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Problem 4.1: Hafele-Keating experiment

[**Oral** | 4 pt(s)]

ID: ex_hafele_keating_experiment:rt2526

Learning objective

Time dilation is a relativistic effect far removed from our everyday experience. It is, however, an experimentally established fact. A famous experiment measuring time dilation explicitly was the *Hafele-Keating experiment*, where portable atomic clocks were flown on commercial airliners around the world twice: once eastward and once westward. The clocks were then compared to stationary reference clocks on the ground to verify the predictions of time dilation quantitatively.

The experimental results were reported in

https://doi.itp3.info/10.1126/science.177.4044.168

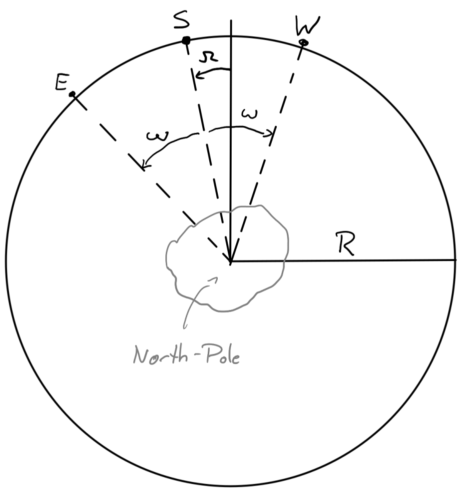
and the theory was developed in

https://doi.itp3.info/10.1126/science.177.4044.166

In this exercise, you derive the contribution of time dilation to explain the result of the experiment (which you can find in the theory paper above).

We consider the following setup:

Imagine an observer in a space station above the north pole; the space station follows earth on its orbit around the sun, but does not follow the rotation of earth (i.e., the observer sees earth slowly rotating beneath the space station). Such an observer is approximately inertial for the relevant timescales of the experiment (see sketch on the right); in particular, he is allowed to use the formalism developed in the lecture to compute the proper time along (potentially accelerated) trajectories.



Now consider three identical atomic clocks located on the equator with radius $R \approx 6.4 \times 10^6$ m. The first clock (labeled S) is stationary with respect to earth, this is our "reference clock." The second clock (E) flies eastward around earth with angular velocity ω (with respect to earth), and the third (W) westward with the same angular velocity ω (also with respect to earth). Both clocks go around earth once and meet again with the reference clock. Note that the rest system of the reference clock S is *not* inertial because earth rotates with angular velocity $\Omega = \frac{2\pi}{24\,\mathrm{h}}$.

a) Parametrize the trajectories of the three clocks $\boldsymbol{x}_S(t)$, $\boldsymbol{x}_E(t)$ and $\boldsymbol{x}_W(t)$ in the inertial system of the space station. 1pt(s)

b) The proper time τ accumulated by a clock can be calculated as shown in the lecture: **1**pt(s)

$$\tau_i = \int \sqrt{1 - \frac{\dot{\boldsymbol{x}}_i^{\,2}}{c^2}}\,\mathrm{d}t \qquad \text{for} \quad i \in \{S, E, W\}\,. \tag{1}$$

Evaluate this integral for the three clocks.

c) Calculate $\Delta\tau_E = \tau_E - \tau_S$ and $\Delta\tau_W = \tau_W - \tau_S$. **1**pt(s)

Since the angular velocities are small, expand the results in $\frac{R\Omega}{c}$ up to second order.

d) Assume the clocks travel around earth with $200\,\mathrm{m\,s^{-1}}$ relative to the ground (the speed of a typical airliner). **1**pt(s)

What are the time differences measured? Compare them to the numbers reported in the original publications.

Note: For a complete description of the experiment, effects of *general relativity* must be taken into account as well. Thus, we will complete our analysis of the Hafele-Keating experiment in the next semester.

Problem 4.2: Tensor Calculus [**Written** | 11 pt(s)]

ID: ex_tensor_calculus:rt2526

> **Learning objective**
>
> Tensor calculus is a crucial toolkit for special and general relativity. In this exercise, you practice calculating with tensor fields and prove some useful rules for the construction of tensor fields.

Consider a D-dimensional differentiable manifold M and an arbitrary coordinate transformation $\bar{x} = \varphi(x)$ from one chart with coordinates $x \in \mathbb{R}^D$ to another chart with coordinates $\bar{x} \in \mathbb{R}^D$.

As motivated in the lecture, we define the transformation of *contravariant* and *covariant* vector (fields) as follows:

$$\textit{Contravariant}\text{ vector field:} \quad \bar{A}^i(\bar{x}) = \sum_{k=1}^{D} \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \equiv \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \tag{2}$$

$$\textit{Covariant}\text{ vector field:} \quad \bar{B}_i(\bar{x}) = \sum_{k=1}^{D} \frac{\partial x^k}{\partial \bar{x}^i} B_k(x) \equiv \frac{\partial x^k}{\partial \bar{x}^i} B_k(x)\,, \tag{3}$$

Here we use the *Einstein sum convention*: Sums over pairs of repeated up and down indices are implied but not explicitly written.

a) Prove that the *contraction* $\Phi(x) := A^i(x)B_i(x)$ of a contravariant vector field $A^i(x)$ with a covariant vector field $B_i(x)$ is invariant under coordinate transformations; i.e., show that it transforms like a *scalar field*. **1**pt(s)

The generalization of co- and contravariant vector fields are (mixed) (p,q) *tensor fields* $T^{m_1,\dots,m_p}{}_{n_1,\dots,n_q}(x)$ with $r = p+q$ indices (called *rank*). Like vector fields, tensor fields are defined by their transformation under coordinate transformations:

$$\bar{T}^{i_1,\dots,i_p}{}_{j_1,\dots,j_q}(\bar{x}) = \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}} \quad \frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}} \quad T^{m_1,\dots,m_p}{}_{n_1,\dots,n_q}(x)\,, \tag{4}$$

where p and q are the number of contravariant and covariant indices, respectively.

b) Show that the following combinations of the tensor fields $A^{ij}{}_k$, $B^{ij}{}_k$, C^{ij}, $D^k{}_l$, E_m and the scalar field Φ are again tensor fields (we suppress the x-dependency): **5pt(s)**

i) $V^{ij}{}_k := A^{ij}{}_k + B^{ij}{}_k$ ii) $W^{ij}{}_k := \Phi A^{ij}{}_k$ iii) $X^{ijk}{}_l := C^{ij} D^k{}_l$
iv) $Y^i := A^{ij} E_j$ v) $Z^i := A^{ij}{}_j$

c) Let C_{ij} be a collection of D^2 fields ($i, j = 1, \ldots, D$). **1pt(s)**

Prove that *if* $B_i := C_{ij} A^j$ is a covariant vector field for *every* contravariant vector field A^i, *then* C_{ij} transforms like a covariant tensor field of rank 2.

Note: This theorem is called *quotient law*, a quite useful tool in tensor calculus.

The *covariant derivative* of a contravariant vector field is defined as

$$A^i{}_{;k} := \partial_k A^i + \Gamma^i{}_{kl} A^l \,, \tag{5}$$

with $\partial_k \equiv \frac{\partial}{\partial x^k}$ and where the *Christoffel symbol* $\Gamma^i{}_{kl}$ is defined as

$$\Gamma^i{}_{kl} := \frac{1}{2} g^{im} \left(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl} \right) \,. \tag{6}$$

$g_{ij} = g_{ij}(x)$ is a given, symmetric ($g_{ij} = g_{ji}$) covariant tensor field of rank 2 called the *metric*.

d) Show that the covariant derivative $A^i{}_{;k}$ of a contravariant vector field A^i transforms as a mixed $(1,1)$ tensor field. **3pt(s)**

Hint: First, prove the transformation law for the Christoffel symbol

$$\bar{\Gamma}^i{}_{kl}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l} \Gamma^j{}_{mn}(x) + \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} \,, \tag{7}$$

and show that

$$\frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} = - \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l} \,. \tag{8}$$

Use this to derive the transformation law for the covariant derivative.

You might want to use the shortcut notations $\alpha^i_k := \frac{\partial \bar{x}^i}{\partial x^k}$ and $\beta^i_k := \frac{\partial x^i}{\partial \bar{x}^k}$ with $\alpha^i_k \beta^k_j = \delta^i_j$.

In the lecture, the general *Levi-Civita symbol* $\epsilon^{i_1 \cdots i_D}$ was introduced. Here we want to focus on the most important case of a $D = 4$ dimensional manifold.

The Levi-Civita symbol is defined as

$$\epsilon^{ijkl} := \begin{cases} +1 & \text{if } (i,j,k,l) \text{ is an even permutation of } (0,1,2,3) \\ -1 & \text{if } (i,j,k,l) \text{ is an odd permutation of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases} \,. \tag{9}$$

This definition is independent of the coordinate system, meaning $\bar{\epsilon}^{ijkl} = \epsilon^{ijkl}$.

e) Show that with this definition the Levi-Civita symbol is a *relative tensor* of rank 4 with weight $w = +1$ (which we call a *tensor density*), i.e. show that 1pt(s)

$$\epsilon^{ijkl} = \left|\frac{\partial x}{\partial \bar{x}}\right| \frac{\partial \bar{x}^i}{\partial x^a}\frac{\partial \bar{x}^j}{\partial x^b}\frac{\partial \bar{x}^k}{\partial x^c}\frac{\partial \bar{x}^l}{\partial x^d}\epsilon^{abcd} \quad \text{with } \textit{Jacobian determinant } \left|\frac{\partial \bar{x}}{\partial x}\right|. \tag{10}$$

Hint: Use that determinants can be calculated via the Leibniz formula using the Levi-Civita symbol:

$$\left|\frac{\partial \bar{x}}{\partial x}\right| = \epsilon^{ijkl}\frac{\partial \bar{x}^0}{\partial x^i}\frac{\partial \bar{x}^1}{\partial x^j}\frac{\partial \bar{x}^2}{\partial x^k}\frac{\partial \bar{x}^3}{\partial x^l}\,. \tag{11}$$

Problem 4.3: The metric tensor [**Oral** | 5 pt(s)]

ID: ex_srt_metric_coordinate_transformation:rt2526

> **Learning objective**
>
> The metric tensor (field) is crucial for Riemannian geometry, i.e., the mathematical framework needed to described curved spaces. Both in special relativity and in general relativity, the metric tensor on the spacetime manifold determines durations (measured by clocks) and lengths (measured by rods) in spacetime.
>
> In this exercise, you show that the components of the metric tensor indeed transform like a covariant tensor field of rank 2. To familiarize yourself with the concept, you study the simple example of Euclidean space (no time!) in two dimensions $\mathbb{R}^2$ and calculate the components of the metric tensor in polar coordinates.

Consider a D-dimensional differentiable manifold M and an arbitrary coordinate transformation $\bar{x} = \varphi(x)$ from one chart with coordinates $x \in \mathbb{R}^D$ to another chart with coordinates $\bar{x} \in \mathbb{R}^D$.

a) We start by showing that the components $g_{ij}(x)$ of the metric transform like a covariant tensor field of rank 2. To this end, use that the "line element" ds^2 is a tensor field and hence does not depend on the coordinate system: 1pt(s)

$$g_{ij}(x)dx^i dx^j = ds^2 = \bar{g}_{ij}(\bar{x})d\bar{x}^i d\bar{x}^j \quad \text{(Einstein summation!)} \tag{12}$$

Hint: Compute the total differential $d\bar{x}^i$ and use that $\{dx^i\}$ is a basis of the cotangent space T_p^*M. The transformation law for a covariant tensor of rank 2 is given in Problem 4.2.

As an example, we consider Euclidean space $M = \mathbb{R}^2$ in $D = 2$ dimensions for the rest of this exercise. In Cartesian coordinates $x^1 = x$ and $x^2 = y$, the components of the metric tensor are

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with line element} \quad ds^2 = g_{ij}(x)dx^i dx^j = dx^2 + dy^2\,. \tag{13}$$

This particular metric tensor ds^2 characterizes the flat, Euclidean plane you already encountered in school.

b) We want to calculate the components of the metric tensor in polar coordinates $\bar{x}^1 = r$ and 2pt(s)

$\bar{x}^2 = \theta$. The coordinate transformation $\bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x})$ between Cartesian and polar coordinates is given by

$$\varphi^{-1} : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} . \tag{14}$$

First, use Eq. (12) and the rule for computing the total differentials dx and dy to compute $\bar{g}_{ij}(\bar{x})$.

Then, derive the same components directly by using the transformation law for a covariant tensor of rank 2.

The physical length of a curve γ can be calculated via the metric tensor by

$$L[\gamma] = \int_\gamma ds := \int_a^b dt \sqrt{g_{ij}(\boldsymbol{\gamma}(t))\, \dot{\gamma}^i(t)\dot{\gamma}^j(t)}\,, \tag{15}$$

where $\boldsymbol{\gamma}(t)$ is the parametrization of the curve with $t \in [a, b]$ in some coordinate system.

c) Let γ be the circle in $\mathbb{R}^2$ with radius R and center in the origin. **2**pt(s)

Calculate the circumference $L[\gamma]$ of this circle once in Cartesian coordinates and then again in polar coordinates. Use the components of the metric tensor given above and computed in subtask b).