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Problem 4.1: Hafele-Keating experiment

[**Oral** | 4 pt(s)]

ID: ex_hafele_keating_experiment:rt2526

Learning objective

Time dilation is a relativistic effect far removed from our everyday experience. It is, however, an experimentally established fact. A famous experiment measuring time dilation explicitly was the *Hafele-Keating experiment*, where portable atomic clocks were flown on commercial airliners around the world twice: once eastward and once westward. The clocks were then compared to stationary reference clocks on the ground to verify the predictions of time dilation quantitatively.

The experimental results were reported in

https://doi.itp3.info/10.1126/science.177.4044.168

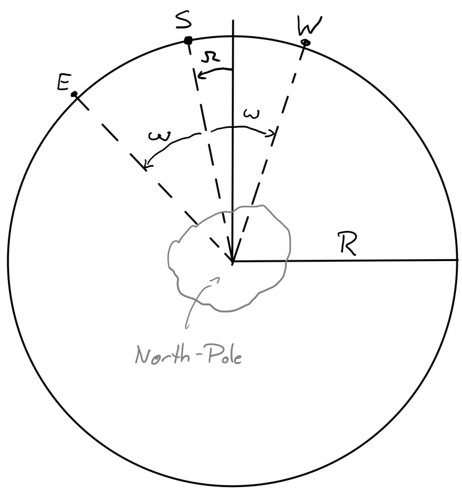
and the theory was developed in

https://doi.itp3.info/10.1126/science.177.4044.166

In this exercise, you derive the contribution of time dilation to explain the result of the experiment (which you can find in the theory paper above).

We consider the following setup:

Imagine an observer in a space station above the north pole; the space station follows earth on its orbit around the sun, but does not follow the rotation of earth (i.e., the observer sees earth slowly rotating beneath the space station). Such an observer is approximately inertial for the relevant timescales of the experiment (see sketch on the right); in particular, he is allowed to use the formalism developed in the lecture to compute the proper time along (potentially accelerated) trajectories.



Now consider three identical atomic clocks located on the equator with radius $R \approx 6.4 \times 10^6$ m. The first clock (labeled S) is stationary with respect to earth, this is our "reference clock." The second clock (E) flies eastward around earth with angular velocity ω (with respect to earth), and the third (W) westward with the same angular velocity ω (also with respect to earth). Both clocks go around earth once and meet again with the reference clock. Note that the rest system of the reference clock S is *not* inertial because earth rotates with angular velocity $\Omega = \frac{2\pi}{24\,\mathrm{h}}$.

a) Parametrize the trajectories of the three clocks $\boldsymbol{x}_S(t)$, $\boldsymbol{x}_E(t)$ and $\boldsymbol{x}_W(t)$ in the inertial system of the space station. 1pt(s)

b) The proper time τ accumulated by a clock can be calculated as shown in the lecture: **1**pt(s)

$$\tau_i = \int \sqrt{1 - \frac{\dot{\boldsymbol{x}}_i^2}{c^2}}\,\mathrm{d}t \qquad \text{for} \quad i \in \{S, E, W\}\,. \tag{1}$$

Evaluate this integral for the three clocks.

c) Calculate $\Delta\tau_E = \tau_E - \tau_S$ and $\Delta\tau_W = \tau_W - \tau_S$. **1**pt(s)

Since the angular velocities are small, expand the results in $\frac{R\Omega}{c}$ up to second order.

d) Assume the clocks travel around earth with $200\,\mathrm{m\,s^{-1}}$ relative to the ground (the speed of a typical airliner). **1**pt(s)

What are the time differences measured? Compare them to the numbers reported in the original publications.

Note: For a complete description of the experiment, effects of *general relativity* must be taken into account as well. Thus, we will complete our analysis of the Hafele-Keating experiment in the next semester.

Problem 4.2: Tensor Calculus [**Written** | 11 pt(s)]

ID: ex_tensor_calculus:rt2526

Learning objective

Tensor calculus is a crucial toolkit for special and general relativity. In this exercise, you practice calculating with tensor fields and prove some useful rules for the construction of tensor fields.

Consider a D-dimensional differentiable manifold M and an arbitrary coordinate transformation $\bar{x} = \varphi(x)$ from one chart with coordinates $x \in \mathbb{R}^D$ to another chart with coordinates $\bar{x} \in \mathbb{R}^D$.

As motivated in the lecture, we define the transformation of *contravariant* and *covariant* vector (fields) as follows:

$$\textit{Contravariant}\text{ vector field:} \quad \bar{A}^i(\bar{x}) = \sum_{k=1}^{D} \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \equiv \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \tag{2}$$

$$\textit{Covariant}\text{ vector field:} \quad \bar{B}_i(\bar{x}) = \sum_{k=1}^{D} \frac{\partial x^k}{\partial \bar{x}^i} B_k(x) \equiv \frac{\partial x^k}{\partial \bar{x}^i} B_k(x)\,, \tag{3}$$

Here we use the *Einstein sum convention*: Sums over pairs of repeated up and down indices are implied but not explicitly written.

a) Prove that the *contraction* $\Phi(x) := A^i(x)B_i(x)$ of a contravariant vector field $A^i(x)$ with a covariant vector field $B_i(x)$ is invariant under coordinate transformations; i.e., show that it transforms like a *scalar field*. **1**pt(s)

The generalization of co- and contravariant vector fields are (mixed) (p, q) *tensor fields* $T^{m_1,\dots,m_p}{}_{n_1,\dots,n_q}(x)$ with $r = p+q$ indices (called *rank*). Like vector fields, tensor fields are defined by their transformation under coordinate transformations:

$$\bar{T}^{i_1,\dots,i_p}{}_{j_1,\dots,j_q}(\bar{x}) = \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}} \quad \frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}} \quad T^{m_1,\dots,m_p}{}_{n_1,\dots,n_q}(x)\,, \tag{4}$$

where p and q are the number of contravariant and covariant indices, respectively.

b) Show that the following combinations of the tensor fields ${A^{ij}}_k$, ${B^{ij}}_k$, C^{ij}, ${D^k}_l$, E_m and the scalar field Φ are again tensor fields (we suppress the x-dependency): **5**pt(s)

i) ${V^{ij}}_k := {A^{ij}}_k + {B^{ij}}_k$ ii) ${W^{ij}}_k := \Phi {A^{ij}}_k$ iii) ${X^{ijk}}_l := C^{ij}{D^k}_l$
iv) $Y^i := A^{ij}E_j$ v) $Z^i := {A^{ij}}_j$

c) Let C_{ij} be a collection of D^2 fields $(i, j = 1, \ldots, D)$. **1**pt(s)

Prove that *if* $B_i := C_{ij}A^j$ is a covariant vector field for *every* contravariant vector field A^i, *then* C_{ij} transforms like a covariant tensor field of rank 2.

Note: This theorem is called *quotient law*, a quite useful tool in tensor calculus.

The *covariant derivative* of a contravariant vector field is defined as

$$ {A^i}_{;k} := \partial_k A^i + {\Gamma^i}_{kl} A^l \,, \tag{5} $$

with $\partial_k \equiv \frac{\partial}{\partial x^k}$ and where the *Christoffel symbol* ${\Gamma^i}_{kl}$ is defined as

$$ {\Gamma^i}_{kl} := \frac{1}{2} g^{im} \left(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl} \right) \,. \tag{6} $$

$g_{ij} = g_{ij}(x)$ is a given, symmetric ($g_{ij} = g_{ji}$) covariant tensor field of rank 2 called the *metric.*

d) Show that the covariant derivative ${A^i}_{;k}$ of a contravariant vector field A^i transforms as a mixed $(1, 1)$ tensor field. **3**pt(s)

Hint: First, prove the transformation law for the Christoffel symbol

$$ {\bar{\Gamma}^i}_{kl}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l} {\Gamma^j}_{mn}(x) + \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} \,, \tag{7} $$

and show that

$$ \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} = -\frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l} \,. \tag{8} $$

Use this to derive the transformation law for the covariant derivative.

You might want to use the shortcut notations $\alpha^i_k := \frac{\partial \bar{x}^i}{\partial x^k}$ and $\beta^i_k := \frac{\partial x^i}{\partial \bar{x}^k}$ with $\alpha^i_k \beta^k_j = \delta^i_j$.

In the lecture, the general *Levi-Civita symbol* $\epsilon^{i_1 \cdots i_D}$ was introduced. Here we want to focus on the most important case of a $D = 4$ dimensional manifold.

The Levi-Civita symbol is defined as

$$ \epsilon^{ijkl} := \begin{cases} +1 & \text{if } (i,j,k,l) \text{ is an even permutation of } (0,1,2,3) \\ -1 & \text{if } (i,j,k,l) \text{ is an odd permutation of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases} \,. \tag{9} $$

This definition is independent of the coordinate system, meaning $\bar{\epsilon}^{ijkl} = \epsilon^{ijkl}$.

e) Show that with this definition the Levi-Civita symbol is a *relative tensor* of rank 4 with weight $w = +1$ (which we call a *tensor density*), i.e. show that 1pt(s)

$$\epsilon^{ijkl} = \left|\frac{\partial x}{\partial \bar{x}}\right| \frac{\partial \bar{x}^i}{\partial x^a}\frac{\partial \bar{x}^j}{\partial x^b}\frac{\partial \bar{x}^k}{\partial x^c}\frac{\partial \bar{x}^l}{\partial x^d}\epsilon^{abcd} \quad \text{with } \textit{Jacobian determinant } \left|\frac{\partial \bar{x}}{\partial x}\right| . \tag{10}$$

Hint: Use that determinants can be calculated via the Leibniz formula using the Levi-Civita symbol:

$$\left|\frac{\partial \bar{x}}{\partial x}\right| = \epsilon^{ijkl}\frac{\partial \bar{x}^0}{\partial x^i}\frac{\partial \bar{x}^1}{\partial x^j}\frac{\partial \bar{x}^2}{\partial x^k}\frac{\partial \bar{x}^3}{\partial x^l} . \tag{11}$$

Problem 4.3: The metric tensor [**Oral** | 5 pt(s)]

ID: ex_srt_metric_coordinate_transformation:rt2526

Learning objective

The metric tensor (field) is crucial for Riemannian geometry, i.e., the mathematical framework needed to described curved spaces. Both in special relativity and in general relativity, the metric tensor on the spacetime manifold determines durations (measured by clocks) and lengths (measured by rods) in spacetime.

In this exercise, you show that the components of the metric tensor indeed transform like a covariant tensor field of rank 2. To familiarize yourself with the concept, you study the simple example of Euclidean space (no time!) in two dimensions $\mathbb{R}^2$ and calculate the components of the metric tensor in polar coordinates.

Consider a D-dimensional differentiable manifold M and an arbitrary coordinate transformation $\bar{x} = \varphi(x)$ from one chart with coordinates $x \in \mathbb{R}^D$ to another chart with coordinates $\bar{x} \in \mathbb{R}^D$.

a) We start by showing that the components $g_{ij}(x)$ of the metric transform like a covariant tensor field of rank 2. To this end, use that the "line element" ds^2 is a tensor field and hence does not depend on the coordinate system: 1pt(s)

$$g_{ij}(x)dx^i dx^j = ds^2 = \bar{g}_{ij}(\bar{x})d\bar{x}^i d\bar{x}^j \quad \text{(Einstein summation!)} \tag{12}$$

Hint: Compute the total differential $d\bar{x}^i$ and use that $\{dx^i\}$ is a basis of the cotangent space T_p^*M. The transformation law for a covariant tensor of rank 2 is given in Problem 4.2.

As an example, we consider Euclidean space $M = \mathbb{R}^2$ in $D = 2$ dimensions for the rest of this exercise. In Cartesian coordinates $x^1 = x$ and $x^2 = y$, the components of the metric tensor are

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with line element} \quad ds^2 = g_{ij}(x)dx^i dx^j = dx^2 + dy^2 . \tag{13}$$

This particular metric tensor ds^2 characterizes the flat, Euclidean plane you already encountered in school.

b) We want to calculate the components of the metric tensor in polar coordinates $\bar{x}^1 = r$ and 2pt(s)

$\bar{x}^2 = \theta$. The coordinate transformation $\bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x})$ between Cartesian and polar coordinates is given by

$$\varphi^{-1} : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} . \tag{14}$$

First, use Eq. (12) and the rule for computing the total differentials dx and dy to compute $\bar{g}_{ij}(\bar{x})$.

Then, derive the same components directly by using the transformation law for a covariant tensor of rank 2.

The physical length of a curve γ can be calculated via the metric tensor by

$$L[\gamma] = \int_\gamma ds := \int_a^b dt \sqrt{g_{ij}(\boldsymbol{\gamma}(t))\, \dot{\gamma}^i(t)\dot{\gamma}^j(t)}\,, \tag{15}$$

where $\boldsymbol{\gamma}(t)$ is the parametrization of the curve with $t \in [a, b]$ in some coordinate system.

c) Let γ be the circle in $\mathbb{R}^2$ with radius R and center in the origin. **2**pt(s)

Calculate the circumference $L[\gamma]$ of this circle once in Cartesian coordinates and then again in polar coordinates. Use the components of the metric tensor given above and computed in subtask b).

Solutions to Problem Set 4
Relativitätstheorie 1

Codex GPT-5.4

Prompt: *"Solve the exercise tasks in the pdf. Write your solutions in a latex document and compile it. Write solutions like a good physics student, with all important steps. Do not skip tiresome calculations. Do not assume deeper knowledge of tensor calculus."*

Problem 4.1: Hafele–Keating experiment

We work in the approximately inertial frame of the space station above the north pole. We choose Cartesian coordinates (x, y, z) with the origin at the center of the earth and the equatorial plane given by $z = 0$. Eastward motion is taken to be the positive direction of rotation.

a) Trajectories in the inertial frame

The reference clock S is fixed on the rotating earth, hence in the inertial frame it rotates with angular velocity Ω. The eastward clock E has angular velocity $\Omega + \omega$, because its angular velocity relative to the ground is $+\omega$. The westward clock W has angular velocity $\Omega - \omega$.

If all three clocks meet at $t = 0$ at the point $(R, 0, 0)$, then their trajectories are

$$\begin{aligned}
\mathbf{x}_S(t) &= \big(R\cos(\Omega t),\, R\sin(\Omega t),\, 0\big),\\
\mathbf{x}_E(t) &= \big(R\cos((\Omega+\omega)t),\, R\sin((\Omega+\omega)t),\, 0\big),\\
\mathbf{x}_W(t) &= \big(R\cos((\Omega-\omega)t),\, R\sin((\Omega-\omega)t),\, 0\big).
\end{aligned}$$

The flying clocks return to the reference clock after one revolution *relative to the earth.* Therefore

$$\omega T = 2\pi \qquad \Longrightarrow \qquad T = \frac{2\pi}{\omega}.$$

This is the flight time in the inertial frame for both round trips.

b) Proper times

For circular motion with constant angular velocity Ω_i, the speed is constant:

$$v_i = |\dot{\mathbf{x}}_i| = R\Omega_i.$$

For the three clocks we therefore have

$$v_S = R\Omega, \qquad v_E = R(\Omega+\omega), \qquad v_W = R(\Omega-\omega).$$

The proper time is

$$\tau_i = \int_0^T \sqrt{1 - \frac{v_i^2}{c^2}}\, dt.$$

Since the integrand is constant for each clock, the integral is immediate:

$$\tau_S = T\sqrt{1 - \frac{R^2\Omega^2}{c^2}} = \frac{2\pi}{\omega}\sqrt{1 - \frac{R^2\Omega^2}{c^2}},$$

$$\tau_E = T\sqrt{1 - \frac{R^2(\Omega+\omega)^2}{c^2}} = \frac{2\pi}{\omega}\sqrt{1 - \frac{R^2(\Omega+\omega)^2}{c^2}},$$

$$\tau_W = T\sqrt{1 - \frac{R^2(\Omega-\omega)^2}{c^2}} = \frac{2\pi}{\omega}\sqrt{1 - \frac{R^2(\Omega-\omega)^2}{c^2}}.$$

c) Time differences and expansion

By definition,

$$\Delta\tau_E = \tau_E - \tau_S = \frac{2\pi}{\omega}\left[\sqrt{1 - \frac{R^2(\Omega+\omega)^2}{c^2}} - \sqrt{1 - \frac{R^2\Omega^2}{c^2}}\right],$$

$$\Delta\tau_W = \tau_W - \tau_S = \frac{2\pi}{\omega}\left[\sqrt{1 - \frac{R^2(\Omega-\omega)^2}{c^2}} - \sqrt{1 - \frac{R^2\Omega^2}{c^2}}\right].$$

For small velocities we use

$$\sqrt{1-\varepsilon} = 1 - \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2), \qquad \varepsilon \sim \frac{R^2\Omega^2}{c^2}.$$

Keeping terms up to second order in the small ratios $R\Omega/c$ and $R\omega/c$, we obtain

$$\tau_S \simeq T\left(1 - \frac{R^2\Omega^2}{2c^2}\right),$$

$$\tau_E \simeq T\left(1 - \frac{R^2(\Omega+\omega)^2}{2c^2}\right),$$

$$\tau_W \simeq T\left(1 - \frac{R^2(\Omega-\omega)^2}{2c^2}\right).$$

Hence

$$\Delta\tau_E \simeq -\frac{TR^2}{2c^2}\left[(\Omega+\omega)^2 - \Omega^2\right] = -\frac{TR^2}{2c^2}(2\Omega\omega + \omega^2),$$

$$\Delta\tau_W \simeq -\frac{TR^2}{2c^2}\left[(\Omega-\omega)^2 - \Omega^2\right] = \frac{TR^2}{2c^2}(2\Omega\omega - \omega^2).$$

Using $T = 2\pi/\omega$, this can also be written as

$$\Delta\tau_E \simeq -\frac{\pi R^2}{c^2}(2\Omega + \omega),$$

$$\Delta\tau_W \simeq \frac{\pi R^2}{c^2}(2\Omega - \omega).$$

d) Numerical values for $v = 200\,\mathrm{m\,s^{-1}}$ relative to the ground

The angular velocity relative to the earth is

$$\omega = \frac{v}{R} = \frac{200\,\mathrm{m\,s^{-1}}}{6.4 \times 10^6\,\mathrm{m}} = 3.125 \times 10^{-5}\,\mathrm{s^{-1}}.$$

For the earth's rotation we use

$$\Omega = \frac{2\pi}{24\,\mathrm{h}} = \frac{2\pi}{86400\,\mathrm{s}} \approx 7.27 \times 10^{-5}\,\mathrm{s^{-1}}.$$

Thus

$$T = \frac{2\pi}{\omega} \approx 2.01 \times 10^5\,\mathrm{s} \approx 55.9\,\mathrm{h}.$$

With the exact expressions above one finds

$$\begin{aligned} \Delta\tau_E &\approx -2.5297 \times 10^{-7}\,\mathrm{s} = -252.97\,\mathrm{ns}, \\ \Delta\tau_W &\approx 1.6353 \times 10^{-7}\,\mathrm{s} = 163.53\,\mathrm{ns}. \end{aligned}$$

The second-order approximation gives practically the same values:

$$\Delta\tau_E \approx -252.98\,\mathrm{ns}, \qquad \Delta\tau_W \approx 163.50\,\mathrm{ns}.$$

Comparison with the original Hafele–Keating publications:

- In the theory paper, the *kinematic* contributions for the actual flight paths were reported as $-184 \pm 18\,\mathrm{ns}$ (eastward) and $+96 \pm 10\,\mathrm{ns}$ (westward).[1]
- Including gravity, the same paper gives the *net* predictions $-40 \pm 23\,\mathrm{ns}$ and $+275 \pm 21\,\mathrm{ns}$.
- The measured values in the companion experimental paper were $-59 \pm 10\,\mathrm{ns}$ and $+273 \pm 7\,\mathrm{ns}$.[2]

Our simplified special-relativistic model therefore gives the correct *signs* and the correct qualitative asymmetry between eastward and westward flights, but it does not reproduce the published numbers quantitatively. That is expected: we assumed a constant speed, a full equatorial trip, and ignored the gravitational contribution.

Problem 4.2: Tensor Calculus

We use the abbreviations

$$\alpha^i{}_k := \frac{\partial \bar{x}^i}{\partial x^k}, \qquad \beta^k{}_i := \frac{\partial x^k}{\partial \bar{x}^i},$$

so that

$$\alpha^i{}_k \beta^k{}_j = \delta^i_j, \qquad \beta^k{}_i \alpha^i{}_l = \delta^k_l.$$

a) Contraction of a contra- and covariant vector

Under a coordinate transformation,

$$\bar{A}^i(\bar{x}) = \alpha^i{}_k A^k(x), \qquad \bar{B}_i(\bar{x}) = \beta^k{}_i B_k(x).$$

[1] J. C. Hafele and R. E. Keating, *Around-the-World Atomic Clocks: Predicted Relativistic Time Gains*, Science **177** (1972) 166–168.

[2] J. C. Hafele and R. E. Keating, *Around-the-World Atomic Clocks: Observed Relativistic Time Gains*, Science **177** (1972) 168–170.

Therefore

$$\begin{aligned}\bar{\Phi}(\bar{x}) &= \bar{A}^i(\bar{x})\,\bar{B}_i(\bar{x}) \\ &= \alpha^i{}_k A^k(x)\,\beta^l{}_i B_l(x) \\ &= \delta^l_k\,A^k(x) B_l(x) \\ &= A^k(x) B_k(x) \\ &= \Phi(x).\end{aligned}$$

So Φ is invariant under coordinate transformations and therefore a scalar field.

b) Construction of new tensor fields

We check each expression explicitly.

(i) $V^{ij}{}_k := A^{ij}{}_k + B^{ij}{}_k$ Since $A^{ij}{}_k$ and $B^{ij}{}_k$ have the same tensor type,

$$\begin{aligned}\bar{V}^{ij}{}_k &= \bar{A}^{ij}{}_k + \bar{B}^{ij}{}_k \\ &= \alpha^i{}_m \alpha^j{}_n \beta^p{}_k A^{mn}{}_p + \alpha^i{}_m \alpha^j{}_n \beta^p{}_k B^{mn}{}_p \\ &= \alpha^i{}_m \alpha^j{}_n \beta^p{}_k (A^{mn}{}_p + B^{mn}{}_p) \\ &= \alpha^i{}_m \alpha^j{}_n \beta^p{}_k V^{mn}{}_p.\end{aligned}$$

Hence $V^{ij}{}_k$ is again a tensor field of type $(2,1)$.

(ii) $W^{ij}{}_k := \Phi\,A^{ij}{}_k$ Because Φ is a scalar, $\bar{\Phi}(\bar{x}) = \Phi(x)$. Thus

$$\begin{aligned}\bar{W}^{ij}{}_k &= \bar{\Phi}\,\bar{A}^{ij}{}_k \\ &= \Phi\,\alpha^i{}_m \alpha^j{}_n \beta^p{}_k A^{mn}{}_p \\ &= \alpha^i{}_m \alpha^j{}_n \beta^p{}_k W^{mn}{}_p.\end{aligned}$$

So $W^{ij}{}_k$ is also a tensor field of type $(2,1)$.

(iii) $X^{ijk}{}_l := C^{ij} D^k{}_l$ Using the transformation laws of C^{ij} and $D^k{}_l$,

$$\begin{aligned}\bar{X}^{ijk}{}_l &= \bar{C}^{ij}\,\bar{D}^k{}_l \\ &= (\alpha^i{}_m \alpha^j{}_n C^{mn})(\alpha^k{}_p \beta^q{}_l D^p{}_q) \\ &= \alpha^i{}_m \alpha^j{}_n \alpha^k{}_p \beta^q{}_l (C^{mn} D^p{}_q) \\ &= \alpha^i{}_m \alpha^j{}_n \alpha^k{}_p \beta^q{}_l X^{mnp}{}_q.\end{aligned}$$

Therefore $X^{ijk}{}_l$ is a tensor field of type $(3,1)$.

(iv) $Y^i := A^{ij} E_j$ The sheet writes this expression with a rank-2 object A^{ij}. Treated exactly as written, we get

$$\begin{aligned}\bar{Y}^i &= \bar{A}^{ij}\,\bar{E}_j \\ &= (\alpha^i{}_m \alpha^j{}_n A^{mn})(\beta^p{}_j E_p) \\ &= \alpha^i{}_m\,\delta^p_n\,A^{mn} E_p \\ &= \alpha^i{}_m A^{mn} E_n \\ &= \alpha^i{}_m Y^m.\end{aligned}$$

So Y^i transforms as a contravariant vector field.

(v) $Z^i := A^{ij}{}_j$ Here we contract one upper and one lower index of $A^{ij}{}_k$:

$$\begin{aligned}\bar{Z}^i &= \bar{A}^{ij}{}_j \\ &= \alpha^i{}_m \alpha^j{}_n \beta^p{}_j A^{mn}{}_p \\ &= \alpha^i{}_m \, \delta^p_n \, A^{mn}{}_p \\ &= \alpha^i{}_m A^{mn}{}_n \\ &= \alpha^i{}_m Z^m.\end{aligned}$$

Hence Z^i is a contravariant vector field.

c) Quotient law

We are given a collection of fields C_{ij} and the statement that

$$B_i := C_{ij} A^j$$

is a covariant vector field for *every* contravariant vector field A^j.

Since B_i is covariant,

$$\bar{B}_i = \beta^k{}_i B_k.$$

Using the definition of B_i, this becomes

$$\bar{B}_i = \beta^k{}_i B_k = \beta^k{}_i C_{kl} A^l.$$

On the other hand, by definition in the barred system,

$$\bar{B}_i = \bar{C}_{ij} \bar{A}^j.$$

Now write A^l in terms of barred components:

$$A^l = \beta^l{}_m \bar{A}^m.$$

Then

$$\bar{C}_{ij} \bar{A}^j = \beta^k{}_i C_{kl} \beta^l{}_m \bar{A}^m.$$

Since this must hold for *every* vector $\bar{A}^m$, the coefficients of $\bar{A}^m$ must agree:

$$\bar{C}_{im} = \beta^k{}_i \beta^l{}_m C_{kl}.$$

This is exactly the transformation law of a covariant rank-2 tensor. Hence C_{ij} is a tensor field.

d) Transformation law of the covariant derivative

We want to show that

$$A^i{}_{;k} := \partial_k A^i + \Gamma^i{}_{kl} A^l$$

transforms as a mixed $(1,1)$ tensor field.

Step 1: Transformation of the Christoffel symbols. Let $e_k = \partial_k$ and $\bar{e}_k = \bar{\partial}_k$ be the coordinate basis vectors. They transform as

$$\bar{e}_k = \beta^m{}_k e_m.$$

The Christoffel symbols are the coefficients of the covariant derivative of the basis vectors:

$$\nabla_{e_l} e_k = \Gamma^i{}_{kl} e_i, \qquad \nabla_{\bar{e}_l} \bar{e}_k = \bar{\Gamma}^i{}_{kl} \bar{e}_i.$$

Now compute $\nabla_{\bar{e}_l}\bar{e}_k$ in the unbarred basis:

$$\begin{aligned}
\nabla_{\bar{e}_l}\bar{e}_k &= \nabla_{\beta^n{}_l e_n}(\beta^m{}_k e_m) \\
&= \beta^n{}_l\,(\partial_n\beta^m{}_k)e_m + \beta^n{}_l\beta^m{}_k\,\nabla_{e_n}e_m \\
&= \beta^n{}_l\,(\partial_n\beta^m{}_k)e_m + \beta^n{}_l\beta^m{}_k\,\Gamma^j{}_{mn}e_j.
\end{aligned}$$

Next we rewrite $e_j = \alpha^i{}_j\bar{e}_i$ and $e_m = \alpha^i{}_m\bar{e}_i$:

$$\nabla_{\bar{e}_l}\bar{e}_k = \Big[\alpha^i{}_j\beta^m{}_k\beta^n{}_l\,\Gamma^j{}_{mn} + \alpha^i{}_m\,\beta^n{}_l(\partial_n\beta^m{}_k)\Big]\,\bar{e}_i.$$

Since $\beta^n{}_l\partial_n = \bar{\partial}_l$, the second term is

$$\beta^n{}_l(\partial_n\beta^m{}_k) = \bar{\partial}_l\beta^m{}_k = \frac{\partial^2 x^m}{\partial\bar{x}^l\,\partial\bar{x}^k}.$$

Comparing coefficients of $\bar{e}_i$, we get

$$\boxed{\bar{\Gamma}^i{}_{kl} = \alpha^i{}_j\beta^m{}_k\beta^n{}_l\,\Gamma^j{}_{mn} + \alpha^i{}_m\frac{\partial^2 x^m}{\partial\bar{x}^k\,\partial\bar{x}^l}}$$

which is exactly Eq. (7).

Step 2: Proof of Eq. (8). Start from the identity

$$\alpha^i{}_m\beta^m{}_k = \delta^i_k.$$

Differentiate with respect to $\bar{x}^l$:

$$\bar{\partial}_l(\alpha^i{}_m\beta^m{}_k) = 0.$$

Using the product rule,

$$(\bar{\partial}_l\alpha^i{}_m)\beta^m{}_k + \alpha^i{}_m(\bar{\partial}_l\beta^m{}_k) = 0.$$

Now

$$\bar{\partial}_l\alpha^i{}_m = \frac{\partial}{\partial\bar{x}^l}\left(\frac{\partial\bar{x}^i}{\partial x^m}\right) = \frac{\partial^2\bar{x}^i}{\partial x^m\partial x^n}\,\frac{\partial x^n}{\partial\bar{x}^l} = \frac{\partial^2\bar{x}^i}{\partial x^m\partial x^n}\beta^n{}_l.$$

Also,

$$\bar{\partial}_l\beta^m{}_k = \frac{\partial^2 x^m}{\partial\bar{x}^l\partial\bar{x}^k}.$$

Therefore

$$\frac{\partial^2\bar{x}^i}{\partial x^m\partial x^n}\beta^n{}_l\beta^m{}_k + \alpha^i{}_m\frac{\partial^2 x^m}{\partial\bar{x}^l\partial\bar{x}^k} = 0,$$

or equivalently

$$\boxed{\alpha^i{}_m\frac{\partial^2 x^m}{\partial\bar{x}^k\partial\bar{x}^l} = -\frac{\partial^2\bar{x}^i}{\partial x^m\partial x^n}\beta^m{}_k\beta^n{}_l}$$

which is Eq. (8).

Step 3: Transformation of $A^i{}_{;k}$. In the barred system,

$$\bar{A}^i{}_{;k} = \bar{\partial}_k\bar{A}^i + \bar{\Gamma}^i{}_{kl}\bar{A}^l.$$

We now insert $\bar{A}^i = \alpha^i{}_m A^m$ and the transformation law for $\bar{\Gamma}^i{}_{kl}$:

$$\begin{aligned}
\bar{A}^i{}_{;k} &= \beta^n{}_k\partial_n(\alpha^i{}_m A^m) + \left(\alpha^i{}_j\beta^r{}_k\beta^s{}_l\Gamma^j{}_{rs} + \alpha^i{}_p\frac{\partial^2 x^p}{\partial\bar{x}^k\partial\bar{x}^l}\right)\alpha^l{}_m A^m \\
&= \beta^n{}_k(\partial_n\alpha^i{}_m)A^m + \beta^n{}_k\alpha^i{}_m\partial_n A^m \\
&\quad + \alpha^i{}_j\beta^r{}_k\delta^s_m\Gamma^j{}_{rs}A^m + \alpha^i{}_p\frac{\partial^2 x^p}{\partial\bar{x}^k\partial\bar{x}^l}\alpha^l{}_m A^m.
\end{aligned}$$

The first and last term cancel. Indeed, Eq. (8) implies

$$\alpha^i{}_p \frac{\partial^2 x^p}{\partial\bar{x}^k \partial\bar{x}^l} \alpha^l{}_m = -\beta^n{}_k \, \partial_n \alpha^i{}_m.$$

Hence

$$\begin{aligned}
\bar{A}^i{}_{;k} &= \alpha^i{}_m \beta^n{}_k \, \partial_n A^m + \alpha^i{}_j \beta^r{}_k \, \Gamma^j{}_{rm} A^m \\
&= \alpha^i{}_j \beta^r{}_k \big(\partial_r A^j + \Gamma^j{}_{rm} A^m\big) \\
&= \alpha^i{}_j \beta^r{}_k \, A^j{}_{;r}.
\end{aligned}$$

This is exactly the transformation law of a mixed tensor of type $(1,1)$. Therefore

$$\boxed{\bar{A}^i{}_{;k} = \alpha^i{}_j \beta^l{}_k A^j{}_{;l}.}$$

e) Levi-Civita symbol as a tensor density

We have to show

$$\epsilon^{ijkl} = \frac{\partial x}{\partial \bar{x}} \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial \bar{x}^j}{\partial x^b} \frac{\partial \bar{x}^k}{\partial x^c} \frac{\partial \bar{x}^l}{\partial x^d} \epsilon^{abcd}.$$

Equivalently, it is enough to prove

$$\epsilon^{abcd} \alpha^i{}_a \alpha^j{}_b \alpha^k{}_c \alpha^l{}_d = \frac{\partial \bar{x}}{\partial x} \, \epsilon^{ijkl}.$$

Now we use the Leibniz formula for the determinant:

$$\det(\alpha) = \frac{\partial \bar{x}}{\partial x} = \epsilon^{abcd} \alpha^0{}_a \alpha^1{}_b \alpha^2{}_c \alpha^3{}_d.$$

There are two cases.

- If two of the indices i, j, k, l are equal, then the left-hand side vanishes because two columns of the determinant are equal. The right-hand side also vanishes because $\epsilon^{ijkl} = 0$.
- If i, j, k, l are all different, then (i, j, k, l) is a permutation σ of $(0, 1, 2, 3)$. Reordering the columns gives

$$\begin{aligned}
\epsilon^{abcd} \alpha^i{}_a \alpha^j{}_b \alpha^k{}_c \alpha^l{}_d &= \operatorname{sgn}(\sigma) \, \epsilon^{abcd} \alpha^0{}_a \alpha^1{}_b \alpha^2{}_c \alpha^3{}_d \\
&= \operatorname{sgn}(\sigma) \, \det(\alpha) \\
&= \det(\alpha) \, \epsilon^{ijkl}.
\end{aligned}$$

Thus in all cases,

$$\epsilon^{abcd} \alpha^i{}_a \alpha^j{}_b \alpha^k{}_c \alpha^l{}_d = \frac{\partial \bar{x}}{\partial x} \, \epsilon^{ijkl}.$$

Multiplying both sides by $\partial x / \partial \bar{x}$ gives Eq. (10). Therefore the Levi-Civita symbol is not an ordinary tensor, but a *relative tensor* (tensor density) of rank 4 and weight $w = +1$.

Problem 4.3: The metric tensor

a) Transformation law of the metric

The line element is coordinate independent:

$$ds^2 = g_{ij}(x) \, dx^i dx^j = \bar{g}_{ij}(\bar{x}) \, d\bar{x}^i d\bar{x}^j.$$

The total differential of the transformed coordinates is

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^m} dx^m = \alpha^i{}_m dx^m.$$

Insert this into the barred expression:

$$\begin{aligned} ds^2 &= \bar{g}_{ij}(\bar{x})\, d\bar{x}^i d\bar{x}^j \\ &= \bar{g}_{ij}(\bar{x})\, \alpha^i{}_m \alpha^j{}_n\, dx^m dx^n. \end{aligned}$$

Since this must equal $g_{mn}(x)\, dx^m dx^n$ for arbitrary differentials dx^m, the coefficients must agree:

$$g_{mn}(x) = \bar{g}_{ij}(\bar{x})\, \alpha^i{}_m \alpha^j{}_n.$$

Multiply by $\beta^m{}_k \beta^n{}_l$:

$$\begin{aligned} \beta^m{}_k \beta^n{}_l g_{mn} &= \beta^m{}_k \beta^n{}_l \bar{g}_{ij} \alpha^i{}_m \alpha^j{}_n \\ &= \bar{g}_{ij}\, \delta^i_k \delta^j_l \\ &= \bar{g}_{kl}. \end{aligned}$$

Therefore

$$\boxed{\bar{g}_{kl}(\bar{x}) = \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l} g_{mn}(x)}$$

which is exactly the transformation law of a covariant tensor of rank 2.

b) Metric in polar coordinates

We now switch to $D = 2$ with Cartesian coordinates (x, y) and polar coordinates (r, θ):

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Method 1: Use the line element directly. First compute the differentials:

$$\begin{aligned} dx &= \cos\theta\, dr - r\sin\theta\, d\theta, \\ dy &= \sin\theta\, dr + r\cos\theta\, d\theta. \end{aligned}$$

Then

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= (\cos\theta\, dr - r\sin\theta\, d\theta)^2 + (\sin\theta\, dr + r\cos\theta\, d\theta)^2. \end{aligned}$$

Expand both squares:

$$\begin{aligned} ds^2 &= \cos^2\theta\, dr^2 - 2r\sin\theta\cos\theta\, dr\, d\theta + r^2\sin^2\theta\, d\theta^2 \\ &\qquad + \sin^2\theta\, dr^2 + 2r\sin\theta\cos\theta\, dr\, d\theta + r^2\cos^2\theta\, d\theta^2 \\ &= (\cos^2\theta + \sin^2\theta)\, dr^2 + r^2(\sin^2\theta + \cos^2\theta)\, d\theta^2 \\ &= dr^2 + r^2 d\theta^2. \end{aligned}$$

Therefore the metric components in polar coordinates are

$$\boxed{\bar{g}_{ij}(r, \theta) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.}$$

Method 2: Use the tensor transformation law. In Cartesian coordinates,

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The needed derivatives are

$$\frac{\partial x}{\partial r} = \cos\theta, \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta, \qquad \frac{\partial y}{\partial r} = \sin\theta, \qquad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

Now compute the transformed components:

$$\begin{aligned}
\bar{g}_{rr} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} g_{yy} \\
&= \cos^2\theta + \sin^2\theta = 1, \\
\bar{g}_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} g_{xx} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} g_{yy} \\
&= \cos\theta(-r\sin\theta) + \sin\theta(r\cos\theta) = 0, \\
\bar{g}_{\theta\theta} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} g_{xx} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} g_{yy} \\
&= r^2\sin^2\theta + r^2\cos^2\theta = r^2.
\end{aligned}$$

So again

$$\bar{g}_{ij}(r,\theta) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

c) Circumference of a circle

Let γ be the circle of radius R centered at the origin.

In Cartesian coordinates. A convenient parametrization is

$$\gamma(t) = (x(t), y(t)) = (R\cos t,\ R\sin t), \qquad 0 \le t \le 2\pi.$$

Then

$$\dot{x}(t) = -R\sin t, \qquad \dot{y}(t) = R\cos t.$$

Using $g_{ij} = \delta_{ij}$, the length is

$$\begin{aligned}
L[\gamma] &= \int_0^{2\pi} \sqrt{g_{ij}\dot{\gamma}^i\dot{\gamma}^j}\, dt \\
&= \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt \\
&= \int_0^{2\pi} \sqrt{R^2\sin^2 t + R^2\cos^2 t}\, dt \\
&= \int_0^{2\pi} R\, dt \\
&= 2\pi R.
\end{aligned}$$

In polar coordinates. The same circle is simply

$$\gamma(t) = (r(t), \theta(t)) = (R, t), \qquad 0 \le t \le 2\pi.$$

Hence

$$\dot{r}(t) = 0, \qquad \dot{\theta}(t) = 1.$$

Using the polar metric,

$$\begin{aligned} L[\gamma] &= \int_0^{2\pi} \sqrt{\bar{g}_{ij}\dot{\gamma}^i\dot{\gamma}^j}\, dt \\ &= \int_0^{2\pi} \sqrt{\bar{g}_{rr}\dot{r}^2 + \bar{g}_{\theta\theta}\dot{\theta}^2}\, dt \\ &= \int_0^{2\pi} \sqrt{1 \cdot 0^2 + R^2 \cdot 1^2}\, dt \\ &= \int_0^{2\pi} R\, dt \\ &= 2\pi R. \end{aligned}$$

Both coordinate systems therefore give the same physical circumference, as they must.