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October 15th, 2025
WS 2025/26

Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Relativitätstheorie 1"***:

- The C@MPUS-ID of this course is 043700001.
- You can find detailed information on lecture and tutorials on the website of our institute:

  `https://itp3.info/rt2526`
- You can also find detailed information on lecture and tutorials on ILIAS:

  `https://ilias3.uni-stuttgart.de/go/crs/4202436`
- **Written** problems have to be handed in via ILIAS and will be corrected by the tutors. You must earn at least **66 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **1** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Einstein Synchronization

[**Oral** | 3 pt(s)]

ID: ex_einstein_synchronization:rt2526

Learning objective

In this exercise, you study the conventional procedure to synchronize distant clocks in inertial frames known as *Einstein synchronization*. Using some assumptions, you show that it defines an equivalence relation (which makes it a reasonable synchronization convention). The procedure is described in Einsteins famous paper "Zur Elektrodynamik bewegter Körper," which you are encouraged to have a look at.

Suppose there are two identical clocks A and B at distant positions. You are at clock A and your goal is to synchronize it with clock B (which we take as a reference clock).

The procedure of Einstein synchronization (ES) is then defined as follows:

1. Clock A emits a light pulse at its time t_{Ae} to clock B, which receives the signal at its time t_B.
2. Clock B immediately reflects the signal together with the timestamp t_B (encoded in the reflected light signal).
3. Finally, clock A receives this signal at its time t_{Ar}.

Located at clock A, you are now in possession of the three times t_{Ae}, t_B, and t_{Ar}. We declare that clock A is synchronized with the reference clock B (write $A \sim_{\text{ES}} B$), if and only if

$$t_B - t_{Ae} = t_{Ar} - t_B \,. \tag{1}$$

Can you explain the idea behind this condition? How would you measure the speed of light in one direction, e.g. from A to B?

Now show that $A \sim_{\text{ES}} B$ is an *equivalence relation*; i.e., show that

$$\text{Reflexivity:} \quad A \sim_{\text{ES}} A \,, \tag{2a}$$
$$\text{Symmetry:} \quad A \sim_{\text{ES}} B \;\Leftrightarrow\; B \sim_{\text{ES}} A \,, \tag{2b}$$
$$\text{Transitivity:} \quad A \sim_{\text{ES}} B \;\wedge\; B \sim_{\text{ES}} C \;\Rightarrow A \sim_{\text{ES}} C \,. \tag{2c}$$

Hint: Use the following assumptions:

(1) Two light pulses sent from A with a time difference Δt_A arrive at B with the same time difference of $\Delta t_B = \Delta t_A$.

(2) Light takes the same time to travel the triangular path $A \to B \to C \to A$ as the reversed path $A \to C \to B \to A$ between three clocks A, B, and C.

Problem 1.2: Groups and Representations

[**Oral** | 7 pt(s)]

ID: ex_groups_and_representations:rt2526

Learning objective

In this exercise, you repeat the important mathematical concepts of groups and their representations. Via simple examples, you study basic properties of groups and learn how to combine them to build larger groups from smaller ones. Groups and their representations are important concepts in all areas of physics as they are the tools to describe symmetries of physical systems. In this course on relativity, we will first encounter the Galilei group, and later the Lorentz- and Poincaré groups (which describe the symmetries of the flat spacetime of special relativity).

We shall first recall some basic facts about groups and their representations.

A set $\mathcal{G}$ together with a binary map (usually called multiplication) $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is called a group iff the following three axioms are fulfilled:

- **Associativity:** For any three elements $g, g', g'' \in \mathcal{G}$ it holds that $g \cdot (g' \cdot g'') = (g \cdot g') \cdot g''$.
- **Neutral Element:** There is an element $e \in \mathcal{G}$ with the property $g \cdot e = g = e \cdot g$ for any $g \in \mathcal{G}$. This element is called the *neutral element.*
- **Inverse Element:** For every element $g \in \mathcal{G}$ there exists an element denoted $g^{-1} \in \mathcal{G}$ with the property $g^{-1} \cdot g = e = g \cdot g^{-1}$. The element g^{-1} is called the *inverse* of g.

A group is called **abelian** if $g \cdot g' = g' \cdot g$ holds for all $g, g' \in \mathcal{G}$.

Groups are rather abstract objects, defined completely in terms of their multiplication rules. In physics, however, groups typically represent some transformation of the states of a system (like the translation of a quantum state or the rotation of a field configuration). This is why we are interested in *representations* of groups:

A **representation** ρ on a given vector space V is a map $\rho : \mathcal{G} \to \mathrm{GL}(V)$ with the property

$$\rho(g \cdot g') = \rho(g)\rho(g') \,. \tag{3}$$

Here $\mathrm{GL}(V)$ is the set of invertible matrices acting on the vector space V by matrix-vector multiplication (which is also a group in itself).

a) Consider the sets $\mathcal{G} = [0, 2\pi)$ and $\mathcal{H} = \{e, h\}$ with binary maps defined as **2pt(s)**

$$\theta \oplus \theta' = \theta + \theta' \bmod 2\pi \quad \text{and} \quad \begin{array}{c|cc} * & e & h \\ \hline e & e & h \\ h & h & e \end{array} \,. \tag{4}$$

Show that $(\mathcal{G}, \oplus)$ and $(\mathcal{H}, *)$ are groups. Are they abelian?

b) We now consider the two-dimensional vector space $V = \mathbb{R}^2$. Show that the matrices **3pt(s)**

$$G(\theta) := \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \quad \text{and} \quad \left\{ E := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}, \tag{5}$$

define a representation of $(\mathcal{G}, \oplus)$ and $(\mathcal{H}, *)$, respectively.

Can you find a different representation of $(\mathcal{H}, *)$ on this vector space?

We can construct a new group from any two groups $(\mathcal{G}, \oplus)$ and $(\mathcal{H}, *)$ using the direct product $\times$. A new element of this group is simply the tuple $(g, h) \in \mathcal{G} \times \mathcal{H}$ and the multiplication in this new group is defined via $(g, h) \circ (g', h') = (g \oplus g', h * h')$. Unfortunately, many groups (including the Galilei and the Poincaré group) are not of this simple form.

The (outer) *semidirect product* $\mathcal{G} \rtimes_\varphi \mathcal{H}$ is a more versatile generalization of the direct product of groups. Here the multiplication is defined via

$$(g, h) \bullet (g', h') := (g \oplus \varphi_h(g'), h * h') \tag{6}$$

with a fixed function $\varphi_h(g')$ that, for each group element $h \in \mathcal{H}$, yields a map from the group $\mathcal{G}$ into itself (which is consistent with the multiplication on $\mathcal{G}$, a so-called *group automorphism*).

To illustrate this rather abstract concept, let us use our two simple groups $\mathcal{G}$ and $\mathcal{H}$ from above:

c) Intuitively, we can combine the two representations by simple matrix multiplication: **2pt(s)**

$$\mathcal{L} = \{ G(\theta)E, G(\theta)H \,|\, \theta \in [0, 2\pi) \} \,. \tag{7}$$

Show that this set of matrices is (a representation of) the semidirect product $\mathcal{G} \rtimes_\varphi \mathcal{H}$ with the map

$$\varphi_e(\theta) = \theta, \quad \varphi_h(\theta) = 2\pi - \theta, \tag{8}$$

by showing that the multiplication law is fulfilled.

Note: With this you have shown that the orthogonal group $\mathcal{L} \cong \mathrm{O}(2)$ of rotations and reflections in the 2D plane is given by the semidirect product $\mathrm{SO}(2) \rtimes \mathbb{Z}_2$ of the group $\mathcal{G} \cong \mathrm{SO}(2)$ of rotations and the two-element group $\mathcal{H} \cong \mathbb{Z}_2$ of reflections (at one axis).

Problem 1.3: Galilei group and Galiei covariance

[**Written** | 8 (+5 bonus) pt(s)]

ID: ex_galilei_group_and_invariance:rt2526

> **Learning objective**
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> The goal of this exercise is to get familiar with the *Galilei group* – the spacetime symmetry group of Newtonian mechanics. First, you show that it is indeed a group. Then you focus on the representation of the group on four-dimensional Galilean spacetime and show that under these coordinate transformations Newton's equation and the Schrödinger equation are *form-invariant* (they are Galilei-covariant). By contrast, you find that the Maxwell equations do not have this property (we will of course discover later that they are instead Lorentz-covariant).

All elements g of the (proper orthochronous) Galilei group $\mathcal{G}_+^\uparrow$ can be uniquely expressed as a combination of a rotation $R \in SO(3)$, a boost with velocity $\boldsymbol{v} \in \mathbb{R}^3$, a time shift $s \in \mathbb{R}$, and a translation $\boldsymbol{b} \in \mathbb{R}^3$ in space.

The multiplication law of the Galilei group is then given by

$$g(R', \boldsymbol{v}', s', \boldsymbol{b}') \cdot g(R, \boldsymbol{v}, s, \boldsymbol{b}) = g(R'R,\ R'\boldsymbol{v} + \boldsymbol{v}',\ s' + s,\ R'\boldsymbol{b} + \boldsymbol{b}' + \boldsymbol{v}'s). \tag{9}$$

Note: This multiplication law makes the Galilei group a semidirect product of the form

$$\mathcal{G}_+^\uparrow \cong \mathbb{R}^4 \rtimes \left(\mathbb{R}^3 \rtimes SO(3)\right), \tag{10}$$

where $\mathbb{R}^4$ denotes the translation group (in spacetime), $\mathbb{R}^3$ describes the boosts, and SO(3) is the group of rotations in 3D space. The term "proper orthochronous" refers to the fact that reflections in space and time are *not* included.

a) Show that the Galilei group is actually a group. Is it abelian? **3**pt(s)

Consider two inertial systems K and K' with coordinates $(t, \boldsymbol{x})$ and $(t', \boldsymbol{x}')$, respectively.

In Newtonian mechanics, the two systems are related via a Galilei transformation

$$\begin{pmatrix} t \\ \boldsymbol{x} \end{pmatrix} \mapsto \begin{pmatrix} t' \\ \boldsymbol{x}' \end{pmatrix} = \begin{pmatrix} t + s \\ R\boldsymbol{x} + \boldsymbol{v}_0 t + \boldsymbol{b} \end{pmatrix}. \tag{11}$$

This transformation is a representation of the group $\mathcal{G}_+^\uparrow$ on the four-dimensional Galilean spacetime $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$.

b) Newton's equations of motion for N particles read **2**pt(s)

$$m_i \frac{d^2 \boldsymbol{x}_i}{dt^2} = \boldsymbol{F}_i(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N) = \sum_{j \neq i} \frac{\boldsymbol{x}_i - \boldsymbol{x}_j}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} f(|\boldsymbol{x}_i - \boldsymbol{x}_j|). \tag{12}$$

Show that in the new frame K' the equations have the same form; that is, show that Newton's equations are *covariant* under Galilei transformations.

For the sake of simplicity, we consider only boosts from now on (i.e., set $R = \mathbb{1}$, $\boldsymbol{b} = \boldsymbol{0}$, and $s = 0$).

c) The Schrödinger equation for a free particle of mass m in the coordinate system K is given by **3**pt(s)

$$i\hbar\partial_t\Psi(\boldsymbol{x},t) = -\frac{\hbar^2}{2m}\Delta\Psi(\boldsymbol{x},t). \tag{13}$$

The Galilei boost $K \xrightarrow{\boldsymbol{v}_0} K'$ is represented on the wave function as

$$\Psi(\boldsymbol{x},t) \longrightarrow \Psi'(\boldsymbol{x}',t') = \mathrm{e}^{if_{\boldsymbol{v}_0}(\boldsymbol{x}',t')}\,\Psi(\boldsymbol{x}'-\boldsymbol{v}_0t',t'), \tag{14}$$

where f is a real-valued function that depends on space, time and the velocity of the boost.

Determine the function f such that $\Psi'(\boldsymbol{x}',t')$ fulfills the Schrödinger equation in K'. Make sure that Eq. (14) with the f you found constitutes a representation of Galilei boosts.

With this you have shown that the Schrödinger equation transforms Galilei-covariant, just as the equations of classical mechanics.

Finally, we want to show that Maxwell's equations are *not* form-invariant under Galilei transformations. To this end, we must first figure out how the magnetic and electric fields transform under Galilei transformations.

We would like classical mechanics and electrodynamics to transform consistently. In classical mechanics, forces transform trivially under Galilei boosts: $\boldsymbol{F} = \boldsymbol{F}'$ (note that we do not rotate, $R = \mathbb{1}$). Thus, we demand that the Lorentz force

$$\boldsymbol{F} = q(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}) \tag{15}$$

on a charged particle with velocity $\boldsymbol{v}$ in frame K is also Galilei invariant.

*d) Show that these assumptions demand the transformation laws **+1**pt(s)

$$\boldsymbol{B}'(\boldsymbol{x}',t') = \boldsymbol{B}(\boldsymbol{x},t), \quad \boldsymbol{E}'(\boldsymbol{x}',t') = \boldsymbol{E}(\boldsymbol{x},t) - \frac{\boldsymbol{v}_0}{c} \times \boldsymbol{B}(\boldsymbol{x},t), \tag{16}$$

for the electric and magnetic fields under a Galilei boost $K \xrightarrow{\boldsymbol{v}_0} K'$.

*e) Show that Maxwell's equations in vacuum **+4**pt(s)

$$\text{Gauss's law (electric):} \quad \nabla\cdot\boldsymbol{E} = 0 \tag{17a}$$

$$\text{Gauss's law (magnetic):} \quad \nabla\cdot\boldsymbol{B} = 0 \tag{17b}$$

$$\text{Law of induction:} \quad \nabla\times\boldsymbol{E} = -\frac{1}{c}\partial_t\boldsymbol{B} \tag{17c}$$

$$\text{Ampère's circuital law:} \quad \nabla\times\boldsymbol{B} = \frac{1}{c}\partial_t\boldsymbol{E} \tag{17d}$$

are *not* form-invariant under Galilei boosts. Is this true for all four equations?