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Problem 2.1: Useful relations for general relativity [**Written** | 9 pt(s)]

ID: ex_useful_relations_for_gr:

> **Learning objective**
>
> In this exercise you derive some useful relations for calculations involving the metric tensor, covariant derivatives, and the Christoffel symbols. This prepares you for calculations in the lecture and hones you skills in tensor calculus.
>
> For example, you show that $d^n x$ is not a scalar under arbitrary coordinate transformations so that integration over a scalar function is not invariant as well. By contrast, the combination $\sqrt{g}\, d^n x$ is a scalar, which makes this a useful quantity that you will encounter throughout this course.

Recall from our discussion of tensor calculus in the last semester that the Christoffel symbols are defined as

$$\Gamma^i{}_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right) , \tag{1}$$

where g_{ij} is a given Riemannian metric tensor, and a comma denotes a partial derivative, e.g., $g_{ij,k} = \partial_k g_{ij}$. The inverse of the metric tensor g_{ij} is denoted by g^{ij}, with

$$g_{ij} g^{jk} = \delta_i^k , \tag{2}$$

and can be calculated explicitly via Cramer's rule

$$g^{ij} = \frac{1}{g} \Delta_{ji} . \tag{3}$$

Here we used the determinant of the metric tensor $g = -\det(g_{ij})$ and the cofactor Δ_{ij}, i.e., the determinant of the matrix obtained by removing the i-th row and j-th column of the matrix g_{ij}, multiplied by $(-1)^{i+j}$.

Using the Laplace expansion, we can also express the determinant of the metric via the cofactors

$$g = \sum_m g_{im} \Delta_{im} , \tag{4}$$

where we only sum over m with a fixed index i.

Note: The minus sign in the definition of $g = -\det(g_{ij})$ is conventional, such that $g > 0$ for the metric tensors of general relativity. Also note, that the cofactor matrix Δ is not a tensor, which is why we will always write it with lower indices and no Einstein summation is implied.

a) Show that the contraction of the Christoffel symbols $\Gamma^i{}_{ki}$ can be expressed as **3** pt(s)

$$\Gamma^i{}_{ki} = \frac{1}{2g} \frac{\partial g}{\partial x^k} = \partial_k \ln \sqrt{g} . \tag{5}$$

Hint: Show that $g^{ij} = \frac{1}{g} \frac{\partial g}{\partial g_{ji}}$ by differentiating Eq. (4) with respect to g_{ij} and use this to simplify $\Gamma^i{}_{ki}$.

b) Similarly, show that **2**pt(s)

$$g^{kl}\Gamma^{i}{}_{kl} = -\partial_m g^{im} - \frac{1}{\sqrt{g}}\partial^i\sqrt{g} = -\frac{1}{\sqrt{g}}\partial_m\left(\sqrt{g}g^{im}\right) . \tag{6}$$

Hint: Prove and use the relation $g_{ik}g^{kj,l} = -g_{ik}{}^{,l}g^{kj}$.

The *covariant derivative* of a Lorentz scalar ϕ, a contravariant vector A^i, and a contravariant tensor B^{ij} are defined as follows:

$$\phi_{;l} := \phi_{,l} , \tag{7a}$$

$$A^{i}{}_{;l} := A^{i}{}_{,l} + \Gamma^{i}{}_{lm}A^{m} , \tag{7b}$$

$$B^{ij}{}_{;l} := B^{ij}{}_{,l} + \Gamma^{i}{}_{lm}B^{mj} + \Gamma^{j}{}_{lm}B^{im} . \tag{7c}$$

c) Use your results from a) to derive the following relations for the covariant divergence of a vector field A^i and a tensor field B^{ij}, as well as the covariant Laplacian applied to a scalar field ϕ: **3**pt(s)

$$A^{i}{}_{;i} = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}A^i\right) , \tag{8a}$$

$$B^{ij}{}_{;j} = \frac{1}{\sqrt{g}}\partial_j\left(\sqrt{g}B^{ij}\right) \quad \text{for an antisymmetric tensor } B^{ij} = -B^{ji} , \tag{8b}$$

$$\phi^{;i}{}_{;i} = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{im}\phi_{,m}\right) . \tag{8c}$$

Finally, we also want to examine the volume element $d^n x$, which is not a scalar but transforms as

$$d^n x = \left|\frac{\partial x}{\partial \bar{x}}\right| d^n\bar{x} , \tag{9}$$

where $\left|\frac{\partial x}{\partial \bar{x}}\right|$ is the determinant of the Jacobian matrix $\frac{\partial x^i}{\partial \bar{x}^j}$ of the coordinate transformation.

d) Use the transformation of the metric tensor $\bar{g}_{ij}(\bar{x})$ to replace the determinant $\left|\frac{\partial x}{\partial \bar{x}}\right|$ and show **1**pt(s)

$$\sqrt{g}\, d^n x = \sqrt{\bar{g}}\, d^n\bar{x} , \tag{10}$$

i.e., show that $\sqrt{g}\, d^n x$ is a scalar (invariant under coordinate transformations).

One can now use the modified volume element $\sqrt{g}\, d^n x$ to define an integral $\int d^n x\sqrt{g}\,\phi(x)$ that, for a scalar field $\phi(x)$ as integrand, is form-invariant under coordinate transformations as well. This is why the modified volume element $\sqrt{g}\, d^n x$ shows up in generally covariant actions of general relativity.

With this volume element [and your result from part c)], one immediately finds the useful form of Gauss's theorem

$$\int_V d^n x\sqrt{g}\, A^{i}{}_{;i} \overset{\text{c)}}{=} \int_V d^n x\, \partial_i\left(\sqrt{g}A^i\right) \overset{\text{Gauss}}{=} \oint_{\partial V} d^{n-1}\sigma_i\, \sqrt{g}A^i , \tag{11}$$

where $d^{n-1}\sigma_i$ denotes the integration over the surface normal.

Problem 2.2: Curved Riemannian manifolds – The sphere

[**Oral** | 6 pt(s)]

ID: ex_connection_curvature_examples:

Learning objective

In this exercise you practice calculations on curved Riemannian manifolds by a simple (though non-trivial) example, namely the sphere. The point is to illustrate the usefulness of the curvature tensor. You will see that it is a non-trivial task to determine the "curvedness" of a given Riemannian manifold from its metric, and that the curvature tensor (and its descendants) are the tool to solve it.

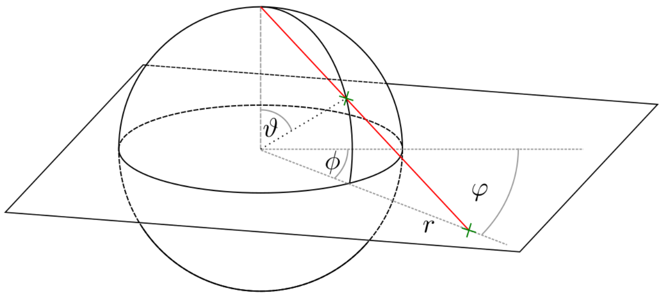
A sphere of radius ρ can be parameterized by two angles, $\vartheta \in [0, \pi]$ and $\phi \in [0, 2\pi)$.

In these coordinates, the metric of the sphere is given by

$$ds^2 = \rho^2 d\vartheta^2 + \rho^2 \sin^2(\vartheta)\, d\phi^2 \,. \tag{12}$$

We consider now a coordinate transformation from (θ, ϕ) to a new chart (r, φ) defined by

$$\cos(\vartheta) = \frac{r^2 - 1}{r^2 + 1} \quad \text{and} \quad \varphi = \phi \tag{13}$$



with $r \in [0, \infty)$.

This corresponds to a stereographic projection of the sphere through the north pole onto the plane $\mathbb{R}^2$ which bisects it (see the sketch on the right).

a) Determine the metric in the new coordinates r and φ. 1pt(s)

Consider now two different metrics defined as follows:

$$ds_A^2 = \frac{4\rho^2}{(r^2+1)^2}\left(dr^2 + r^2 d\varphi^2\right) \,, \quad \text{and} \quad ds_B^2 = dr^2 + r^2 d\varphi^2 \,. \tag{14}$$

The first metric $ds_A^2 = ds^2$ is the result from a), i.e., the metric of the sphere (in unusual coordinates). By contrast, ds_B^2 is the metric of flat Euclidean space $\mathbb{R}^2$ (in polar coordinates).

The two manifolds are clearly very different: the sphere is curved while the plane is not. However, this differentiating feature is hard to extract from the expressions in Eq. (14).

In general, it is almost impossible to look at a metric and decide whether the manifold it describes is curved or not because the choice of curvilinear coordinates can make even the simplest metric look very complicated.

Here the Riemann curvature tensor (and its traces) come in handy as a tool to distinguish between curved and flat spaces:

b) To compute the curvature tensor of both manifolds, first determine the Christoffel symbols 2pt(s)

$$\Gamma^i{}_{kl} = \frac{1}{2} g^{im}\left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right) \,, \tag{15}$$

for both metrics.

Hint: Use a previous exercise for the Christoffel symbols of the Euclidean plane in polar coordinates.

c) The Riemann curvature tensor is given by **2**pt(s)

$$R^l{}_{ijk} = -\left[\Gamma^l{}_{ij,k} - \Gamma^l{}_{ik,j} + \Gamma^m{}_{ij}\Gamma^l{}_{km} - \Gamma^m{}_{ik}\Gamma^l{}_{jm}\right] . \tag{16}$$

Show that this tensor vanishes for the flat space ds_B^2 but not for the sphere ds_A^2.

Hint: Since we are in two dimensions, there is only one independent nonzero component of the curvature tensor due to the symmetries

$$R_{lijk} = -R_{iljk} , \quad \text{and} \quad R_{lijk} = -R_{likj} . \tag{17}$$

We will have a closer look at the symmetries of the curvature tensor in a future exercise.

d) Finally, compute the Ricci scalar **1**pt(s)

$$R = g^{ij} R^k{}_{ijk} , \tag{18}$$

for both metrics. How does the curvature R_A of the sphere change with varying radius ρ?

Problem 2.3: Equivalence priciple - Local inertial systems [**Oral** | 3 pt(s)]

ID: ex_local_inertial_system:

Learning objective

The Einstein Equivalence principle (EEP) is the foundational principle of general relativity. It states that in a small enough laboratory there is no experiment that can detect the presence of a gravitational field. In other words, *locally* (in space and time) special relativity is sufficient to describe the world.

In this exercise, you study generic properties of Lorentzian manifolds that allow for a natural incorporation of the EEP into the formalism of general relativity. In particular, you show that at every point there is a coordinate system in which a Lorentzian metric takes the Minkowskian form of special relativity.

Let us first set the stage: We consider a manifold M (= spacetime) and select a point $P \in M$ (the center of our lab at a specific time). We focus on a small neighborhood $\mathcal{V}$ of P (the lab during some small time interval). In this neighborhood $\mathcal{V}$ we can define a local coordinate system x^μ.

In special relativity, we considered a particular combination of manifold (Minkowski space) and coordinates (inertial coordinate systems) where the metric tensor had the special form

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)_{\mu\nu} ,$$

and the Christoffel symbols vanished identically.

In the presence of an inhomogeneous gravitational field, general relativity forces us to work on a more general 4D spacetime manifold with an arbitrary Lorentzian metric $g_{\mu\nu}$ of signature $(1, 3)$ (and its corresponding Christoffel symbols).

In the following, you show how one can locally recover the spacetime structure of special relativity on such a manifold:

a) Recall from last semester (or the lecture) that Christoffel symbols change under an arbitrary coordinate transformation in a non-tensorial way: 1pt(s)

$$\bar{\Gamma}^{\kappa}{}_{\mu\nu}(\bar{x}) = \frac{\partial \bar{x}^{\kappa}}{\partial x^{\sigma}} \frac{\partial x^{\alpha}}{\partial \bar{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \bar{x}^{\nu}} \Gamma^{\sigma}{}_{\alpha\beta}(x) + \frac{\partial \bar{x}^{\kappa}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial \bar{x}^{\mu} \partial \bar{x}^{\nu}} \,. \tag{19}$$

Show that there exist coordinates $\bar{x}^{\mu}$ such that the Christoffel symbols vanish at the point P:

$$\bar{\Gamma}^{\alpha}{}_{\mu\nu}(P) = 0 \,. \tag{20}$$

Hint: Consider a non-linear coordinate transformation of the form

$$x^{\mu} - x^{\mu}_P = \bar{x}^{\mu} + \frac{1}{2} Q^{\mu}{}_{\alpha\beta} \bar{x}^{\alpha} \bar{x}^{\beta} \,, \tag{21}$$

where $Q^{\mu}{}_{\alpha\beta}$ is w.l.o.g. symmetric in the two lower indices and x^{μ}_P are the coordinates of the point P.

b) Show that there exists a linear coordinate transformation, which transforms the coordinates $\bar{x}$ to new coordinates $\tilde{x}$ such that the metric at the point P takes the Minkowski form: 1pt(s)

$$\tilde{g}_{\mu\nu}(P) = \eta_{\mu\nu} \,. \tag{22}$$

Why is it important that the transformation $\tilde{x} = \tilde{x}(\bar{x})$ is linear? What does this imply for the Christoffel symbols in these new coordinates?

The two properties

$$\tilde{\Gamma}^{\kappa}{}_{\mu\nu}(P) = 0 \quad \text{and} \quad \tilde{g}_{\mu\nu}(P) = \eta_{\mu\nu} \tag{23}$$

characterize a local inertial coordinate system in P.

(Is this coordinate system unique? Which coordinate transformations leave Eq. (23) invariant?)

Note: Conceptually this is very important because this tells us how to implement the EEP: We can use any Lorentzian metric to describe spacetime, as long as we obey the constraint that our equations reduce to their special relativistic form when we transform into a local inertial system where $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^{\kappa}{}_{\mu\nu} = 0$.

c) Show that the properties (23) imply that 1pt(s)

$$\tilde{\partial}_{\alpha} \tilde{g}_{\mu\nu} \Big|_P = 0 \,. \tag{24}$$

Combining our results, show that close to the point P we can expand the metric as

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2} \frac{\partial^2 g_{\mu\nu}}{\partial \tilde{x}^{\alpha} \partial \tilde{x}^{\beta}} \tilde{x}^{\alpha} \tilde{x}^{\beta} + \mathcal{O}(\tilde{x}^3) \,. \tag{25}$$