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December 10th, 2025
WS 2025/26

Problem 5.1: Christoffel symbols and the covariant derivative

[**Oral** | 7 pt(s)]

ID: ex_christoffel_symbols:

Learning objective

This exercise is a sequel to Problem 4.3. Here you study the Christoffel symbols – induced by the metric tensor – which are needed to define the covariant derivative.

In particular, you use a simple example in flat (Euclidean) space to get familiar with the covariant derivative. You will explicitly see that the "normal" partial derivative of a vector field does not transform as a tensor whereas the covariant derivative does.

We consider the same setting as in Problem 4.3:

Our manifold is the two-dimensional Euclidean plane $M = \mathbb{R}^2$ and we consider two charts: Cartesian coordinates (x^1, x^2) and polar coordinates $(\bar{x}^1 = r, \bar{x}^2 = \theta)$. The components of the metric tensor in these charts are given by

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{g}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} . \tag{1}$$

We now want to study the Christoffel symbols induced by this metric.

a) As a first step, calculate the inverse metric tensor g^{ij} by using the fact that $g^{ij} g_{jk} = \delta^i_k$. **2**pt(s)

Then compute the Christoffel symbols

$$\Gamma^i{}_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{lm,k} - g_{kl,m} \right) . \tag{2}$$

Do this for both coordinate systems.

We see that even though the metric describes flat Euclidean space $\mathbb{R}^2$, not all Christoffel symbols vanish – depending on the coordinates used to express the metric tensor.

We now want to get more familiar with the Christoffel symbols and the covariant derivative. To this end, consider the vector field A with contravariant components A^i:

$$A := \partial_r = \bar{A}^i \partial_{\bar{x}^i} \quad \text{with} \quad \bar{A}^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{3}$$

b) Use the transformation law for tensors to calculate A^i (the components of A in Cartesian coordinates). Draw or plot the vector field in both coordinate systems. **1**pt(s)

c) Now compute the partial derivative $A^i{}_{,j}$ and the covariant derivative $A^i{}_{;j}$ of the vector field in both coordinate systems. **2**pt(s)

d) Explicitly check that the covariant derivative transforms like a $(1,1)$-tensor, i.e., show that **2**pt(s)

$$A^{i}{}_{;j} = \frac{\partial x^i}{\partial \bar{x}^k}\frac{\partial \bar{x}^l}{\partial x^j}\,\bar{A}^{k}{}_{;l} \tag{4}$$

holds. Demonstrate that this fails for the partial derivative.

Problem 5.2: Covariant form of the Lagrangian of classical electrodynamics [**Oral** | 3 pt(s)]

ID: ex_covariant_em_lagrangian:

Learning objective

Maxwell's electrodynamics is the prototypical example of a *relativistic field theory*. In this exercise, you use tensor calculus [recall Problem 4.2], specialized to the inertial coordinate systems of special relativity, to show that the Lagrangian and the action of classical electrodynamics are invariant under Lorentz transformations.

Here we consider Minkowski space $M = \mathbb{R}^{1,3} \simeq \mathbb{R}^4$ as spacetime manifold and focus on inertial coordinate systems and the transformations between them (Lorentz transformations). The metric tensor is the Minkowski metric with components $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$ in every inertial coordinate system. In the following, Greek indices run from 0 to 3.

The action of classical electrodynamics in vacuum (with speed of light $c = 1$) is given in some inertial coordinate system by

$$S = \int d^4x\,\mathcal{L} = \int d^4x\,\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{5}$$

with the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\pi}F_{\rho\pi}$ (where $\eta^{\mu\nu}$ denotes the inverse of the metric $\eta_{\mu\nu}$).

Here, A^μ is the gauge potential of electrodynamics; however, for this exercise you only need to know that A^μ transforms like a contravariant vector field [a $(1,0)$ tensor] and $A_\mu = \eta_{\mu\nu}A^\nu$ like a covariant vector field [a $(0,1)$ tensor].

a) Show that $F^{\mu\nu}$ is a contravariant tensor field of rank 2, and that the Lagrangian (density) $\mathcal{L}$ is a Lorentz scalar. **2**pt(s)

Hint: Use the results from Problem 4.2 and your knowledge about Lorentz transformations (which are the only coordinate transformations considered in special relativity).

b) Show that Eq. (5) is invariant under Lorentz transformations, i.e., show that **1**pt(s)

$$S = \int d^4x\,\mathcal{L} = \int d^4\bar{x}\,\bar{\mathcal{L}} = \bar{S}\,. \tag{6}$$

In summary, you have shown that the Lagrangian (density) $\mathcal{L}$ and the action (5) are Lorentz scalars. To prove that classical electrodynamics (the Maxwell equations) is Lorentz covariant (= the equations take the same form in all inertial systems), we still must show that the *Euler-Lagrange equations* that follow from varying the action (5) produce the Maxwell equations in their conventional form (you will do this in another exercise at the end of this course).

Problem 5.3: The Lorentz group

[**Written** | 7 (+1 bonus) pt(s)]

ID: ex_lorentz_group:

Learning objective

The goal of this exercise is to get familiar with the Lorentz group and its continuous and discrete generators. Since the Lorentz group is a Lie group, we can study its Lie algebra. Knowing the Lie algebra of a group is very helpful to construct other representations of the group. This is particularly important in the context of relativistic quantum mechanics, where one has to replace the Schrödinger equation by the Dirac equation.

The Lorentz group $O(1,3)$ is defined as group of real 4×4 matrices Λ that leave the Minkowski metric $\eta = \text{diag}(1,-1,-1,-1)$ invariant, i.e.

$$O(1,3) = \{\Lambda \in \mathbb{R}^{4\times 4} \,|\, \Lambda^{\mu}{}_{\alpha}\eta_{\mu\nu}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta} \iff \Lambda^T\eta\Lambda = \eta\}\,. \tag{7}$$

a) Show that $O(1,3)$ is a group [use only the definition in Eq. (7)]. **2**pt(s)

Specifically, show the following properties for every $\Lambda_1, \Lambda_2 \in O(1,3)$:

(i) $\Lambda_1\Lambda_2 \in O(1,3)$

(ii) The inverse Λ_1^{-1} exists and is in $O(1,3)$.

Since the Lorentz group is a *Lie group* (i.e., a group that is also a differentiable manifold), we can study its *Lie algebra* (a vector space with an additional multiplication known as *Lie bracket*).

To this end, we consider an infinitesimal Lorentz transformation

$$\Lambda_{\boldsymbol{\xi}} = \exp\left(-i\xi_i X_i\right) \overset{\xi_i \ll 1}{\approx} \mathbb{1} - i\xi_i X_i\,, \tag{8}$$

where the matrices $X_i \in \mathbb{R}^{4\times 4}$ belong to the Lie algebra and are the generators of the Lorentz group; the $\xi_i \in \mathbb{R}$ are the corresponding coefficients that define the group element $\Lambda_{\boldsymbol{\xi}}$ in terms of these generators (and a sum over i is implied).

b) How does the defining condition of the Lorentz group $\Lambda^T\eta\Lambda = \eta$ translate to the generators X_i? **2**pt(s)

Show that one can write the generators X_i in the general form

$$(X_i)^{\rho}{}_{\sigma} = \begin{pmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & -d & 0 & f \\ c & -e & -f & 0 \end{pmatrix}\,, \tag{9}$$

with only 6 degrees of freedom left.

Since the Lie algebra is a vector space, one can choose a set of basis vectors (= generators) and write every element of the algebra as a linear combination of these.

A conventional basis of the Lie algebra of the Lorentz group is

$$(\mathcal{J}^{\mu\nu})^{\rho}{}_{\sigma} = i\left(\eta^{\mu\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\eta^{\nu\rho}\right)\,, \tag{10}$$

with

$$\Lambda_{\boldsymbol{\omega}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) , \tag{11}$$

where both, the set of coefficients $\omega_{\mu\nu}$ and the set of generators $\mathcal{J}^{\mu\nu}$ are antisymmetric in their indices.

Note: For every pair μ, ν, the object $\mathcal{J}^{\mu\nu}$ is a real 4×4 matrix, given by Eq. (10).

Convince yourself that this basis can be used to construct all matrices of the form (9) by setting one of the parameters to one and all others to zero. Therefore, the antisymmetric coefficient tensor $\omega_{\mu\nu}$ takes the place of ξ_i and encodes 6 degrees of freedom.

*c) Show that the commutator (Lie bracket) of the generators is **+1**pt(s)

$$([\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}])^{\rho}{}_{\sigma} = i\left(\eta^{\nu\alpha}\mathcal{J}^{\mu\beta} - \eta^{\nu\beta}\mathcal{J}^{\mu\alpha} - \eta^{\mu\alpha}\mathcal{J}^{\nu\beta} + \eta^{\mu\beta}\mathcal{J}^{\nu\alpha}\right)^{\rho}{}_{\sigma} . \tag{12}$$

This is the *Lie algebra* of the Lorentz group; it has to be the same for all its representations.

d) Calculate the determinant of Λ first from the definition (7), and then from the expression (8) using the form of the generators. **1**pt(s)

Can the exponential form $\Lambda_{\boldsymbol{\xi}}$ (equivalently: $\Lambda_{\boldsymbol{\omega}}$) be used to generate the complete group $O(1,3)$?

e) Convince yourself that time reversal T and space inversion P are part of the Lorentz group (7) but *cannot* be continuously connected to the identity $\Lambda_{\boldsymbol{\xi}=\mathbf{0}} = \mathbb{1}$: **2**pt(s)

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{13}$$

Finally, prove that T and P cannot be continuously connected to each other as well.

Interpret your result by sketching the structure of the Lorentz group.

Hint: To prove that T and P are not connected consider the metric element η_{00} and its transformation under the Lorentz group.