

ii | Remember: Quantum mechanics

In quantum mechanics, the state of a physical system is described by \downarrow *state vectors* in some Hilbert space \mathcal{H} (which is a special kind of vector space). Vectors in this space are written as \downarrow *kets*: $|\Psi\rangle \in \mathcal{H}$. You can produce a \downarrow *bra* $\langle\Psi| = |\Psi\rangle^\dagger$ by applying the complex transpose operator. As in the example above, the bra $\langle\Psi|$ is a covector from the dual space \mathcal{H}^* ; indeed, it acts as a linear functional on state vectors via the inner product of the Hilbert space:

$$\langle\Psi||\Phi\rangle := \langle\Psi|\Phi\rangle \in \mathbb{C}. \tag{3.12}$$

This is the gist of the famous \downarrow *Dirac bra-ket notation*.

\downarrow **Lecture 9** [10.12.25]

iii | Hopefully these examples convinced you that the dual space is just as important and useful as the vector space itself.

→ Dual space of the tangent space $T_p M$?

Given a coordinate basis $\{\partial_i\} \in T_p M$ of a vector space, there is a standard way to define a basis of the dual space $T_p^* M$:

\downarrow *Dual basis* $\{dx^i\}$ with

$$dx^i(\partial_j) := \delta_j^i = \frac{\partial x^i}{\partial x^j} \tag{3.13}$$

→ $\{dx^i\}$ is a basis of the $\ast\ast$ *Cotangent space* $T_p^* M$

$T_p^* M$ is the dual space of $T_p M$; it is common to write $T_p^* M$ and not $(T_p M)^\ast$.

13 | Since $T_p^* M$ is just another vector space for each point p of the manifold M , we can again define *fields* on M that map into this space:

$\ast\ast$ *Covector field*: $B(p) = \sum_{i=1}^D B_i(x) dx^i$ with $x = u(p)$

14 | Just like the coordinate basis, the dual coordinate basis depends on the chart and changes under coordinate transformations:

\triangleleft Coordinate transformation $\bar{x} = \varphi(x)$:

$$d\bar{x}^i = \sum_{k=1}^D \frac{\partial \bar{x}^i}{\partial x^k} dx^k \tag{3.14}$$

- Check that this is the correct transformation for the dual coordinate basis:

$$\begin{aligned} d\bar{x}^i(\bar{\partial}_j) &= \left[\sum_k \frac{\partial \bar{x}^i}{\partial x^k} dx^k \right] \left(\sum_l \frac{\partial x^l}{\partial \bar{x}^j} \partial_l \right) \\ &= \sum_{k,l} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^j} \underbrace{dx^k(\partial_l)}_{\delta_l^k} = \sum_k \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^j} = \delta_j^i \quad \odot \end{aligned} \tag{3.15}$$

- You might recognize Eq. (3.14): This is simply the rule to compute the \downarrow *total differential* of the function $\bar{x} = \varphi(x)$. This is no coincidence and explains why we use the differential notation dx^i for the dual vectors: The objects dx^i that we physicists like to illustrate as “infinitesimal shifts” in x^i are actually *linear functionals* (\uparrow *1-forms*).

15 | Now we can play the same game on T_p^*M as before on T_pM :

◁ Covector field B and expand it in different dual coordinate bases:

$$\sum_i B_i(x) dx^i = B(p) = \sum_i \bar{B}_i(\bar{x}) d\bar{x}^i \quad (3.16)$$

with $x = u(p)$ and $\bar{x} = v(p)$.

! The covector field B is another geometric object, just as the vector field A was. That it does *not* depend on the chosen chart is the statement of this equation.

Eq. (3.16) →

$$B = \sum_i B_i(x) dx^i \stackrel{!}{=} \sum_i \bar{B}_i(\bar{x}) d\bar{x}^i \stackrel{\text{Eq. (3.14)}}{=} \sum_k \underbrace{\left[\sum_i \frac{\partial \bar{x}^i}{\partial x^k} \bar{B}_i(\bar{x}) \right]}_{\stackrel{!}{=} B_k(x)} dx^k \quad (3.17)$$

This motivates the following definition (we replace $x \leftrightarrow \bar{x}$ and the indices $i \leftrightarrow k$):

16 | ◁ D -tuple $\{B_i(x)\}$ of fields (in some chart with coordinates x):

$$** \text{ Covariant vector field } \{B_i(x)\} \quad :\Leftrightarrow \quad \bar{B}_i(\bar{x}) = \sum_{k=1}^D \frac{\partial x^k}{\partial \bar{x}^i} B_k(x) \quad (3.18)$$

Covariant vector (field) → *Subscript* indices!

- The terms *covariant vector (field)* and *covector (field)* are used synonymously.
- The rationale of Eq. (3.18) is the same as that of Eq. (3.8): Whenever we find a family of fields that transform under coordinate transformations as Eq. (3.18), we immediately know that together they encode a geometric, chart-independent object on the manifold that can be used to describe a physical quantity. To indicate that this object is a *covariant* vector field, we use *subscript* indices.

17 | Example:

First, let us introduce an even shorter notation for partial derivatives: $\Phi_{,i} \equiv \partial_i \Phi$

Following our index convention, the lower index in these expressions is only warranted *if* the field transforms as a covariant vector field according to Eq. (3.18). Let us check this:

$$\bar{\Phi}_{,i}(\bar{x}) = \bar{\partial}_i \bar{\Phi}(\bar{x}) \stackrel{\text{Eq. (3.4)}}{\stackrel{\text{Eq. (3.5)}}{=}} \sum_{k=1}^D \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial \Phi(x)}{\partial x^k} = \sum_{k=1}^D \frac{\partial x^k}{\partial \bar{x}^i} \Phi_{,k}(x) \quad (3.19)$$

→ The gradient of a scalar is a covariant vector field.

18 | What happens if we apply a covector field on a vector field at each point $p \in M$?

$$\phi(p) := B(p)A(p) = \sum_{i,j} B_i(x) A^j(x) \underbrace{dx^i(\partial_j)}_{\delta_j^i} = \sum_i A^i(x) B_i(x) =: \Phi(x) \quad (3.20)$$

→ $\Phi(x)$ must be a scalar!

This is a good point to introduce a new (and very convenient) notation:

** *Einstein sum convention*:

$$\sum_{i=1}^D A^i(x) B_i(x) \equiv \underbrace{A^i(x) B_i(x)}_{\substack{\text{** Einstein summation} \\ \text{** Contraction}}} = A^I(x) B_I(x) \quad (3.21)$$

The *Einstein sum convention* or *Einstein summation* is a syntactic convention according to which a sum is automatically implied (but not written) whenever two indices show up twice in an expression and one is up (contravariant) and one down (covariant). Note that such indices are “dummy indices” in the sense that you can rename them to whatever you want (as long as you do not use the same letter for other indices already!). The sum over one co- and one contravariant index is called a *contraction*.

With this new notation it is straightforward to check that Φ transforms according to Eq. (3.4) by using the transformations Eq. (3.8) and Eq. (3.18):

$$\bar{\Phi}(\bar{x}) = \bar{A}^i(\bar{x}) \bar{B}_i(\bar{x}) = \left[\frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \right] \left[\frac{\partial x^l}{\partial \bar{x}^i} B_l(x) \right] \quad (3.22a)$$

$$= \underbrace{\frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^i}}_{\text{Chain rule} \rightarrow \delta_k^l} A^k(x) B_l(x) = A^I(x) B_I(x) = \Phi(x) \quad (3.22b)$$

The intermediate expression contains *three* sums over the colored indices (which we don’t write)!

→ The contraction of a contra- and a covariant vector field yields a scalar field.

19 | Note on nomenclature:

- If you compare Eq. (3.18) with Eq. (3.5) you find that the *components* B_i of a covector field transform like the *basis vectors* ∂_i of the tangent space. We say the components *covary* (“vary together”) with the basis. This is why they are called *covariant*.
- A comparison of Eq. (3.8) and Eq. (3.14) shows that the components A^i of a vector field transform like the basis dx^i of the cotangent space – which is the *inverse* (“opposite”) transformation as for the basis of the tangent space ∂_i . Thus we say the components A^i *contravary* (“vary opposite to”) the basis ∂_i . This is why they are called *contravariant*.

3.4. Higher-rank tensors

You learned in your linear algebra course that two vector spaces V and W can be used to construct a new vector space $V \otimes W$ called the \downarrow *tensor product*. This allows us to generalize the notion of contra- and covariant *vector* fields to *tensor* fields, all of which are geometric, chart-independent objects defined on the manifold that are needed to describe physical quantities:

20 | An ** (*absolute*) (p, q) -*tensor (field)* T of rank $r = p + q$

$$T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \equiv T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) \quad \text{or} \quad T^I_J \equiv T^I_J(x), \quad (3.23)$$

with \downarrow *multi-indices* $I = (i_1 \dots i_p)$ and $J = (j_1 \dots j_q)$,

transforms like the tensor product of p contravariant and q covariant vector fields:

$$\overbrace{\bar{T}^{i_1 \dots i_p}_{j_1 \dots j_q}(\bar{x})}^{=:\bar{T}^I_J(\bar{x})} = \underbrace{\left[\frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \dots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}} \right]}_{=:\frac{\partial \bar{x}^I}{\partial x^M}} \underbrace{\left[\frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \dots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}} \right]}_{=:\frac{\partial x^N}{\partial \bar{x}^J}} \underbrace{T^{m_1 \dots m_p}_{n_1 \dots n_q}(x)}_{=:T^M_N(x)} \quad (3.24)$$

There are $r = p + q$ sums in this transformation rule (Einstein summation!).

- $!$ It is important that we do *not* write contra- and covariant indices above each other like so: T_j^i (at least not with additional knowledge about the tensor). This will become important below.
- Henceforth we always encode tensor fields by their chart-dependent *components*. The actual tensor field is of course chart-independent and maps each point $p \in M$ to an element of the tensor product

$$\underbrace{T_p M \otimes \dots \otimes T_p M}_{p \text{ factors}} \otimes \underbrace{T_p^* M \otimes \dots \otimes T_p^* M}_{q \text{ factors}}. \quad (3.25)$$

like so

$$T(p) = \sum_{I,J} T^{i_1 \dots i_p}_{j_1 \dots j_q}(x) \partial_{i_1} \otimes \dots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}. \quad (3.26)$$

- Note that while tensors (more precisely: tensor components) are indicated by upper and lower indices (corresponding to their rank), not every object that is conventionally written with upper and lower indices does encode a tensor. For example, the transformation matrices $\frac{\partial \bar{x}^i}{\partial x^m}$, which describe a basis change on $T_p^* M$, do not encode a tensor field.

21 | Examples:

- Scalar $\Phi(x) \rightarrow (0, 0)$ -tensor
- Contravariant vector $A^i(x) \rightarrow (1, 0)$ -tensor
- Covariant vector $B_i(x) \rightarrow (0, 1)$ -tensor
- Tensor product $T^i_j(x) := A^i(x)B_j(x) \rightarrow (1, 1)$ -tensor (Check this!)

22 | Properties:

- *Equality:*

$$A = B \quad :\Leftrightarrow \quad \forall_{i_1 \dots i_p} \forall_{j_1 \dots j_q} : A^{i_1 \dots i_p}_{j_1 \dots j_q} = B^{i_1 \dots i_p}_{j_1 \dots j_q} \quad (3.27)$$

- *Symmetry:*

$$T \text{ (anti-)symmetric in } k \text{ and } l \quad :\Leftrightarrow \quad T^{\dots k \dots l \dots} = (-1) T^{\dots l \dots k \dots} \quad (3.28)$$

Every contra- or covariant rank-2 tensor can be decomposed into a sum of symmetric and antisymmetric tensors:

$$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{=:T_{(ij)}} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{=:T_{[ij]}} = T_{(ij)} + T_{[ij]}. \quad (3.29)$$

23 | Constructing tensors:

New tensors can be constructed from known tensors as follows (Proofs: → Problemset 4):

- Sum of (p, q) -tensors A and B yields (p, q) -tensor C :

$$C^{i_1 \dots i_p}_{j_1 \dots j_q} := A^{i_1 \dots i_p}_{j_1 \dots j_q} + B^{i_1 \dots i_p}_{j_1 \dots j_q} \quad (3.30a)$$

$$\text{or } C^I_J := A^I_J + B^I_J \quad (3.30b)$$

- Product of (p, q) -tensor A and scalar Φ yields (p, q) -tensor C :

$$C^I_J := \Phi A^I_J \quad (3.31)$$

- Tensor product of (p, q) -tensor A and (r, s) -tensor B yields $(p + r, q + s)$ -tensor C :

$$C^{IK}_{JL} := A^I_J \cdot B^K_L \quad (3.32)$$

- Contractions:

Summing over a pair of contra- and covariant indices yields a tensor of rank $(p - 1, q - 1)$:

$$\tilde{A}^{i_1 \dots \bullet \dots i_p}_{j_1 \dots \bullet \dots j_q} := A^{i_1 \dots k \dots i_p}_{j_1 \dots k \dots j_q} \quad (3.33)$$

The \bullet indicates that the index summed over on the right side is missing in the list.

Proof: → Problemset 4

A special case of a contraction (in combination with a tensor product) is the scalar obtained from a contra- and a covariant vector field above:

$$\Phi = C^i_i = A^i B_i. \quad (3.34)$$

- Quotient theorem:

$$\overline{AB} = C \text{ tensor for all tensors } B \Rightarrow A \text{ is tensor} \quad (3.35)$$

Here, \overline{AB} denotes (potentially multiple) contractions between indices of A and B (but not within A and B).

- As an example, rewrite an arbitrary contravariant vector A^i as $A^i = \delta^i_j A^j$ with Kronecker symbol δ^i_j . The above theorem then implies that δ^i_j transforms as a $(1, 1)$ -tensor (verify this using the definition!). Hence we actually should write δ^i_j instead of δ^i_j . However, because the Kronecker symbol is symmetric in its indices, this simplified notation is allowed (→ later).

- Special case:

$$A_{ik} B^k = C_i \text{ covector for all vectors } B^k \Rightarrow A_{ik} \text{ is } (0, 2)\text{-tensor} \quad (3.36)$$

Proof: → Problemset 4

24 | Relative tensors:

- i | Relative tensors are a generalization of the (absolute) tensors defined above. This generalization is useful because most of the rules for computing with tensors discussed so far carry over to relative tensors.

A $\star\star$ relative tensor of weight $w \in \mathbb{Z}$ picks up an additional power w of the \downarrow Jacobian determinant under coordinate transformations:

$$\bar{R}^I{}_J(\bar{x}) = \det\left(\frac{\partial x}{\partial \bar{x}}\right)^w \frac{\partial \bar{x}^I}{\partial x^M} \frac{\partial x^N}{\partial \bar{x}^J} R^M{}_N(x) \quad \text{with weight } w \in \mathbb{Z} \quad (3.37)$$

and Jacobian determinant

$$\det\left(\frac{\partial x}{\partial \bar{x}}\right) := \sum_{\sigma \in S_D} (-1)^\sigma \prod_{i=1}^D \frac{\partial x^i}{\partial \bar{x}^{\sigma_j}}. \quad (3.38)$$

Here S_D is the group of permutations σ on D elements.

Since $\bar{x} = \varphi(x)$ is invertible, $x = \varphi^{-1}(\bar{x})$, it is $\frac{\partial \bar{x}}{\partial x} = \left(\frac{\partial x}{\partial \bar{x}}\right)^{-1}$ and therefore $\det\left(\frac{\partial \bar{x}}{\partial x}\right) = \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{-1}$.

ii | Examples:

- (Absolute) tensors \equiv Relative tensors of weight $w = 0$
- Volume form: Relative tensor of weight $w = -1$:

$$d^{D\bar{x}} = d^Dx \det\left(\frac{\partial \bar{x}}{\partial x}\right) = d^Dx \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{-1} \quad (3.39)$$

Remember the rule for integration by substitution with multiple variables!

- $\star\star$ Tensor density $\mathcal{L}(x) :=$ Relative tensor of weight $w = +1 \rightarrow$

$$S = \int \underbrace{d^Dx \mathcal{L}(x)}_{\text{Absolute tensor}} = \int d^{D\bar{x}} \bar{\mathcal{L}}(\bar{x}) \quad (3.40)$$

In this example, we assume that $\mathcal{L}(x)$ is a scalar tensor density such that its integral is a (absolute) scalar quantity.

In \uparrow relativistic field theories (like electrodynamics, \ominus Problemset 5), the Lagrangian density $\mathcal{L}(x)$ is a scalar tensor density such that the \downarrow action S becomes a scalar.

- Let $i_1, i_2, \dots, i_D \in \{1, 2, \dots, D\}$ and define the $\star\star$ Levi-Civita symbol as

$$\varepsilon^I \equiv \varepsilon^{i_1 i_2 \dots i_D} := \begin{cases} +1 & I \text{ even permutation of } 1, 2, \dots, D \\ -1 & I \text{ odd permutation of } 1, 2, \dots, D \\ 0 & \text{(at least) two indices equal} \end{cases} \quad (3.41)$$

An even (odd) permutation of $1, 2, \dots, D$ is constructed by an even (odd) number of transpositions (= exchanges of only two indices).

→

$$\bar{\varepsilon}^I = \varepsilon^I \stackrel{\circ}{=} \det \left(\frac{\partial x}{\partial \bar{x}} \right)^{+1} \frac{\partial \bar{x}^I}{\partial x^J} \varepsilon^J \tag{3.42}$$

→ $\varepsilon^I = \varepsilon^{i_1 i_2 \dots i_D}$ is a $(D, 0)$ -tensor density

- ¡! $\bar{\varepsilon}^I = \varepsilon^I$ is true by definition: ε is a *symbol* defined by Eq. (3.41); this definition is independent of the coordinate system. In Eq. (3.42) we compare this trivial transformation with that of a (relative) tensor and conclude that it is equivalent to the statement that ε^I transforms as a $(D, 0)$ -tensor density of weight $w = +1$. This knowledge is helpful in tensor calculus to construct covariant expressions that contain Levi-Civita symbols (→ *below*).
- To show this, note that the Levi-Civita symbol can be used to compute determinants:

$$\det \left(\frac{\partial \bar{x}}{\partial x} \right) = \sum_{\sigma \in S_D} (-1)^\sigma \prod_{i=1}^D \frac{\partial \bar{x}^i}{\partial x^{\sigma_j}} = \frac{\partial \bar{x}^1}{\partial x^{j_1}} \dots \frac{\partial \bar{x}^D}{\partial x^{j_D}} \varepsilon^{j_1 \dots j_D}. \tag{3.43}$$

Details: ➔ Problemset 4

3.5. The metric tensor

A differentiable manifold M does not automatically allow us to measure the length of curves, the angles of intersecting lines, or the area/volume of subsets of the manifold; to do so, we need a *metric* on M (which is an additional piece of information). While the continuity structure (an atlas) that comes with M determines its *topology*, the metric determines its *geometry* (= shape). The same manifold M can be equipped with *different* metrics; this corresponds to different geometries of the same topology (a potato and an egg both have the topology of a sphere, nonetheless they are geometrically distinct).

A differentiable manifold together with a (pseudo-)metric is called a *(pseudo-)Riemannian manifold*. In SPECIAL RELATIVITY and GENERAL RELATIVITY, spacetime is modeled by such (pseudo-)Riemannian manifolds where the metric is used to represent spatial and temporal distances between events.

25 | Motivation:

On linear spaces V , it is convenient to define an *inner product* (like in quantum mechanics where you consider Hilbert spaces and use their inner product to compute probabilities and transition amplitudes).

Recall the definition of a (real) inner product:

$$\langle \bullet | \bullet \rangle : V \times V \rightarrow \mathbb{R} \quad \text{with ...} \tag{3.44a}$$

$$\text{Symmetry: } \langle x | y \rangle = \langle y | x \rangle \tag{3.44b}$$

$$\text{(Bi)linearity: } \langle ax + by | z \rangle = a \langle x | z \rangle + b \langle y | z \rangle \tag{3.44c}$$

$$\text{Positive-definiteness: } x \neq 0 \Rightarrow \langle x | x \rangle > 0 \tag{3.44d}$$

Once you have an inner product, you get a norm, and subsequently a metric for free:

$$\underbrace{\langle x | y \rangle}_{\text{Inner product}} \Rightarrow \underbrace{\|x\| := \sqrt{\langle x | x \rangle}}_{\text{Norm}} \Rightarrow \underbrace{d(x, y) := \|x - y\|}_{\text{Metric}} \tag{3.45}$$

Thus an inner product is a rather versatile structure and nice to have!

Problem: We cannot define an inner product on the manifold directly because M is not a linear space.

However: We can introduce an inner product on each of its tangent spaces $T_p M!$ →

26 | **** Riemannian (Pseudo-)Metric** $ds^2 :=$ Symmetric, non-degenerate (0, 2)-tensor field:

$$ds^2 : M \ni p \mapsto \underbrace{(ds_p^2 : T_p M \times T_p M \rightarrow \mathbb{R})}_{\text{Bilinear \& symmetric \& non-degenerate}} \quad (3.46a)$$

$$\begin{aligned} ds_p^2 \text{ bilinear} &\Rightarrow ds_p^2 \in T_p^* M \otimes T_p^* M \\ &\Rightarrow ds_p^2 = \sum_{i,j=1}^D g_{ij}(x) dx^i \otimes dx^j \equiv g_{ij}(x) dx^i dx^j \end{aligned} \quad (3.46b)$$

with $g_{ij} = g_{ji}$ (symmetry) and $g = \det(g_{ij}) \neq 0$ (non-degeneracy).

- The tensor product is *non-commutative*: $dx^i \otimes dx^j \neq dx^j \otimes dx^i$. However, you can always decompose a tensor product as

$$dx^i \otimes dx^j = \underbrace{\frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)}_{=: dx^i \vee dx^j} + \underbrace{\frac{1}{2}(dx^i \otimes dx^j - dx^j \otimes dx^i)}_{=: dx^i \wedge dx^j} \quad (3.47)$$

with the symmetrized tensor product $dx^i \vee dx^j$ and the anti-symmetrized tensor product $dx^i \wedge dx^j$ (↑ *wedge product*).

Since g_{ij} is assumed to be symmetric, only the symmetric component survives:

$$g_{ij}(x) dx^i \otimes dx^j = g_{ij}(x) dx^i \vee dx^j \equiv g_{ij}(x) dx^i dx^j \quad (3.48)$$

This means that when writing $dx^i dx^j$ in the above formula, you can be sloppy and either mean $dx^i \otimes dx^j$ or, equivalently, $dx^i \vee dx^j$. You will find both conventions in the literature. I will use $dx^i dx^j \equiv dx^i \vee dx^j$ so that $dx^i dx^j = dx^j dx^i$.

- It would be more appropriate to write $g = g_{ij} dx^i dx^j$ for the metric (0, 2)-tensor; it is conventional, however, to reserve g for the determinant $\det(g_{ij})$ so that we are stuck with ds^2 for the metric. Note that the d in ds^2 does *not* refer to an ↑ *exterior derivative*, it is purely symbolical.
- To define a proper ↓ *inner product* on $T_p M$, we should demand ↓ *positive-definiteness* instead of *non-degeneracy*. This, however, is often (for example in RELATIVITY) too restrictive; as it turns out, non-degeneracy is all we need for an isomorphism between $T_p M$ and $T_p^* M$ (“pulling indices up and down”, → *below*). This is why *negative* eigenvalues of g_{ij} are fine for many purposes, and motivates the concept of a → *signature*:

27 | Signature:

Since $g_{ij}(x) = g_{ji}(x)$ and $\det(g_{ij}(x)) \neq 0$

→ $g_{ij}(x)$ has r *positive* and s *negative* real eigenvalues for all $p \in M$

Since $\det(g_{ij}(x)) \neq 0$, these numbers must be the *same* for all $p \in M$.

→ (r, s) : **** Signature** of the metric ds^2

This classification does not depend on the coordinate basis (↑ *Sylvester’s law of inertia*).