

Dr. Nicolai Lang
 Institute for Theoretical Physics III, University of Stuttgart

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Problem 4.1: Hafele-Keating experiment

[Oral | 4 pt(s)]

ID: ex_hafele_keating_experiment:rt2526

Learning objective

Time dilation is a relativistic effect far removed from our everyday experience. It is, however, an experimentally established fact. A famous experiment measuring time dilation explicitly was the *Hafele-Keating experiment*, where portable atomic clocks were flown on commercial airliners around the world twice: once eastward and once westward. The clocks were then compared to stationary reference clocks on the ground to verify the predictions of time dilation quantitatively.

The experimental results were reported in

<https://doi.itp3.info/10.1126/science.177.4044.168>

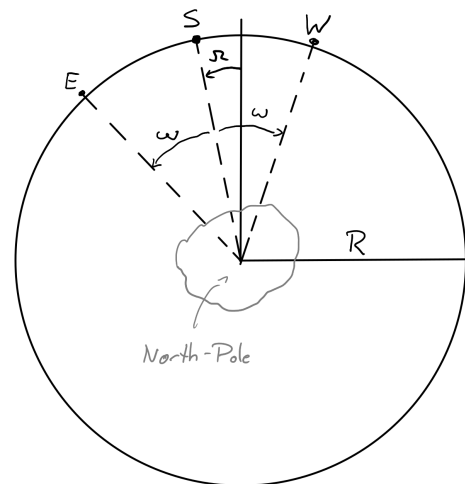
and the theory was developed in

<https://doi.itp3.info/10.1126/science.177.4044.166>

In this exercise, you derive the contribution of time dilation to explain the result of the experiment (which you can find in the theory paper above).

We consider the following setup:

Imagine an observer in a space station above the north pole; the space station follows earth on its orbit around the sun, but does not follow the rotation of earth (i.e., the observer sees earth slowly rotating beneath the space station). Such an observer is approximately inertial for the relevant timescales of the experiment (see sketch on the right); in particular, he is allowed to use the formalism developed in the lecture to compute the proper time along (potentially accelerated) trajectories.



Now consider three identical atomic clocks located on the equator with radius $R \approx 6.4 \times 10^6$ m. The first clock (labeled S) is stationary with respect to earth, this is our “reference clock.” The second clock (E) flies eastward around earth with angular velocity ω (with respect to earth), and the third (W) westward with the same angular velocity ω (also with respect to earth). Both clocks go around earth once and meet again with the reference clock. Note that the rest system of the reference clock S is *not* inertial because earth rotates with angular velocity $\Omega = \frac{2\pi}{24\text{h}}$.

- a) Parametrize the trajectories of the three clocks $\mathbf{x}_S(t)$, $\mathbf{x}_E(t)$ and $\mathbf{x}_W(t)$ in the inertial system of the space station. 1^{pt(s)}

- b) The proper time τ accumulated by a clock can be calculated as shown in the lecture: 1pt(s)

$$\tau_i = \int \sqrt{1 - \frac{\dot{x}_i^2}{c^2}} dt \quad \text{for } i \in \{S, E, W\}. \quad (1)$$

Evaluate this integral for the three clocks.

- c) Calculate $\Delta\tau_E = \tau_E - \tau_S$ and $\Delta\tau_W = \tau_W - \tau_S$. 1pt(s)

Since the angular velocities are small, expand the results in $\frac{R\Omega}{c}$ up to second order.

- d) Assume the clocks travel around earth with 200 m s^{-1} relative to the ground (the speed of a typical airliner). 1pt(s)

What are the time differences measured? Compare them to the numbers reported in the original publications.

Note: For a complete description of the experiment, effects of *general relativity* must be taken into account as well. Thus, we will complete our analysis of the Hafele-Keating experiment in the next semester.

Problem 4.2: Tensor Calculus

[Written | 11 pt(s)]

ID: ex_tensor_calculus:rt2526

Learning objective

Tensor calculus is a crucial toolkit for special and general relativity. In this exercise, you practice calculating with tensor fields and prove some useful rules for the construction of tensor fields.

Consider a D -dimensional differentiable manifold M and an arbitrary coordinate transformation $\bar{x} = \varphi(x)$ from one chart with coordinates $x \in \mathbb{R}^D$ to another chart with coordinates $\bar{x} \in \mathbb{R}^D$.

As motivated in the lecture, we define the transformation of *contravariant* and *covariant* vector (fields) as follows:

$$\text{Contravariant vector field: } \bar{A}^i(\bar{x}) = \sum_{k=1}^D \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \equiv \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \quad (2)$$

$$\text{Covariant vector field: } \bar{B}_i(\bar{x}) = \sum_{k=1}^D \frac{\partial x^k}{\partial \bar{x}^i} B_k(x) \equiv \frac{\partial x^k}{\partial \bar{x}^i} B_k(x), \quad (3)$$

Here we use the *Einstein sum convention*: Sums over pairs of repeated up and down indices are implied but not explicitly written.

- a) Prove that the *contraction* $\Phi(x) := A^i(x)B_i(x)$ of a contravariant vector field $A^i(x)$ with a covariant vector field $B_i(x)$ is invariant under coordinate transformations; i.e., show that it transforms like a *scalar field*. 1pt(s)

The generalization of co- and contravariant vector fields are (mixed) (p, q) *tensor fields* $T^{m_1, \dots, m_p}_{n_1, \dots, n_q}(x)$ with $r = p+q$ indices (called *rank*). Like vector fields, tensor fields are defined by their transformation under coordinate transformations:

$$\bar{T}^{i_1, \dots, i_p}_{j_1, \dots, j_q}(\bar{x}) = \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}} \frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}} T^{m_1, \dots, m_p}_{n_1, \dots, n_q}(x), \quad (4)$$

where p and q are the number of contravariant and covariant indices, respectively.

b) Show that the following combinations of the tensor fields $A^{ij}_k, B^{ij}_k, C^{ij}, D^k_l, E_m$ and the scalar field Φ are again tensor fields (we suppress the x -dependency): 5pt(s)

$$\begin{array}{lll} \text{i)} & V^{ij}_k := A^{ij}_k + B^{ij}_k & \text{ii)} & W^{ij}_k := \Phi A^{ij}_k & \text{iii)} & X^{ijk}_l := C^{ij} D^k_l \\ \text{iv)} & Y^i := A^{ij} E_j & \text{v)} & Z^i := A^{ij}_j \end{array}$$

c) Let C_{ij} be a collection of D^2 fields ($i, j = 1, \dots, D$). 1pt(s)

Prove that if $B_i := C_{ij} A^j$ is a covariant vector field for every contravariant vector field A^i , then C_{ij} transforms like a covariant tensor field of rank 2.

Note: This theorem is called *quotient law*, a quite useful tool in tensor calculus.

The *covariant derivative* of a contravariant vector field is defined as

$$A^i_{;k} := \partial_k A^i + \Gamma^i_{kl} A^l, \tag{5}$$

with $\partial_k \equiv \frac{\partial}{\partial x^k}$ and where the *Christoffel symbol* Γ^i_{kl} is defined as

$$\Gamma^i_{kl} := \frac{1}{2} g^{im} (\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}). \tag{6}$$

$g_{ij} = g_{ij}(x)$ is a given, symmetric ($g_{ij} = g_{ji}$) covariant tensor field of rank 2 called the *metric*.

d) Show that the covariant derivative $A^i_{;k}$ of a contravariant vector field A^i transforms as a mixed $(1, 1)$ tensor field. 3pt(s)

Hint: First, prove the transformation law for the Christoffel symbol

$$\bar{\Gamma}^i_{kl}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l} \Gamma^j_{mn}(x) + \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l}, \tag{7}$$

and show that

$$\frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} = - \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l}. \tag{8}$$

Use this to derive the transformation law for the covariant derivative.

You might want to use the shortcut notations $\alpha^i_k := \frac{\partial \bar{x}^i}{\partial x^k}$ and $\beta^i_k := \frac{\partial x^i}{\partial \bar{x}^k}$ with $\alpha^i_k \beta^k_j = \delta^i_j$.

In the lecture, the general *Levi-Civita symbol* $\epsilon^{i_1 \dots i_D}$ was introduced. Here we want to focus on the most important case of a $D = 4$ dimensional manifold.

The Levi-Civita symbol is defined as

$$\epsilon^{ijkl} := \begin{cases} +1 & \text{if } (i, j, k, l) \text{ is an even permutation of } (0, 1, 2, 3) \\ -1 & \text{if } (i, j, k, l) \text{ is an odd permutation of } (0, 1, 2, 3) \\ 0 & \text{otherwise} \end{cases}. \tag{9}$$

This definition is independent of the coordinate system, meaning $\bar{\epsilon}^{ijkl} = \epsilon^{ijkl}$.

- e) Show that with this definition the Levi-Civita symbol is a *relative tensor* of rank 4 with weight $w = +1$ (which we call a *tensor density*), i.e. show that 1pt(s)

$$\epsilon^{ijkl} = \left| \frac{\partial x}{\partial \bar{x}} \right| \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial \bar{x}^j}{\partial x^b} \frac{\partial \bar{x}^k}{\partial x^c} \frac{\partial \bar{x}^l}{\partial x^d} \epsilon^{abcd} \quad \text{with Jacobian determinant} \quad \left| \frac{\partial \bar{x}}{\partial x} \right|. \quad (10)$$

Hint: Use that determinants can be calculated via the Leibniz formula using the Levi-Civita symbol:

$$\left| \frac{\partial \bar{x}}{\partial x} \right| = \epsilon^{ijkl} \frac{\partial \bar{x}^0}{\partial x^i} \frac{\partial \bar{x}^1}{\partial x^j} \frac{\partial \bar{x}^2}{\partial x^k} \frac{\partial \bar{x}^3}{\partial x^l}. \quad (11)$$

Problem 4.3: The metric tensor

[Oral | 5 pt(s)]

ID: ex_srt_metric_coordinate_transformation:rt2526

Learning objective

The metric tensor (field) is crucial for Riemannian geometry, i.e., the mathematical framework needed to describe curved spaces. Both in special relativity and in general relativity, the metric tensor on the spacetime manifold determines durations (measured by clocks) and lengths (measured by rods) in spacetime.

In this exercise, you show that the components of the metric tensor indeed transform like a covariant tensor field of rank 2. To familiarize yourself with the concept, you study the simple example of Euclidean space (no time!) in two dimensions \mathbb{R}^2 and calculate the components of the metric tensor in polar coordinates.

Consider a D -dimensional differentiable manifold M and an arbitrary coordinate transformation $\bar{x} = \varphi(x)$ from one chart with coordinates $x \in \mathbb{R}^D$ to another chart with coordinates $\bar{x} \in \mathbb{R}^D$.

- a) We start by showing that the components $g_{ij}(x)$ of the metric transform like a covariant tensor field of rank 2. To this end, use that the “line element” ds^2 is a tensor field and hence does not depend on the coordinate system: 1pt(s)

$$g_{ij}(x) dx^i dx^j = ds^2 = \bar{g}_{ij}(\bar{x}) d\bar{x}^i d\bar{x}^j \quad (\text{Einstein summation!}) \quad (12)$$

Hint: Compute the total differential $d\bar{x}^i$ and use that $\{dx^i\}$ is a basis of the cotangent space T_p^*M . The transformation law for a covariant tensor of rank 2 is given in Problem 4.2.

As an example, we consider Euclidean space $M = \mathbb{R}^2$ in $D = 2$ dimensions for the rest of this exercise. In Cartesian coordinates $x^1 = x$ and $x^2 = y$, the components of the metric tensor are

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with line element} \quad ds^2 = g_{ij}(x) dx^i dx^j = dx^2 + dy^2. \quad (13)$$

This particular metric tensor ds^2 characterizes the flat, Euclidean plane you already encountered in school.

- b) We want to calculate the components of the metric tensor in polar coordinates $\bar{x}^1 = r$ and 2pt(s)

$\bar{x}^2 = \theta$. The coordinate transformation $\bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x})$ between Cartesian and polar coordinates is given by

$$\varphi^{-1} : \begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} . \quad (14)$$

First, use Eq. (12) and the rule for computing the total differentials dx and dy to compute $\bar{g}_{ij}(\bar{x})$. Then, derive the same components directly by using the transformation law for a covariant tensor of rank 2.

The physical length of a curve γ can be calculated via the metric tensor by

$$L[\gamma] = \int_{\gamma} ds := \int_a^b dt \sqrt{g_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t)} , \quad (15)$$

where $\gamma(t)$ is the parametrization of the curve with $t \in [a, b]$ in some coordinate system.

c) Let γ be the circle in \mathbb{R}^2 with radius R and center in the origin.

2pt(s)

Calculate the circumference $L[\gamma]$ of this circle once in Cartesian coordinates and then again in polar coordinates. Use the components of the metric tensor given above and computed in subtask b).

Solutions to Problem Set 4

Relativitätstheorie 1, WS 2025/26

Claude Opus 4.6

Prompt: “Solve the exercise tasks in the pdf. Write your solutions in a latex document and compile it. Write solutions like a good physics student, with all important steps. Do not skip tiresome calculations. Do not assume deeper knowledge of tensor calculus.”

Problem 4.1: Hafele–Keating experiment

(a) Parametrization of the trajectories

We work in the inertial frame of the space station observer above the North Pole. In this frame, the Earth rotates eastward with angular velocity $\Omega = \frac{2\pi}{24\text{h}}$. All three clocks move on circles of radius R (the equator) in the equatorial plane.

Clock S (stationary on Earth): In the inertial frame, it co-rotates with the Earth at angular velocity Ω .

$$\mathbf{x}_S(t) = R \begin{pmatrix} \cos(\Omega t) \\ \sin(\Omega t) \end{pmatrix}.$$

Clock E (eastward): It moves eastward with angular velocity ω relative to Earth. Since Earth already rotates with Ω , the total angular velocity in the inertial frame is $\Omega + \omega$:

$$\mathbf{x}_E(t) = R \begin{pmatrix} \cos((\Omega + \omega)t) \\ \sin((\Omega + \omega)t) \end{pmatrix}.$$

Clock W (westward): It moves westward with angular velocity ω relative to Earth, so the inertial angular velocity is $\Omega - \omega$:

$$\mathbf{x}_W(t) = R \begin{pmatrix} \cos((\Omega - \omega)t) \\ \sin((\Omega - \omega)t) \end{pmatrix}.$$

Here t is the coordinate time of the inertial frame. All clocks start at the same point at $t = 0$.

(b) Proper times

The speed of each clock in the inertial frame is constant:

$$|\dot{\mathbf{x}}_S|^2 = R^2\Omega^2, \quad |\dot{\mathbf{x}}_E|^2 = R^2(\Omega + \omega)^2, \quad |\dot{\mathbf{x}}_W|^2 = R^2(\Omega - \omega)^2.$$

Since each speed is constant in time, the proper-time integral simplifies. Both the E and W clocks go around the Earth once relative to the surface, which takes coordinate time

$$T = \frac{2\pi}{\omega}.$$

(To see this: the E clock must gain exactly 2π of angle over S, i.e. $(\Omega + \omega)T - \Omega T = \omega T = 2\pi$. Similarly for W.)

The proper times accumulated during this interval are:

$$\tau_S = \int_0^T \sqrt{1 - \frac{R^2 \Omega^2}{c^2}} dt = \frac{2\pi}{\omega} \sqrt{1 - \frac{R^2 \Omega^2}{c^2}}, \quad (1)$$

$$\tau_E = \int_0^T \sqrt{1 - \frac{R^2 (\Omega + \omega)^2}{c^2}} dt = \frac{2\pi}{\omega} \sqrt{1 - \frac{R^2 (\Omega + \omega)^2}{c^2}}, \quad (2)$$

$$\tau_W = \int_0^T \sqrt{1 - \frac{R^2 (\Omega - \omega)^2}{c^2}} dt = \frac{2\pi}{\omega} \sqrt{1 - \frac{R^2 (\Omega - \omega)^2}{c^2}}. \quad (3)$$

(c) Time differences (expanded to second order)

We compute $\Delta\tau_E = \tau_E - \tau_S$ and $\Delta\tau_W = \tau_W - \tau_S$. Since $R\Omega/c \sim 10^{-6} \ll 1$ and $R\omega/c$ is of similar order, we expand $\sqrt{1 - \epsilon} \approx 1 - \frac{\epsilon}{2}$ to leading (second) order.

Eastward:

$$\begin{aligned} \Delta\tau_E &= \frac{2\pi}{\omega} \left[\sqrt{1 - \frac{R^2 (\Omega + \omega)^2}{c^2}} - \sqrt{1 - \frac{R^2 \Omega^2}{c^2}} \right] \\ &\approx \frac{2\pi}{\omega} \left[\left(1 - \frac{R^2 (\Omega + \omega)^2}{2c^2} \right) - \left(1 - \frac{R^2 \Omega^2}{2c^2} \right) \right] \\ &= -\frac{2\pi}{2\omega c^2} R^2 \left[(\Omega + \omega)^2 - \Omega^2 \right] \\ &= -\frac{\pi R^2}{\omega c^2} \left[2\Omega\omega + \omega^2 \right] \\ &= \boxed{-\frac{\pi R^2}{c^2} (2\Omega + \omega)}. \end{aligned} \quad (4)$$

Westward:

$$\begin{aligned} \Delta\tau_W &= \frac{2\pi}{\omega} \left[\sqrt{1 - \frac{R^2 (\Omega - \omega)^2}{c^2}} - \sqrt{1 - \frac{R^2 \Omega^2}{c^2}} \right] \\ &\approx -\frac{\pi R^2}{\omega c^2} \left[(\Omega - \omega)^2 - \Omega^2 \right] \\ &= -\frac{\pi R^2}{\omega c^2} \left[-2\Omega\omega + \omega^2 \right] \\ &= \boxed{+\frac{\pi R^2}{c^2} (2\Omega - \omega)}. \end{aligned} \quad (5)$$

Physical interpretation: The eastward clock moves faster in the inertial frame ($v_E = R(\Omega + \omega) > R\Omega = v_S$), so it ages *less* than S ($\Delta\tau_E < 0$). The westward clock moves slower in the inertial frame (for $\omega < \Omega$: $v_W = R(\Omega - \omega) < v_S$), so it ages *more* than S ($\Delta\tau_W > 0$).

(d) Numerical evaluation

Given: $v = R\omega = 200$ m/s, $R = 6.4 \times 10^6$ m, $c = 3 \times 10^8$ m/s, $\Omega = 2\pi/(86400)$ s.

First, compute the relevant velocities:

$$R\Omega = \frac{2\pi \times 6.4 \times 10^6}{86400} \text{ m/s} \approx 465.4 \text{ m/s}.$$

The flight time (coordinate time):

$$T = \frac{2\pi R}{v} = \frac{2\pi \times 6.4 \times 10^6}{200} \text{ s} \approx 2.011 \times 10^5 \text{ s} \approx 55.9 \text{ h.}$$

The common prefactor:

$$\frac{\pi R^2}{c^2} = \frac{\pi \times (6.4 \times 10^6)^2}{(3 \times 10^8)^2} = \frac{\pi \times 4.096 \times 10^{13}}{9 \times 10^{16}} \approx 1.430 \times 10^{-3} \text{ s.}$$

The angular velocity factors:

$$2\Omega + \omega = 2\Omega + \frac{v}{R} = \frac{4\pi}{86400} + \frac{200}{6.4 \times 10^6} \approx 1.454 \times 10^{-4} + 3.125 \times 10^{-5} = 1.767 \times 10^{-4} \text{ s}^{-1},$$

$$2\Omega - \omega = 2\Omega - \frac{v}{R} \approx 1.454 \times 10^{-4} - 3.125 \times 10^{-5} = 1.142 \times 10^{-4} \text{ s}^{-1}.$$

Therefore:

$$\Delta\tau_E = -1.430 \times 10^{-3} \times 1.767 \times 10^{-4} \text{ s} \approx \boxed{-253 \text{ ns}}, \quad (6)$$

$$\Delta\tau_W = +1.430 \times 10^{-3} \times 1.142 \times 10^{-4} \text{ s} \approx \boxed{+163 \text{ ns}}. \quad (7)$$

Comparison with the original publication: The theory paper by Hafele & Keating (Science **177**, 166–168, 1972) reports for the *kinematic* (velocity-dependent, special-relativistic) contribution:

$$\Delta\tau_E^{\text{kin}} = -184 \pm 18 \text{ ns}, \quad \Delta\tau_W^{\text{kin}} = +96 \pm 10 \text{ ns.}$$

Our idealized values (−253 ns and +163 ns) are of the correct sign and order of magnitude, but are larger because we assumed a perfect equatorial circumnavigation at constant speed. The actual experiment involved non-equatorial flight paths, varying speeds, and layovers, which reduce the effective time dilation. Additionally, the complete description requires gravitational time dilation (a general-relativistic effect), which contributes +144±14 ns (eastward) and +179±18 ns (westward) according to the original paper.

Problem 4.2: Tensor Calculus

Throughout this problem, we use the Einstein summation convention and the coordinate transformation $\bar{x} = \varphi(x)$ with the shorthand notations

$$\alpha_k^i := \frac{\partial \bar{x}^i}{\partial x^k}, \quad \beta_k^i := \frac{\partial x^i}{\partial \bar{x}^k}, \quad \text{satisfying} \quad \alpha_k^i \beta_j^k = \delta_j^i.$$

(a) The contraction $\Phi(x) = A^i(x) B_i(x)$ is a scalar field

We compute the contraction in the barred coordinate system using the transformation laws (2) and (3):

$$\begin{aligned} \bar{\Phi}(\bar{x}) &= \bar{A}^i(\bar{x}) \bar{B}_i(\bar{x}) = \left(\frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \right) \left(\frac{\partial x^l}{\partial \bar{x}^i} B_l(x) \right) \\ &= \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^i} A^k(x) B_l(x) = \delta_k^l A^k(x) B_l(x) \\ &= A^k(x) B_k(x) = \Phi(x). \end{aligned} \tag{8}$$

Here we used the chain rule identity $\frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^i} = \frac{\partial x^l}{\partial x^k} = \delta_k^l$. Since $\bar{\Phi}(\bar{x}) = \Phi(x)$, the contraction transforms as a scalar (invariant under coordinate transformations). \square

(b) Tensor field combinations

We use the general transformation law (4) for a (p, q) -tensor field. Using our shorthand, a $(2, 1)$ -tensor A^{ij}_k transforms as $\bar{A}^{ij}_k = \alpha_a^i \alpha_b^j \beta_k^c A^{ab}_c$.

(i) $V^{ij}_k := A^{ij}_k + B^{ij}_k$ is a $(2, 1)$ -tensor.

Since A^{ij}_k and B^{ij}_k are both $(2, 1)$ -tensors, they transform with the same Jacobian prefactors:

$$\begin{aligned} \bar{V}^{ij}_k &= \bar{A}^{ij}_k + \bar{B}^{ij}_k = \alpha_a^i \alpha_b^j \beta_k^c A^{ab}_c + \alpha_a^i \alpha_b^j \beta_k^c B^{ab}_c \\ &= \alpha_a^i \alpha_b^j \beta_k^c (A^{ab}_c + B^{ab}_c) = \alpha_a^i \alpha_b^j \beta_k^c V^{ab}_c. \end{aligned} \tag{9}$$

This is the transformation law of a $(2, 1)$ -tensor. \square

(ii) $W^{ij}_k := \Phi A^{ij}_k$ is a $(2, 1)$ -tensor.

Since Φ is a scalar field, $\bar{\Phi}(\bar{x}) = \Phi(x)$. Therefore:

$$\begin{aligned} \bar{W}^{ij}_k &= \bar{\Phi}(\bar{x}) \bar{A}^{ij}_k = \Phi(x) \alpha_a^i \alpha_b^j \beta_k^c A^{ab}_c \\ &= \alpha_a^i \alpha_b^j \beta_k^c \Phi(x) A^{ab}_c = \alpha_a^i \alpha_b^j \beta_k^c W^{ab}_c. \end{aligned} \tag{10}$$

This is the transformation law of a $(2, 1)$ -tensor. \square

(iii) $X^{ijk}_l := C^{ij} D^k_l$ is a $(3, 1)$ -tensor.

C^{ij} is a $(2, 0)$ -tensor: $\bar{C}^{ij} = \alpha_a^i \alpha_b^j C^{ab}$. D^k_l is a $(1, 1)$ -tensor: $\bar{D}^k_l = \alpha_c^k \beta_l^d D^c_d$.

Their product transforms as:

$$\begin{aligned} \bar{X}^{ijk}_l &= \bar{C}^{ij} \bar{D}^k_l = \alpha_a^i \alpha_b^j C^{ab} \cdot \alpha_c^k \beta_l^d D^c_d \\ &= \alpha_a^i \alpha_b^j \alpha_c^k \beta_l^d C^{ab} D^c_d = \alpha_a^i \alpha_b^j \alpha_c^k \beta_l^d X^{abc}_d. \end{aligned} \tag{11}$$

This is the transformation law of a $(3, 1)$ -tensor. \square

(iv) $Y^i := C^{ij} E_j$ is a $(1, 0)$ -tensor (contravariant vector).

This is a contraction of the $(2, 0)$ -tensor C^{ij} with the covariant vector E_j :

$$\begin{aligned}\bar{Y}^i &= \bar{C}^{ij} \bar{E}_j = \alpha_a^i \alpha_b^j C^{ab} \cdot \beta_j^c E_c \\ &= \alpha_a^i \underbrace{\alpha_b^j \beta_j^c}_{\delta_b^c} C^{ab} E_c = \alpha_a^i C^{ab} E_b = \alpha_a^i Y^a = \frac{\partial \bar{x}^i}{\partial x^a} Y^a.\end{aligned}\quad (12)$$

This is the transformation law of a contravariant vector. \square

(v) $Z^i := A^{ij}{}_j$ is a $(1, 0)$ -tensor (contravariant vector).

This is a trace (contraction of the second upper index with the lower index) of the $(2, 1)$ -tensor $A^{ij}{}_k$:

$$\begin{aligned}\bar{Z}^i &= \bar{A}^{ij}{}_j = \alpha_a^i \alpha_b^j \beta_j^c A^{ab}{}_c \\ &= \alpha_a^i \underbrace{\alpha_b^j \beta_j^c}_{\delta_b^c} A^{ab}{}_c = \alpha_a^i A^{ab}{}_b = \alpha_a^i Z^a = \frac{\partial \bar{x}^i}{\partial x^a} Z^a.\end{aligned}\quad (13)$$

This is the transformation law of a contravariant vector. \square

(c) Quotient law

Claim: If $B_i := C_{ij} A^j$ is a covariant vector field for every contravariant vector field A^j , then C_{ij} transforms like a covariant tensor field of rank 2.

Proof. Since B_i is a covariant vector field, it transforms as

$$\bar{B}_i(\bar{x}) = \beta_i^k B_k(x) = \beta_i^k C_{kl}(x) A^l(x).$$

On the other hand, we can also write in the barred system:

$$\bar{B}_i(\bar{x}) = \bar{C}_{ij}(\bar{x}) \bar{A}^j(\bar{x}) = \bar{C}_{ij}(\bar{x}) \alpha_l^j A^l(x).$$

Here we used that A^j is a contravariant vector, so $\bar{A}^j = \alpha_l^j A^l$.

Equating the two expressions:

$$\bar{C}_{ij}(\bar{x}) \alpha_l^j A^l(x) = \beta_i^k C_{kl}(x) A^l(x).$$

Since this must hold for *every* contravariant vector field A^l (and the transformation $x \mapsto \bar{x}$ is invertible, so arbitrary A^l corresponds to arbitrary \bar{A}^j), we can conclude that the coefficients of A^l must agree:

$$\bar{C}_{ij}(\bar{x}) \alpha_l^j = \beta_i^k C_{kl}(x).$$

Multiplying both sides by β_m^l (i.e., contracting with $\partial x^l / \partial \bar{x}^m$):

$$\bar{C}_{ij}(\bar{x}) \underbrace{\alpha_l^j \beta_m^l}_{\delta_m^j} = \beta_i^k \beta_m^l C_{kl}(x),$$

which gives

$$\bar{C}_{im}(\bar{x}) = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^m} C_{kl}(x).$$

This is precisely the transformation law of a covariant tensor of rank 2. \square

(d) Covariant derivative transforms as a (1, 1)-tensor

We need to show that $A^i{}_{;k} = \partial_k A^i + \Gamma_{kl}^i A^l$ transforms as a mixed (1, 1)-tensor. We proceed in three steps as suggested.

Step 1: Proof of the identity (8).

We differentiate the identity $\alpha_m^i \beta_l^m = \delta_l^i$ with respect to \bar{x}^k :

$$\frac{\partial \alpha_m^i}{\partial \bar{x}^k} \beta_l^m + \alpha_m^i \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} = 0.$$

For the first term, we use the chain rule $\frac{\partial}{\partial \bar{x}^k} = \beta_k^n \frac{\partial}{\partial x^n}$ to write:

$$\frac{\partial \alpha_m^i}{\partial \bar{x}^k} = \frac{\partial}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^m} = \beta_k^n \frac{\partial^2 \bar{x}^i}{\partial x^n \partial x^m}.$$

Substituting back:

$$\beta_k^n \frac{\partial^2 \bar{x}^i}{\partial x^n \partial x^m} \beta_l^m + \alpha_m^i \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} = 0.$$

Solving for the second term and relabeling $n \rightarrow m$, $m \rightarrow n$ in the first term:

$$\boxed{\alpha_m^i \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} = - \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^n} \beta_k^m \beta_l^n.}$$

This is equation (8).

Step 2: Proof of the Christoffel symbol transformation law (7).

The metric g_{ij} is a covariant (0, 2)-tensor, so $\bar{g}_{mk}(\bar{x}) = \beta_m^a \beta_k^b g_{ab}(x)$. The inverse metric transforms as $\bar{g}^{im}(\bar{x}) = \alpha_r^i \alpha_s^m g^{rs}(x)$.

Starting from the Christoffel symbol definition in barred coordinates,

$$\bar{\Gamma}_{kl}^i = \frac{1}{2} \bar{g}^{im} \left(\frac{\partial \bar{g}_{mk}}{\partial \bar{x}^l} + \frac{\partial \bar{g}_{ml}}{\partial \bar{x}^k} - \frac{\partial \bar{g}_{kl}}{\partial \bar{x}^m} \right),$$

we need to compute $S_{mkl} := \frac{\partial \bar{g}_{mk}}{\partial \bar{x}^l} + \frac{\partial \bar{g}_{ml}}{\partial \bar{x}^k} - \frac{\partial \bar{g}_{kl}}{\partial \bar{x}^m}$.

Differentiating $\bar{g}_{mk} = \beta_m^a \beta_k^b g_{ab}$ using $\frac{\partial}{\partial \bar{x}^l} = \beta_l^c \partial_c$:

$$\frac{\partial \bar{g}_{mk}}{\partial \bar{x}^l} = \frac{\partial^2 x^a}{\partial \bar{x}^l \partial \bar{x}^m} \beta_k^b g_{ab} + \beta_m^a \frac{\partial^2 x^b}{\partial \bar{x}^l \partial \bar{x}^k} g_{ab} + \beta_m^a \beta_k^b \beta_l^c \partial_c g_{ab}. \quad (14)$$

Writing out all three terms in S_{mkl} using (14) (with appropriate permutations of k, l, m), we collect two types of contributions:

Second-derivative terms: From the six second-derivative terms (two from each of the three derivatives), four cancel in pairs due to the symmetry $\partial^2 x^a / \partial \bar{x}^k \partial \bar{x}^m = \partial^2 x^a / \partial \bar{x}^m \partial \bar{x}^k$. Specifically, the terms containing $\beta_l^b \frac{\partial^2 x^a}{\partial \bar{x}^k \partial \bar{x}^m}$ from $\frac{\partial \bar{g}_{ml}}{\partial \bar{x}^k}$ and $-\frac{\partial \bar{g}_{kl}}{\partial \bar{x}^m}$ cancel each other. Among the remaining four, renaming dummy indices $a \leftrightarrow b$ (using $g_{ab} = g_{ba}$) in one pair shows that two more cancel, leaving only:

$$\text{second-derivative terms} = 2 \beta_m^a g_{ab} \frac{\partial^2 x^b}{\partial \bar{x}^k \partial \bar{x}^l}.$$

Metric-derivative terms: After relabeling dummy indices to factor out $\beta_m^a \beta_k^b \beta_l^c$, the three metric-derivative contributions combine to:

$$\beta_m^a \beta_k^b \beta_l^c (\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}) = 2 \beta_m^a \beta_k^b \beta_l^c g_{cd} \Gamma_{ab}^d,$$

where we used $\frac{1}{2}(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}) = g_{cd} \Gamma_{ab}^d$ (the Christoffel symbol of the first kind).

Combining, we have:

$$\frac{1}{2} S_{mkl} = \beta_m^a \beta_k^b \beta_l^c g_{cd} \Gamma_{ab}^d + \beta_m^a g_{ab} \frac{\partial^2 x^b}{\partial \bar{x}^k \partial \bar{x}^l}.$$

Contracting with $\bar{g}^{im} = \alpha_r^i \alpha_s^m g^{rs}$ and using $\alpha_s^m \beta_m^a = \delta_s^a$:

$$\bar{\Gamma}_{kl}^i = \alpha_r^i g^{rs} \beta_k^b \beta_l^c g_{cd} \Gamma_{sb}^d + \alpha_r^i \underbrace{g^{rs} g_{sb}}_{\delta_b^r} \frac{\partial^2 x^b}{\partial \bar{x}^k \partial \bar{x}^l}.$$

For the first term, we use $g^{rs} g_{cd} \Gamma_{sb}^d = g^{rs} \frac{1}{2}(\partial_b g_{sc} + \partial_s g_{bc} - \partial_c g_{sb}) = \Gamma_{bc}^r$, and for the second term, δ_b^r collapses the sum. Renaming dummy indices ($r \rightarrow j$, $b \rightarrow m$, $c \rightarrow n$):

$$\bar{\Gamma}_{kl}^i(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l} \Gamma_{mn}^j(x) + \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l}.$$

This is equation (7). The first term is the ‘‘tensor part’’ and the second is the inhomogeneous part that makes Γ_{kl}^i *not* a tensor.

Step 3: Transformation of the covariant derivative.

We compute $\bar{A}^i{}_{;k} = \bar{\partial}_k \bar{A}^i + \bar{\Gamma}_{kl}^i \bar{A}^l$ in barred coordinates (where $\bar{\partial}_k \equiv \partial/\partial \bar{x}^k$).

Term 1: $\bar{\partial}_k \bar{A}^i$.

Since $\bar{A}^i = \alpha_j^i A^j$:

$$\begin{aligned} \bar{\partial}_k \bar{A}^i &= \bar{\partial}_k (\alpha_j^i A^j) = (\bar{\partial}_k \alpha_j^i) A^j + \alpha_j^i \bar{\partial}_k A^j \\ &= \beta_k^m \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^j} A^j + \alpha_j^i \beta_k^m \partial_m A^j. \end{aligned} \quad (15)$$

Term 2: $\bar{\Gamma}_{kl}^i \bar{A}^l$.

Using the transformation law (7) for $\bar{\Gamma}_{kl}^i$ and $\bar{A}^l = \alpha_n^l A^n$:

$$\bar{\Gamma}_{kl}^i \bar{A}^l = \left(\alpha_j^i \beta_k^m \beta_l^n \Gamma_{mn}^j + \alpha_s^i \frac{\partial^2 x^s}{\partial \bar{x}^k \partial \bar{x}^l} \right) \alpha_r^l A^r. \quad (16)$$

The first part of (16): using $\beta_l^n \alpha_r^l = \delta_r^n$,

$$\alpha_j^i \beta_k^m \underbrace{\beta_l^n \alpha_r^l}_{\delta_r^n} \Gamma_{mn}^j A^r = \alpha_j^i \beta_k^m \Gamma_{mr}^j A^r. \quad (17)$$

The second part of (16): we use identity (8) applied to $\alpha_s^i \frac{\partial^2 x^s}{\partial \bar{x}^k \partial \bar{x}^l} = -\frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^n} \beta_k^m \beta_l^n$, then contract with α_r^l :

$$\alpha_s^i \frac{\partial^2 x^s}{\partial \bar{x}^k \partial \bar{x}^l} \alpha_r^l A^r = -\frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^n} \beta_k^m \underbrace{\beta_l^n \alpha_r^l}_{\delta_r^n} A^r = -\beta_k^m \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^r} A^r. \quad (18)$$

Combining: Adding (15), (17), and (18):

$$\bar{A}^i{}_{;k} = \underbrace{\beta_k^m \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^j} A^j}_{(I)} + \underbrace{\alpha_j^i \beta_k^m \partial_m A^j}_{(II)} + \underbrace{\alpha_j^i \beta_k^m \Gamma_{mr}^j A^r}_{(III)} - \underbrace{\beta_k^m \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^r} A^r}_{(IV)}.$$

In term (I), rename the dummy index $j \rightarrow r$: $(I) = \beta_k^m \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^r} A^r$. This is identical to term (IV), so **terms (I) and (IV) cancel**.

The remaining terms give:

$$\bar{A}^i{}_{;k} = \alpha_j^i \beta_k^m (\partial_m A^j + \Gamma_{mr}^j A^r) = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^m}{\partial \bar{x}^k} A^j{}_{;m}. \quad (19)$$

This is precisely the transformation law of a mixed (1,1)-tensor field. \square

(e) **Levi-Civita symbol as a relative tensor of weight $w = +1$**

We need to show that

$$\epsilon^{ijkl} = \left| \frac{\partial x}{\partial \bar{x}} \right| \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial \bar{x}^j}{\partial x^b} \frac{\partial \bar{x}^k}{\partial x^c} \frac{\partial \bar{x}^l}{\partial x^d} \epsilon^{abcd}.$$

Consider the right-hand side. Using the Leibniz formula (11) for the Jacobian determinant:

$$\left| \frac{\partial \bar{x}}{\partial x} \right| = \epsilon^{abcd} \frac{\partial \bar{x}^0}{\partial x^a} \frac{\partial \bar{x}^1}{\partial x^b} \frac{\partial \bar{x}^2}{\partial x^c} \frac{\partial \bar{x}^3}{\partial x^d}.$$

This is the determinant of the Jacobian matrix J , where $J_{pa} = \frac{\partial \bar{x}^p}{\partial x^a}$. Now consider the expression

$$\mathcal{J}^{ijkl} := \epsilon^{abcd} \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial \bar{x}^j}{\partial x^b} \frac{\partial \bar{x}^k}{\partial x^c} \frac{\partial \bar{x}^l}{\partial x^d} = \epsilon^{abcd} J_{ia} J_{jb} J_{kc} J_{ld}.$$

This is the determinant of the 4×4 matrix with rows $(J_{i\cdot}, J_{j\cdot}, J_{k\cdot}, J_{l\cdot})$, i.e., the rows of J rearranged according to (i, j, k, l) .

- If any two of i, j, k, l are equal, two rows of this matrix coincide, so the determinant vanishes: $\mathcal{J}^{ijkl} = 0 = \epsilon^{ijkl} \det J$.
- If (i, j, k, l) is a permutation σ of $(0, 1, 2, 3)$, then the rows have been permuted by σ relative to the standard order. A permutation of rows changes the determinant by $\text{sgn}(\sigma)$:

$$\mathcal{J}^{ijkl} = \text{sgn}(\sigma) \det J = \epsilon^{ijkl} \left| \frac{\partial \bar{x}}{\partial x} \right|.$$

In both cases, $\mathcal{J}^{ijkl} = \epsilon^{ijkl} \left| \frac{\partial \bar{x}}{\partial x} \right|$. Therefore:

$$\left| \frac{\partial x}{\partial \bar{x}} \right| \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial \bar{x}^j}{\partial x^b} \frac{\partial \bar{x}^k}{\partial x^c} \frac{\partial \bar{x}^l}{\partial x^d} \epsilon^{abcd} = \left| \frac{\partial x}{\partial \bar{x}} \right| \epsilon^{ijkl} \left| \frac{\partial \bar{x}}{\partial x} \right| = \epsilon^{ijkl}, \quad (20)$$

where we used $\left| \frac{\partial x}{\partial \bar{x}} \right| \cdot \left| \frac{\partial \bar{x}}{\partial x} \right| = 1$ (the inverse function theorem).

This is exactly equation (10), confirming that ϵ^{ijkl} is a relative tensor (tensor density) of rank 4 with weight $w = +1$. \square

Problem 4.3: The metric tensor

(a) The metric transforms as a covariant tensor of rank 2

The line element ds^2 is invariant under coordinate transformations:

$$g_{ij}(x) dx^i dx^j = ds^2 = \bar{g}_{ij}(\bar{x}) d\bar{x}^i d\bar{x}^j.$$

Compute the total differential of the new coordinates:

$$d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^k} dx^k.$$

Substituting into the right-hand side:

$$\bar{g}_{ij}(\bar{x}) d\bar{x}^i d\bar{x}^j = \bar{g}_{ij}(\bar{x}) \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l} dx^k dx^l. \quad (21)$$

By the invariance of ds^2 , this must equal $g_{kl}(x) dx^k dx^l$. Since the differentials $\{dx^k\}$ form a basis of the cotangent space T_p^*M (and hence are linearly independent), we can compare coefficients¹:

$$g_{kl}(x) = \bar{g}_{ij}(\bar{x}) \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial \bar{x}^j}{\partial x^l}.$$

Multiplying both sides by $\frac{\partial x^k}{\partial \bar{x}^m} \frac{\partial x^l}{\partial \bar{x}^n}$ and using $\frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^m} = \delta_m^i$:

$$\boxed{\bar{g}_{mn}(\bar{x}) = \frac{\partial x^k}{\partial \bar{x}^m} \frac{\partial x^l}{\partial \bar{x}^n} g_{kl}(x).}$$

This is the transformation law of a covariant tensor field of rank 2. □

(b) Metric in polar coordinates

The coordinate transformation from Cartesian $(x^1, x^2) = (x, y)$ to polar $(\bar{x}^1, \bar{x}^2) = (r, \theta)$ is given by the inverse:

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Method 1: Via total differentials.

Compute the differentials:

$$\begin{aligned} dx &= \cos \theta dr - r \sin \theta d\theta, \\ dy &= \sin \theta dr + r \cos \theta d\theta. \end{aligned}$$

The line element in Cartesian coordinates is $ds^2 = dx^2 + dy^2$:

$$\begin{aligned} ds^2 &= (\cos \theta dr - r \sin \theta d\theta)^2 + (\sin \theta dr + r \cos \theta d\theta)^2 \\ &= \cos^2 \theta dr^2 - 2r \sin \theta \cos \theta dr d\theta + r^2 \sin^2 \theta d\theta^2 \\ &\quad + \sin^2 \theta dr^2 + 2r \sin \theta \cos \theta dr d\theta + r^2 \cos^2 \theta d\theta^2 \\ &= (\cos^2 \theta + \sin^2 \theta) dr^2 + r^2 (\sin^2 \theta + \cos^2 \theta) d\theta^2 \\ &= dr^2 + r^2 d\theta^2. \end{aligned} \quad (22)$$

¹More precisely, since $dx^k dx^l$ is symmetric in k, l , we compare the symmetrized coefficients, but since g_{kl} and \bar{g}_{ij} are already symmetric, this is immediate.

Reading off the metric components from $ds^2 = \bar{g}_{ij} d\bar{x}^i d\bar{x}^j$:

$$\bar{g}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

Method 2: Via the transformation law.

The Jacobian matrix of the inverse transformation is:

$$\frac{\partial x^k}{\partial \bar{x}^i} = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}.$$

Using the transformation law $\bar{g}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial x^l}{\partial \bar{x}^j} g_{kl}$ with $g_{kl} = \delta_{kl}$:

$$\begin{aligned} \bar{g}_{11} &= \frac{\partial x^k}{\partial r} \frac{\partial x^k}{\partial r} = \cos^2 \theta + \sin^2 \theta = 1, \\ \bar{g}_{12} &= \frac{\partial x^k}{\partial r} \frac{\partial x^k}{\partial \theta} = \cos \theta (-r \sin \theta) + \sin \theta (r \cos \theta) = 0, \\ \bar{g}_{22} &= \frac{\partial x^k}{\partial \theta} \frac{\partial x^k}{\partial \theta} = r^2 \sin^2 \theta + r^2 \cos^2 \theta = r^2. \end{aligned}$$

This confirms $\bar{g}_{ij} = \text{diag}(1, r^2)$, in agreement with Method 1.

(c) Circumference of a circle

Let γ be the circle of radius R centered at the origin. The circumference is

$$L[\gamma] = \int_{\gamma} ds = \int_a^b dt \sqrt{g_{ij}(\gamma(t)) \dot{\gamma}^i(t) \dot{\gamma}^j(t)}.$$

In Cartesian coordinates:

Parametrize: $\gamma(t) = (R \cos t, R \sin t)$, $t \in [0, 2\pi]$.

Velocity: $\dot{\gamma}(t) = (-R \sin t, R \cos t)$.

With $g_{ij} = \delta_{ij}$:

$$g_{ij} \dot{\gamma}^i \dot{\gamma}^j = (-R \sin t)^2 + (R \cos t)^2 = R^2(\sin^2 t + \cos^2 t) = R^2.$$

Therefore:

$$L[\gamma] = \int_0^{2\pi} \sqrt{R^2} dt = R \int_0^{2\pi} dt = 2\pi R.$$

In polar coordinates:

Parametrize: $\gamma(t) = (r, \theta) = (R, t)$, $t \in [0, 2\pi]$.

Velocity: $\dot{\gamma}(t) = (\dot{r}, \dot{\theta}) = (0, 1)$.

With $\bar{g}_{ij} = \text{diag}(1, r^2)$:

$$\bar{g}_{ij} \dot{\gamma}^i \dot{\gamma}^j = 1 \cdot 0^2 + r^2 \cdot 1^2 = R^2.$$

Therefore:

$$L[\gamma] = \int_0^{2\pi} \sqrt{R^2} dt = R \int_0^{2\pi} dt = 2\pi R.$$

Both coordinate systems yield the same result $L[\gamma] = 2\pi R$, as expected since the physical length is a scalar quantity (coordinate-independent).