$$

This is the lowest order of the \checkmark *multipole expansion*, which yields the dominant contribution to the field far away from the source. To include higher multipole moments (beyond quadrupole radiation, \rightarrow *below*), the phase factor must be expanded in higher orders of $\delta \vec{r}$. For a compendium on the multipole expansion of gravitational radiation: \uparrow Ref. [284].

- iv | To proceed, we need a few Relations:
 - Hilbert gauge Eq. (13.145) $\xrightarrow{\text{Eq. (13.153)}}$

$$\frac{1}{c}\partial_t \Phi^{\mu 0} = -\partial_i \Phi^{\mu i} \quad \Rightarrow \quad \tilde{\Phi}^{\mu 0} = \frac{ic}{\omega}\partial_i \tilde{\Phi}^{\mu i} \tag{13.156}$$



We can use this to compute time components from spatial components:

$$\{\tilde{\Phi}^{ij}\} \xrightarrow{13.156} \{\tilde{\Phi}^{i0}\} \xrightarrow{13.156} \tilde{\Phi}^{00}$$
(13.157)

 \rightarrow We only need to study the *spatial* components of the metric.

• Energy-momentum conservation Eq. (13.151) \rightarrow

$$\frac{1}{c}\partial_t \hat{T}^{\mu 0} = -\partial_k \hat{T}^{\mu k} \quad \Rightarrow \quad \partial_k \tilde{\hat{T}}^{\mu k} = -i\frac{\omega}{c}\tilde{\hat{T}}^{\mu 0} \tag{13.158}$$

v | With these preliminary steps, we find:

$$\tilde{\Phi}_{ij}(\omega,\vec{x}) \stackrel{13.155}{\approx} -\frac{\kappa}{2\pi} \frac{-e^{i\frac{\omega}{c}r}}{r} \int_{V} \mathrm{d}^{3}y \tilde{\hat{T}}^{ij}(\omega,\vec{y})$$
(13.159a)

$$= -\frac{\kappa}{2\pi} \frac{-e^{i\frac{\omega}{c}r}}{r} \Big[\underbrace{\int_{V} d^{3}y \partial_{k} \left(y^{i}\tilde{\tilde{T}}^{kj} \right)}_{= 0 \text{ since } \tilde{\tilde{T}}^{kj}|_{\partial V} = 0} - \int_{V} d^{3}y y^{i} \left(\partial_{k}\tilde{\tilde{T}}^{kj} \right) \Big] \quad (13.159b)$$

$$= \frac{\kappa}{4\pi} \frac{-e^{i\frac{\omega}{c}r}}{r} \int_{V} \mathrm{d}^{3}y \Big[y^{i} \Big(\partial_{k}\tilde{\tilde{T}}^{kj}\Big) + y^{j} \Big(\partial_{k}\tilde{\tilde{T}}^{ki}\Big) \Big]$$
(13.159c)

$$\stackrel{13.158}{=} -i \frac{\kappa \omega}{4\pi c} \frac{e^{-i \,\overline{c} \, r}}{r} \int_{V} \mathrm{d}^{3} y \Big[y^{i} \,\tilde{\tilde{T}}^{0j} + y^{j} \,\tilde{\tilde{T}}^{0i} \Big]$$
(13.159d)

$$= -i\frac{\kappa\omega}{4\pi c}\frac{e^{-i\frac{\omega}{c}r}}{r}\int_{V} d^{3}y \Big[\underbrace{\partial_{l}\left(y^{i}y^{j}\tilde{T}^{0l}\right)}_{=0\ (\epsilon\ above)} - y^{i}y^{j}\left(\partial_{l}\tilde{T}^{0l}\right)\Big]$$
(13.159e)

$$\stackrel{13.158}{=} \frac{\kappa \omega^2}{4\pi c^2} \frac{e^{-i\frac{\omega}{c}r}}{r} \int_V \mathrm{d}^3 y \left(y^i y^j \tilde{\hat{T}}^{00} \right) \tag{13.159f}$$

Here we used $\tilde{T}_{ij} = \tilde{T}^{ij}$ (because we use $\eta_{\mu\nu}$ to pull indices) and that $\tilde{\Phi}_{ij} = \tilde{\Phi}_{ji}$. We can now undo the Fourier transform in time:

$$\Phi_{ij}(t,\vec{x}) = \frac{1}{\sqrt{2\pi}} \int d\omega \, e^{i\,\omega t} \,\tilde{\Phi}_{ij}(\omega,\vec{x}) \tag{13.160a}$$

$$\stackrel{13.159}{\approx} \frac{\kappa}{4\pi r} \frac{1}{c^2} \int_V \mathrm{d}^3 y \, y^i \, y^j \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\omega \, \underbrace{e^{i(t-r/c)\omega}\omega^2}_{-\partial_t^2 e^{i(t-r/c)\omega}} \tilde{T}^{00}(\omega,\vec{y}) \quad (13.160b)$$

$$= -\frac{\kappa}{4\pi r} \left. \frac{\mathrm{d}^2}{\mathrm{d}t^2} \right|_{t=t_{\mathrm{ret}}} \frac{1}{c^2} \int_V \mathrm{d}^3 y \, y^i \, y^j \, \hat{T}^{00}(t, \vec{y}) \tag{13.160c}$$

with retarded time $t_{ret} = t - r/c$.

vi | This motivates the definition of the ...

** Quadrupole moment tensor

$$I_{ij}(t) := \frac{1}{c^2} \int_V \mathrm{d}^3 y \, y^j \, y^j \, \hat{T}^{00}(t, \vec{y}) \tag{13.161}$$

Note that \hat{T}^{00} is an *energy* density so that \hat{T}^{00}/c^2 is a *mass* density.



which leads to the ...

** Quadrupole formula

$$\Phi_{ij}(t,\vec{x}) \approx -\frac{\kappa}{4\pi r} \left. \frac{\mathrm{d}^2 I_{ij}}{\mathrm{d}t^2} \right|_{t=t_{\mathrm{ret}}}$$
(13.162)

with retarded time $t_{\text{ret}} = t - \frac{r}{c}$ and r the distance between \vec{x} and the source.

In conclusion: The gravitational waves produced by a localized & non-relativistic object are proportional to the *second derivative* of the *quadrupole moment* of the energy density of this object (evaluated at the retarded time t_{ret}). This retarded time $t_{ret} = t - \frac{r}{c}$ shows that gravitational waves propagate with the *speed of light c*.

 \rightarrow Conclusions:

- Gravitational waves (GWs) propagate with the speed of light.
- GWs in the far-field are dominated by quadrupole radiation.

vii Comments:

• Here is a comparison of gravitational and electromagnetic radiation:

Moment	Gravity	Electrodynamics
Monopole	×	×
	(Energy conservation)	(Charge conservation)
Dipole	×	 Image: A start of the start of
	(Momentum conservation)	
Quadrupole	 Image: A second s	✓

• We can directly check that the *monopole* moment of the energy density

$$M(t) := \frac{1}{c^2} \int_V \hat{T}^{00}(t, \vec{y}) \,\mathrm{d}^3 y \tag{13.163}$$

must be constant due to energy conservation:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{1}{c^2} \int_V \partial_t \hat{T}^{00}(t, \vec{y}) \,\mathrm{d}^3 y \stackrel{\mathbf{13.151}}{=} -\frac{1}{c} \int_V \partial_k \hat{T}^{k0}(t, \vec{y}) \,\mathrm{d}^3 y = 0 \,. \tag{13.164}$$

Hence there can be no gravitational monopole radiation (this also follows from \leftarrow *Birkhoff's theorem*, Section 13.1.2). In the case of electrodynamics, it is U(1) charge conservation that forbids monopole radiation. But this exhausts all charge-related symmetries of electrodynamics, which is why electromagnetic dipole (quadrupole, ...) radiation is possible.

However, in the case of gravitational waves *mass* is the charge, and there are more conservation laws that can be exploited to pin down its higher-order moments. For example, the conservation of linear momentum (coming from translation symmetry) implies that the static *dipole* moment

$$D_i(t) := \frac{1}{c^2} \int_V y^i \hat{T}^{00}(t, \vec{y}) \,\mathrm{d}^3 y \tag{13.165}$$



cannot oscillate either:

$$\frac{d^2 D_i}{dt^2} = \frac{1}{c^2} \frac{d}{dt} \int_V y^i \partial_t \hat{T}^{00}(t, \vec{y}) \, \mathrm{d}^3 y \stackrel{\mathrm{13.151}}{=} -\frac{1}{c} \frac{d}{dt} \int_V y^i \partial_k \hat{T}^{k0}(t, \vec{y}) \, \mathrm{d}^3 y \quad (13.166a)$$

$$\stackrel{\circ}{=} \frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\frac{1}{c}}_{V} \int_{V} \hat{T}^{i0}(t, \vec{y}) \,\mathrm{d}^{3}y \stackrel{\mathbf{13.151}}{=} - \int_{V} \partial_{k} \hat{T}^{ik}(t, \vec{y}) \,\mathrm{d}^{3}y = 0.$$
(13.166b)

Total momentum $P^{i}(t)$

This shows that *momentum conservation* is the root cause for the absence of gravitational dipole radiation.

The argument that is sometimes made that "there are only positive masses" whereas there are "positive and negative electric charges" is beside the point (one does not need positive and negative electric charges to produce dipole radiation; \rightarrow *below*). The difference is that there is a symmetry (momentum conservation) that forbids the oscillation of the center of *mass*, whereas there is *no* symmetry that prevents the oscillation of the center of *charge*:



- Einstein predicted gravitational waves first in 1916 when he studied the linearized version of his field equations [285]. (At the time, Einstein still struggled with the gauge invariance of his theory: He later realized that some of his solutions were actually unphysical "coordinate waves"; see the addendum on p. 696 of Ref. [285].) He derived the quadrupole formula Eq. (13.162) two years later in his paper "*Über Gravitationswellen*"; see Eq. (23) and (25) in Ref. [16].
- At the time it was widely believed that gravitational waves do exist. (They were, after all, a direct consequence of GENERAL RELATIVITY, other predictions of which had been verified; although there was no experimental evidence for gravitational waves.) However, Einstein later changed his view and doubted their existence. In 1936, he wrote in a letter to MAX BORN [286]:

Together with a young collaborator [Nathan Rosen], I arrived at the interesting result that gravitational waves do not exist, though they had been assumed a certainty to the first approximation. This shows that the non-linear general relativistic field equations can tell us more or, rather, limit us more than we have believed up to now.

At the same time, Einstein wrote a paper (together with NATHAN ROSEN) with the title "Do Gravitational Waves Exist?" (the answer was "No"). The paper has an interesting history, because its original version no longer exists, and the version that was eventually published in 1937 comes to a completely different conclusion (the paper was then titled "On gravitational Waves" [287]). On the unusual history of this particular paper (and Einstein's first confrontation with the modern peer review system) see Ref. [288].



3 | Homogeneous solutions:

Let us now focus on the homogeneous solutions of the linearized field equations Eq. (13.148); i.e., the solutions of the linearized *vacuum* field equations Eq. (13.150). Gravitational waves far away from their sources can be well approximated by these (plane wave) solutions:

i | We want so solve the linear wave equation

$$\Box h^{\mu\nu} \stackrel{13.150}{=} 0 \quad \text{with constraint} \quad h^{\mu\alpha}_{,\alpha} \stackrel{13.145}{=} \frac{1}{2} \eta^{\mu\alpha} h_{,\alpha} \tag{13.167}$$

ii | General form of solution \rightarrow Plane waves:

$$h^{\mu\nu}(x) = \operatorname{Re}\left[A^{\mu\nu}e^{ik\cdot x}\right]$$
(13.168)

with *wave vector* k^{μ} and (symmetric) *amplitude* (tensor) $A^{\mu\nu} = A^{\nu\mu}$. We drop the real part in the following to simplify the notation.

- iii | <u>Constraints:</u>
 - Dispersion relation $\xrightarrow{13.167}$

$$k^{\mu}k_{\mu} \stackrel{!}{=} 0 \tag{13.169}$$

This is a constraint on the wave vector to solve the wave equation.

• Hilbert gauge $\xrightarrow{13.167}$

$$A^{\mu\alpha}k_{\alpha} \stackrel{!}{=} \frac{1}{2}Ak^{\mu}$$
 with $A = A^{\alpha}{}_{\alpha}$ (13.170)

This is a constraint on the amplitude to satisfy the Hilbert gauge.

iv $A^{\mu\nu} = A^{\nu\mu}$ symmetric $\rightarrow 10$ components

Our next goal is to show that only 6 of them can be chose independently:

 $\triangleleft \mathfrak{w}.l.o.g. \ k^{\mu} = (k, 0, 0, k) \text{ with } k \equiv \omega/c > 0$

That is, choose an inertial coordinate system in which the wave propagates in z-direction. Eq. (13.170) \rightarrow

$$A^{00} - A^{03} = \frac{1}{2}A \tag{13.171a}$$

$$A^{10} - A^{13} = 0 (13.171b)$$

$$A^{20} - A^{23} = 0 \tag{13.171c}$$

$$A^{30} - A^{33} = \frac{1}{2}A \tag{13.171d}$$

 $\stackrel{\circ}{\rightarrow}$ Only 6 independent components: A^{00} , A^{11} , A^{33} , A^{12} , A^{13} , A^{23}

The remaining 4 are given as follows:

$$A^{10} = A^{13}, \ A^{20} = A^{23}, \ A^{30} = \frac{1}{2} \left(A^{00} + A^{33} \right), \ A^{22} = -A^{11}.$$
 (13.172)



• We show now that only 2 (!) of these 6 parameters are *physical* degrees of freedom: Remember that $h_{\mu\nu}$ is a gauge field [Eq. (13.140)].

⊲ Gauge transformation:

$$\xi^{\mu} := -i\varepsilon^{\mu}e^{ik\cdot x} \quad \text{with} \quad \Box\xi^{\mu} \stackrel{13.169}{=} 0 \tag{13.173}$$

 ε^{μ} : arbitrary (small, constant) vector (with $|\xi^{\mu}| = |\varepsilon^{\mu}| \sim |h_{\mu\nu}| \ll 1$)

i! Because of Eq. (13.144), this gauge transformation keeps the Hilbert gauge Eq. (13.145) intact. Thus imposing the Hilbert gauge did not fix all gauge degrees of freedom. This is similar to electrodynamics, where the Lorenz gauge allows for \checkmark *residual gauge transformations* that satisfy $\Box \lambda = 0$.

Eqs. (13.140), (13.168) and (13.173) \rightarrow

$$\bar{A}^{\mu\nu} = A^{\mu\nu} - \varepsilon^{\mu}k^{\nu} - \varepsilon^{\nu}k^{\mu}$$
(13.174)

vi | Since ε^{μ} is arbitrary (note that ε^{μ} and $A^{\mu\nu}$ must be both of the order of $h^{\mu\nu}$), we can use this residual gauge freedom to gauge-fix the amplitudes further:

Eq. (13.174) \rightarrow [use again $k^{\mu} = (k, 0, 0, k)$]

$$\bar{A}^{11} = A^{11}, \qquad \bar{A}^{12} = A^{12}$$
 (13.175a)

$$\bar{A}^{13} = A^{13} - \varepsilon^1 k$$
, $\bar{A}^{23} = A^{23} - \varepsilon^2 k$ (13.175b)

$$\bar{A}^{00} = A^{00} - 2\varepsilon^0 k$$
, $\bar{A}^{33} = A^{33} - 2\varepsilon^3 k$ (13.175c)

This shows that by choosing ε^{μ} , we can "tweak" the 6 independent components of the amplitude without affecting the physical content of the solution.

Consider the trivial solution $A^{\mu\nu} = 0$, which corresponds to flat Minkowski space (in inertial coordinates). The gauge transformation Eq. (13.175) can now be used to find other representations of the same flat metric, since the (linearized) Riemann curvature is invariant under these transformations [Eq. (13.141)]. These gauge transformations are therefore residuals of the general covariance of the (non-linear) Einstein field equations.

Fix gauge (and drop the bars: $\overline{A} \mapsto A$) $\rightarrow \mathfrak{w.l.o.g.} A^{\mu\nu} = 0$ except for

$$A^{11} = -A^{22} =: A_+ \text{ and } A^{12} = A^{21} =: A_{\times}$$
 (13.176)

Choose ε^{μ} appropriately to zero four of the six independent components in Eq. (13.175) and use Eq. (13.172) to show that then three of the remaining four components vanish as well.

This gauge is called *** TT*-gauge (transverse traceless gauge)

In the TT-gauge the polarization tensor is traceless $(A^{\mu}{}_{\mu} = 0)$ so that $h = h^{\mu}{}_{\mu} = 0$.

 \rightarrow

Gravitational waves have two physical polarizations (degrees of freedom).

This is analogous to electrodynamics, where one starts with *four* gauge fields A^{μ} and ends up with only *two* transversal polarizations. The rule is that every generator of a gauge symmetry cuts down the degrees of freedom by *two*. So in electrodynamics, the single U(1) gauge

symmetry yields $4-2 \times 1 = 2$ physical degrees of freedom. In GENERAL RELATIVITY gauge transformations are diffeomorphisms, which are generated by *four* infinitesimal translations [\leftarrow Eq. (11.90)]; so one expects again $10 - 2 \times 4 = 2$ physical degrees of freedom (there are 10 components in the metric).

vii | <u>Polarizations:</u>

Write $A^{\mu\nu} = A_+ e_+^{\mu\nu} + A_\times e_\times^{\mu\nu}$ with the two * linear polarization tensors

$$e_{+}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} \text{ and } e_{\times}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu}.$$
 (13.177)

With this we can write the Plane gravitational wave solutions $\xrightarrow{\text{Eq. (13.168)}}$

$$h_{k,A_{+},A_{\times}}^{\mu\nu}(x) = \operatorname{Re}\left[A_{+}e_{+}^{\mu\nu}e^{ik\cdot x} + A_{\times}e_{\times}^{\mu\nu}e^{ik\cdot x}\right]$$
(13.178)

with wave vector \vec{k}^{μ} $(k^{\mu}k_{\mu}=0)$ and linear polarizations $A_{+}, A_{\times} \in \mathbb{C}$.

• Compare this to the + *plane wave solutions of electrodynamics* in Lorenz gauge:

$$A^{\mu}(x) = \operatorname{Re}\left[a_{1} \varepsilon_{1}^{\mu} e^{ik \cdot x} + a_{2} \varepsilon_{2}^{\mu} e^{ik \cdot x}\right]$$
(13.179)

Here, the wave vector k^{μ} satisfies also a linear dispersion $(k^{\mu}k_{\mu} = 0)$; $\varepsilon_{1,2}^{\mu}$ are the two space-like $(\|\varepsilon_i\|^2 = -1)$ and transversal $(\varepsilon_i^{\mu}k_{\mu} = 0)$ polarization vectors.

• In analogy to electrodynamics, we can also introduce ** circular polarization tensors

$$e_{R/L}^{\mu\nu} := \frac{1}{\sqrt{2}} \left(e_{+}^{\mu\nu} \pm i \, e_{\times}^{\mu\nu} \right) \tag{13.180}$$

as an alternative basis; it is then $A^{\mu\nu} = A_R \, e_R^{\mu\nu} + A_L \, e_L^{\mu\nu}$ with circular amplitudes

$$A_{R/L} = \frac{1}{\sqrt{2}} \left(A_+ \mp i A_{\times} \right) \,. \tag{13.181}$$

viii | Helicity: (Details:) Problemset 6)

Consider spatial rotations by φ about the propagation direction (*z*-axis):

$$(R_{\varphi})^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{\mu\nu}.$$
 (13.182)

We say that a tensorial object has $\overset{*}{\Rightarrow}$ helicity $h \in \mathbb{R}$, if it is an eigenvector under this transformation (or its tensor products) with eigenvalue $e^{ih\varphi}$.

The circular polarization tensors Eq. (13.180) transform according to the rules of rank-2 tensors:

$$(R_{\varphi})^{\mu}{}_{\alpha}(R_{\varphi})^{\nu}{}_{\beta}e^{\alpha\beta}_{R/L} \stackrel{\circ}{=} e^{\pm 2i\varphi}e^{\mu\nu}_{R/L}.$$
(13.183)



(Use $\cos \varphi + i \sin \varphi = e^{i\varphi}$ to show this.)

This reveals that circularly polarized gravitational waves have helicity $h = \pm 2$. As Eq. (13.183) shows, this manifests as an invariance of gravitational waves under rotations by only $\varphi = \pi = 180^{\circ}$ (and not $\varphi = 2\pi = 360^{\circ}$ as for "normal" vectors with helicity $h = \pm 1$).

Gravitational waves have *helicity* $h = \pm 2$.

- Operationally, this means that rotating your gravitational wave detector by only 180° about the axis of incidence makes gravitational waves look the same.
- Mathematically, only *massive* particles transform under *spin* representations [= (projective) representations of SO(3)]; by contrast, *massless* particles like photons and (hypothetical) gravitons transform under *helicity* representations [= representations of ISO(2)]. (This is so because massless particles propagate with the speed of light and do not have a rest frame; consequently, one can only rotate them about their propagation axis.) This is the mathematical reason why photons (and gravitons) have only *two* polarizations, whereas one would expect $2 \times 1 + 1 = 3$ ($2 \times 2 + 1 = 5$) states for a true spin-1 (spin-2) particle. Thus, strictly speaking, neither photons nor gravitons do have *spin* they have *helicity*. For details: \uparrow *Wigner's little groups* and WEINBERG [289] (§2.5, pp. 69-74).

4 | Effects on test particles:

Now that we know that the Einstein field equations allow for wave-like solutions, it is reasonable to ask which *physically observable effects* one should expect if such a wave passes by:

i | First test:

To test whether the wave-like solutions of the metric Eq. (13.178) have physical consequences (and are not just *gauge*), we can evaluate the curvature tensor:

Eqs. (13.137b) and (13.178) \rightarrow

$$\hat{R}_{0mn0} = \frac{1}{2c^2} \frac{\mathrm{d}^2 h_{mn}}{\mathrm{d}t^2} \neq 0 \quad \text{for} \quad m, n = 1, 2$$
 (13.184)

There are also other non-zero components of the curvature tensor [recall the symmetries Eq. (10.107)]. Note also that $h_{\mu\nu,\alpha,\beta} = -k_{\alpha}k_{\beta}h_{\mu\nu}$ for the plane waves Eq. (13.178) and $k_{\mu} = (k, 0, 0, -k)$ (propagation in z-direction). This implies that $h_{\mu\nu,\alpha,\beta}$ either vanishes or is proportional to k^2h_{mn} , which is proportional to Eq. (13.184). Thus all non-zero components can be written as time derivatives of h_{11} and h_{12} .

\rightarrow Oscillations in local spacetime curvature!

- On flat Minkowski space, all components of the curvature tensor vanish (independent of the coordinates). Non-vanishing components therefore prove that spacetime is no longer Minkowskian (again independent of the coordinates).
- Recall that spacetime curvature manifests physically as tidal forces via geodesic deviation Eq. (10.140). Non-vanishing curvature components are therefore always physically (and operationally) significant; they are not "coordinate or gauge effects."

ii | \triangleleft Free-falling test mass \rightarrow Follows geodesic equation:

$$\frac{\mathrm{d}u^{\mu}}{\mathrm{d}\tau} + \hat{\Gamma}^{\mu}_{\ \nu\rho} u^{\nu} u^{\rho} = 0 \quad \text{with} \quad u^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}$$
(13.185)

Eqs. (13.137a) and (13.178) $\rightarrow \hat{\Gamma}^{\mu}_{\ 00} = 0$ (in TT-gauge) $\stackrel{\circ}{\rightarrow}$ Time-like solution:

$$x^{\mu}(\tau) = \begin{pmatrix} c\tau \\ \vec{x}_0 \end{pmatrix}$$
 with $\vec{x}_0 = \text{const}$ (13.186)

\rightarrow Mass remains at rest in chosen coordinates!

This is of course only one of many possible geodesics; namely the one of a test mass placed in \vec{x}_0 at rest in the chosen coordinate system. The crucial point is that it remains at \vec{x}_0 , even though a gravitational wave distorts spacetime.

i! This result might be surprising at first. However, remember the conclusions of Section 9.2 regarding the role of coordinates in GENERAL RELATIVITY: The fact that the spatial coordinates of a particle remain constant over time (proper or coordinate) *has no physical content* a priori. Here is a (reprint of the) sketch to illustrate this point:



iii | $\triangleleft Two$ test masses on x-axis at $\vec{x}_{\pm} = (\pm a, 0, 0)$ (initially at rest)

Eq. $(13.186) \rightarrow$ Geodesics:

$$x_{\pm}^{\mu}(\tau) = \begin{pmatrix} c\tau \\ \vec{x}_{\pm} \end{pmatrix}$$
 with $\vec{x}_{\pm} = \text{const}$ (13.187)

 \rightarrow Test masses have constant *coordinate* distance $\vec{x}_{+}(\tau) - \vec{x}_{-}(\tau) = (2a, 0, 0)!$

i! This result might be even more surprising. But again, remember Section 9.2: If coordinates have no physical meaning, their coordinate distance has neither! Their *physical* distance is a property of the *values* of the metric field $g_{\mu\nu}$ between them and can change with time, even though their coordinates don't. Remember this sketch:



oretical



iv $| \triangleleft \text{Proper length at } x^0 = ct = \text{const} (dt = 0)$

$$dl^2 \stackrel{11.30}{=} -g_{mn}(x) \, dx^m dx^n \tag{13.188a}$$

$$\stackrel{13.135}{=} - \left[\eta_{mn} + h_{mn}(x)\right] \mathrm{d}x^m \mathrm{d}x^n \tag{13.188b}$$

$$\stackrel{13.178}{=} \underbrace{[1 - h_{11}(t, z)]}_{+-\text{Polarization}} dx^2 + \underbrace{[1 - h_{22}(t, z)]}_{+-\text{Polarization}} dy^2 + dz^2 \qquad (13.188c)$$

$$-\underbrace{2h_{12}(t,z)\,\mathrm{d}x\mathrm{d}y}_{\times\text{-Polarization}}$$

Remember that $k^{\mu} = (k, 0, 0, k)$, i.e., the wave propagates in z-direction. Eq. (13.188) shows that such a wave does not change distances in z-direction; this explains the "transverse" in \leftarrow transverse traceless gauge.

 \rightarrow Distance of the two test masses (along the *x*-axis):

$$\Delta l_x = 2a\sqrt{1 - h_{11}(t, z)} \quad \Rightarrow \quad \Delta l_x \approx 2a\left[1 - \frac{1}{2}h_{11}(t, z)\right] \tag{13.189}$$

Here we used again that $|h_{\mu\nu}| \ll 1$, i.e., the deformations of spacetime are small.

v | \triangleleft +-Polarized solution with frequency $\omega/c = k$ and amplitude $|A_+| \ll 1$:

$$h^{\mu\nu} \stackrel{13.178}{=} A_+ e_+^{\mu\nu} \cos\left(\omega t - kz\right)$$
(13.190)

 \triangleleft Position $z = 0 \rightarrow$

$$h^{11} = -h^{22} = A_{+}\cos(\omega t)$$
 and $h^{12} = h^{21} = 0$ (13.191)

Eq. (13.189) \rightarrow [Use that $h_{mn} = h^{mn}$, recall Eq. (13.136).]

$$\Delta l_x \approx 2a \left[1 - \frac{1}{2}A_+ \cos(\omega t) \right]$$

$$\Delta l_y \approx 2a \left[1 + \frac{1}{2}A_+ \cos(\omega t) \right]$$
(13.192a)
(13.192b)

Here, Δl_y is the measured distance between two test masses on the y-axis at $\vec{x}_{\pm} = (0, \pm a, 0)$; due to $h_{11} = -h_{22}$ the length is phase-shifted by π wrt. Δl_x .

- The result Eq. (13.192) shows that a *ring* of free-falling test masses is periodically squeezed and stretched into an ellipse since the deformations in x- and y-direction are phase-shifted by π .
- An analogous analysis for ×-polarized gravitational waves yields a similar dynamics in the xy-plane – only rotated by 45° (→ next). This means that a rotation by ±45° converts a + into a ×-polarized wave and vice versa; this reflects the helicity h = 2 of gravitational waves.
- vi | Conclusion:

Effect of a passing + or × polarized gravitational wave (traveling along the z-axis) on the *measured distances* $\Delta l_{x/y}$ (not coordinate distances!) between a ring of free-falling test masses in the xy-plane:





- One can exploit this effect to detect gravitational waves $(\rightarrow next)$.
- This picture motivates the labels "+" and "×" of the two polarizations.
- Similarly, *circularly* polarized gravitational waves $e_{R/L}^{\mu\nu}$ make the ellipse "rotate" left or right, respectively (note that the particles do not rotate, only the ellipse of their distances to the origin does).
- Let us estimate the length variations to be expected for a typical gravitational wave caused by a black hole merger. To this end, we need an estimate for the amplitude A_+ which depends both on the parameters of the source and the distance d to Earth:

$$\delta(\Delta l_x) \sim aA_+ \sim a|h_{11}| \stackrel{13.162}{\sim} \frac{aG}{dc^4} \frac{\mathrm{d}^2 I_{11}}{\mathrm{d}t^2} \stackrel{13.161}{\sim} \frac{aG}{dc^4} (\Omega^2 R^2 M) \,. \tag{13.193}$$

Here we assumed a binary system of two massive bodies (neutron stars or black holes) of roughly equal masses M that orbit each other with frequency Ω and distance 2R.

Let us assume we monitor the distance of two test masses 2a = 2 km apart, and use the parameters of the binary black hole merger that caused the first gravitational wave event GW150914 detected by LIGO [290] (\Rightarrow below):

$$M \sim 30 M_{\odot}, \quad \Omega \sim 75 \,\text{Hz}, \quad R \sim 175 \,\text{km}, \quad d \sim 400 \,\text{Mpc}.$$
 (13.194)

This yields for the length variations on Earth:

$$\delta(\Delta l_x) \sim 10^{-20} \,\mathrm{m} = 10^{-5} \,\mathrm{fm} \,.$$
 (13.195)

For comparison, the radius of a proton is $r_p \sim 10^{-15}$ m = 1 fm.

This explains the technical complexity of gravitational wave detectors ($\rightarrow next$).



5 | Detecting gravitational waves:

The previous analysis suggest that gravitational waves can be detected by continuously monitoring the distance between a ring of free-falling test masses. The distance between the masses could be measured by bouncing laser pulses between them and measuring the time needed for a round trip with an atomic clock.

Technical difficulties:

• Problem: On Earth, is impossible setup test masses that remain free-falling in all directions.

Solution: It is sufficient to be free-falling *in one direction* to detect deformations of the metric in this direction. For example, the mass of a simple pendulum is (approximately) free-falling in the directions *orthogonal* to the string that holds the mass.

• *Problem:* The length variations due to gravitational waves are so tiny [Eq. (13.195)] that even the best atomic clocks cannot resolve the fluctuations in the round-trip time.

Solution: Use a \downarrow Michelson interferometer to imprint the relative length fluctuations of two orthogonal arms onto the phase of a laser beam. Interference between light traveled along the two arms then reveals miniscule variations in their relative length. By using a \downarrow Fabry-Pérot cavity for each arm, the effective size (and thereby the sensitivity) of the arms can be increased significantly (since light traverses the arms multiple times).

\rightarrow <u>Rationale</u>:



The time spent by the laser light in the interferometer must be small compared to the (inverse) frequency of the gravitational wave. Then the interferometer measures the slowly varying relative length of its arms.





This is the operating principle of gravitational-wave detectors like LIGO (Fig. 13.7) which detected the first gravitational wave in 2015 (Fig. 13.8):

FIGURE 13.7. • LIGO – Laser Interferometer Gravitational-Wave Observatory: (a) The LIGO collaboration operates two nearly identical detectors at two different sites in the United States (3000 km apart), one in (b) Hanford and one in (c) Livingston (each with 4 km arm length). Operating two detectors (or three, including VIRGO in Italy) has several benefits: First, simultaneous detections allow for the rejection of false-positives caused by local perturbations. Second, using that gravitational waves travel with the speed of light allows for the "triangulation" of the source by measuring the time delay between signals at the different sites. Each site consists of a large \checkmark *Fabry–Pérot interferometer* (b,c) with two orthogonal arms (the Fabry–Pérot cavities). The crucial point is that the two mirrors that make up each cavity play the role of *test masses that fall freely* (= are force-free) along the direction of the cavity; i.e., in this direction the mirrors follow geodesics in spacetime. This is achieved by a intricate multi-stage setup of pendula (d) that decouple the mirror from their environment. (e) Each of the test masses (= mirrors) weighs 40 kg and operates in ultra-high vacuum (like most of the equipment) to mitigate noise. Images from https://www.ligo.caltech.edu (© Caltech/MIT/LIGO Lab).



6 Observations:

- The first *indirect* evidence for the existence of gravitational waves was based on the observation of a neutron star circling a pulsar (the ↑ *Hulse-Taylor pulsar*, also known as PSR B1913+16). Over time their orbital period changes, indicating a decay of the orbit [169–171]. This decay matches perfectly with the loss of energy predicted by GENERAL RELATIVITY due to the emission of gravitational waves. These observations earned HULSE and TAYLOR the 1993 Nobel Prize in Physics "for the discovery of a new type of pulsar, a discovery that has opened up new possibilities for the study of gravitation."
- The first *direct* detection of a gravitational wave was made in 2015 (and reported in 2016, Fig. 13.8) by the LIGO and VIRGO collaboration [290] (LIGO = Laser Interferometer Gravitational-Wave Observatory, VIRGO is named after the ↑ *Virgo cluster*). The event GW150914 (detected on 14.09.15) was detected simultaneously by the two interferometers at Hanford and Livingston (both USA), and was caused by the merger of two stellar-mass black holes:





The observed spacetime distortions were miniscule but matched the models derived from GENERAL RELATIVITY [291]. The detection earned WEISS, BARISH and THORNE the 2017 Nobel Prize in Physics "for decisive contributions to the LIGO detector and the observation of gravitational waves." For a comprehensive account on gravitational waves, their detection, the history of the field, and many references, see the three Nobel lectures by Weiss [292], Barish [293] and Thorne [294]; for a technical summary by the Nobel committee see Ref. [295].

• The first detection of a gravitational wave *that was accompanied by electromagnetic signals* was reported in 2017 (Fig. 13.9): The merger of two neutron stars GW170817 was detected independently by all three gravitational wave detectors of the LIGO and VIRGO collaborations and – for the first time – various other observatories in the electromagnetic spectrum; it is therefore a breakthrough in ↑ *multi-messenger astronomy*. The detection was reported in [296]



and the various other "messengers" are listed in [297].

The comparison of gravitational waves and electromagnetic signals were then used to constrain (violations of) the equivalence principle and Lorentz invariance. In particular, the difference $v_g - c$ of the speed of gravitational waves and the speed of light can now be constrained to -3×10^{-15} c and $+7 \times 10^{-16}$ c [298] (in accordance with our result \leftarrow *above*, namely that gravitational waves propagate with the speed of light). Also the dimensionality of spacetime could be verified to be D = 3 + 1 for both gravity and photons [299]; that is, there is no evidence for (non-compact) extra dimensions!





Here is the database of detected events of the LIGO/VIRGO/KAGRA collaboration:

➔ Gravitational-wave Transient Catalog (GWTC)

The list shows that detecting gravitational waves has quickly become "daily routine" (at the time of writing, there are already ~ 100 confidently-detected events).

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Part III.

Excursions




↓Lecture 31 [05.08.24]

Epistemological disclaimer

In Part I and Part II we studied *widely accepted* and *experimentally tested* theories of nature: SPECIAL RELATIVITY and GENERAL RELATIVITY.

Here in Part III, we enter the realm of theories that are the brainchilds of theoretical physicists only – *without any experimental evidence supporting these theories*! We do not even know whether gravity is a quantum phenomenon to begin with ...

Plan for this Excursion

This is a brief outlook on the fascinating but vast and complicated subject of *quantum gravity*; it is neither a comprehensive review nor a replacement for dedicated courses on the various subjects.

In this excursion, we address the following questions:

- Chapter 14:
 - Why to quantize gravity in the first place?
 - How do we quantized non-gravitational theories?
 - Why does this procedure fail for gravity?
 - How to circumvent these problems?
- Chapter 15:
 - What is the rationale of string theory?
 - Why does the quantization of the bosonic string only work in D = 26 spacetime dimensions?
 - Why is string theory a theory of quantum gravity?
 - Where does supersymmetry enter the picture?

Warning

The following conventions are widely used in the quantum gravity literature:

In this partwe work in units where c = 1 and $\hbar = 1$we use the sign convention $\eta_{\mu\nu} = (-1, +1, \dots, +1)$.

For example, the dispersion of a massive particle reads no longer $p^2 = m^2 c^2$ but $p^2 = -m^2$.



14. Why is quantizing Gravity hard?

i! Note that the Lorentz symmetry of SPECIAL RELATIVITY is not a problem for quantum theory (\leftarrow Chapter 7). For example, the quantum field theories that constitute the Standard Model of particle physics are all Poincaré invariant and fully consistent with SPECIAL RELATIVITY. \rightarrow

The problem of quantum gravity is the quantization of the metric tensor field $g_{\mu\nu}$ of GENERAL RELATIVITY.

1 Why to quantize gravity in the first place?

• Simple answer: Because everything else can be quantized!

Quantum electrodynamics	$\xrightarrow{h \to 0}$	Maxwell's electrodynamics
Quantum mechanics	$\xrightarrow{\hbar \to 0}$	Newton's classical mechanics
What?	$\xrightarrow{\hbar \to 0}$	Einstein's GENERAL RELATIVITY

The fact that every classical theory – *except* GENERAL RELATIVITY– can be understood as the classical limit of an underlying quantum theory suggests that the superposition principle is a fundamental feature of reality, and motivates the quest for a quantum theory of the gravitational field (= the metric).

- Extrapolation of GENERAL RELATIVITY and quantum theory \rightarrow Inconsistencies:
 - i | Quantum mechanics:

Heisenberg uncertainty: $\Delta x \Delta p \ge \frac{\hbar}{2} \to \langle p^2 \rangle \ge (\Delta p)^2 \ge (\hbar/2\Delta x)^2$ \triangleleft Relativistic particle: $E \sim cp \to E^2 \ge (\hbar c/2\Delta x)^2$

In words: Probing small distances requires high energies (e.g., particle colliders).

ii | GENERAL RELATIVITY:

... Energy = Gravitational mass: $M \sim E/c^2$

... Mass *M* concentrated in region $r_s = \frac{2GM}{c^2} \rightarrow$ Black hole & Event horizon

iii | Combing GENERAL RELATIVITY and quantum mechanics yields:

$$r_s \ge \frac{\hbar G}{\Delta x c^3} \tag{14.1}$$

Imagine you want to mark a point in space with precision δl by placing a particle there. Then the particle must have position uncertainty $\Delta x \sim \delta l$. For $\delta l \rightarrow 0$, the particle requires more an more energy until a black hole forms and its event horizon prevents you from interacting with the particle. This happens when $\delta l \leq r_s$, i.e., latest when

$$\delta l \leq \frac{\hbar G}{\delta l c^3} \quad \Rightarrow \quad \delta l \leq l_{\text{Planck}} := \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-35} \,\mathrm{m} \,.$$
 (14.2)



 \rightarrow One cannot localize anything beyond the Planck length.

What this semiclassical argument shows is the impossibility to "zoom in" on the Planck scale and find a world in which both GENERAL RELATIVITY and quantum mechanics remain valid unmodified.

The concept of space (and time) itself becomes inconsistent on the Planck scale l_{Planck} .

This argument goes back to MATVEI BRONSTEIN [300, 301]; he writes in 1936 in Ref. [301] (§4, p. 150):

Ohne eine tiefgehende Umarbeitung der klassischen Begriffe scheint es daher wohl kaum möglich, die Quantentheorie der Gravitation auch auf dieses Gebiet [der kleinen Abstände] auszudehnen.

For a historical account on the early days of quantum gravity and the role played by Bronstein see Ref. [302].

• An argument from reductionism:

Physics follows a reductionist approach to explain the world around us: All entities are split into smaller and smaller pieces that follow simpler and simpler laws (molecules \rightarrow atoms \rightarrow nuclei \rightarrow quarks). The complexity of macroscopic phenomena is then explained as the *emergent* behavior of many simple constituents. This approach has worked remarkably well in compressing the apparent complexity of the world into a few simple fundamental laws.

According to this view, the realm of the very small (studied by atomic and particle physics) is *fundamental*, everything else is *emergent*. But every single experiment that explored the realm of atomic or subatomic physics revealed a world goverened by the laws of quantum mechanics. There is no classical behaviour on subatomic scales! Thus, if we take the reductionist stance, we are forced to accept that *quantum mechanics rules the world*, and that our classical, macrospopic world is only an emergent perspective on this reality. Consequently, gravity should emerge from underlying quantum phenomena as well.

The fly in the ointment is that no one has ever observed *any* effect of gravity – be it classical or quantum – in any experiment small enough to be clearly dominated by quantum effects *because gravity is such a weak force*: To see quantum effects, the studied systems must be extremely small (on atomic scales); but then the involved masses are also tiny. Since the gravitational coupling constant G is orders of magnitudes smaller than the electromagnetic coupling, every experiment on atomic scales is dominated by electromagnetic forces, while the gravitational force is practically absent.

To date, the *smallest* object that showed measurable gravitational effects had a mass of $m \sim 0.5 \times 10^{-9}$ kg [303]. While this might seem light on everyday scales, it is still heavy on atomic scales: The *heaviest* object that showed quantum interference effects (\checkmark *double-slit experiment*) had a mass of $m \sim 4 \times 10^{-23}$ kg [304]. Because of decoherence (= coupling to the environment), it is experimentally extremely challenging to conduct experiments with relevant gravitational coupling while remaining coherent.

[You might wonder: $m \sim 0.5 \times 10^{-9}$ kg = 0.5 mg is quite "heavy". If one drops a particle of dust (which certainly weighs much less than 0.5 mg) in vacuum, it certainly falls to the ground. Doesn't this show that it interacts gravitationally? The answer is negative: This experiment only verifies the weak equivalence principle WEP, namely that everything – independent of its mass – is accelerated by g on the surface of Earth. The experiment reveals the spacetime curvature due to *Earth* by using the dust particle as a *test mass*. What is meant by



"interacts gravitationally" is really "acts as a *source* of gravity," i.e., creates its own curvature of spacetime.]

• Could gravity be intrinsically classical?

While it is certainly a majority view among physicists that gravity emerges from an underlying quantum theory, not everyone agrees on this. Roger Penrose, for example, advocates that "quantum mechanics must be gravitized." He denies that gravity has a quantum nature at all, and that the collapse of the wavefunction is an objective dynamical process – induced by gravity – that makes a unique, classical, macroscopic world emerge out of a microscopic quantum world [173]. This view is in direct contradiction to most other interpretations of quantum mechanics (collapse theories are not interpretations but *modifications* of quantum mechanics) like \uparrow *Everett's many-worlds interpretation* or \uparrow *decoherence theory*.

Recent proposals suggest methods to experimentally probe the relation between gravity and the quantum-classical boundary [305–308] (see Ref. [309] for a review). These proposals are based on recent (and foreseeable) technological advances in the control of quantum systems and precision measurement techniques. Since there will be no experiments on the Planck scale anytime soon, this alternative approach to assess the quantum nature of gravity is perhaps the most promising route forward.

Ignoring the lack of experimental evidence, let us henceforth assume that gravity emerges from an underlying quantum theory.

2 How do we quantized non-gravitational theories?

To understand why physicists struggle to quantize the field theory called GENERAL RELATIVITY, we must first understand how all the other fields are quantized:

Details: Any course on \uparrow Quantum field theory [20].

 $i \mid \triangleleft$ Relativistic field theory given by a Lagrangian:

$$\mathcal{L}(\phi, \partial \phi) = \underbrace{\underbrace{(\partial^{\mu} \phi)(\partial_{\mu} \phi) - m^{2} \phi^{2}}_{\text{Quadratic part}} - \underbrace{\lambda \phi^{n \ge 4} + \dots}_{\text{Non-quadratic part}}_{\text{Non-quadratic part}}$$
(14.3)

We use here exemparily a scalar field ϕ ; the fields in the Standard Model are more complicated, but the concepts are the same.

 \rightarrow Action:

$$S[\phi] = \int d^d x \, \mathcal{L}(\phi, \partial \phi) \equiv S_0[\phi] + S_{\text{int}}[\phi]$$
(14.4)

For now, we consider arbitrary spacetime dimensions d.

So far, this defines a *classical* field theory with equations of motion $\delta_{\phi} S \stackrel{!}{=} 0$.

ii | The corresponding quantum theory is most conveniently defined via a \downarrow path integral:



Define \checkmark Scattering amplitudes via \checkmark path integrals:

$$\mathcal{M} = \underbrace{\langle \phi_{\text{out}} | \phi_{\text{in}} \rangle}_{\text{Scattering}} \sim \underbrace{\int_{\phi_{\text{in}}}^{\phi_{\text{out}}} \mathcal{D}\phi}_{\text{amplitude}} \underbrace{\overbrace{e^{\frac{i}{\hbar}S[\phi]}}^{\text{Phase}}}_{\text{Sum over all evolutions}}$$
(14.5)

- You can think of the initial (final) field configuration φ_{in} (φ_{out}) as state that encodes the positions and momenta of many particles long before (after) they interact/collide. The scattering amplitude *M* is then the probability amplitude of this particular process happening. Such quantities can therefore be measured at particle colliders where such scattering experiment are performed.
- The path integral makes Feynman's interpretation of quantum mechanics explit, according to which all possible evolutions that connect an initial state φ_{in} with a final state φ_{out} happen simultaneously. The probability of the transition φ_{in} → φ_{out} is then given by the (modulus square) of the sum of phases, each of which depends on the action of the particular path taken. Why? Because for ħ → 0 this construction yields as allowed transitions only the ones connected by trajectories that satisfy the classical equations of motion. The path integral therefore has the correspondence principle that connects quantum with classical physics built in.
- The benefit of the path integral approach over a canonical quantization via a Hamiltonian operator is that the former is based on the Lagrangian which, for a relativistic field theory, is a Lorentz scalar. This makes the path integral quantization manifestly Lorentz covariant. By contrast, this symmetry is not manifest in a canonical quantization scheme, since the Hamiltonian is not a scalar but the zero-component of a 4-vector (the energy-momentum vector).

Problem:

Without specifying the path integral, this is not a mathematically well-defined theory; it is only a (physically motivated) *sketch* of a theory.

This means that one must operationally define how exactly the "sum over all trajectories" is to be evaluated:

- iii | How to compute the path integral Eq. (14.5)?
 - It is hard to mathematically implement an integral over "all smooth functions φ". A first step is therefore to Fourier transform all fields and parametrize them by their Fourier components (a countable infinite set of real numbers). The path integral can then be performed by integrating over each of these Fourier components separately:
 - \rightarrow Fourier transform:

$$\begin{cases} \phi(x) \\ \mathcal{D}\phi \end{cases} \longrightarrow \begin{cases} \hat{\phi}_k \\ \prod_k \mathrm{d}\hat{\phi}_k \end{cases}$$
(14.6)



(2) If $S_{int} \equiv 0$, the exponential includes only contributions from S_0 which are, by definition, *quadratic* in the fields (and the Fourier components). The integrals to be evaluated are then *Gaussian* and can be computed exactly. Such theories describe the propagation of particles that do not interact, hence they are called *free theories*. How the particles propagate is determined by their \checkmark *propagator*, which can be directly computed from the Gaussian path integral and the free action S_0 .

Interesting physics (= scattering) happens only when $S_{int} \neq 0$; in this case, the integrals are no longer Gaussian and one must resort to *perturbative methods* to find approximate solutions of the path integral:

 \rightarrow Expand exponential in coupling parameter λ of interactions:

$$\mathcal{M} \sim \int_{\phi_{\text{in}}}^{\phi_{\text{out}}} \mathcal{D}\phi \underbrace{\underbrace{e^{\frac{i}{\hbar}S_{0}[\phi]}}_{Gaussian}}_{\Rightarrow \text{ Propagator}} \underbrace{\left[1 + \frac{i}{\hbar}S_{\text{int}} + \left(\frac{i}{\hbar}S_{\text{int}}\right)^{2} + \dots\right]}_{Perturbation \text{ expansion in }\lambda} (14.7)$$

If the coupling constant $\lambda \propto S_{int}$ of the interaction is small, this approximation yields good results in low orders of the expansion.

This is true for quantum electrodynamics (QED), but not in the low-energy regime of quantum chromodynamics (QCD) which makes the latter much harder to work with.

In summary, the path integral can be evaluated as a perturbation series. These calculations are complicated, first because of the combinatorial problem to identify the different terms of the expansion that must be evaluated, and second, because evaluating the integrations associated to each of these terms is hard.

The first problem (writing down all terms up to a given order of the expansion) can be significantly simplified by using the technique of \uparrow *Fenyman diagrams*:

 \rightarrow Perturbation theory \rightarrow Feynman diagrams:



- Each Feynman diagram can be translated via a dictionary of ↑ *Feynman rules* into a mathematical expression (typically including integrals) that must be evaluated. The infinite sum of all these expressions converges to the scattering amplitude.
- The amplitude is specified by the "legs" of the diagram: Say two particles with momenta $k_{\rm in}$ and $q_{\rm in}$ collide and scatter into two (potentially different) particles with momenta $k_{\rm out}$ and $q_{\rm out}$. Quantum mechanics (via the path integral) tells us that the total amplitude of this process is the sum of the amplitudes of all possible processes consistent with these boundary conditions. The interaction Lagrangian $\mathcal{L}_{\rm int}$ of the theory specifies the rules for allowed processes; these rules can be condensed into a set of \uparrow *Feynman rules* specific for the theory. In general, each Feynman graph consists of *vertices* that



come from S_{int} and contribute one coupling constant λ to the overal expression. The links between the vertices (excluding the "legs" that stick out and are fixed by the boundary conditions) correspond to particles propagating between these interactions. Mathematically, each link corresponds to a \checkmark *propagator* of the free theory.

iv Momentum conservation demands that the sum of in- and outgoing momenta at each vertex of a Feynman diagram add up to zero. It is now easy to check that these constraints, together with the fixed external momenta k_{in} , q_{in} , k_{out} , q_{out} do *not* fix the momenta of all propagators (links) if Feynman diagrams contain *loops*. This makes sense: the particle propagating along the loop can have *any* momentum without violating energy-momentum conservation at the vertices. Since we do not measure these particles, the path integral tells us that we must add up all possible values of these "loop momenta":

Path integral \rightarrow Integrate over all *undetermined* momenta \rightarrow *Loop* integrals

- \rightarrow *Problem:* Divergent expressions for $k \rightarrow \infty$ in loops \rightarrow UV-divergences
 - Remember that large momenta k → ∞ correspond to small distances and high energies; the limit k → ∞ is therefore called *** UV-limit* and the corresponding divergences *** UV-divergences*.
 - The occurence of these divergences is rather generic, and not specific to particularly "problematic" quantum field theories. One can interpret UV-divergences as indicators for their breakdown at very small distances (= high energies); i.e., quantum field theories are presumably *effective* descriptions of some other (UV-finite) theory that we do not know. In this regard, they are similar to the singularities of GENERAL RELATIVITY in that both can be interpreted as mathematical artifacts that signal the inconsistency (and thereby invalidity) of the theory in some domain.

The crucial question is whether these UV-divergencies make the whole endeavour (to describe particles by quantum field theories) a lost cause? After all, if computations yield only infinite results, we cannot make predictions about anything ©.

 \rightarrow Temporary fix: Introduce momentum cutoff $\Lambda < \infty$ in all divergent integrals:

(This is called a \uparrow *regularization*.)

$$\int_0^\infty \mathrm{d}k \quad \mapsto \quad \int_0^\Lambda \mathrm{d}k \tag{14.8}$$

This certainly removes all UV-divergencies and makes your results (scattering amplitudes) finite. The problem is that these now depend on the *unphysical* cutoff Λ , so that they cannot be measurable quantities anymore! We clearly didn't solve the problem but only masked it.

 \rightarrow *Idea*:

Can we "hide" the terms that diverge for $\Lambda \to \infty$ in unphysical parameters?

The answer is "Yes" and the procedure is called \uparrow *renormalization*. To get a feeling for the conditions that must be met for this to work, we must quantify the divergence of Feynman diagrams a bit more carefully:

v | Superficial degree of divergence:

The following line of arguments might seem sloppy; there are more rigorous derivations that come to the same conclusion but require more input from \uparrow *quantum field theory* [20].

a | Recall: $\hbar = c = 1 \rightarrow \text{Compton wavelength: } \lambda_c = \frac{h}{mc} = \frac{2\pi}{m}$ $\rightarrow \text{Dimension of length: } [\lambda_c] = M^{-1} (M: \text{dimension of mass})$



- **b** | Dimension of action: [S] = 1 (since $\hbar = 1$)
- $c \mid S = \int d^d x \, \mathcal{L}$ and $[d^d x] = M^{-d} \to \text{Dimension of Lagrangian: } [\mathcal{L}] = M^d$ Since all dimensions can be expressed in M, we say that " \mathcal{L} has (mass) dimension d".
- **d** | From Eq. (14.3) follows (use $[\partial] = M$):

$$[\phi] = M^{\frac{d-2}{2}}$$
 and $[\lambda] = M^{d-n\frac{d-2}{2}}$ (14.9)

e | ⊲ Amplitude *F* of *single Feynman diagram F* with *N* external lines: Has same dimension as (hypothetical) single interaction $ηφ^N → [η] = M^{d-N\frac{d-2}{2}}$:



$$\to [\mathcal{F}] = [\eta] = M^{d - N\frac{d-2}{2}}$$

f | Let the Feynman diagram F have V interaction vertices \rightarrow

$$\mathcal{F} \stackrel{\Lambda \to \infty}{\sim} \lambda^V \Lambda^D$$
 (14.10)

D: ** Superficial degree of divergence of F

After performing the integrals to compute the amplitude \mathcal{F} from the Feynman diagram F, the only dimensionful quantities left are V powers of the coupling constant λ (one for each interaction vertex of the diagram) and D orders of the momentum cutoff Λ . In the limit $\Lambda \to \infty$, the asymptotic expression of \mathcal{F} must therefore scale as $\lambda^V \Lambda^D$; note that this is an implicit *definition* of D. If D > 0, the contribution \mathcal{F} has a UV-divergence.

 \rightarrow (use $[\Lambda] = M$)

$$[\lambda]^{V}[\Lambda]^{D} = [\mathcal{F}] = M^{d-N\frac{d-2}{2}}$$
(14.11a)

$$\Rightarrow V \log_{M} [\lambda] + D = d - N \frac{d-2}{2}$$
(14.11b)

 $\log_M [\lambda]$: ** Mass dimension of the coupling constant λ

 \rightarrow We find for the superficial degree of divergence of $\mathcal{F}\colon$

$$D = d - \underbrace{\underbrace{\log_{M} [\lambda] \cdot V}_{\text{Eq. (14.9):}}_{d - n\frac{d-2}{2}} } \underbrace{\underbrace{\log_{M} [\lambda] \cdot V}_{\text{Depends on}}_{amplitude M} (14.12)$$

We can conclude for d > 2: $(D < 0 \Rightarrow \text{Amplitude } \mathcal{F} \text{ converges for } \Lambda \rightarrow \infty.)$

- If the mass dimension of *λ* is zero or positive, diagrams with more "legs" (in- and out-going particles) become less divergent and eventually converge.
- Whether diagrams with more interaction vertices (= higher order of perturbation theory) start to converge or diverge depends on the *sign* of the mass dimension of the coupling constant.

$\mathbf{g} \mid \rightarrow \underline{\text{Classification:}}$

This lead to the following classification of interacting quantum field theories:

- $\log_{M}[\lambda] > 0$ (Coupling constant has *positive* mass dimension.)
 - \rightarrow Only a finite number of *Feynman diagrams* (superficially) diverge \bigcirc \bigcirc .
 - \rightarrow ** Super-renormalizable theory

While this case is in some sense optimal, it is less relevant for interesting quantum field theories like the Standard Model; thus it plays no role in the following.

• $\log_{M}[\lambda] = 0$ (Coupling constant is *dimensionless*.)

 \rightarrow Only a finite number of *amplitudes* (superficially) diverge \odot .

Here "amplitudes" refer to infinite sums of Feynman diagrams, classified by their number of external "legs" *N*.

$\rightarrow *$ Renormalizable theory

Most interesting quantum field theories (like the Standard Model) are of this type.

- $\log_{M}[\lambda] < 0$ (Coupling constant has *negative* mass dimension.)
 - \rightarrow All amplitudes diverge at sufficiently high order in perturbation theory \odot .

This follows because every amplitude has contributions from Feynman diagrams with arbitrary many vertices V so that – independent of N – the superficial degree of divergence D becomes positive for high-enough orders of perturbation theory.

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\rightarrow ** Non-renormalizable theory
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h <u>Renormalization:</u>

The following procedure of renormalization works (provably) for renormalizable (and super-renormalizable) theories because it assumes that a finite number of *amplitudes* are UV-divergent. One can then show rigorously that all UV-divergencies of such theories can be traced back to this finite set of divergent amplitudes (\uparrow *Weinberg theorem*). This implies that if these UV-divergences can be "cured", all scattering amplitudes of the theory become UV-finite. The procedure to "cure" a finite number of UV-divergent amplitudes is called *renormalization* and goes as follows:

(1) Start with a regularized (UV-cutoff Λ) (super-)renormalizable theory.

 \rightarrow There is only a finite number of UV-divergent amplitudes.

(2) For each divergent amplitude, "add" a *counter term* with unphysical bare parameter to the Lagrangian.

Strictly speaking you don't *add* the counterterms: You *split* the terms with bare parameters into (fixed) physical parameters and (UV-divergent) unphysical parameters; the latter are the counter terms.

(3) One can then show that all UV-divergences can be absorbed by these (unobservable & unphysical) *bare* parameters by fixing their corresponding (observable & physical) *renormalized* parameters. (4) You end up with a theory that yields finite scattering amplitudes in the limit $\Lambda \to \infty$ and reproduces the observed physical parameters every order of perturbation theory \odot .

The UV-divergencies are now "hidden" in the *bare* parameters and make them diverge in the limit $\Lambda \to \infty$. But this is not a problem because they do not affecet observable quantitites.

i | This means in particular:

<u>Renormalizable</u> quantum field theories allow for the computation of predictions by fixing a <u>finite</u> number of physical low-energy parameters.

- For QED this would be the physical electron mass *m* and charge *e*; in the Standard Model there are about 18 such parameters that determine the masses and interactions of elementary particles. These must be *measured* and can then be used to make predictions about scattering amplitudes. This is why the Standard Model cannot *predict* the masses of elementary particles (e.g., the Higgs boson).
- You might wonder: If we must use masses and interaction strengths of particles as *input* of our theories, what is it actually good for? Remember that the predictions of quantum field theories are *scattering amplitudes* \mathcal{M} . These are complicated functions of the momenta (both absolute value and direction) of the in- and outgoing particles. It is this highly non-trivial functional form that is predicted by the theory which can be compared to scattering experiments. In the case of the Standard Model, theory and experiment match perfectly!
- Here is a very accessible explanation of renormalization by John Baez:

https://math.ucr.edu/home/baez/renormalization.html

But conversely:

We do not know how to define and/or extract predictions from <u>non-renormalizable</u> quantum field theories.

For a non-renormalizable QFT we would have to add infinitely many counter terms and fix infinitely many physical parameters to absorb the infinitely many UV-divergent amplitudes. This makes such theories useseless and conceptually ill-defined.

- vi | Examples:
 - \triangleleft Scalar field Eq. (14.3) in d = 4 with n = 4:

$$\log_{M}[\lambda] \stackrel{14.9}{=} d - n \frac{d-2}{2} = 0$$
(14.13)

 $\rightarrow \phi^4$ -theory is renormalizable in d = 3 + 1 spacetime dimensions.

• \triangleleft Quantum electrodynamics (QED) in d = 4:

$$\mathcal{L}_{\text{QED}}(A, \partial A, \Psi, \partial \Psi) = \underbrace{\overline{\Psi}(i\partial - m)\Psi}_{\text{Free fermions}} - \underbrace{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{Free photons}} - \underbrace{e\overline{\Psi}\gamma^{\mu}\Psi A_{\mu}}_{\text{Free photons}}$$
(14.14)



- e: Coupling constant (= electric charge of fermion Ψ)
- \rightarrow Dimensional analysis:

$$[A] \stackrel{\circ}{=} M^{\frac{d-2}{2}} = M^1, \ [\Psi] \stackrel{\circ}{=} M^{\frac{d-1}{2}} = M^{3/2} \xrightarrow{14.14} \log_M[e] \stackrel{\circ}{=} 0$$
(14.15)

- \rightarrow QED is (superficially) renormalizable in d = 3 + 1 spacetime dimensions \odot .
 - Note that we only showed that QED is *superficially* renormalizable by essentially dimensional anlysis (= power counting). This is not a rigorous proof that QED really is renormalizable to all orders of perturbation theory it is only suggestive that it might be. However, one can show rigorously that QED is renormalizable to all orders of perturbation theory, although such proofs are very technical [310].
 - The same is true for the strong interactions of quantum chromodynamics (QCD) and the electroweak interactions: One (more precisely: GERARD 'T HOOFT) can prove (to the standards of theoretical physicists) that the full Standard Model is renormalizable to all orders of perturbation theory [311, 312]. This yields operationally well-defined quantum field theories for three of the four fundamental forces of nature:
 - * Electromagnetic force \checkmark
 - ∗ Weak force ✓
 - * Strong force \checkmark
- **3** Why does this procedure fail for gravity?

After this preliminary work, the question to answer is clear:

Is GENERAL RELATIVITY- defined by the Einstein-Hilbert action - renormalizable?

i \triangleleft Pure gravity \rightarrow Einstein-Hilbert action Eq. (12.54) with $\kappa := \sqrt{16\pi G}$:

$$S_{\rm EH}[g] = \frac{1}{\kappa^2} \int d^4x \sqrt{g} R \tag{14.16}$$

Problem: This is not in the form $S_0 + S_{int}$ required for perturbation theory!

This means that we cannot simply declare the gravitational constant $\kappa^2 \propto G$ as the coupling parameter and draw conclusions from its mass dimension.

Note that due to the non-linearity of the Einstein field equations, pure gravity (without matter) is already an interacting field theory that must be solved perturvatively.

ii | Expand Eq. (14.16) around static background spacetime:

$$g_{\mu\nu} \equiv \underbrace{\eta_{\mu\nu}}_{\text{Background}} + \underbrace{\kappa h_{\mu\nu}}_{\text{Quantum}}$$
(14.17)

The choice to rescale the field by κ is covenient to bring the action \rightarrow *below* into the standard form needed for perturbation theory.

Eq. (14.17) \rightarrow

$$\sqrt{g} \stackrel{\circ}{=} 1 + \frac{\kappa}{2} h_{\mu}{}^{\mu} + \frac{\kappa^2}{8} h_{\mu}{}^{\mu} h_{\nu}{}^{\nu} - \frac{\kappa^2}{4} h_{\mu\nu} h^{\mu\nu} + \mathcal{O}(h^3)$$
(14.18a)

$$R \stackrel{\circ}{=} \kappa \partial^2 h^{\mu}{}_{\mu} - \kappa \partial_{\mu} \partial_{\nu} h^{\mu\nu} + \mathcal{O}(h^2) \tag{14.18b}$$



Expanding the Einstein-Hilbert action in the fluctuations $h_{\mu\nu}$ yields:

$$S_{\rm EH}[h] \stackrel{\circ}{=} \int d^4x \left[\underbrace{\begin{cases} \frac{1}{2} \partial^{\nu} h_{\mu\nu} \partial_{\sigma} h^{\mu\sigma} - \frac{1}{2} \partial^{\nu} h_{\mu\nu} \partial^{\mu} h_{\sigma}^{\sigma}}{+\frac{1}{4} \partial^{\mu} h_{\nu}^{\nu} \partial_{\mu} h_{\sigma}^{\sigma} - \frac{1}{4} \partial_{\sigma} h_{\mu\nu} \partial^{\sigma} h^{\mu\nu}} \right] + \underbrace{\kappa(\partial h)^2 h + \dots}_{\mathscr{X}_{\rm int} \to \, {\rm Interactions}} \right]$$
(14.19)
$$\mathscr{X}_0 \to {\rm Graviton \, propagator}$$

- Note that the κ^2 cancels in the quadratic terms, while coupling constants survive in the higher-order interaction terms (this is why we rescaled the field in the first place).
- L₀ should be familiar: You studied this theory on ⇒ Problemset 1 as a first attempt at a relativistic theory of gravity.
- Here we only write one of the lowest-order interaction terms exemplarily (omitting indices); it is useful to derive the mass dimension of κ from the mass dimension of h_{µν} (which, in turn, is fixed by the non-interacting quadratic part). That an interaction term of this form exists follows from Eq. (14.18) via partial integration; see Ref. [313] for details.
- iii | In principle you can start now to derive the Feynman rules from Eq. (14.19) to compute scattering amplitudes of the Einstein-Hilbert quantum gravity.

When evaluating the path integral (e.g., to compute the propagator), a complication arises: Eq. (14.19) is a gauge theory [due to the diffeomorphism invariance of Eq. (14.16); \bigcirc Problemset 6 and \leftarrow Section 13.4 and also Eq. (11.103)]. If one naïvely calculates the path integral of a gauge theory, all expressions blow up because the gauge orbits don't oscillate and produce infinities. To count physically distinct field configuration only once, one has to add a gauge-fixing term to the Lagrangian. In doing so, one encounters a functional determinant that leads to new, artificial fields called \uparrow *Fadeev-Popov ghosts*. They are a necessary mathematical nuisance and expand the list of Feynman rules & diagrams. Because of this, the interactions of the Einstein-Hilbert action, and the fact that $h_{\mu\nu}$ is a rank-2 tensor field, enumerating and evaluating Feynman diagrams of this theory is not fun (even ignoring potential UV-divergences).

By fixing the gauge appropriately, one can compute the graviton \uparrow *Feynman propagator* from the quadratic part \mathcal{L}_0 of Eq. (14.19) (\uparrow Ref. [313]) and finds

$$D^{F}_{\mu\nu\alpha\beta}(k) \stackrel{*}{=} \frac{\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}}{k^2 + i\varepsilon} \,. \tag{14.20}$$

This is nice but eventually futile because the theory contains infinitely many UV-divergencies that one cannot control (\rightarrow *next*).

iv | In Eq. (14.19), κ plays the role of the coupling constant that controls graviton-graviton scattering. Hence its mass dimension determines the renormalizability of GENERAL RELATIVITY:

Dimensional analysis of Eq. (14.19):

$$[h] \stackrel{\circ}{=} M^{\frac{d-2}{2}} = M^1 \quad \xrightarrow{14.19} \quad \log_M [\kappa] = -1 < 0$$

 $\rightarrow \kappa$ has negative mass dimension!

Here is a sanity check: The gravitational constant can be written as $G = \hbar c/m_{\text{Planck}}^2$ with Planck mass m_{Planck} . With $\hbar = 1 = c$ it follows that G has dimensions of M^{-2} , consistent with our result above for $\kappa = \sqrt{16\pi G}$.



v | Conclusion:

GENERAL RELATIVITY is superficially non-renormalizable 🙁 .

i! This does not *prove* that the Einstein-Hilbert action is not renormalizable; but if it were, some sort of unexpected cancellations/symmetries would be necessary (\rightarrow next).

- vi | Known results:
 - One can show that at one-loop level, *pure* Einstein gravity (no matter fields) has quite unexpectedly! no UV-divergences [314].
 - However, when matter is involved, the one-loop diagrams of the Einstein-Hilbert action become UV-divergent, see Ref. [314] for the example of a scalar field (see also references in Ref. [315]).
 - Unfortunately, pure Einstein gravity is proven to be UV-divergent at two-loop level [315, 316]. This suggests that no unexpected cancellations/symmetries make the theory renormalizable.
 - It is therefore widely believed (though, to my knowledge, not proven) that no unexpected cancellations occur beyond two-loop order; therefore, Einstein gravity seems to be perturbatively non-renormalizable.
 - For an alternative (and pedagogic) explanation for the non-renormalizability of GEN-ERAL RELATIVITY see Ref. [317].
- vii | In a nutshell:
 - GENERAL RELATIVITY is different from the field theories of the Standard Model in that its coupling constant has *negative* mass dimension.
 - As a consequence, the only systematic procedure to operationally define quantum field theories (namely: renormalization) does *not* work for GENERAL RELATIVITY.
 - However, there is no rigorous proof that GENERAL RELATIVITY *cannot* be quantized by another (non-perturbative?) method.

4 | How to circumvent this problem?

Since the conventional method to study quantum field theories fails for GENERAL RELATIVITY, one needs new methods to tackle the problem. There are two very different approaches:

• Approach 1:

Try to "rediscover" GENERAL RELATIVITY in some limit of a UV-finite quantum field theory.

Most prominent contender: \rightarrow *String theory* (Chapter 15)

String theory does not only claim to provide a path for quantizing the gravitational field, but also strives to explain the existence and interactions of all other particles ("matter") in a single, consistent framework. Its aspiration is therefore not only to be a theory of quantum gravity but a "theory of everything" (ToE). The emergence of GENERAL RELATIVITY is only one of its aspects.



• Approach 2:

Try to come up with an alternative (non-perturbative?) method to quantize the metric field of GENERAL RELATIVITY.

Quantum loop gravity "takes GENERAL RELATIVITY seriously" and directly tries to quantize the theory by discretizing its degrees of freedom and proposing a UV-finite action & path integral that determine the dynamics of the geometry of spacetime. Quantum loop gravity (in its basic incarnation) does not contain matter fields; it is "just" a theory of quantum gravity. In contrast to string theory, quantum loop gravity does not claim to be a "theory of everything". ↓Lecture 32 [06.08.24]



15. Sneak Peek: Bosonic String Theory

This primer on bosonic string theory is an amalgamation of various sources, mostly lecture scripts (by Carmen A. Núñez, Arthur Hebecker, and David Tong), and the introductory textbook by BARTON ZWIEBACH [7].

- 1 | What is the rationale of string theory?
 - Hypotheses:
 - \triangleleft Fixed background spacetime $g_{\mu\nu}$

In contrast to GENERAL RELATIVITY, string theory has *no* manifestly background independent formulation. The dynamics of the spacetime geometry is described by quantum fluctuations (of gravitons) on top of a classical, static background metric.

- Postulate elementary entities: Relativistic strings

The strings of string theory are *elementary entities* that propagate (and interact) on the fixed background spacetime; think of them as "rubber bands," i.e., they can be stretched. These strings can be closed (= loops) or open (\rightarrow *later*). Note that strings are not emergent from other degrees of freedom – string theory does not explain were strings come from.

- Postulate an action that determines ...
 - * ... dynamics of single string and
 - * ... interaction between strings.

This action is motivated as a generalization of the action of a free point particle.

- Hope:
 - * Quantized excitations of strings = Fundamental particles
 - * Joining/splitting of strings = Fundamental interactions between particles
 - * No point-like particles \rightarrow No UV-divergences for $\Lambda \rightarrow \infty$
 - * Classical limit \rightarrow general relativity
- Intuition:

Closed strings can oscillate. Their lowest-frequency modes look as follows:





Note that the left oscillation is invariant under rotations about the symmetry axis, whereas the two modes on the right transform into each other under rotations by $\pm 45^{\circ}$ – just as a gravitational wave with helicity ± 2 would (\leftarrow Section 13.4).

\rightarrow We should expect a graviton state from *closed* strings!

- At this point, it is unclear whether these modes are truly *massless* after quantization (as required for excitations of a long-range interaction like gravity).
- The "breathing mode" corresponds to a scalar particle called → *dilaton* which comes along with the graviton in string theory; this means that string theory actually predicts a *scalar-tensor theory* of gravity (< Section 12.3). For consistence with reality (in GEN-ERAL RELATIVITY there is no dilaton), there must be a mechanism to render the dilaton massive (= short-ranged).

2 | How to identify gravity?

Since string theory follows Approach 1, we will not start from GENERAL RELATIVITY and the Einstein-Hilbert action. But how do we know then that string theory is actually a quantum theory of *gravity*? How do we identify the "gravity" part? We could of course hope that the Einstein field equations fall into our lap, but this is naïve. String theory is a *quantum* theory and the EFEs are *classical* – and the classical limit of a quantum theory is often not evident at all.

A quantum theory of gravity should somehow quantize the gravitational field of GENERAL RELA-TIVITY, i.e., the metric tensor field $g_{\mu\nu}$ that describes the geometry of spacetime. To identify gravity is then tantamount to identifying the (excitations/quanta of the) metric. Hence we arrive at the fundamental question what makes a field "the metric" in the first place? Up to now we always postulated the existence of a Riemannian manifold equipped with a metric tensor. We show below that this is not necessary. A field does not become "the metric" by *declaration*, but by the way it *interacts* with other fields. This yields an operational method to identify gravity in any theory:

- i Observation (Section 13.4): Gravitational waves ...
 - ... propagate with the speed of light.
 - ... have helicity ± 2 .
 - \rightarrow Gravitons should be *massless spin-2* particles.
 - \rightarrow If we want to "find gravity" we should search for *massless rank-2* tensor fields $h_{\mu\nu}$.

Since $1 \otimes 1 = 0 \oplus 1 \oplus 2$ (irreducible representations of SO(3), \checkmark angular momentum coupling), spin-2 particles are described by rank-2 tensor fields: $h_{\mu\nu}$. This makes sense if you recall Eq. (13.183) which directly links the two spacetime-indices of the tensor field with the helicity ± 2 under spatial rotations. It also makes sense because a metric tensor $g_{\mu\nu}$ is a (symmetric) rank-2 tensor field. That the field is *massless* is also directly related to the fact that gravity is a *long-range* interaction (like electromagnetism).



- \rightarrow Question: What makes a massless rank-2 tensor field "the metric"?
- ii | \triangleleft Massless rank-2 tensor field $h_{\mu\nu}$ on static Minkowski space $\eta_{\mu\nu}$:

The field $h_{\mu\nu}$ is not (yet) the metric – it is just an ordinary tensor field on Minkowski space! Massless \rightarrow Only quadratic derivatives allowed $\stackrel{\circ}{\rightarrow}$ Four possible terms: (Details: \uparrow Ref. [313])

$$\partial^{\nu}h_{\mu\nu}\partial_{\sigma}h^{\mu\sigma}, \quad \partial^{\nu}h_{\mu\nu}\partial^{\mu}h_{\sigma}^{\ \sigma}, \quad \partial^{\mu}h_{\nu}^{\ \nu}\partial_{\mu}h_{\sigma}^{\ \sigma}, \quad \partial_{\sigma}h_{\mu\nu}\partial^{\sigma}h^{\mu\nu}$$
(15.1)

- All other conceivable contractions are related to these terms modulo partial integration (= total derivatives).
- Terms without derivative like $h_{\mu\nu}h^{\mu\nu}$ lead to a constant shift in the relativistic dispersion of the field theory, i.e., to *massive* excitations.
- iii | \triangleleft Additional matter fields ϕ : $\mathcal{L}_{Matter}(\phi, \partial \phi) \rightarrow \text{HEMT } T_{Matter}^{\mu\nu}$ with $\partial_{\nu} T_{Matter}^{\mu\nu} \doteq 0$ Assumption:

 $h_{\mu\nu}$ couples to energy & momentum via $T^{\mu\nu}_{\text{matter}}$.

This is our *only* assumption that makes the massless rank-2 tensor field "special". We will see that this assumption (plus some self-consistency condition) is all that is needed to elevate $h_{\mu\nu}$ from an ordinary tensor field to "the metric".

 \rightarrow Most general action:

$$S[h,\phi] := \int d^{4}x \left[\begin{cases} a \,\partial^{\nu}h_{\mu\nu}\partial_{\sigma}h^{\mu\sigma} + b \,\partial^{\nu}h_{\mu\nu}\partial^{\mu}h_{\sigma}^{\ \sigma} \\ +c \,\partial^{\mu}h_{\nu}^{\ \nu}\partial_{\mu}h_{\sigma}^{\ \sigma} + d \,\partial_{\sigma}h_{\mu\nu}\partial^{\sigma}h^{\mu\nu} \end{cases} + \frac{\kappa}{2}h_{\mu\nu}T_{\text{Matter}}^{\mu\nu} + \mathcal{L}_{\text{Matter}} \right]$$
(15.2)

with arbitrary couplings $a, b, c, d, \kappa \in \mathbb{R}$.

Since $T_{\text{Matter}}^{\mu\nu}$ is symmetric, the tensor field is *w.l.o.g.* symmetric as well: $h_{\mu\nu} = h_{\nu\mu}$.

iv $| \stackrel{\circ}{\rightarrow}$ Equation of motion for $h_{\mu\nu}$:

$$\underbrace{[\dots\partial^2 h\dots]^{\mu\nu}}_{\text{Depends on } a, b, c, d} = \frac{\kappa}{2} T_{\text{Matter}}^{\mu\nu}$$
(15.3)

This EOM is linear in $h_{\mu\nu}$ since the action is quadratic.

 \rightarrow Energy-momentum conservation:

$$\partial_{\nu}[\dots\partial^2 h\dots]^{\mu\nu} = \frac{\kappa}{2}\partial_{\nu}T^{\mu\nu}_{\text{Matter}} \doteq 0$$
 (15.4)

We want this to be *identically* satisfied for $h_{\mu\nu}$:

$$\partial_{\nu}[\dots\partial^2 h\dots]^{\mu\nu} \equiv 0 \quad \stackrel{\circ}{\Rightarrow} \quad a = \frac{1}{2}, \ b = -\frac{1}{2}, \ c = \frac{1}{4}, \ d = -\frac{1}{4}$$
 (15.5)

The solution is unique up to a global rescaling that can be absorbed into κ via a rescaling of the tensor field.

This means that energy momentum conservation for the matter fields is *enforced* by the coupling to $h_{\mu\nu}$ rather than a constraint on the dynamics of $h_{\mu\nu}$ itself.



v | Hence we end up with the most general action that meets our requirements:

$$S[h,\phi] = \int d^4x \Big[\underbrace{\mathcal{L}_0(h,\partial h)}_{\epsilon \text{ Eq. (14.19)}} + \underbrace{\kappa}_2 h_{\mu\nu} T_{\text{Matter}}^{\mu\nu} + \mathscr{L}_{\text{Matter}} \Big]$$
(15.6)

i! The quadratic Lagrangian of the tensor field is identical to \mathcal{L}_0 in Eq. (14.19) (in which $h_{\mu\nu}$ is the deviation of the metric $g_{\mu\nu}$ from Minkowski space $\eta_{\mu\nu}$). Recall that Eq. (14.19) was derived from the Einstein-Hilbert action Eq. (14.16). This result shows that the seemingly arbitrary structure of \mathcal{L}_0 in Eq. (14.19) is actually not arbitrary at all – it is the *only* possible quadratic action for a massless rank-2 tensor field that couples to a conserved current.

vi | Inconsistency of Eq. (15.6):

Eq. (15.6) is conceptually inconsistent because it implies the existence of two "types" of energy & momentum: The first type is the energy & momentum of matter fields, which couples to $h_{\mu\nu}$. But $h_{\mu\nu}$ is a dynamical field and therefore carries energy & momentum of its own – to this second type $h_{\mu\nu}$ does *not* couple. This doesn't make sense and we should get rid of this two-class society of energy & momentum:

This is not just a conceptual inconsistency: Enforcing energy-momentum conservation on a subsystem (the matter fields) while coupling this subsystem to another dynamical field (the tensor field) cannot be consistent (i.e., allow for solutions of the combined EOMs). You studied this on \bigcirc Problemset 1.

\rightarrow Assumption (updated):

 $h_{\mu\nu}$ couples to energy & momentum of all fields (including itself!).

 \rightarrow We therefore should replace the matter HEMT by the total HEMT of the theory:

$$T_{\text{Matter}}^{\mu\nu} \mapsto T^{\mu\nu} := T_{\text{Matter}}^{\mu\nu} + T_h^{\mu\nu}$$
(15.7)

But this makes the Lagrangian self-referential: $T_h^{\mu\nu}$ is computed from the part of the Lagrangian that includes $h_{\mu\nu}$ – which includes $T_h^{\mu\nu}$. Thus you will be forced to add higher and higher order terms of $h_{\mu\nu}$ to make $h_{\mu\nu}$ couple to its own energy-momentum tensor. This infinite series can be summed and yields a new, non-linear theory for the tensor field $h_{\mu\nu}$: the Einstein-Hilbert action!

One can show [102] $\stackrel{*}{\rightarrow}$

 $S[h, \phi]$ becomes the Einstein-Hilbert action of the field $g_{\mu\nu} := \eta_{\mu\nu} + h_{\mu\nu}$ which couples minimally to \mathcal{L}_{Matter} :

Eq.
$$(15.6) \mapsto \text{Eq.} (12.65)$$

This means that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ becomes the metric – while the static background $\eta_{\mu\nu}$ becomes unobservable – simply by demanding that the massless tensor field $h_{\mu\nu}$ couples to the energy & momentum of all fields.

vii | Conclusion:

Massless tensor
$$h_{\mu\nu}$$

... that couples to $T^{\mu\nu}$ $\Rightarrow \underbrace{\begin{cases} g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu} : \text{Metric} \\ \& \text{Einstein-Hilbert action} \\ \hline \\ GENERAL RELATIVITY \end{cases}}$



• So in principle, we should look for quantized, massless excitations that transform under a spin-2 representation (have two symmetric spacetime-indices). If these excitations couple to energy & momentum, they are the excitations of the metric field, i.e., gravitons.

We will not do the latter in string theory, as it requires studying the interactions of strings. However, we will use a different argument to show that the gravitons of string theory have metric meaning.

• Let me reformulate the conclusion of this part, as its importance cannot be overstated:

Imagine you are given the action of everything Eq. (12.65), but all fields (metric and matter alike) have been labeled X^i were *i* runs through all components of all fields; for good measure, all interactions are given by one big sum of many terms. To interpret this theory, do you need to know which field plays the role of the metric of spacetime? According to our findings above, the surprising answer is "No": The metric field $g_{\mu\nu}$ is not the metric *by declaration* – it behaves as the metric because it can be interpreted as a massless rank-2 tensor that couples to the total energy-momentum tensor. Being the metric means that the values of the field correlate with the relational properties we call "length" and "time", and these correlations are established by its coupling to the energy-momentum tensor (which, unsurprisingly, generates local translations in space and time).

- The above line of arguments shows again that the Einstein field equations are very *generic*: One does not need much input to end up with GENERAL RELATIVITY; recall Section 12.1.
- **3** | Reminder (Section 5.3): Relativistic point particle:

i! Throughout this chapter we consider objects on D dimensional Minkowski space.

i | \triangleleft Action on *D*-dimensional Minkowski space $\mathbb{R}^{1,D-1}$:

$$S[X] = -m \int d\tau \sqrt{-\dot{X}^{\mu} \dot{X}_{\mu}} \quad \propto \quad \text{Proper time along trajectory } X^{\mu} \tag{15.8}$$

This is a functional of time-like trajectories $X^{\mu} : \mathbb{R} \to \mathbb{R}^{1,D-1}$:

In string theory, points in spacetime are conventionally denoted by capital letters: X^{μ} .



This action Eq. (15.8) is ...

- Poincaré/Lorentz invariant
- Reparametrization invariant $[\tau \mapsto \overline{\tau} = \overline{\tau}(\tau)]$ (\leftarrow Section 5.4)



$$p_{\mu} := \frac{\partial L}{\partial \dot{X}^{\mu}} = \frac{m \dot{X}_{\mu}}{\sqrt{-\dot{X}^{\mu} \dot{X}_{\mu}}} \tag{15.9}$$

Note: These *D* momenta are not independent! \rightarrow Constraint:

$$p^2 = -m^2$$
 (Mass shell condition) (15.10)

- iii | The action Eq. (15.8) is not easy to work with because of the square root. Can one get rid of it?
 - $\rightarrow \triangleleft$ Alternative action with *auxiliary variable* $e = e(\tau)$:

$$S[e, X] := \frac{1}{2} \int d\tau \left[e^{-1} \dot{X}^{\mu} \dot{X}_{\mu} - em^2 \right]$$
(15.11)

 \rightarrow Classically equivalent to Eq. (15.8)

To check this, compute the EOM for the auxiliary variable e,

$$\frac{\partial L}{\partial e} = 0 \quad \Leftrightarrow \quad e^2 = -\frac{1}{m^2} \dot{X}^{\mu} \dot{X}_{\mu} \,, \tag{15.12}$$

and plug this back into Eq. (15.11) which immediately yields Eq. (15.8).

Benefits of Eq. (15.11):

- No square root.
- Well-defined for massless particles (= null trajectories).
- Quadratic in derivatives \rightarrow Quantization via path integral straightforward.

For these reasons, we will use a similar construction for the relativistic string \rightarrow *below*.

15.1. The classical relativistic string

4 | Relativistic string:

We generalize the relativistic point particle (which traces out a 1D world line in spacetime) to a relativistic string (which traces out a 2D world *sheet*):

i! This is still classical, relativistic physics; there is no quantum mechanics involved!

 $i \mid \triangleleft$ Parametrization of 2D world sheet in D spacetime dimensions:

$$X^{\mu}: \underbrace{\mathbb{R} \times I}_{\text{World sheet}} \to \underbrace{\mathbb{R}^{1,D-1}}_{\text{Spacetime}} \quad \text{with} \quad (\tau,\sigma) \mapsto X^{\mu}(\tau,\sigma)$$
(15.13)

 $I \subseteq \mathbb{R}$: Some interval for σ (will be specified later)

Interpretation of world sheet coordinates:

- τ: Time (coordinate) along trajectory of string
- σ : Point (coordinate) on string





i! You should think of the parameter range as the *base space* and spacetime as the *target space*; the string positions X^{μ} are then *D fields* on the 2D base space (the world sheet). This implies that the first-quantized theory of a relativistic string will be a 1+1-dimensional *quantum field theory*.

ii | What is a reasonable String action?

We do not derive this action but motivate it as generalization of the relativistic point particle:

Tangent vectors to world sheet embedded in spacetime:

$$X_{\tau} := \underbrace{\partial_{\tau} X^{\mu}}_{=: \dot{X}^{\mu}} \partial_{\mu} \quad \text{and} \quad X_{\sigma} := \underbrace{\partial_{\sigma} X^{\mu}}_{=: X'^{\mu}} \partial_{\mu}$$
(15.14)

 \rightarrow Induced ** world sheet metric:

$$g_{ab} := \eta(X_a, X_b) = \partial_a X^\mu \partial_b X_\mu \tag{15.15}$$

with $a, b \in \{\tau, \sigma\} \equiv \{0, 1\}$.

The world sheet is a 2D submanifold of *D*-dimensional spacetime. The Minkowski metric $\eta_{\mu\nu}$ then induces a metric on the world sheet; just like the surface of a ball inherits a metric from the Euclidean space in which it is embedded.

 \rightarrow World sheet area element:

$$\mathrm{d}A = \sqrt{|\det g_{ab}|} \,\mathrm{d}\sigma\mathrm{d}\tau \tag{15.16}$$

- Integrating this over *τ* and *σ* yields the surface area of the world sheet wrt. the Minkowski metric of spacetime.
- This is a mathematical fact from Riemannian geometry; it has nothing to do with string theory. Remember that the determinant of a 2 × 2-matrix is the area of a parallelogram determined by the four numbers of the matrix. You can also think of the worldsheet as a two-dimensional Riemannian manifold. From Eq. (10.101) it follows then that the coordinate-independent volume form on such a manifold is $dV = \sqrt{g} d^d x = \sqrt{g} d^d \bar{x}$; in d = 2 dimensions this is simply the area: $dA = \sqrt{g} d^2 x$.

For details: ↑ ZWIEBACH [7] (§6.1-§6.3, pp. 100–110).

Remember: Relativistic particle action (15.8) \propto Length of world line \rightarrow Idea: Relativistic string action \propto Area of world sheet



 $\rightarrow ** Nambu-Goto action:$

$$S_{\rm NG}[X] := -T \int dA \stackrel{15.15}{=} -T \int \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \, d\sigma d\tau \qquad (15.17)$$

Note that \dot{X} is time-like whereas X' is space-like, so that the expression under the square root is positive. Expressions like $\dot{X} \cdot X'$ are short for $\dot{X}^{\mu}X'_{\mu}$ etc.

i! This is a classical 2D field theory on the world sheet with D fields: $X^{\mu}(\tau, \sigma)$.

There is only one parameter that will show up in various permutations:

- T: ** String tension
- $\alpha' \equiv \frac{1}{2\pi T}$: ** Regge slope

•
$$\ell \equiv \sqrt{2\alpha'} = \frac{1}{\sqrt{\pi T}}$$
: ** String length

iii | Symmetries of S_{NG} :

It is always good to know the symmetries of a theory, both global and local (gauge):

• D-dimensional Poincaré invariance:

3

$$\bar{X}^{\mu}(\tau,\sigma) = \Lambda^{\mu}_{\ \nu} X^{\nu}(\tau,\sigma) + a^{\mu}$$
(15.18)

- This follows from Eq. (15.17) because the integrand is a scalar and Poincaré transformations are isometries of Minkowski space. This symmetry is therefore a consequence of our chosen background spacetime.
- This is a *global* symmetry on the world sheet as it transforms the fields independent of the point (τ, σ) on the world sheet. It is also an *internal* symmetry, in that it mixes only the components of the fields and does not mess with the world sheet points. (Recall that the world sheet and not spacetime is the base space of our field theory!)
- Reparametrization invariance = 2D diffeomorphism invariance:

$$\bar{X}^{\mu}(\bar{\tau},\bar{\sigma}) := X^{\mu}(\tau,\sigma) \quad \text{with} \quad \bar{\tau} = \bar{\tau}(\tau,\sigma) \text{ and } \bar{\sigma} = \bar{\sigma}(\tau,\sigma) \quad (15.19)$$

- This symmetry reflects the geometric nature of the Nambu-Goto action: The area of the world sheet traced out by the string is independent of the coordinates (τ, σ) used to parametrize the world sheet. It is therefore a *local gauge* symmetry on the world sheet.
- The transformation Eq. (15.19) marks X^μ as *scalar fields* on the world sheet (the Lorentz index only labels different fields!).
- This symmetry follows from the invariance of the area element Eq. (15.16) under reparametrizations (coordinate transformations on the world sheet); cf. Eq. (10.101) for D = 2.
- iv | Define the following quantities (P^{τ}_{μ} are the canonical momenta of the field theory):

$$P_{\mu}^{\tau} := \frac{\partial L}{\partial \dot{X}^{\mu}} \stackrel{\circ}{=} -T \frac{(X \cdot X')X'_{\mu} - (X')^2 X_{\mu}}{\sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}}$$
(15.20a)

$$P^{\sigma}_{\mu} := \frac{\partial L}{\partial X'^{\mu}} \stackrel{\circ}{=} -T \frac{(\dot{X} \cdot X')\dot{X}_{\mu} - (\dot{X})^{2}X'_{\mu}}{\sqrt{(\dot{X} \cdot X')^{2} - (\dot{X})^{2}(X')^{2}}}$$
(15.20b)



 $\stackrel{\circ}{\rightarrow}$ Equation of motion (for arbitrary boundary conditions):

$$\frac{\partial P^{\tau}_{\mu}}{\partial \tau} + \frac{\partial P^{\sigma}_{\mu}}{\partial \sigma} = 0.$$
(15.21)

This is the equation of motion of a relativistic string on a D-dimensional Minkowski spacetime. It looks deceptively simple, but is actually extremely complicated due to Eq. (15.20). Luckily we will not have to solve it in this form.

v | <u>Alternative action:</u>

Similar to the point particle Eq. (15.8), The Nambu-Goto action is not well-suited for quantization due to the square root. We can find the analog of Eq. (15.11) by introducing an auxiliary field that makes the metric on the world sheet dynamical:

 $\rightarrow **$ Polyakov action:

$$S_{\mathrm{P}}[h, X] := -\frac{T}{2} \int \sqrt{h} \underbrace{h^{ab} \partial_a X_{\mu} \partial_b X^{\mu}}_{=: L_{\mathrm{P}}/(^{-T}/2)} \,\mathrm{d}\sigma \,\mathrm{d}\tau \tag{15.22}$$

with dynamical world sheet metric h_{ab} $(a, b \in \{0, 1\})$ and $h = |\det(h_{ab})|$.

• The Polyakov action (15.22) is classically equivalent to the Nambu-Goto action (15.17).

To check this, calculate the EOMs for h^{ab} from Eq. (15.22) and plug them back into Eq. (15.22) to obtain Eq. (15.17).

- We denote the world sheet metric by h_{ab} (and not by g_{ab}) to emphasize that h_{ab} is *dynamical* and not necessarily the metric g_{ab} induced on the world sheet by the static spacetime.
- The metric h^{ab} has Lorentzian signature (-, +).
- i! There are now two metric tensors involved: h_{ab} is the dynamical metric on the two-dimensional string world sheet, whereas η_{μν} (hidden in the contraction of the μ-indices with μ = 0, · · · , D − 1) is the static background metric of the D-dimensional spacetime on which the string propagates (here the Minkowski metric):

$$\partial_a X_{\mu} \partial^a X^{\mu} \equiv h^{ab} \partial_a X_{\mu} \partial_b X^{\mu} \equiv h^{ab} \eta_{\mu\nu} \partial_a X^{\mu} \partial_b X^{\nu}$$
(15.23)
World sheet metric (dynamic) Spacetime metric (static)

- vi | Symmetries of S_P :
 - <u>D-dimensional Poincaré invariance:</u> (global & internal symmetry)

$$\bar{X}^{\mu}(\tau,\sigma) := \Lambda^{\mu}_{\nu} X^{\nu}(\tau,\sigma) + a^{\mu} \quad \text{and} \quad \bar{h}_{ab}(\tau,\sigma) := h_{ab}(\tau,\sigma) \quad (15.24)$$

This symmetry is inherited from the Nambu-Goto action and reflects the fact that the integrand of the Polyakov action is still a spacetime scalar (and that Poincaré transformations are isometries of Minkowski space). Note that the world sheet metric transforms as a *scalar* under these transformations.



- Local gauge symmetries: (on the world sheet)
 - Diffeomorphism invariance:
 - a | Let us first define an alternative notation: σ⁰ := τ and σ¹ := σ
 A reparametrization/diffeomorphism can then be written in a compact form:

$$\left. \begin{array}{c} \bar{\tau} := \bar{\tau}(\tau, \sigma) \\ \bar{\sigma} := \bar{\sigma}(\tau, \sigma) \end{array} \right\} \quad \Leftrightarrow \quad \bar{\sigma}^a := \bar{\varphi}^a(\sigma) \tag{15.25}$$

Here we use the shortcut $\sigma \equiv {\sigma^0, \sigma^1} = {\tau, \sigma}$.

The fields transform then as follows under diffeomorphisms:

$$\bar{X}^{\mu}(\bar{\sigma}) := X^{\mu}(\sigma) \qquad (\text{Scalar}) \tag{15.26a}$$

$$\bar{h}_{ab}(\bar{\sigma}) := \frac{\partial \sigma^c}{\partial \bar{\sigma}^a} \frac{\partial \sigma^d}{\partial \bar{\sigma}^b} h_{cd}(\sigma) \quad \text{(Covariant rank-2 tensor)} \tag{15.26b}$$

Again, this reflects the fact that the parametrization of the string world sheet is unphysical and therefore a gauge symmetry.

i! This transformation tells us that the *D* components X^{μ} are *scalar fields* on the world sheet. By contrast, they transform as *vector components* on Minkowski space (\leftarrow *Poincaré symmetry*).

b | There is an important special class of diffeomorphisms:

 $\bar{\sigma}^a = \varphi^a(\sigma)$ is a * conformal diffeomorphism (or \uparrow conformal map) iff

$$\bar{h}_{ab}(\bar{\sigma}) = \frac{\partial \sigma^c}{\partial \bar{\sigma}^a} \frac{\partial \sigma^d}{\partial \bar{\sigma}^b} h_{cd}(\sigma) = \Omega(\sigma) h_{ab}(\sigma)$$
(15.27)

for some $\Omega(\sigma) > 0$.

- Conformal diffeomorphisms do not change angles. This is apparent from the rescaling of the metric tensor by Ω(σ), which only changes the length of tangent vectors, recall Eqs. (10.9) and (10.10).
- Conformal diffeomorphisms include ← *isometries* of the world sheet metric,
 i.e., coordinate transformations that do not change the components of the metric at all: Ω(σ) = 1.

- Weyl invariance:

The Polyakov action is invariant under the transformation

$$\tilde{X}^{\mu}(\tau,\sigma) := X^{\mu}(\tau,\sigma) \quad \text{and} \quad \underbrace{\tilde{h}_{ab}(\tau,\sigma) := e^{2\rho(\tau,\sigma)} h_{ab}(\tau,\sigma)}_{** Weyl transformation}$$
(15.28)

for some real-valued function $\rho(\tau, \sigma)$.

- * A transformation of this form is called *** Weyl transformation.
- * i! Note that Weyl transformations are active transformations of the world sheet metric, they are not coordinate transformations. Hence, Weyl transformations are not conformal diffeomorphisms.
- * Use $\tilde{h}^{ab} = e^{-2\rho(\tau,\sigma)}h^{ab}$ to show the invariance of Eq. (15.22).



* The rescaling by $e^{2\rho}$ is a convenient way to make sure that the prefactor is positive for all functions ρ (which it must be for the new metric to remain regular everywhere).

To sum up:

- Conformal diffeomorphisms are a special class of diffeomorphisms that change the components of the world sheet metric only by a local factor (i.e., they act on points of the world sheet and move them around).
- Weyl transformations are a particular class of transformation of the *values* of the metric field by rescaling it locally *without moving points on the manifold around*.

Since the Polyakov action has both, Weyl invariance and diffeomorphism invariance ...

Diffeo. invariance: $(h, X) \xrightarrow{\varphi} (\bar{h}, \bar{X}) \Rightarrow S_{P}[\bar{h}, \bar{X}] = S_{P}[h, X]$ (15.29a) Weyl invariance: $h \xrightarrow{\rho} \tilde{h} \Rightarrow S_{P}[\tilde{h}, X] = S_{P}[h, X]$ (15.29b)

... we can combine them:

 \triangleleft *Conformal* diffeomorphism $\varphi \rightarrow$

$$S_{\mathrm{P}}[h, X] \stackrel{\mathrm{Diff}}{=} S_{\mathrm{P}}[\bar{h}, \bar{X}] \stackrel{\mathrm{Conf}}{=} S_{\mathrm{P}}[\tilde{h}, \bar{X}] \stackrel{\mathrm{Weyl}}{=} S_{\mathrm{P}}[h, \bar{X}]$$
(15.30)

That is, we can use the Weyl symmetry to "undo" the effect of a conformal diffeomorphism on the metric, such that only the fields are affected by the conformal map. We call such transformations of the fields *conformal transformations*.

 \rightarrow Conformal transformation φ is symmetry of S_P on fixed background metric h_{ab} .

The Polyakov action is a ** conformal field theory.

(15.31)

- Note that a particular class of conformal transformations are *global rescalings* of the world sheet: (τ, σ) → (λτ, λσ); i.e., conformal field theories are *scale invariant*. This makes such theories (though not the Polyakov action) useful tools in condensed matter physics to describe second-order phase transitions (where systems become scale invariant due to fluctuations).
- The conformal symmetry will not survive the quantization of the Polyakov action in general; this is called \uparrow *conformal/trace/Weyl anomaly*. Since the conformal symmetry is a (unphysical) gauge symmetry of the relativistic string, this poses a fundamental problem. The conformal symmetry can only be restored if (1) the spacetime dimension is D = 26 and (2) the metric of *spacetime* satisfies the Einstein field equations.



↓Lecture 33 [07.08.24]

5 \triangleleft Hilbert energy-momentum tensor:

$$T_{ab}(\tau,\sigma) \stackrel{11.106}{=} -\frac{2}{\sqrt{h}} \frac{\delta(\sqrt{hL_{\rm P}})}{\delta h^{ab}} \stackrel{11.118}{=} T\left(\partial_a X^{\mu} \partial_b X_{\mu} - \frac{1}{2} h_{ab} \partial_c X^{\mu} \partial^c X_{\mu}\right)$$
(15.32)

Use Eq. (11.118) or your result from O Problemset 4 for the Klein-Gordon field to show this. Note that we defined the HEMT here with the opposite sign [cf. Eq. (11.106)]; this is because we use the opposite signature (-, +) for the metric h_{ab} . The sign makes the energy density T_{00} positive (which is how the sign of the HEMT is conventionally chosen).

Symmetries constrain the HEMT as follows:

Diffeomorphism invariance (15.26)
$$\xrightarrow{11.109} T^{ab}_{;b} \doteq 0$$
 (divergence-free) (15.33a)

Weyl invariance (15.28) $\stackrel{\circ}{\rightarrow}$ $T^a_{\ a} \equiv 0$ (traceless) (15.33b)

To show that the trace vanishes due to Weyl invariance is straightforward. First, define

$$h_{ab} := e^{2\rho} h_{ab} \,, \tag{15.34}$$

and then compute the variational derivative of $\mathcal{L}_{P}(\tilde{h}_{ab}, X^{\mu})$ wrt. ρ :

$$0 \stackrel{\text{Weyl}}{=} \frac{\delta \mathcal{L}_{\text{P}}}{\delta \rho} \stackrel{\text{15.28}}{=} \frac{\delta \mathcal{L}_{\text{P}}}{\delta \tilde{h}_{ab}} \frac{\delta h_{ab}}{\delta \rho} \stackrel{\text{11.100}}{=} -\sqrt{h} T^{ab} e^{2\rho} h_{ab} = -\sqrt{h} T^{a}_{\ a} e^{2\rho} \,. \tag{15.35}$$

This implies $T^a_{\ a} \equiv 0$ without imposing the equations of motion; it is an *identity*. [This is a consequence of the fact that the fields X^{μ} do not change under Weyl transformations.]

6 | Equations of motion:

• Varying Eq. (15.22) wrt. the world sheet metric h_{ab} yields the HEMT:

$$\delta_h S_{\rm P} \stackrel{!}{=} 0 \quad \stackrel{15.32}{\longleftrightarrow} \quad T_{ab} \stackrel{!}{=} 0 \tag{15.36}$$

 T_{ab} has no derivatives of the metric [Eq. (15.32)] \rightarrow Constraint

• Varying Eq. (15.22) wrt. the fields X^{μ} yields:

$$\delta_X S_{\rm P} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \partial_a \left(\sqrt{h} h^{ab} \partial_b X^{\mu} \right) \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \Delta X^{\mu} \stackrel{!}{=} 0 \tag{15.37}$$

with Laplace-Beltrami operator $\Delta = \nabla^a \nabla_a [\leftarrow \text{Eq. (10.97)}].$

Recall → Problemset 4 and Eq. (11.38).

This EOM looks much more tracktable than the EOM (15.21) of the Nambu-Goto action. But we should not forget that it must be augmented by the constraint Eq. (15.36).

7 | Boundary conditions:

If the world sheet is finite (here in σ direction), the variation of the action has boundary terms that must also vanish (in addition to the EOMs above):



 \triangleleft World sheet with $\tau \in \mathbb{R}$ and $\sigma \in I = [0, l]$ for $0 < l < \infty$:

$$\delta_{X} S_{P} \stackrel{a}{=} \int_{-\infty}^{+\infty} d\tau \int_{0}^{l} d\sigma \underbrace{\frac{\delta(\sqrt{h}L_{P})}{\delta X_{\mu}}}_{\rightarrow \text{ EOM (15.37)}} \delta X_{\mu} - T \int_{-\infty}^{+\infty} d\tau \left[\underbrace{\frac{\frac{1}{\sqrt{h}\delta X_{\mu}} \partial^{\sigma} X^{\mu}}_{\text{Boundary term}}}_{\text{Boundary term}}\right]_{\sigma=0}^{\sigma=l}$$
(15.38)

To show this, use Eqs. (11.98) and (11.100) (for X_{μ} instead of $g^{\mu\nu}$) and apply them to the Polyakov action Eq. (15.22).

There are three possibilities to make the boundary term vanish:



• Closed string:

A closed string requires that the points $\sigma = 0$ and $\sigma = l$ in I are identified; in particular:

$$X^{\mu}(\tau,0) \stackrel{!}{=} X^{\mu}(\tau,l) \quad \text{and} \quad \partial^{\sigma} X^{\mu}(\tau,0) \stackrel{!}{=} \partial^{\sigma} X^{\mu}(\tau,l)$$
(15.39)

These conditions make the difference of the two boundary terms in Eq. (15.38) vanish. For consistency, also the metric must be periodic: $h_{ab}(\tau, 0) \stackrel{!}{=} h_{ab}(\tau, l)$.

- \rightarrow All fields are periodic in σ -direction
- \rightarrow String = Closed loop
- Open string:

An open string has endpoints; there are two possibilities to make each of the two boundary terms in Eq. (15.38) vanish separately on these endpoints:

- <u>Neumann</u> boundary conditions:

$$\partial^{\sigma} X^{\mu}(\tau, 0) \stackrel{!}{=} 0 \stackrel{!}{=} \partial^{\sigma} X^{\mu}(\tau, l) \quad \Leftrightarrow \quad \underbrace{n^{a} \partial_{a} X^{\mu}}_{\text{Coordinate independent}} \qquad (15.40)$$

Here \mathcal{M} denotes the 2D world sheet and n^a is the normal on the boundary 1D $\partial \mathcal{M}$.

 \rightarrow Ends of string move *freely* in spacetime

Note that the *positions* $X^{\mu}(\tau, 0)$ and $X^{\mu}(\tau, l)$ are not fixed.

- Dirichlet boundary conditions:

The constraints Eq. (15.39) and Eq. (15.40) are Poincaré/Lorentz covariant equations. With these boundary conditions, the internal Poincaré invariance of the theory remains



in tact. If we allow for a *violation* of this symmetry, there is a third possibility to make the boundary terms vanish:

$$\begin{bmatrix} \delta X_{\mu} \end{bmatrix}_{\sigma=0}^{\sigma=l} \stackrel{!}{=} 0 \Rightarrow \begin{cases} \partial_{\tau} X^{\mu}(\tau, 0) = 0\\ \partial_{\tau} X^{\mu}(\tau, l) = 0 \end{cases} \Rightarrow \begin{cases} X^{\mu}(\tau, 0) = \text{const}\\ X^{\mu}(\tau, l) = \text{const} \end{cases}$$
(15.41)

 \rightarrow String ends are *fixed* (here: in *spacetime*)

For an open string, one can mix Neumann and Dirichlet boundary conditions for the different components X^{μ} because the boundary terms in Eq. (15.38) are a sum over $\mu = 0, ..., D-1$. If X^0 and p of the spatial components X^i satisfy *Neumann* boundary conditions, the string can move freely on a p-dimensional hyperplane in space; this hyperplane (extended by one dimension in time) is called a $\uparrow Dp$ -brane.

 \rightarrow Strings can be attached to a \uparrow *D-branes* (D = Dirichlet)

After quantizing strings attached to a D-brane, one finds that some of their oscillator modes can be interpreted as quantum fluctuations *of the D-brane itself* (their coherent states determine the expectation value of the D-brane position in spacetime). Hence one finds, quite surprisingly, that D-branes are actually *dynamical objects* – and not static & classical background structures.

In the following we only consier closed strings and open strings with Neumann boundary conditions.

8 | Flat gauge: (also called *conformal gauge*)

Mathematical fact: Every two-dimensional pseudo-Riemannian manifold is conformally flat:

 $\rightarrow \forall h_{ab} \exists$ Coordinates such that

$$h_{ab} = \Omega^2(\tau, \sigma)\eta_{ab} = \Omega^2(\tau, \sigma) \begin{pmatrix} -1 & 0\\ 0 & 1 \end{pmatrix}_{ab}$$
(15.42)

for some non-vanishing *conformal factor* $\Omega(\tau, \sigma)$.

i! Conformal flatness does not imply the vanishing of the Riemann curvature tensor.

This is a peculiar feature of two dimensions: On a d-dimensional manifold the metric tensor has d(d + 1)/2 independent components. The diffeomorphism group (coordinate transformations) has d generators [\leftarrow Eq. (11.101)], which leaves d(d - 1)/2 degrees of freedom of the metric that cannot be fixed by coordinate transformations. In d = 2 this is exactly one degree of freedom, namely the conformal factor in Eq. (15.42).

We can now use the Weyl invariance (15.28) of the Polyakov action to drop the conformal factor:

Weyl invariance $\rightarrow h_{ab} = \eta_{ab}$ ** Flat gauge (15.43)

All calculations that follow are perfomed in flat gauge:

9 | Conjugate momentum & Poisson algebra:

In flat gauge, the Polyakov action & Lagrangian are quite simple:

$$S_{\rm P}^{\rm flat}[X] \stackrel{15.22}{=} \frac{T}{2} \int d\sigma d\tau \underbrace{\left[(\dot{X})^2 - (X')^2 \right]}_{=: L_{\rm p}^{\rm flat}/(T/2)}$$
(15.44)



To prepare for canonical quantization, we need the conjugate momentum of X^{μ} :

\rightarrow Conjugate momentum:

$$\Pi_{\mu}(\tau,\sigma) := \frac{\partial L_{\rm P}^{\rm flat}}{\partial \dot{X}^{\mu}} = T \dot{X}_{\mu}$$
(15.45)

with satisfies the \checkmark canonical Poisson algebra: (defined at equal time!)

$$\left\{X^{\mu}(\tau,\sigma), \Pi_{\nu}(\tau,\sigma')\right\} = \delta(\sigma - \sigma')\,\delta^{\mu}_{\nu} \tag{15.46}$$

This is the field-theory analog of $\{x_i, p_j\} = \delta_{ij}$ that you encountered in your course on classical mechanics. The Poisson bracket for fields is defined via functional derivatives. However, we will expand the fields into a discrete set of modes \rightarrow *below* anyway, so that we can impose this Poisson algebra directly on the modes (without the need for functional derivatives).

10 | <u>Classical solutions</u> of EOM (15.37) for X^{μ} :

In flat gauge, the Laplace-Beltrami operator yields a simple wave equation:

$$\underbrace{\left(\partial_{\tau}^{2} - \partial_{\sigma}^{2}\right)}_{\Box} X^{\mu} = 0 \tag{15.47}$$

We will now write down the general solutions of this EOM for a closed and an open string.

i! Do not forget that this is only one of the EOMs; it must be augmented by the constraint Eq. (15.36). We will study the implementation of this constraint on the solutions \rightarrow *later*.

 $\stackrel{\circ}{\rightarrow}$ General solution:

$$X^{\mu}(\tau,\sigma) = \underbrace{X^{\mu}_{R}(\tau-\sigma)}_{\text{"Right mover"}} + \underbrace{X^{\mu}_{L}(\tau+\sigma)}_{\text{"Left mover"}}$$
(15.48)

Here $X_{R/L}^{\mu}$ are arbitrary (differentiable) functions of a single variable.

In \rightarrow *light-cone coordinates* $\sigma^{\pm} = \tau \pm \sigma$ the EOM (15.47) reads $\partial_{+}\partial_{-}X^{\mu} = 0$. Integrating twice yields the general solution $X^{\mu} = X^{\mu}_{R}(\sigma^{-}) + X^{\mu}_{L}(\sigma^{+})$.

 $i \mid \triangleleft$ Closed string:



a | We must implement the boundary conditions Eq. (15.39) on the solutions Eq. (15.48). Let *w.l.o.g.* $l = \pi$: (this can always be achieved by reparametrizing the world sheet)

$$X^{\mu} \in \mathbb{R}$$
 and $X^{\mu}(\tau, \sigma + \pi) = X^{\mu}(\tau, \sigma)$. (15.49)

 \rightarrow We want to parametrize *real-valued*, σ *-periodic* and *differentiable* functions.

 \rightarrow Fourier series



 $\mathbf{b} \mid \stackrel{\circ}{\rightarrow} \text{Most general solution:}$

$$X_{R}^{\mu} = \frac{1}{2}x^{\mu} + \alpha' p^{\mu}(\tau - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_{n}^{\mu}}{n} \exp\left[-2in(\tau - \sigma)\right] \quad (15.50a)$$

$$X_{L}^{\mu} = \frac{1}{2}x^{\mu} + \alpha' p^{\mu}(\tau + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{\tilde{\alpha}_{n}^{\mu}}{n} \exp\left[-2in(\tau + \sigma)\right] \quad (15.50b)$$

Here we use the slope parameter $\alpha' = (2\pi T)^{-1}$ for convenience. Note that we extracted the n = 0 component as $\frac{1}{2}x^{\mu}$ from the sum. The linear part $\propto p^{\mu}$ is not periodic in σ but becomes so in the sum Eq. (15.48). All prefactors are chosen for convenience.

The solutions are parametrized by the following free parameters:

• $x^{\mu}, p^{\mu} \in \mathbb{R}$: Center of mass initial position & momentum of string

The interpretation of x^{μ} and p^{μ} is easily confirmed:

$$x^{\mu} \stackrel{\circ}{=} \frac{1}{\pi} \int_0^{\pi} \mathrm{d}\sigma \; X^{\mu}(0,\sigma) \tag{15.51}$$

and

$$p^{\mu} \stackrel{\circ}{=} \frac{1}{2\pi\alpha'} \int_0^{\pi} d\sigma \, \partial_{\tau} X^{\mu}(\tau, \sigma) \stackrel{15.45}{=} \int_0^{\pi} d\sigma \, \Pi^{\mu}(\tau, \sigma) \,. \tag{15.52}$$

α_n^μ, α̃_n^μ ∈ C: Fourier components of string oscillation modes
 Reality condition: X^μ ∈ ℝ ⇔

$$\alpha_{-n}^{\mu} = (\alpha_{n}^{\mu})^{*}$$
 and $\tilde{\alpha}_{-n}^{\mu} = (\tilde{\alpha}_{n}^{\mu})^{*}$ (15.53)

When checking this, do not forget that $n \in \mathbb{Z}$ so that $n \mapsto -n$.

It will be convenient to define for the closed string: $\alpha_0^{\mu} \equiv \tilde{\alpha}_0^{\mu} \equiv \sqrt{\frac{\alpha'}{2}} p^{\mu}$.

c | Poisson algebra Eq. (15.46) in mode space \Leftrightarrow

$$\{x^{\mu}, p^{\nu}\} = \eta^{\mu\nu} \tag{15.54a}$$

$${}^{\mu}_{m}, \alpha^{\nu}_{n} \} = im\delta_{m+n}\eta^{\mu\nu} \tag{15.54b}$$

$$\{\tilde{\alpha}_{m}^{\mu}, \tilde{\alpha}_{n}^{\nu}\} = im\delta_{m+n}\eta^{\mu\nu}$$

$$\{\alpha_{m}^{\mu}, \tilde{\alpha}_{n}^{\nu}\} = 0$$
(15.54c)
(15.54d)

- i! Note the complex *i* in the Poisson algebra of the Fourier modes. After quantization, this will make pairs (α^μ_n, α^μ_{-n}) into creation and annihilation operators of a harmonic oscillator mode.
- The Poisson algebra is the starting point for a canonical quantization procedure (→ *below*). So what is the point of the *classical* solutions Eq. (15.50)? There are two aspects to consider:



First, for $\tau = 0$, the expansion Eq. (15.50) is a completely general parametrization of configurations X^{μ} of the string that are consistent with its boundary conditions. This makes the Fourier coefficients α_n^{μ} , $\tilde{\alpha}_n^{\mu}$ [together with x^{μ} and the reality constraint Eq. (15.53)] a convenient (and discrete) set of dynamical variables to encode the field X^{μ} . The Fourier expansion exploits the symmetry of the problem under translations along σ , and leads to a decoupling of the Poisson brackets between different modes. (Note that only brackets of the form { α_n^{μ} , α_{-n}^{μ} } do *not* vanish.)

Second, eventually we want to quantize the fields X^{μ} . Since the Heisenberg field operators of free fields obey the *classical* equations of motion (\uparrow *Quantum field theory* [20]), we can simply quantize the mode operators and plug them into Eq. (15.50) to obtain the Heisenberg field operators for $\tau \neq 0$ (thereby skipping the solution of the Heisenberg equation, i.e., the application of the time-evolution operator).

ii | < Open string & Neumann boundary conditions: (no D-branes!)



a | We must implement the boundary conditions Eq. (15.40) on the solutions Eq. (15.48). Let again *w.l.o.g.* $l = \pi$:

$$X^{\mu} \in \mathbb{R} \quad \text{and} \quad \partial_{\sigma} X^{\mu}(\tau, 0) = 0 = \partial_{\sigma} X^{\mu}(\tau, \pi)$$
 (15.55)

b $| \xrightarrow{\circ}$ General solutions:

$$X^{\mu}(\tau,\sigma) = x^{\mu} + 2\alpha' p^{\mu}\tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^{\mu}}{n} \exp\left[-in\tau\right] \cos\left(n\sigma\right) \quad (15.56)$$

You can derive this from the closed string solutions Eq. (15.50) by imposing the constraint Eq. (15.55) which cuts the degrees of freedom in half.

The solutions are parametrized by the following free parameters:

- $x^{\mu}, p^{\mu} \in \mathbb{R}$: Center of mass initial position & momentum of string
- $\alpha_n^{\mu} \in \mathbb{C}$: Fourier components of string oscillation modes
 - \rightarrow Only one set $\{\alpha_n^{\mu}\}$ of oscillator modes!

Reality condition: $X^{\mu} \in \mathbb{R} \iff$

$$\alpha_{-n}^{\mu} = (\alpha_{n}^{\mu})^{*} \tag{15.57}$$

It will be convenient to define for the closed string: $\alpha_0^{\mu} \equiv \sqrt{2\alpha'} p^{\mu}$ (Note that this definition is different from the closed string!)



$$\{x^{\mu}, p^{\nu}\} = \eta^{\mu\nu} \quad \text{and} \quad \{\alpha^{\mu}_{m}, \alpha^{\nu}_{n}\} = im\delta_{m+n}\eta^{\mu\nu}$$
(15.58)

This is the subset of Eq. (15.54) where the modes $\tilde{\alpha}_n^{\mu}$ have been dropped.

11 | \triangleleft Constraint Eq. (15.36):

Now that we have the solutions of the EOM (15.37) (for open and closed strings), we should also impose the constraint Eq. (15.36) on them. Here we only simplify the contraint in flat gauge, but do not enforce it yet on the level of oscillator modes. We do this \rightarrow *later* after some more gauge fixing.

$$T_{ab}(\tau,\sigma) \stackrel{\overset{15,32}{=}}{=} T\left(\partial_a X^{\mu} \partial_b X_{\mu} - \frac{1}{2}\eta_{ab}\eta^{cd} \partial_c X^{\mu} \partial_d X_{\mu}\right) \stackrel{!}{=} 0$$
(15.59)

 $\stackrel{\circ}{\rightarrow}$ In components this reads:

$$T_{00} = T_{11} = \frac{1}{2} \left[(\dot{X})^2 + (X')^2 \right] \stackrel{!}{=} 0$$
(15.60a)

$$T_{01} = T_{10} = \dot{X} \cdot X' \stackrel{!}{=} 0 \tag{15.60b}$$

We can now check Eq. (15.33b) explicitly:

$$T^{a}_{\ a} = \eta^{ab} T_{ab} = T_{11} - T_{00} = 0 \tag{15.61}$$

As explained above, this is a consequence of the Weyl invariance of the Polyakov action.

The constraint equations can be combined in a convenient form:

Eq. (15.60)
$$\Leftrightarrow$$
 $(\dot{X} \pm X')^2 \stackrel{!}{=} 0$ (15.62)

This will be our starting point to enforce the constraint \rightarrow *later*.

12 | Conserved quantities:

As preparation for \rightarrow *later*, let us briefly discuss the conserved quantities that follow from the global Poincaré symmetry of the Polyakov action:

Poincaré symmetry Eq. (15.24) \rightarrow Noether currents = * World sheet currents

[Remember: Poincaré transformations = Translations + Rotations + Boosts]

i! The Poincaré symmetry is an *internal* symmetry, and the corresponding Noether currents live on the 2D *world sheet* (not on spacetime!). This means that latin indices a, b, ... label the *components* of the currents, whereas greek (spacetime) indices $\mu, \nu, ...$ label different *types* of currents, corresponding to different spacetime symmetries.

• μ -Translations: $\delta_{\nu} X^{\mu} = \delta_{\nu}^{\mu}$

(Recall Eqs. (6.79) and (6.89) and note that a μ -translation shifts the *value* of the field X^{μ} .)

Eq. (6.84)
$$\xrightarrow{\circ}$$
 $P_a^{\mu} = T \partial_a X^{\mu}$ with $\partial_a P^{a\mu} \doteq 0$ (15.63)

 \rightarrow Conserved charge: Total 4-momentum:

$$P^{\mu} \stackrel{6.86}{:=} \int_0^{\pi} \mathrm{d}\sigma P_0^{\mu} = T \int_0^{\pi} \mathrm{d}\sigma \dot{X}^{\mu} \stackrel{15.52}{=} p^{\mu}$$
(15.64)

oret



When using Eq. (6.84) to derive this, be very careful: Here the symmetry is labeled by a spacetime index $(a \mapsto v)$ whereas "spacetime" is now the world sheet $(\mu \mapsto a)$. The field is still a scalar, but there are *D* of them labeled by another spacetime index $(\phi \mapsto X^{\mu})$. Since the Poincaré symmetry is an *internal* symmetry, it is $\delta_a x^{\mu} \mapsto \delta_v \sigma^a = 0$, i.e., it does not transform the world sheet coordinates.

• $\mu\nu$ -Rotations: $\delta^{\alpha\beta}X^{\mu} = \eta^{\alpha\mu}X^{\beta} - \eta^{\beta\mu}X^{\alpha} \ [\in Eq. (6.78), we drop the arbitrary <math>\frac{1}{2}]$ ("Rotations" here refers to both spatial rotations and boosts.)

Eq. (6.84)
$$\stackrel{\circ}{\rightarrow}$$
 $J_a^{\mu\nu} = T\left(X^{\mu}\partial_a X^{\nu} - X^{\nu}\partial_a X^{\mu}\right)$ with $\partial_a J^{a\mu\nu} \doteq 0$
(15.65)

 \rightarrow Conserved charge: Total 4-angular momentum:

$$J^{\mu\nu} = \int_0^{\pi} d\sigma J_0^{\mu\nu} = T \int_0^{\pi} d\sigma \left(X^{\mu} \dot{X}^{\nu} - X^{\nu} \dot{X}^{\mu} \right)$$
(15.66)

After quantization, this charge becomes an *operator* that generates rotations & boosts on the Hilbert space of the string (just like the momentum operator generates translations). It will be crucial to determine the critical dimension of bosonic string theory.

13 | <u>Hamiltonian</u>:

In flat gauge, and with the mode expansion at hand, it is now straightforward to derive the Hamiltonian of the Polyakov action:

i | As usual, we get the Hamiltonian via Legendre transformation from the Polyakov Lagrangian:

$$H = \int_0^{\pi} d\sigma \left[\dot{X} \cdot \Pi - L_P^{\text{flat}} \right] \stackrel{15.44}{=} \frac{T}{2} \int_0^{\pi} d\sigma \left[(\dot{X})^2 + (X')^2 \right]$$
(15.67)

Using the Fourier expansion of the fields, this can be rewritten in terms of oscillator modes:

Open string:
$$H \stackrel{15.56}{=} \frac{1}{2} \sum_{n} \alpha_{-n} \cdot \alpha_n$$
 (15.68a)

Closed string:
$$H \stackrel{15.48}{=} \frac{1}{2} \sum_{n} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n)$$
 (15.68b)

- Here we introduced the shorthand notation $\alpha_n \cdot \alpha_{-n} \equiv \eta_{\mu\nu} \alpha_n^{\mu} \alpha_{-n}^{\nu}$.
- Note that these sums include the n = 0 mode, i.e., the momentum p^{μ} of the string:

Open string
$$(\alpha_0^{\mu} = \sqrt{2\alpha'}p^{\mu})$$
: $\frac{1}{2}\alpha_0 \cdot \alpha_0 = \alpha'p^2$ (15.69a)
Closed string $(\tilde{\alpha}_0^{\mu} = \sqrt{\alpha'/2}p^{\mu})$: $\frac{1}{2}(\alpha_0 \cdot \alpha_0 + \tilde{\alpha}_0 \cdot \tilde{\alpha}_0) = \frac{1}{2}\alpha'p^2$ (15.69b)

These terms account for the kinetic energy of the string.

• To derive Eq. (15.68), use that for $n, m \in \mathbb{N}$

$$\int_0^{\pi} d\sigma \cos(n\sigma) \cos(m\sigma) = \frac{\pi}{2} \delta_{n,m} = \int_0^{\pi} d\sigma \sin(n\sigma) \sin(m\sigma) .$$
 (15.70)



ii | The constraint equation implies that the Hamiltonian vanishes on-shell:

Eqs. (15.60a) and (15.67)
$$\Rightarrow H \doteq 0$$
 (15.71)

This is similar to Section 5.4 [in particular Eq. (5.93)] were we found the Hamiltonian of the relativistic particle to vanish as well. We identified the reparametrization invariance as the root cause, which is a local (gauge) symmetry that produces constraints via Noether's second theorem. Here, the Hamiltonian generates translations in τ – but τ is only one of many possible time-like parametrizations (due to the diffeomorphism invariance on the world sheet); it has no physical interpretation. Consequently, the Hamiltonian that generates translations in this parameter has no physical significance either.

iii | Eq. (15.71) \rightarrow <u>Mass shell condition</u>:

We study open and closed strings separately:

• *< Open* string: Combining our previous results implies:

$$\frac{1}{2}\alpha_0 \cdot \alpha_0 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n \stackrel{15.68a}{=} 0 \quad \Leftrightarrow \quad \alpha' p^2 \stackrel{15.69a}{=} - \sum_{n>0} \alpha_{-n} \cdot \alpha_n \quad (15.72)$$

Thus the norm of the 4-momentum of the string is determined by its oscillation modes.

 \rightarrow Recall that the norm of a 4-momentum is a Lorentz scalar called *(rest)* mass:

$$p^2 \stackrel{5.4}{=} -M^2 \quad \Rightarrow \quad M^2 = \frac{1}{\alpha'} \sum_{n>0} \alpha_{-n} \cdot \alpha_n$$
 (15.73)

M: Rest mass of the open string

- If you think about it, this result makes sense: The oscillations of the string contribute to its *internal* energy. And in Section 5.2 we argued that in a relativistic theory, any type of internal energy contributes to the rest mass of an object.
- Note that $(\alpha_n^{\mu})^* = \alpha_{-n}^{\mu}$ makes terms like $\alpha_{-n}^{\mu} \alpha_n^{\mu} = |\alpha_n^{\mu}|^2$ non-negative. However, not also that $\alpha_{-n} \cdot \alpha_n = \eta_{\mu\nu} \alpha_{-n}^{\mu} \alpha_n^{\nu}$, so that the Lorentzian signature of $\eta_{\mu\nu}$ produces positive and negative terms in the sum. The current form of Eq. (15.73) is therefore potentially problematic, since the left-hand side is the mass squared.
- < *Closed* string: Along the same lines, one finds for the closed string the constraint:

$$\frac{15.68b}{15.71}_{\frac{15}{2}\alpha'p^2} \stackrel{15.69b}{=} -\sum_{n>0} \left(\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n\right)$$
(15.74)

... so that the rest mass of the string is given by:

$$p^{2} \stackrel{5.4}{=} -M^{2} \quad \Rightarrow \quad M^{2} = \frac{2}{\alpha'} \sum_{n>0} \left(\alpha_{-n} \cdot \alpha_{n} + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_{n} \right)$$
(15.75)

M: Rest mass of the *closed* string



↓Lecture 34 [08.08.24]

15.1.1. Light-cone gauge

We could quantize the string right away by replacing the Poisson brackets of the oscillator modes with commutators. However, our theory still has unfixed residual gauge degrees of freedom that lead to problems after quantization. Here we identify and fix these gauge degrees of freedom, and finally impose the constraint $T_{ab} \stackrel{!}{=} 0$. The result will be a classical formulation of the relativistic string that can be canonically quantized without any issues (almost ...):

14 | Remember: We are in Flat gauge Eq. (15.43): $h_{ab} = \eta_{ab} = \text{diag}(-1, +1)_{ab}$

But remember also: Polyakov action has ← *conformal symmetry* Eq. (15.30)

Conformal symmetries are *residual gauge symmetries* on the world sheet that allow for the transformation of the fields X^{μ} without modifying the world sheet metric h_{ab} .

- \rightarrow \triangleleft Combinations of diffeomorphisms & Weyl transformations consistent with flat gauge:
 - \triangleleft *Infinitesimal* diffeomorphism: $\bar{\sigma}^a = \sigma^a + \varepsilon^a(\sigma) [\leftarrow \text{Eq. (11.90)}]$

Eq. (11.103)
$$\rightarrow \delta h_{ab}^{\text{Diff}} = -(\partial_b \varepsilon_a + \partial_a \varepsilon_b) - \underbrace{\varepsilon^c \partial_c \eta_{ab}}_{=0}$$
 (15.76)

The signs are different from Eq. (11.103) because the tensor is covariant.

• \triangleleft *Infinitesimal* Weyl transformation (15.28): ($|\rho| \ll 1$)

$$\tilde{h}_{ab} = e^{2\rho} \eta_{ab} \approx (1+2\rho) \eta_{ab} \quad \Rightarrow \quad \delta h_{ab}^{\text{Weyl}} = 2\rho \eta_{ab} \tag{15.77}$$

15 \rightarrow *Infinitesimal* conformal transformation:

$$\delta h_{ab}^{\text{Diff}} + \delta h_{ab}^{\text{Weyl}} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad 2\rho \eta_{ab} = \partial_b \varepsilon_a + \partial_a \varepsilon_b \tag{15.78}$$

This differential equation must be solved for ε^a and ρ .

16 | To proceed, it is convenient to introduce new coordinates on the world sheet and on spacetime:



• On spacetime, introduce ** *light-cone fields*:

$$X^{\pm} := \frac{1}{\sqrt{2}} (X^0 \pm X^{D-1})$$
(15.79)



and analogously $p^{\pm} := \frac{1}{\sqrt{2}}(p^0 \pm p^{D-1})$ etc.

The choice of the space-like components X^{D-1} is arbitrary. Rewriting the theory in these variables singles out the direction $\mu = D - 1$ and breaks manifest Lorentz covariance. This is the price we have to pay for a quantization without gauge-degrees of freedom (\Rightarrow *below*).

• On the world sheet, define *** light-cone coordinates*:

$$\sigma^{\pm} := \tau \pm \sigma \quad \stackrel{\circ}{\Rightarrow} \quad \begin{cases} \partial_{\pm} = \frac{1}{2}(\partial_{\tau} \pm \partial_{\sigma}) \\ \Box = -4\partial_{+}\partial_{-} \\ ds^{2} = -d\sigma^{+}d\sigma^{-} \\ ds^{2} = -d\sigma^{+}d\sigma^{-} \\ \begin{pmatrix} \eta_{++} & \eta_{+-} \\ \eta_{--} & \eta_{--} \end{pmatrix} = \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix} \\ \begin{pmatrix} \eta^{++} & \eta^{+-} \\ \eta^{--} & \eta^{--} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} \end{cases}$$
(15.80)

Light-cone coordinates are simply inertial coordinates rotated by $\pm 45^{\circ}$; i.e., both coordinate vectors point along null cones and are therefore light-like.

17 | Eq. (15.78) $\stackrel{\circ}{\rightarrow}$ Constraints on conformal transformations in light-cone coordinates:

$$\partial_{-}\varepsilon^{-} + \partial_{+}\varepsilon^{+} = 2\rho \quad \Rightarrow \quad \rho = \rho(\varepsilon)$$
 (15.81a)

$$\partial_{+}\varepsilon^{-} = 0 \quad \Rightarrow \quad \varepsilon^{-} = \varepsilon^{-}(\sigma^{-})$$
 (15.81b)

$$\partial_{-}\varepsilon^{+} = 0 \quad \Rightarrow \quad \varepsilon^{+} = \varepsilon^{+}(\sigma^{+})$$
 (15.81c)

 \rightarrow Non-infinitesimal conformal transformation:

$$\bar{\sigma}^+ = \bar{\sigma}^+(\sigma^+) \quad \text{and} \quad \bar{\sigma}^+ = \bar{\sigma}^-(\sigma^-)$$
 (15.82)

The above derivation shows that for each such diffeomorphism a Weyl transformation ρ exists to keep the metric in flat gauge ($h_{ab} = \eta_{ab}$). We do not need to known the specific form of ρ because we drop the conformal scaling factor anyway. So the point is that we can make any transformation of the form Eq. (15.82) *while keeping the flat gauge fixed*.

18 | Define a rescaled time coordinate:

$$\bar{\tau}(\sigma^+,\sigma^-) := \frac{1}{2} \left[\bar{\sigma}^+(\sigma^+) + \bar{\sigma}^-(\sigma^-) \right] \quad \Leftrightarrow \quad \Box \bar{\tau} = -4\partial_+\partial_-\bar{\tau} = 0 \tag{15.83}$$

Note that the two expressions are *equivalent*. But this implies that the only constraint on the new world sheet coordinate $\bar{\tau} = \bar{\tau}(\sigma^+, \sigma^-) = \bar{\tau}(\tau, \sigma)$ is that is satisfies the wave equation. Conversely, whenever we have a function on the world sheet that satisfies the wave equation, we can *m.l.o.g.* set it equal to (an affine function of) τ .

Compare this to the EOM (15.47) that the fields X^{μ} satisfy:

$$\Box X^{\mu}(\tau,\sigma) = 0 \quad \Rightarrow \quad \Box X^{+}(\tau,\sigma) = 0 \tag{15.84a}$$

$$\Rightarrow \quad X^+(\tau,\sigma) = X^+_R(\sigma^-) + X^+_L(\sigma^+) \tag{15.84b}$$

That we focus on the light-cone field X^+ is arbitrary; it becomes useful below.
19 | Thus we can always choose world sheet coordinates (τ, σ) (we omit bars) such that ...

$$X^{+}(\tau,\sigma) = 2\alpha' p^{+}\tau \qquad \stackrel{*}{\ast} Light-cone\ gauge \qquad (15.85)$$

- In this gauge, the +-oscillator modes of the string are not excited and "frozen."
- That we pick out X^+ seems arbitrary at this point; it becomes useful \rightarrow below. [We could have chosen any (linear combination of) X^{μ} to be an affine function of τ .]
- **20** | We are finally in the position to Enforce the constraint Eq. (15.62):

$$T_{ab} \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \left(\dot{X} \pm X'\right)^2 \stackrel{!}{=} 0 \tag{15.86}$$

Expand the contraction in the square in light-cone fields:

$$-2 \underbrace{\left(\dot{X} \pm X'\right)^{+}}_{\text{Light-cone gauge } \odot} \left(\dot{X} \pm X'\right)^{-} + \underbrace{\sum_{i=1}^{D-2} \left(\dot{X} \pm X'\right)^{i} \left(\dot{X} \pm X'\right)^{i}}_{= \left(\dot{X} \pm X'\right)_{\perp}^{2}} \stackrel{!}{=} 0 \qquad (15.87)$$

We omit transversal sum symbols over i = 1, ..., D - 2 in the following.

 \rightarrow The constraint can be satisfied by setting:

$$(\dot{X} \pm X')^{-} := \frac{1}{4\alpha' p^{+}} (\dot{X} \pm X')^{2}_{\perp}$$
 (15.88)

- \rightarrow Also the X⁻ degrees of freedom are no longer dynamically independent.
- \rightarrow We must quantize only the D 2 transversal components X^i (and p^+).
- **21** $| \triangleleft Open$ string (for simplicity, similar arguments hold for the closed string)

Recall that the mode expansion for the open string reads:

Eq. (15.56)
$$\rightarrow X^{-} = x^{-} + 2\alpha' p^{-} \tau + i \sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n}{n} e^{-in\tau} \cos(n\sigma)$$
 (15.89)

We use this (and the mode expansions for X^i) to express Eq. (15.88) in terms of modes: Eq. (15.88) $\stackrel{\circ}{\rightarrow}$ (to show this set $\tau = 0$)

$$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+} \left(\frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i \right) \equiv \frac{1}{p^+} L_n^\perp$$
(15.90)

(Note that this includes $\sqrt{2\alpha'}p^- = \alpha_0^-$ [for the open string].) with * Transversal Virasoro modes:

$$L_n^{\perp} := \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i$$
(15.91)

That $\alpha_n^- \propto L_n^\perp$ is *quadratic* in the transversal modes α_n^i is evident from Eq. (15.88).



22 | To sum up, we have fixed the *flat gauge* and the *light-cone gauge*. In these gauges, the dynamics of the classical relativistic string is described by the following canonical pairs of variables:

Transversal modes:	X^{i} and Π^{i} $(i = 1,, D - 2)$
	Equivalently: x^i , p^i , α^i_m $(m \neq 0)$
Light-cone position & momentum:	x^- and p^+

Note that $x^+ = 0$ is frozen [Eq. (15.85)] and $p^- \propto \alpha_0^-$ is determined via Eq. (15.90) and therefore also no longer dynamical.

These variables satisfy the Poisson algebra Eq. (15.58) for $i = 1, \dots, D-2$:

$$\{x^{i}, p^{j}\} = \delta^{ij}$$
(15.92a)

$$\{p^{+}, x^{-}\} = 1$$
(15.92b)

$$\{\alpha^{i}_{m}, \alpha^{j}_{n}\} = im\delta_{m+n}\delta^{ij}$$
(15.92c)

The generalization to the closed string is straightforward and will not be shown in detail.

With this we are ready to quantize the open string! But before we do this, one last thing ...

23 | Witt algebra:

The Poisson algebra of the transversal oscillator modes determines the Poisson algebra of the transversal Virasoro modes (using the bilinearity and product rule for the Poisson bracket):

Eqs. (15.91) and (15.92) $\stackrel{\circ}{\rightarrow}$

$$\left\{L_m^{\perp}, L_n^{\perp}\right\} = i(m-n)L_{m+n}^{\perp} \qquad \text{** Witt algebra}$$
(15.93)

- Canonical quantization is the prescription to replace classical Poisson brackets of phase space functions by the commutators of operators on a Hilbert space. However, this prescription is not well-defined for quadratic functions like the Virasoro modes. We will find that after quantization (and a suitable definition of quantized Virasoro *operators*) the Witt algebra will be *modified* by a ↑ *central extension*. This unexpected modification signifies a ↑ *quantum anomaly*, and is directly linked to the critical dimension D = 26 of bosonic string theory.
- The Witt algebra shows up due to the conformal symmetry Eq. (15.82) of the Polyakov action. That the Witt algebra (interpreted as an abstract Lie algebra) describes conformal transformations in two dimensions can be seen as follows:

Remember that conformal transformations on (some region of) $\mathbb{R}^2 \simeq \mathbb{C}$ are given by \checkmark *mero-morphic functions* f(z) on \mathbb{C} ; these can be expanded in a Laurent series:

$$\tilde{z} \equiv f(z) = z - \sum_{n=-\infty}^{\infty} a_n z^n \,. \tag{15.94}$$

An infinitesimal conformal transformation $(|a_n| \ll 1)$ changes a scalar field $\phi(z)$ (for simplicity assumed to be holomorphic) as follows:

 $\phi(z) \stackrel{\text{Scalar}}{=} \tilde{\phi}(\tilde{z}) \stackrel{\text{Taylor}}{\approx} \tilde{\phi}(z) - \left(\sum_{n} a_{n} z^{n}\right) \partial_{z} \tilde{\phi}(z) \,. \tag{15.95}$



Thus the generators of such transformations have the form

$$\delta_a \phi \equiv \tilde{\phi}(z) - \phi(z) \approx \left(\sum_n a_n z^n \partial_z\right) \phi(z) \equiv \left(\sum_n a_n L_{1-n}\right) \phi(z) \tag{15.96}$$

with generator basis $L_n := z^{1-n} \partial_z$. The Lie algebra of these generators is:

$$[L_m, L_n]\phi(z) = (z^{1-m}\partial_z)(z^{1-n}\partial_z)\phi(z) - (z^{1-n}\partial_z)(z^{1-m}\partial_z)\phi(z)$$
(15.97a)

$$= (m-n)z^{1-m-n}\partial_z \phi(z) \tag{15.97b}$$

$$= (m-n)L_{m+n}\phi(z).$$
(15.97c)

That is, the Witt algebra is the Lie algebra of the "group" of conformal transformations. (The missing *i* can be obtained by redefining $L_m \mapsto -iL_m$.)

15.2. Quantization of the relativistic string

We quantize the string canonically, by replacing phase-space variables by operators and the Poisson algebra by a commutator algebra. The result will be a "first quantized" string, i.e., a relativistic quantum theory that describes a single string. Mathematically, this is achieved by techniques of "second quantization" because the string is described by a field theory.

There are three approaches to quantize the bosonic string:

• ↑ Covariant canonical quantization

Pros: Manifestly Lorentz covariant | Cons: Unphysical states & ghosts (= negative norm states)

This route starts by canonically quantizing Eq. (15.58) *without* fixing the light-cone gauge and enforcing the constraint Eq. (15.62) on the classical level. It is akin to \uparrow *Gupta-Bleuler quantization* of the electromagnetic field.

• Light-cone quantization (\rightarrow Section 15.2.1)

Pros: No unphysical states & ghosts | Cons: Not manifestly Lorentz covariant

This is the approach taken below; it is akin to the quantization of the electromagnetic field in Coulomb gauge usually presented in courses on advanced quantum mechanics.

• ↑ Covariant path integral quantization

This is the modern approach used in string theory (it is more abstract & versatile, but less suited for a first introduction).

This approaches leverages the full machinery of quantum field theory and is akin to the \uparrow *Faddeev-Popov quantization* of the electrommagnetic field [20].

Based on our preliminary work in Section 15.1 we can already conclude:

The "first quantized" string is described by a *quantum field theory* of D scalar fields X^{μ} that live on the 1+1-dimensional world sheet.

There is also a "second" quantization of string theory: ↑ *string field theory*.

15.2.1. Light-cone quantization

: We focus again on the *open* string for simplicity and state results for the closed string \rightarrow *later*.



1 | Remember: ↓ *Canonical quantization*:

Poisson bracket:
$$\{\bullet, \bullet\} \rightarrow \frac{1}{i\hbar} [\hat{\bullet}, \hat{\bullet}]$$
 : Commutator (15.98)

In the following we set $\hbar = 1$ an omit hats $\hat{}$ for operators.

2 | Eq. (15.92) \rightarrow Operator algebra for *open* string:

$$\begin{bmatrix} x^{i}, p^{j} \end{bmatrix} = i \delta^{ij}$$
(15.99a)
$$\begin{bmatrix} p^{+}, x^{-} \end{bmatrix} = i$$
(15.99b)
$$\begin{bmatrix} \alpha_{m}^{i}, \alpha_{n}^{j} \end{bmatrix} = m \delta_{m+n} \delta^{ij}$$
(15.99c)

$$\begin{bmatrix} \alpha_m^i, \alpha_n^j \end{bmatrix} = m\delta_{m+n}\delta^{ij}$$
(15.99c)

For the *closed* string, this algebra is extended by modes $\tilde{\alpha}_m^i$ in a straightforward way, cf. Eq. (15.54). $\triangleleft m > 0 \rightarrow$ Only non-vanishing commutator:

$$\underbrace{\begin{bmatrix} \alpha_m^i, \alpha_{-m}^j \end{bmatrix} = m\delta^{ij}}_{\text{Harmonic oscillator}?} \rightarrow \begin{cases} a_m^i := \frac{1}{\sqrt{m}} \alpha_m^i \\ a_m^{i\dagger} := \frac{1}{\sqrt{m}} \alpha_{-m}^i \end{cases} \rightarrow \underbrace{\begin{bmatrix} a_m^i, a_m^{j\dagger} \end{bmatrix} = \delta^{ij}}_{\text{Harmonic oscillator } \odot}$$
(15.100)

The excitations of an open string are thus described by a set of harmonic oscillator modes, labeled by the (transversal) direction i = 1, ..., D - 2 and mode m = 1, 2, ... of the oscillation.

3 | Virasoro *modes* Eq. (15.91) $\xrightarrow{\text{Quantization}}$ Virasoro *operators*:

Problem: Ordering ambiguity for L_0^{\perp} :

$$L_{n\neq0}^{\perp} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \underbrace{\alpha_{n-m}^{i} \alpha_{m}^{i}}_{\text{Commute!}} \quad \text{but} \quad L_{0}^{\perp} = \frac{1}{2} \sum_{m=-\infty}^{\infty} \underbrace{\alpha_{-m}^{i} \alpha_{m}^{i}}_{\text{Do not}}_{\text{commute!}}$$
(15.101)

What is the correct ordering for quantization?

We do not know! So let us play it safe and not fix the ordering prematurely:

i | To this end, we first *define* the operator L_0^{\perp} as \uparrow normal ordered:

$$L_0^{\perp} := \frac{1}{2}\alpha_0^i \alpha_0^i + \sum_{n=1}^{\infty} \underbrace{\alpha_{-n}^i \alpha_n^i}_{\text{Normal ordered}} \stackrel{15.100}{=} \alpha' \underbrace{p^i p^i}_{=: p_{\perp}^2} + \underbrace{\sum_{n=1}^{\infty} n \, a_n^{i\dagger} a_n^i}_{=: N^{\perp}}$$
(15.102)

N^{\perp} : Transverse ** *level operator*

Normal ordering is a prescription (a \uparrow meta operator) to order strings of non-commuting creation- and annihilation operators such that creation operators are on the left and annihilation operators on the right (this ordering is done without commutation relations). The result is an operator with vanishing expectation value wrt. the vacuum/ground state $|0\rangle$. Normal ordering is often indicated by enclosing an expression by colons: : • : .



- ii | But we do not know the correct ordering for quantization! Conveniently, *all possible* orderings can be brought into the normal ordered form Eq. (15.102) by using the commutator algebra Eq. (15.100); the result is always L_0^{\perp} with some constant offset $A = \text{const} \times \mathbb{1}$.
 - \rightarrow Wherever we used L_0^{\perp} in the *classical* theory, we make the replacement ...

$$L_0^{\perp} \quad \mapsto \quad L_0^{\perp} - A \tag{15.103}$$

... in the *quantized* theory.

 $A = \text{const} \times 1$: Unknown "normal ordering" constant

We henceforth carry the undetermined constant *A* along; maybe we encounter some condition that constrains *A* along the way ...

The appearance of the undetermined normal ordering constant A might be surprising. However, canonical quantization is not always a unique recipe to bootstrap a quantum theory from a given classical theory. This is only true for the most simple models – if they do not contain terms like $x \cdot p$ that lead to ordering ambiguities. Quantization is not a "fire-and-forget" procedure that assigns every classical theory a unique quantum theory that is magically "true". Classical theories are *limits* (= approximations) of underlying quantum theories for macroscopic systems. As such, they often do not contain enough information to recover the quantum theory unambiguously (\uparrow *Groenewold's theorem* [318]).

4 | This implies in particular: (remember that $\alpha_0^{\mu} = \sqrt{2\alpha'} p^{\mu}$ for an open string)

$$2\alpha' p^{-} = \sqrt{2\alpha'} \alpha_0^{-} \stackrel{15.00}{:=} \frac{1}{p^{+}} \left(L_0^{\perp} - A \right)$$
(15.104)

Formally, $1/p^+$ is the inverse operator of p^+ ; it will be canceled \rightarrow *below* anyway, so that a formal definition is not necessary.

5 \triangleleft \triangleleft Mass shell condition:

With these preparations, we find the quantized version of the mass shell condition Eq. (15.73):

$$M^{2} = -p^{2} = 2p^{+}p^{-} - p_{\perp}^{2}$$
(15.105a)
$$\stackrel{15.104}{=} \frac{1}{2} \left(L_{0}^{\perp} - A \right) - p_{\perp}^{2}$$
(15.105b)

$$\stackrel{\alpha'}{=} \frac{1}{\alpha'} \left(N^{\perp} - A \right)$$
(15.105c)

This result has two immediate implications:

- The mass of a string depends on the eigenvalues of the level operator N^{\perp} (= its excitations). This will lead to the identification of various particles in the \rightarrow *string spectrum* (Section 15.2.2).
- The (so far undetermined) normal ordering constant *A is important*! Its value determines the masses of the particles; in particular depending on the value of *A* the mass squared can become *negative* (which would imply a *space-like* 4-momentum).
- **6** $| \triangleleft$ Virasoro operators:

What is the commutator algebra of the transverse Virasoro operators?

Witt algebra Eq. (15.93) $\xrightarrow{\text{Eq. (15.99)}} ** Virasoro algebra:$

$$\left[L_{m}^{\perp}, L_{n}^{\perp}\right] \stackrel{\circ}{=} (m-n)L_{m+n}^{\perp} \underbrace{+ \frac{D-2}{12}m(m^{2}-1)\delta_{m+n}}_{\uparrow Central \ extension}$$
(15.10)

$D-2 \equiv c$: \uparrow Central charge (the prefactor $\frac{1}{12}$ is conventional)

• The Virasoro algebra is the most important algebra in string theory. As it descends from the conformal symmetry of the classical action, it is also the centerpiece of more general \uparrow conformal field theories, where the central charge c is not necessarily linked to the spacetime dimension (this is rather special to string theory).

It is well-known from conformal field theory that a free scalar (boson) has central charge c = 1. Thus, in bosonic string theory, each scalar field X^{μ} contributes c = 1 to the total central charge. In light-cone gauge, there are only D - 2 transversal fields X^i that are dynamical, so that the total central charge is c = D - 2.

- For a detailed derivation of Eq. (15.106) see ↑ ZWIEBACH [7] (§12.4, pp. 254–257).
- We found that the Lie algebra of the quantized generators of conformal transformations is different from their classical Poisson algebra Eq. (15.93). [Put differently: The Lie algebra of Virasoro operators does not follow from their classical algebra via the substitution Eq. (15.98).] This suggests that the original conformal/Weyl symmetry of the classical action might not be shared by the quantized theory. In general, the phenomenon that a classical symmetry does not survive quantization is called a \uparrow (quantum) anomaly. In the case of string theory, it is Weyl symmetry that can be spoiled; this particular anomaly is called \uparrow *Weyl anomaly*.
- Side note:

The additional term in Eq. (15.106) is called a \uparrow *central extension* of the Witt algebra Eq. (15.93) because it extends the old algebra by a new element of the form const $\times 1$ that commutes with all other elements (L_m^{\perp}) ; such elements (of a group or an algebra) are called \uparrow central in mathematics. If one exponentiates a centrally extended Lie algebra, the new central element leads to additional phase factors in the multiplication rules of the corresponding Lie group, so called \uparrow *cocycles*. These modified multiplication rules define \uparrow *projective representations* of the original Lie group (these are essentially group representations "up to phase factors"). Now remember that quantum mechanics is concerned with state vectors in Hilbert spaces up to global phases; mathematically speaking, the physical state spaces of quantum theories are \uparrow projective Hilbert spaces. Physical symmetries on such spaces are then implemented by the aforementioned projective representations. This line of arguments shows that the appearance of central extensions of symmetry algebras in quantum mechanics is directly linked to the fact that global phases are unphysical.



6)



↓Lecture35 [09.08.24]

7 | Lorentz covariance:

Here we finally answer the question:

Why does the quantization of the bosonic string only work in D = 26 spacetime dimensions?

Note that so far there is no restriction on the normalization constant A and/or the spacetime dimension D (= number of scalar fields X^{μ}).

However, remember that we sacrificed manifest Lorentz covariance when fixing the light-cone gauge. (The return of this investment was a ghost-free quantum theory, i.e., a theory without negative-norm states in the Hilbert space; cf. \uparrow *Covariant quantization*.) As our formulation is no longer manifestly Lorentz covariant, we cannot be sure that our *quantum* theory is still relativistic (that is, carries a representation of the Poincaré group).

If there is no representation of the Poincaré group on the Hilbert space of a given quantum theory, it is not relativistic. Remember that representations of symmetry groups are exactly that: they *represent* physical actions in the real world (translations, rotations, boosts, ...) by linear operators on an abstract, mathematical state space (the Hilbert space). The defining feature of a *relativistic* quantum theory is that it specifies how e.g. a boost modifies the quantum state that describes your system, and that the combination of such transformations yields a multiplicative structure called "Poincaré group." (Note that the Hamiltonian of the theory is part of this representation as it is the generator of translations in time.)

So let us manually check whether the Hilbert space of the quantized (open) string is a representation of the Poincaré group:

i | Remember: Lie algebra of Lorentz group Eq. (4.69):
 (➡) Problemset 5 of SPECIAL RELATIVITY course)

$$\left[J^{\mu\nu}, J^{\rho\sigma}\right] = \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho} \,. \tag{15.107}$$

Here on a *D*-dimensional spacetime: $\mu, \nu = 0, 1, ..., D - 1$. Note that $J^{\mu\nu}$ are abstract elements of a Lie algebra, not operators.

 \triangleleft In particular the generator:

$$J^{-i} = \frac{1}{\sqrt{2}} \left(J^{0,i} - J^{D-1,i} \right)$$
(15.108)

If something bad happens to Lorentz symmetry, it most likely is related to this generator because X^- is a rather non-trivial function of the dynamical fields X^i via Eqs. (15.89) and (15.90).

Eq. (15.107) implies $\stackrel{\circ}{\rightarrow}$

$$\left[J^{-i}, J^{-j}\right] = 0 \tag{15.109}$$

The question is now whether there are *operators* J^{i-} (= representations) acting on the Hilbert space of the quantized string that satisfy this commutation relation. If not, we lost Lorentz symmetry and have a problem ...

ii | Noether charges Eq. (15.66) $\xrightarrow{\text{Eq. (15.99)}}$ Representations of Lorentz group (?):



The very fact that the Witt algebra got spoiled by quantization should make us wary; after all, the Lorentz algebra might be affected by an anomaly as well!

(For once we mark these operators with a hat ^ to mark them as *representations*.)

$$\hat{J}^{\mu\nu} \stackrel{15.66}{:=} T \int_{0}^{\pi} d\sigma \underbrace{:(X^{\mu} \dot{X}^{\nu} - X^{\nu} \dot{X}^{\mu}):}_{\text{Normal ordered}}$$

$$\stackrel{\circ}{=} \underbrace{x^{\mu} p^{\nu} - x^{\nu} p^{\mu}}_{\text{Orbital angular momentum}} - i \underbrace{\sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha_{-n}^{\mu} \alpha_{n}^{\nu} - \alpha_{-n}^{\nu} \alpha_{n}^{\mu} \right)}_{\text{Normal ordered}}$$
(15.110)

Internal angular momentum (spin)

- *Ĵ^{μν}* is an operator on the light-cone Hilbert space ℋ spanned by the transversal modes
 (→ Section 15.2.2). This Hilbert space must be a representation of the Lorentz group,
 i.e., the commutator algebra must be Eq. (15.107) and Eq. (15.109) must hold.
- The fact that $\hat{J}^{\mu\nu}$ has a contribution that can be interpreted as internal angular momentum (= spin) already suggests that different excitations of the string describe particles not only with different *masses* but also with different *spin*.
- Eq. (15.110) is a *definition* of the quantization of the classical charge J^{μν}. Due to the occurrence of quadratic terms in oscillator modes, it suffers from an ordering ambiguity similar to the Virasoro mode L₀[⊥]. The operator is then again defined via *ϵ normal ordering*, such that the ground state/vacuum is Lorentz invariant. Note that the second summand in Eq. (15.110) is indeed normal ordered since the modes α^μ_{-n} (α^μ_n) are creation (annihilation) operators [*ϵ* Eq. (15.100)].
- A global, continuous symmetry gives rise to a classically conserved quantity via Noether's theorem (for example: spatial translation symmetry leads to conservation of the total momentum). Quantizing the theory makes this quantity into an operator (for example: the momentum operator). In the absence of quantum anomalies, this operator is the generator of the original symmetry transformation (for example: the momentum operators).
- iii | In particular, we must set for the crucial operator Eq. (15.108):

$$\hat{J}^{-i} := x^{-}p^{i} - \underbrace{\frac{1}{2}\left(x^{i}p^{-} + p^{-}x^{i}\right)}_{\text{Symmetrized} \to \text{Hermitian}} - i\sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{-n}^{-}\alpha_{n}^{i} - \alpha_{-n}^{i}\alpha_{n}^{-}\right) \quad (15.111a)$$

$$\stackrel{15.104}{=} x^{-}p^{i} - \frac{1}{4\alpha' p^{+}} \left[x^{i} (L_{0}^{\perp} - A) + (L_{0}^{\perp} - A) x^{i} \right]$$

$$- \frac{i}{\sqrt{2\alpha'}p^{+}} \sum_{n=1}^{\infty} \frac{1}{n} \left(L_{-n}^{\perp} \alpha_{n}^{i} - \alpha_{-n}^{i} L_{n}^{\perp} \right)$$
(15.111b)

- Due to p⁻, the Virasoro operator L[⊥]₀ enters the stage. To account for its ordering ambiguity, we must augment the expression by the (yet undetermined) normal ordering constant A.
- Note that the generators of a symmetry group should also be *Hermitian*, such that the representation of the group is *unitary* (symmetries must not change the absolute values of overlaps of state vectors; ↑ *Wigner's theorem*). The mode term in Eq. (15.110) is



clearly Hermitian since $\alpha_n^{\mu\dagger} = \alpha_{-n}^{\mu}$. However, without a careful ordering of operators, this is not true for the orbital angular momentum term: since x^i does not commute with p^- [due to Eqs. (15.90) and (15.104)], the second term must be symmetrized. For details: $\uparrow ZWIEBACH$ [7] (§12.5, pp. 260–261).

iv | Plug Eq. (15.111b) in Eq. (15.109):

From Eq. (15.111b) we should expect the normal ordering constant A to show up. The commutator of Virasoro operators also certainly plays a role, so that the Virasoro algebra Eq. (15.106) with its central charge c = D - 2 enters the computation. It is therefore not surprising that the result depends on A and D:

$$\left[\hat{J}^{-i}, \hat{J}^{-j}\right]^{\underbrace{15.11b}_{15\underline{106}}} - \underbrace{\frac{1}{\alpha'(p^+)^2} \sum_{m=1}^{\infty} \Delta_m \left(\alpha^i_{-m} \alpha^j_m - \alpha^j_{-m} \alpha^i_m\right)}_{\neq 0 \to \uparrow \text{ Quantum anomaly } \odot}$$
(15.112)

with anomaly factors

$$\Delta_m = m\left(\frac{26-D}{24}\right) + \frac{1}{m}\left[\frac{D-26}{24} + (1-A)\right].$$
 (15.113)

So like the Witt algebra, the Lorentz algebra suffers from a *quantum anomaly*: the quantization modifies the algebra. Thus, whatever we quantized, it is no longer a relativistic string \odot .

Except ...

 \rightarrow Lorentz symmetry is broken *unless* $\forall m \in \mathbb{N}$: $\Delta_m = 0 \Leftrightarrow$

$$A = 1$$
 and $D = 26$ (15.114)

- This result states that a relativistic string propagating on Minkowski space can only be quantized consistently in *D* = 26 spacetime dimensions. The constant *A* = 1 has consequences of similar importance for the masses of the particles predicted by the theory (→ Section 15.2.2).
- There are two perspectives on this result:
 - String-theory advocate: ©©©

Wow! String theory doesn't leave us any choice – it predicts spacetime to be D = 26 dimensional (or D = 10 dimensional in superstring theory).

String-theory opponent: ©

Bullshit! Our spacetime is not D = 26 but D = 4-dimensional. This theory cannot describe reality; it is a mathematical peculiarity, nothing more.

Unfortunately, as just shown, we cannot simply "tweak" the theory to match D = 4; the quantum version of the relativistic string is *rigid*: D = 26 or we're out of business! A creative cop-out is to keep and "hide" the unwanted 22 spatial dimensions by curling them up into tiny circles (or more complicate \uparrow *Calabi-Yau manifolds*; \uparrow *compactification*). This modification of course affects the dynamics and interaction of strings propagating in the "large" four dimensions of our spacetime. How exactly the string dynamics is modified depends on how exactly one curls up the additional dimensions. Unfortunately, there are many different ways to do



this ("unfortunately" is a word needed quite often in string theory); this leads to the \uparrow *string theory landscape*, the \uparrow *anthropic principle* raising its ugly head, and, eventually, the demise of the scientific principle ...

• In \uparrow covariant (canonical) quantization, Lorentz covariance is manifest throughout the computation (all operators have Lorentz indices), but for $D \neq 26$ the constructed representation is not unitary (= there are ghosts [negative-norm states] in the physical state space). By contrast, in light-cone quantization there are only positive-norm states in the Hilbert space, but for $D \neq 26$ the operators $J^{\mu\nu}$ no longer satisfy the Lorentz algebra and Lorentz covariance is lost.

15.2.2. Bosonic string spectrum

Now that we quantized the (open) bosonic string, we can start to build its Hilbert space [= the representation of the commutator algebra Eq. (15.99)]. The (now quantized) excitations of the string are identified with elementary particles of various masses and spins. Finding the Hilbert space is straightforward since Eq. (15.99) consists of canonical position and momentum operators, together with (multiple copies) of the harmonic oscillator algebra, both of which you studied in your first course on quantum mechanics.

We start with the open string:

- **8** | \triangleleft Open string:
 - i | Eq. (15.99) \rightarrow Canonical pairs (x^-, p^+) and (x^i, p^i)
 - \rightarrow Momentum space representation:

$$|k^+, k_\perp\rangle \equiv |k\rangle$$
 ** String ground states (15.115)

with

$$\forall_{m \ge 1} : a_m^i |k\rangle = 0 \text{ and } (p^+, \vec{p}_\perp) |k\rangle = (k^+, \vec{k}_\perp) |k\rangle$$
 (15.116)

for i = 1, ..., D - 2.

These states describe a single string in its oscillatory ground state with momentum (k^+, \vec{k}_\perp) .

- ii | We can now create excitations of the string by acting with mode creation operators $a_n^{i\dagger}$ on these ground states:
 - \rightarrow Fock space \mathcal{H}_o spanned by open string states

$$|\boldsymbol{\lambda}, \boldsymbol{k}\rangle := \prod_{\substack{n=1\\\text{Oscillation}\\\text{Mode}}}^{\infty} \prod_{\substack{i=1\\\text{Transversal}\\\text{direction}}}^{24} \left(a_n^{i\dagger}\right)^{\lambda_{n,i}} |\boldsymbol{k}\rangle \quad \text{with } \lambda_{n,i} \in \mathbb{N}_0.$$
(15.117)

 $\rightarrow |\lambda, k\rangle$ = State of particle with mass squared (remember that A = 1 is now fixed)

$$M^{2}|\boldsymbol{\lambda},k\rangle \stackrel{15.105}{=} \frac{1}{\alpha'} (N_{\boldsymbol{\lambda}}^{\perp} - 1)|\boldsymbol{\lambda},k\rangle \quad \text{with} \quad N_{\boldsymbol{\lambda}}^{\perp} = \sum_{n=1}^{\infty} \sum_{i=1}^{24} n\lambda_{n,i} \,.$$

(15.118)



- The state |λ = 0, k = 0) describes a string with zero momentum and no oscillations, not the vacuum ("no string").
- Since there are infinitely many levels $N_{\lambda}^{\perp} = 0, 1, 2, ...$ (and the irreducible representations of the Poincaré group live within these levels), string theory predicts *infinitely* many particles!
- The operators J^{µν} generate a representation of the Lorentz group on the Hilbert space spanned by the states Eq. (15.117). This representation decomposes into a direct sum of ↓ *irreducible representations* of states that mix under Lorentz transformations. Since M² = -p² ∝ (N[⊥] 1) is a Lorentz scalar, only states of the same level N[⊥] can transform into each other under Lorentz transformations.

According to \uparrow *Wigner's classification*, physical *particles* correspond to irreducible representations of the Poincaré group. (A particle type is the linear subspace of quantum states that can be transformed into each other by Poincaré transformations. This is why we say that an electron with spin up and an electron with spin down are the same type of particle: If you rotate your experiment, you can make one into the other.)

This means that we should identify particles by the linear subspaces within each level of string the Hilbert space that are invariant under Lorentz transformations. Due to the light-cone gauge, this is a non-trivial task: We have only transversal modes $a_n^{i\dagger}$, but know that *massive* particles need more. These are provided by other modes in the same level, but the identification of the irreducible representations is rather involved for massive particles.

We can now study the particles that arise from the lowest levels of the string spectrum:

iii $| \triangleleft$ <u>Lowest-mass excitations:</u>

a | Level $N^{\perp} = 0$: (this is the particle that corresponds to the string ground state)

 $|k\rangle$ with mass $M^2 = -\frac{1}{\alpha'} < 0 \rightarrow$ ** Tachyon (Scalar) \rightarrow Unstable vacuum \odot

- Since there is only one state |k⟩ for each momentum in the lowest level, the particle
 must be a *scalar* (no internal degrees of freedom).
- Particles with M² < 0 are called *tachyons*. They have a space-like 4-momentum (p² = -M² > 0) and therefore "move faster than the speed of light." This, however, is a misleading interpretation in the context of quantum field theories, where they are symptoms of an *unstable vacuum state* [64, 65] (← Section 4.4). The existence of this state makes bosonic string theory unstable and motivates its extension by fermions and supersymmetry; ↑ *superstring theory*.

To understand the effect of negative square masses in a scalar field theory (here: the quantum field of the tachyons), recall that the generic ϕ^4 -Lagrangian that governs such fields has the form

$$\mathcal{L} = \frac{1}{2} (\partial \phi)^2 - m^2 \phi^2 - \lambda \phi^4 \equiv \frac{1}{2} (\partial \phi)^2 - V(\phi) , \qquad (15.119)$$

with (bare) particle mass *m* and interaction λ ; the potential is $V(\phi) = m^2 \phi^2 + \lambda \phi^4$. For $\lambda = 0$, the classical EOM is the Klein-Gordon equation: $(\partial^2 + m^2)\phi = 0$. Note that for a lower-bounded potential energy it must be $\lambda > 0$.

For *positive* square mass $m^2 > 0$, the classical ground state is clearly $\phi = 0 = \text{const.}$ However, for *negative* square mass $m^2 < 0$ (now ϕ is a tachyon field!), the lowest



potential energy is found for $|\phi| = \sqrt{-m^2/2\lambda} \neq 0$. This indicates \checkmark spontaneous symmetry breaking, and therefore an instability of the (tachyon-)vacuum $\phi = 0$. What then happens is that the system transitions into the new vacuum by "condensing tachyons"; this vacuum has new excitations which have *positive* square mass (the Higgs mode). This is exactly what happens with the Higgs field in the Standard Model: The Higgs symmetry breaking can be understood as "tachyon condensation", and the excitations of the new (symmetry-broken) vacuum are the Higgs bosons.

The bottom line is that tachyons in the spectrum of a quantum field theory are not superluminal "science-fiction particles" but harbingers of a vacuum decay.

b | Level
$$N^{\perp} = 1$$
:

This level can only be reached by applying a single creation operator of the n = 1-mode:

 $\underbrace{\frac{D-2=24 \text{ states}}{|i,k\rangle \equiv a_1^{i\dagger}|k\rangle}_{i\dagger} \text{ with mass } M^2 = 0 \rightarrow \text{Massless vector boson} \rightarrow \text{Photon } \bigcirc$

Since i = 1,..., 24 these states transform under the vector representation of SO(24), as one would expect for massless vector bosons in D = 26 dimensions. This allows us to identify them with the *photons* of electrodynamics in D = 26 dimensions:

$\underbrace{ i,k\rangle = a_1^{i\dagger} k\rangle}_{1}$	\leftrightarrow	$\underbrace{a_k^{i\dagger} 0\rangle}$	(15.120)
String states @ $N^{\perp} = 1$		Photon states in $D = 26$	
`		1	

Same Poincaré representation & momenta & mass

Remember that in D = 4-dimensional electrodynamics there are D - 2 = 2 transverse polarizations for massless photons. These form helicity representations of SO(2). Analogously, the D - 2 = 24 transverse polarizations above form a representation of SO(24), the symmetry group of photons in D = 24 spacetime dimensions.

The fact that there are D - 2 states on the first level is independent of the normal ordering constant A = 1. The latter, however, makes these states massless (M² = 0) and thereby consistent with the representation theory of the Poincaré group: D-2 states can form a vector representation of SO(D-2) - but not of SO(D-1). The latter is a subgroup of the Lorentz group SO(1, D-1) and would be needed for massive particles. By contrast, massless particles transform exactly under SO(D-2) (because you can only rotate them about their momentum vector).

Fun fact: By this line of arguments you can actually infer A = 1 from the requirement of Lorentz covariance even before you know that D = 26.

c | Level $N^{\perp} > 1$: Massive bosonic particles ...

For example, $N^{\perp} = 2$ can be reached by either applying $a_1^{i\dagger}a_1^{j\dagger}$ or $a_2^{i\dagger}$ to the string ground state; the mass of these particles is $M^2 = 1/\alpha' > 0$. In total there are $\frac{1}{2}(D - 2)(D + 1)$ states on level $N^{\perp} = 2$ (324 states for D = 26). These states form the representations of a so called \uparrow massive tensor boson.



9 $| \triangleleft$ Closed string:

We start with the generalization of the results for the quantized open string to the closed string (without derivations). We can then study the closed string spectrum in analogy to the open string spectrum:

- i | Quantized closed string in light-cone gauge:
 - **a** | Eq. (15.54) \rightarrow Operator algebra for *closed* string:

$$\begin{bmatrix} x^{i}, p^{j} \end{bmatrix} = i \delta^{ij}$$
(15.121a)
$$\begin{bmatrix} p^{+}, x^{-} \end{bmatrix} = i$$
(15.121b)

$$\begin{bmatrix} \mu^{i}, \alpha^{j}_{n} \end{bmatrix} = m\delta_{m+n}\delta^{ij}$$
(15.121c)

New!
$$\Rightarrow \left[\tilde{\alpha}_{m}^{i}, \tilde{\alpha}_{n}^{j}\right] = m\delta_{m+n}\delta^{ij}$$
 (15.121d)

- All commutators not shown vanish, of course.
- The only difference to the open string is that there is another independent copy Eq. (15.121d) of oscillator modes.
- According to Eq. (15.50), the interpretation of the two mode sets is that they create right- and left-moving oscillations on the string, respectively.
- **b** | The only subtlety concerning the closed string is that there is a constraint connecting the left- and right-moving zero-modes:

$$\alpha_0^{\mu} \stackrel{\text{def}}{=} \sqrt{\frac{\alpha'}{2}} p^{\mu} \stackrel{\text{def}}{=} \tilde{\alpha}_0^{\mu} \implies \alpha_0^- = \tilde{\alpha}_0^- \implies L_0^\perp = \tilde{L}_0^\perp \quad (15.122)$$

- While α_n^μ and α_n^μ are *independent* oscillator modes for n ≠ 0, they are *the same* center-of-mass mode for n = 0 (there is only one string with one center-of-mass!).
- The analog of Eq. (15.104) for the closed string reads

$$\sqrt{2\alpha'}\alpha_0^- = \frac{2}{p^+}(L_0^\perp - 1) \text{ and } \sqrt{2\alpha'}\tilde{\alpha}_0^- = \frac{2}{p^+}(\tilde{L}_0^\perp - 1), \quad (15.123)$$

i.e. the normal-ordering constants A = 1 and $\tilde{A} = 1$ required for Lorentz covariance are the same for left- and right-movers. (The critical dimension D = 26 is also the same.)

With the analog of Eq. (15.102)

$$L_0^{\perp} = \frac{\alpha'}{4} p_{\perp}^2 + N^{\perp}$$
 and $\tilde{L}_0^{\perp} = \frac{\alpha'}{4} p_{\perp}^2 + \tilde{N}^{\perp}$ (15.124)

this constraint translates into the ...

$$N^{\perp} = \tilde{N}^{\perp}$$
 ** Level-matching condition (15.125)

with level operators

$$N^{\perp} := \sum_{n=1}^{\infty} n \, a_n^{i\dagger} a_n^i \quad \text{and} \quad \tilde{N}^{\perp} := \sum_{n=1}^{\infty} n \, \tilde{a}_n^{i\dagger} \tilde{a}_n^i \,.$$
 (15.126)

 \rightarrow



The level-matching condition Eq. (15.125) excludes some states from the Fock space generated by the oscillator modes $a_n^{i\dagger}$ and $\tilde{a}_n^{i\dagger}$; i.e., only a subspace of the Hilbert space contains *physical states*.

c | The mass shell condition follows in analogy to Eq. (15.105):

$$M^2 = \frac{2}{\alpha'} \left(N^{\perp} + \tilde{N}^{\perp} - 2 \right)$$
 ** Mass shell condition

- Combine Eqs. (15.122) to (15.124) to show this.
- The -2 differs from the open string case Eq. (15.118) since the normal ordering constant contributes twice in the sum (for both left- and right-moving modes).
- ii | Fock space:

The closed string Fock space \mathcal{H}_c is spanned by the states

$$|\boldsymbol{\lambda}, \tilde{\boldsymbol{\lambda}}, k\rangle := \underbrace{\left[\prod_{n=1}^{\infty} \prod_{i=1}^{24} \left(a_{n}^{i\dagger}\right)^{\lambda_{n,i}}\right]}_{\text{Right-moving modes}} \times \underbrace{\left[\prod_{m=1}^{\infty} \prod_{j=1}^{24} \left(\tilde{a}_{m}^{j\dagger}\right)^{\tilde{\lambda}_{m,j}}\right]}_{\text{Left-moving modes}}|k\rangle$$
(15.127)

Here again $\lambda_{n,i}, \lambda_{n,i} \in \mathbb{N}_0$.

if and only if they satisfy the level-matching condition Eq. (15.125):

$$\sum_{n=1}^{\infty} \sum_{i=1}^{24} n\lambda_{n,i} =: N_{\lambda}^{\perp} \stackrel{!}{=} \tilde{N}_{\lambda}^{\perp} := \sum_{n=1}^{\infty} \sum_{i=1}^{24} n\tilde{\lambda}_{n,i} .$$
(15.128)

This equation cannot be solved explicitly, it must be studied on a case-by-case basis!

- iii $| \triangleleft$ <u>Lowest-mass excitations:</u>
 - **a** | Level $N^{\perp} = \tilde{N}^{\perp} = 0$:

 $|k\rangle$ with mass $M^2 = -\frac{4}{\alpha'} < 0 \rightarrow$ Another \leftarrow Tachyon (Scalar)

- The closed string tachyons are mathematically analogous to the open string tachyons.
- Closed string tachyons are less well understood than open string tachonys. ZWIEBACH [7] writes (§13.3, p. 291):

The closed string tachyon is far less understood than the open string tachyon. [...] The instabilities associated with closed string tachyons are expected to be instabilities of spacetime itself. They remain largely mysterious.

Congratulations! You have officially entered physics mystery land ...



b | Level $N^{\perp} = \tilde{N}^{\perp} = 1$:

Such states must have always one left- and one right-moving mode with n = 1 excited:

$$|\Psi,k\rangle := \sum_{i,j} R_{ij} a_1^{i\dagger} \tilde{a}_1^{j\dagger} |k\rangle \text{ with } M^2 |\Psi,k\rangle = 0 \text{ (massless)}$$
(15.129)

 R_{ij} : arbitrary $(D-2) \times (D-2)$ square matrix

 \rightarrow Decompose this matrix *w.l.o.g.* as follows:

$$\underbrace{R_{ij}}_{(D-2)^2} = \underbrace{\underbrace{S_{ij}}_{\substack{\& \text{ traceless}}}_{\substack{ij \\ 1 \\ 2 \\ (D-2)(D-1)-1}} + \underbrace{A_{ij}}_{\substack{ij \\ 2 \\ (D-2)(D-3)}} + \underbrace{\underbrace{D \ \delta_{ij}}_{1}}_{1}$$
(15.130)

The expressions below the matrix components denote the number of degrees of freedom. These three parts transform each as an irreducible representation under Poincaré transformations, i.e., these states correspond to different types of massless particles:

 \rightarrow *Three* massless particle types:

$ \boldsymbol{S},\boldsymbol{k}\rangle := \sum_{i,j} S_{ij} a_1^{i\dagger} \tilde{a}_1^{j\dagger} \boldsymbol{k}\rangle$	Graviton states ©	(15.131a)
$ A,k\rangle := \sum_{i,j} A_{ij} a_1^{i\dagger} \tilde{a}_1^{j\dagger} k\rangle$	* Kalb-Ramond states ©	(15.131b)
$ D,k\rangle := \sum_i D a_1^{i\dagger} \tilde{a}_1^{i\dagger} k\rangle$	* Dilaton state 😟	(15.131c)

• Remember from our discussion of gravitational waves (Section 13.4) that classical excitations of the metric field can be parametrized (after exploiting the gauge symmetry of GENERAL RELATIVITY) by a symmetric, traceless field $h_{\mu\nu}$ with only transverse modes. In D spacetime dimensions, this means that a gravitational wave can be encoded in a $(D-2) \times (D-2)$ matrix h_{ij} that is symmetric and traceless; it has therefore

$$\frac{1}{2}(D-2)(D-1) - 1 = \frac{1}{2}D(D-3)$$
(15.132)

physical degrees of freedom; in D = 4 one finds the *two* degrees of freedom that we identified in Section 13.4 as two possible polarizations.

But Eq. (15.130) shows that there are exactly as many states $|S, k\rangle$ as demanded by Eq. (15.132), which tells us that these are the required massless "spin-2" states needed for a graviton. (Note that it is not obvious that this field couples to the other particles of the theory via the energy-momentum tensor.)

For details: ↑ ZWIEBACH [7] (§10.6, pp. 209–212).

- The Kalb-Ramond states are the excitations of an antisymmetric, massless tensor field B_{µν} that is similar to the electromagnetic gauge potential A_µ. It acts as a source of *ϵ torsion* for the affine connection on spacetime [*ϵ* Eq. (10.41)].
- The dilaton states are the excitations of a massless scalar field, the ↑ *dilaton field*. Intuitively, the dilaton is the quantization of the "breathing" mode mentioned *← earlier*. The dilaton modifies the theory of gravity predicted by string theory to a tensor-scalar type (*←* Section 12.3); it also controls the *interaction strength* of strings.

For details: ↑ ZWIEBACH [7] (§13.4, pp. 294–296).



15.3. ‡ Closing remarks

We conclude our excursion with a few comments on advanced topics:

For more details the reader is referred to David Tong's script on String Theory [319].

• How do strings interact?

So far we described the states of a single string (open or closed) propagating on a D = 26dimensional spacetime.

1 \triangleleft Difference between theories of *point particles* and *strings*:



- The action of a free point particle lives on a 1D worldline and is therefore *undefined* on a vertex where two particles meet (the crossing of two lines is not a manifold).
 - \rightarrow Interaction terms must be added by hand.
- The action of a free string lives on a 2D worldsheet and is well-defined everywhere on the worldsheet of two strings that merge and separate again.
 - \rightarrow The Polyakov action is already well-defined for interacting strings.
- **2** | Given the gauge symmetries of the Polyakov action (= diffeomorphism invariance & Weyl symmetry), are there possible terms that we could add to modify string interactions?

$\stackrel{\circ}{\rightarrow}$ Polyakov action can be extended by only one term:

$$\tilde{S}_{P} := -\int d\tau d\sigma \sqrt{h} \left[\frac{T}{2} h^{ab} \partial_{a} X^{\mu} \partial_{b} X_{\mu} + \underbrace{\frac{\lambda}{4\pi} R(h)}^{\text{invariant}} \right]$$
(15.133a)
$$\equiv S_{P}[h, X] - \lambda \chi[h]$$
(15.133b)

Weyl & Diff.

R(h): Ricci scalar on worldsheet (This is not the curvature of spacetime!)

The general covariance of the new term is obvious. It's Weyl invariance (up to a total derivative!) can be checked by a straightforward calculation.

3 | Interpretation:

 \triangleleft Worldsheet without boundaries \rightarrow

$$\chi[h] = \frac{1}{4\pi} \int d\tau d\sigma \sqrt{h} R(h) \stackrel{\text{Gauss}}{=} 2(1-g) \in \mathbb{Z}$$
(15.134)



- g: Number of "handles" of the worldsheet (g is called the \uparrow genus of the worldsheet.)
- $\rightarrow \chi$ depends only on the *topology* of the worldsheet.
- I.e., the action χ is invariant under *geometrical* deformations of the worldsheet.
- 4 | Superposition principle \rightarrow Sum over all worldsheet *topologies* and *geometries* ...
 - ... consistent with the string scattering process under consideration:

$$\underbrace{\int_{\text{Topologies}}_{\text{Metrics}} e^{i\tilde{S}_{P}} \sim \sum_{\text{Topologies}} \underbrace{e^{2i\lambda(g-1)}}_{\equiv g_{s}^{2i(g-1)}} \int_{\text{Fixed}} \mathcal{D}X \mathcal{D}h e^{iS_{P}[h,X]}$$
(15.135)

 $g_s = e^{\lambda}$: ** String coupling constant

- String theory has therefore only two parameters: The string tension T and the string coupling constant g_s. These two numbers determine the scattering amplitudes of all particles predicted by the theory. This is of course a fascinating prospect: The Standard Model has ~ 18 free parameters (masses and coupling constants) that ask for an explanation. Unfortunately (③), none of these parameters have been derived from (super-)string theory so far.
- String theory calculations typically make use of two perturbative expansions: one in α' (to capture interactions on the worldsheet) and one in g_s (summing over the number of "holes" in the worldsheet).

• Where is GENERAL RELATIVITY? Where are the Einstein field equations?

This was a course on SPECIAL RELATIVITY and GENERAL RELATIVITY, and our most precious result was the Einstein field Eq. (12.10) that controls the geometry of spacetime.

String theory claims to be a theory of quantum gravity; to earn this title, it should reproduce the Einstein field equations in some limit. Since string theory is formulated on a static background metric (\rightarrow *below*) – and not in a background-independent form – GENERAL RELATIVITY emerges in a rather esoteric way:

1 Let us first generalize the Polyakov action to arbitrary cuved spacetimes:

Eq. (15.22)
$$\xrightarrow{\eta_{\mu\nu} \mapsto g_{\mu\nu}(X)} S_{g}^{\text{flat}}[X] := -\frac{T}{2} \underbrace{\int g_{\mu\nu}(X) \partial_{a} X^{\mu} \partial^{a} X^{\nu} \, d\sigma d\tau}_{\uparrow Non-linear \, \sigma - model}$$
(15.136)

 \rightarrow Interacting quantum field theory with infinitely many coupling constants $g_{\mu\nu}(x)$

This explains why used flat Minkowski space $\eta_{\mu\nu}$ to quantize the bosonic string.

Actions like Eq. (15.136) with fields that map into non-linear manifolds (here: curved spacetime) are called \uparrow *non-linear* σ *-models*; a well-studied class of interacting quantum field theories that also have applications in condensed matter physics.

 \rightarrow Question: Is $g_{\mu\nu}(x)$ arbitrary?

Surprising answer: Not if we want the conformal gauge symmetry to be unbroken!



2 | To show this \triangleleft Small fluctuations of the fields around x_0^{μ} :

$$X^{\mu}(\tau,\sigma) \equiv x_{0}^{\mu} + \sqrt{\alpha'}Y^{\mu}(\tau,\sigma)$$
(15.137)

$$\Rightarrow \quad g_{\mu\nu}(X) \stackrel{\circ}{=} \eta_{\mu\nu} - \frac{\alpha}{3} \underbrace{R_{\mu\alpha\nu\beta}(x_0)}_{\text{Riemann curvature}} Y^{\alpha}Y^{\beta} + \mathcal{O}(Y^3) \tag{15.138}$$

Here we use \leftarrow *locally inertial coordinates* [\leftarrow Eq. (10.89)] on spacetime to make the first derivatives of the metric vanish. One can show that the (non-vanishing) quadratic order is then given by the \leftarrow *Riemann curvature tensor* (\uparrow *Riemann normal coordinates*).

Eq. (15.136) \rightarrow Interacting quantum field theory on 2D worldsheet:

$$S_{g}^{\text{flat}}[Y] \approx -\frac{1}{4\pi} \int d\sigma d\tau \left[\underbrace{\eta_{\mu\nu} \partial_{a} Y^{\mu} \partial^{a} Y^{\nu}}_{\text{Free fields}} - \underbrace{\frac{\alpha'}{3} R_{\mu\alpha\nu\beta} Y^{\alpha} Y^{\beta} \partial_{a} Y^{\mu} \partial^{a} Y^{\nu}}_{\text{Interaction}} \right]_{\text{Interaction}}$$
(15.139)

One can now apply the well-honed machinery of quantum field theory to this action:

 \rightarrow Feynman rules & Perturbation theory in $\alpha' \dots$

- **3** | Remember: The conformal symmetry Eq. (15.30) of the Polyakov action is a *gauge* symmetry.
 - \rightarrow Consistency requires that it remains *unbroken* after quantization.

This means that a \leftarrow quantum anomaly that spoils this symmetry cannot be tolerated.

 \rightarrow How to check the conformal symmetry of Eq. (15.139) after quantization?

Idea: Calculate \uparrow *renormalization flow* of couplings $g_{\mu\nu}$:

Conformal symmetry \Leftrightarrow Scale invariance \Leftrightarrow RG fixed point

- Note that conformal transformations include *global scale transformations*; scale invariance is therefore a necessary (and under most circumstances sufficient) condition for conformal symmetry.
- The idea of the \uparrow renormalization group (RG) is to study how the couplings in a Lagrangian change if one "zooms" our. Mathematically, one integrates out a thin shell of high-energy momenta from the partition sum (in perturbation theory for interacting QFTs) and studies how this changes the coupling constants of the obtained effective action (here: $g_{\mu\nu}$). As a result, one obtains differential equations that encode the change of coupling constants with changing length/energy scale λ . This is called the \uparrow renormalization flow, and the function that determines the flow of a coupling constant is called its \uparrow beta function:

$$\beta_{\mu\nu}(g) := \lambda \frac{\partial G_{\mu\nu}(X;\lambda)}{\partial \lambda} \,. \tag{15.140}$$

A vanishing beta functions means that the theory "looks the same" on all length scales, i.e., is *scale invariant*.

4 | Apply standard techniques to compute RG flow of Eq. (15.139) in first order of α' :

$$\beta_{\mu\nu}(g) = \alpha' R_{\mu\nu} + \mathcal{O}(\alpha'^2)$$
(15.141)

$R_{\mu\nu}$: Ricci tensor of $g_{\mu\nu}$

The resulting RG flow is called the \uparrow *Ricci flow*; it is an important concept for the RG analysis of non-linear σ -models in general.



5 | We can finally piece everything together:

No conformal anomaly!	\Rightarrow	$\beta_{\mu u}(g) \stackrel{!}{=} 0$
	\Rightarrow	$R_{\mu u} \stackrel{!}{=} 0$
	\Rightarrow	Vacuum Einstein field equations ©
	\Rightarrow	GENERAL RELATIVITY 😀

- This means that the conformal anomaly only cancels in D = 26 spacetime dimensions and if the *spacetime metric satisfies the Einstein field equations*!
- Similar results can be obtained with matter fields, i.e., also the coupling to the energymomentum tensor follows from the constraint of conformal invariance.
- Computing the beta function to higher orders in α' (= evaluate Feynman diagrams with more than one loop) yields *quantum corrections* to the Einstein field equations (as expected for a theory of quantum gravity).
- How do the gravitons of string theory relate to the spacetime metric?

So far we only found massless states of the closed string that transform under the correct representation of the Poincaré group (that of a symmetric, traceless rank-2 tensor field). We called these states "gravitons" – but it is not clear that (and how) these states relate to the metric of spacetime (which enters string theory not as a dynamical field but as a static background).

Here is a *sketch* (!) how one can establish this relation:

1 \triangleleft String scattering amplitude of i = 1, ..., N string states:

Scattering amplitudes are calculated from the path integral via the insertion of so called \uparrow *Vertex operators*. Each in- and out-going string state corresponds to a particular vertex operator (\uparrow *Operator-state correspondence*):

$$\mathcal{M}(V_1,\ldots,V_N) \sim \int \mathcal{D}X \mathcal{D}h \ e^{iS_g^{\text{flat}}[h,X]} \prod_{i=1}^N V_i[h,X]$$
(15.142)

 V_i : \uparrow Vertex operators

2 One can show that the Vertex operator of *single graviton state* has the form:

$$|\mathbf{S},k\rangle \quad \leftrightarrow \quad V_{\mathbf{S},k} \sim \int \mathrm{d}\sigma^2 \, S_{\mu\nu}(\partial_a X^{\mu})(\partial^a X^{\nu}) \, e^{ik_{\mu}X^{\mu}}$$
(15.143)

Here k^{μ} is the momentum of the graviton and $S^{\mu\nu}$ encodes its polarization; cf. Eq. (15.131a) in light-cone gauge.

3 We are interested in the effect of *quantized* gravitons on the *classical* background metric. Hence we should study graviton states that are "as classical as possible" (= minimize uncertainty relations). These states are *↓ coherent states* that describe superpositions of many graviton excitations; just like classical laser light is not described by single photon states but by coherent states of photons.

Remember that the coherent state of a bosonic mode (e.g., a photon mode) is obtained by exponentiating the creation & annihilation operators:

Coherent state:
$$|\alpha\rangle = e^{\alpha a^{\top} - \alpha^* a} |0\rangle$$
 (15.144)

 $\rightarrow \triangleleft$ Vertex operator for *Coherent state* of gravitons:

$$V_{\boldsymbol{S},k}^{\text{Coherent}} \sim e^{iV_{\boldsymbol{S},k}} \tag{15.145}$$



- **4** We can now study how the presence of such a coherent state affects scattering amplitudes:

$$\mathcal{M} \sim \int \mathcal{D}X \mathcal{D}h \ e^{iS_g^{\text{flat}}} V_{S,k}^{\text{Coherent}} \underbrace{\cdots}_{\text{Other vertex operators}}$$
(15.146)

Observation: The coherent graviton state has the same form as the Polyakov action:

$$e^{iS_{g}^{\text{flat}}}e^{iV_{\mathcal{S},k}} = \exp\left\{i\int \mathrm{d}\sigma^{2}\underbrace{\left[g_{\mu\nu}(X) + S_{\mu\nu}e^{ik_{\mu}X^{\mu}}\right]}_{=:\bar{g}_{\mu\nu}(X)}\partial_{a}X^{\mu}\partial^{a}X^{\nu}\right\}$$
(15.147)

 $\bar{g}_{\mu\nu}(X)$: New background metric

 \rightarrow This demonstrates that ...

Coherent graviton state \equiv Modification of classical background metric

In conclusion, one can think of the static background metric $g_{\mu\nu}$ as a "condensate" of gravitons around which the quantum theory is developed. Gravitons are then the quantum fluctuations on top of this condensate.

• What about Superstring Theory?

Here we studied *bosonic* string theory: We only found particles with *integer* spin that commute when exchanged (*↑ Spin-statistics theorem*). Since our world very much relies on the existence of *fermions* with half-integer spin (electrons etc.), this begs the question:

Where are the fermions?

1 | Answer: They are put in by hand:

$$S_{\text{SS}}^{\text{flat}}[X,\Psi] := -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \underbrace{\left[\underbrace{\partial_a X^{\mu} \partial^a X_{\mu}}_{\text{Polyakov}} + \underbrace{\alpha' \,\bar{\Psi}^{\mu} \gamma^a \partial_a \Psi_{\mu}}_{\text{Worldsheet fermions}}\right]}_{\text{(15.148)}}_{\text{(Bosonic string)}}$$

$$\uparrow$$
 Superstring

$\Psi^{\mu}_{\alpha}(\tau,\sigma)$: \uparrow Majorana fermions

These are real-valued two-component \uparrow Grassmann fields; i.e., $\alpha = 1, 2$ and $\mu = 0, \dots, D - D$ 1 and $\Psi^{\mu}_{\alpha}\Psi^{\nu}_{\beta} = -\Psi^{\nu}_{\beta}\Psi^{\mu}_{\alpha}$.

- You shouldn't be worried about the vector index on the two-component spinors Ψ^{μ} ; they play the same role as for the worldsheet *scalars* X^{μ} .
- We interpreted the bosonic fields X^{μ} as the spacetime positions of the string. The worldsheet fermions Ψ^{μ} do not have such a natural interpretation. They describe internal fermionic degrees of freedom that propagate along the string itself.
- The action Eq. (15.148) has a new continuous symmetry that mixes the bosonic fields X^{μ} with the fermionic Grassmann fields Ψ^{μ} ; this symmetry is called \uparrow supersymmetry and gives \uparrow Superstring Theory its name. (Supersymmetry guarantees the absence of tachyons and is therefore needed for the theories consistency.)



- The Dirac adjoint is $\overline{\Psi}^{\mu} := \Psi^{\mu\dagger}\gamma^0$ and the *two* Dirac matrices (the worldsheet is two-dimensional!) can be chosen as

$$\gamma^{0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ \gamma^{1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ acting on } \Psi^{\mu} = \begin{pmatrix} \Psi^{\mu}_{1} \\ \Psi^{\mu}_{2} \end{pmatrix};$$
 (15.149)

they satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$.

2 | Teaser:

Besides the emergence of fermionic particles (= spacetime fermions) in the spectrum of the superstring, this extended theory has additional advantages over the bosonic string:

	Critical dimension $D = 10 < 26 \odot$
String theory + Supersymmetry \rightarrow	Fermions included ©
Superstring theory	No Tachyon ©

- The bosonic string (= Polyakov action) is essentially unique. This is not so for the superstring: There are *five* distinct ways to define consistent supersymmetric string theories: Type I, Type IIA, Type IIB, Heterotic $E_8 \times E_8$, and Heterotic SO(32). They are conjectured to be limits of a single theory dubbed $\uparrow M$ -Theory.

• Meta-Knowledge:

The calculations in this chapter were rather involved. It is important, however, to keep in mind that this is physics from the 1970s; this version of string theory is not representative for the sophistication of the field today. Here is a sketch to provide perspective:





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