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Notes:

- In the rest frame of the flying *muons* one would measure the usual lifetime $\tau_\mu^0 \approx 2.1948(10)\,\mu\text{s}$. However, in this frame, the *laboratory* is *Lorentz contracted* such that the muon reaches exactly the same point in space where it decays in this "shorter" lifetime. Note how time-dilation and Lorentz contraction provide different explanations for the same experimental observation.
- One can also use different particle species to study time dilation, for example *pions* (a sort of meson, i.e., a hadron with one quark and one antiquark) [49].

• *Hafele-Keating experiment [50, 51]:* (➔ Problemset 4)

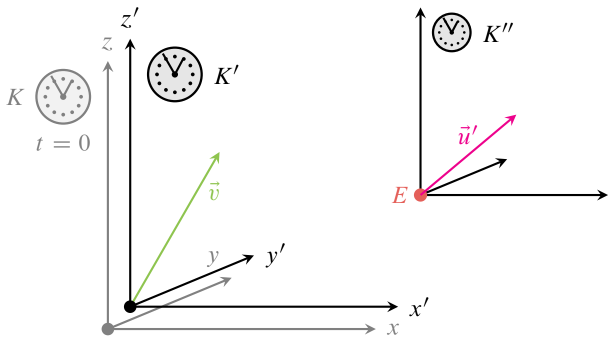
In 1971, J.C. Hafele and R. E. Keating took four Cesium atomic clocks along commerical jet flights around the globe twice: once eastward and once westward. Compared to a reference clock on the ground, the clocks on the eastward flight lost on average $\sim$ 59 ns (= they ran slower) and the clocks on the westward flight gained $\sim$ 273 ns (= they ran faster). To understand this qualitatively, note that the reference clock on the ground is *rotating* (together with earth) and therefore is *not* an inertial clock. Therefore imagine an (approximately) inertial reference system flying along earth around the sun, and from this system look down on the north pole; earth is now slowly rotating beneath you. From this inertial system, the eastward flight has higher velocity than the reference clock, which, in turn, has higher velocity than the westward flight. Thus you find that the eastward clock runs slower than the reference clock which runs slower than the westward clock (this is also true if the clocks are accelerated,→ *below*). These theoretical considerations are explained in [50].

↓ Lecture 7 [26.11.25]

2.3. Addition of velocities

Details: ➔ Problemset 2 and ➔ Problemset 3

1 | $\sphericalangle$ Particle moving with $\vec{u}' = \frac{d\vec{x}'}{dt'}$ in system K' and inertial system K with $K \xrightarrow{\vec{v}} K'$:



2 | Velocity $\vec{u}$ in K:

$$\vec{u} = \frac{d\vec{x}}{dt} \equiv \vec{v} \oplus \vec{u}' \stackrel{\circ}{=} \frac{1}{1+\frac{\vec{v}\cdot\vec{u}'}{c^2}} \left[\vec{v} + \frac{\vec{u}'}{\gamma_v} + \frac{\gamma_v}{c^2(1+\gamma_v)}(\vec{u}'\cdot\vec{v})\,\vec{v}\right] \tag{2.16}$$

Proof: Use Eq. (1.75) (➔ Problemset 2).

¡! The relativistic addition of velocities $\oplus$ is in general not commutative ($\vec{v}\oplus\vec{u} \neq \vec{u}\oplus\vec{v}$) nor associative [$\vec{v}\oplus(\vec{u}\oplus\vec{w}) \neq (\vec{v}\oplus\vec{u})\oplus\vec{w}$]. As you can easily see from Eq. (2.16), it is also not linear:

$(\lambda\vec{v}) \oplus (\lambda\vec{u}) \neq \lambda(\vec{v} \oplus \vec{u})$. Be careful: There are different notations (in particular: orderings) used in the literature.

3 | ⊲ Non-relativistic limit ($c \to \infty \Rightarrow \gamma_v \to 1$):

$$\lim_{c\to\infty} \vec{v} \oplus \vec{u}' = \lim_{c\to\infty} \vec{u}' \oplus \vec{v} = \vec{v} + \vec{u}' \tag{2.17}$$

→ Galilean addition of velocities

4 | Special case: $\vec{v} = (v_x, 0, 0)$ but still $\vec{u}' = (u'_x, u'_y, u'_z)$

$$u_x \overset{\circ}{=} \frac{v_x + u'_x}{1 + \frac{v_x u'_x}{c^2}}, \quad u_y \overset{\circ}{=} \frac{u'_y/\gamma_v}{1 + \frac{v_x u'_x}{c^2}}, \quad u_z \overset{\circ}{=} \frac{u'_z/\gamma_v}{1 + \frac{v_x u'_x}{c^2}}. \tag{2.18}$$

¡! Note that also the transverse components of $\vec{u}'$ are modified, but in a different way than the collinear component u'_x. For $\vec{u}' = (u'_x, 0, 0)$ we get our previous result for collinear velocities Eq. (1.70) back.

5 | Thomas-Wigner rotation [52, 53]:

Remember that for *collinear* addition of velocities the concatenation of two boosts yields another boost: $\Lambda_{v_x}\Lambda_{u_x} = \Lambda_{w_x}$ [recall Eq. (1.57)].

As a straightforward (but tedious) calculation using two general boosts Eq. (1.75) shows, this is *not* true in general: $\Lambda_{\vec{v}}\Lambda_{\vec{u}} \neq \Lambda_{\vec{w}}$ with $\vec{w} = \vec{u} \oplus \vec{v}$. Rather one finds

$$\Lambda_{\vec{v}}\Lambda_{\vec{u}} = \Lambda_{\vec{u}\oplus\vec{v}}\Lambda_{R(\vec{u},\vec{v})} \tag{2.19}$$

with the ⁂ *Thomas-Wigner rotation* $R(\vec{u}, \vec{v}) \in \mathrm{SO}(3)$ (we omit the expliclit form of $R(\vec{u}, \vec{v})$ here).

This is not in contradiction with our general addition for velocities above because there we were only interested in the velocity of a moving particle (which you can identify with the origin of its rest frame K''); we completely ignored the *axes* of K''. The Thomas-Wigner rotation tells you that the concatenation of two *pure* boosts is *not* a pure boost in general.

2.4. Proper time and the twin "paradox"

1 | ⊲ Time-like trajectory $\mathcal{P} \subseteq \mathcal{E}$ of a spaceship with departure $D \in \mathcal{P}$ and arrival $A \in \mathcal{P}$.

A *time-like trajectory* is a spacetime path that can be traversed by a massive particle, i.e., for its coordinate velocity holds $|\dot{\vec{x}}(t)| < c$ at all times t. When we are equipped with → *tensor calculus*, we will see that this means that the tangent vector at the trajectory (in spacetime) is ← *time-like* at all points of the trajectory. The condition $|\dot{\vec{x}}(t)| < c$ will be crucial for our derivation → *below*. You can also characterize a time-like trajectory via local light cones: At every point, the trajectory "enters" from the local *past* light cone and "leaves" into the local *future* light cone.

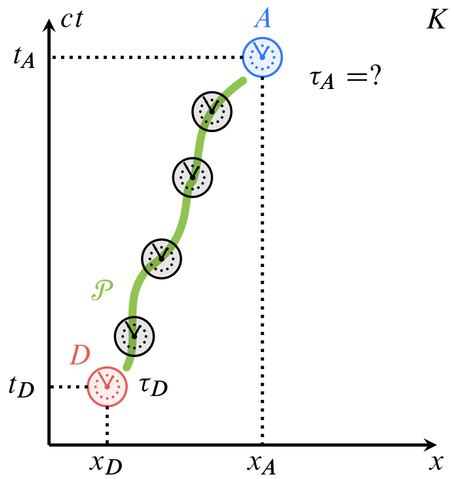
⊲ Coordinate parametrization $\vec{x}(t)$ of $\mathcal{P}$ in system K with

$$\text{departure } [D]_K = (t_D, \vec{x}_D) \quad \text{and} \quad \text{arrival } [A]_K = (t_A, \vec{x}_A): \tag{2.20}$$

Formally, $\mathcal{P}$ is a set of coincidence classes parametrized in K by the clock events $(t, \vec{x}(t))_K$:

$$\mathcal{P} = \{ [(t, \vec{x}(t))_K] \mid t \in [t_D, t_A] \} \subseteq \mathcal{E}. \tag{2.21}$$

This suggests the formal notation $[\mathcal{P}]_K = (t, \vec{x}(t))$.



2 | Thought experiment:

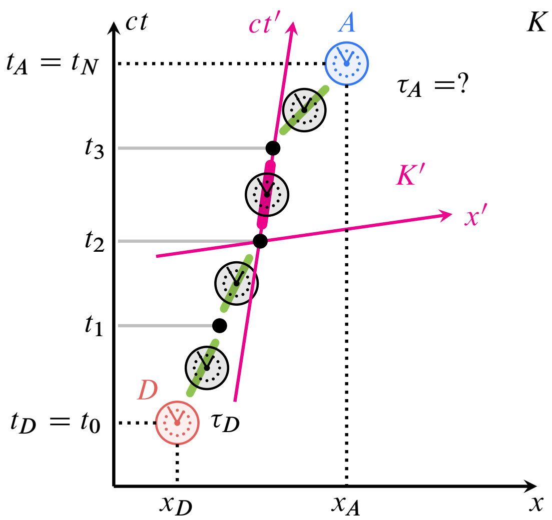
The spaceship takes a clock along and resets it to $\tau_D = \tau(t_D)$ at departure D.

What is the reading $\tau_A = \tau(t_A)$ of the clock at arrival A?

We assume that the clock in the spaceship is type-identical to the clocks used for inertial observers.

3 | *Idea:*

Approximate the trajectory by a *polygon* of N segments $i = 1, \dots, N$ separated by time steps t_i (with $t_0 := t_D$ and $t_N := t_A$):



i | Let $\Delta t_i := t_{i-1} - t_i$ and $\Delta\vec{x}_i := \vec{x}(t_{i-1}) - \vec{x}(t_i)$

For each segment, there is an inertial frame K' with a t'-axis that follows the spacetime segment (because all segments are time-like!). This is the instantaneous rest frame of the spaceship where the clock in the spaceship and the origin clock of K' are at the same place and at rest relative to each other. Since the clocks are type-identical, the time $\Delta\tau_i$ accumulated by the spaceship clock on this segment is identical to the time $\Delta t'_i$ elapsed for the origin clock of K' on this segment: $\Delta\tau_i = \Delta t'_i$. This time is equal to the spacetime interval $(\Delta s'_i)^2 = (c\Delta t'_i)^2 - 0$ because the origin clock is at rest in K' (so that $\Delta\vec{x}'_i = \vec{0}$). But remember that the spacetime interval $(\Delta s'_i)^2$ is Lorentz invariant so that we can calculate *the same number* in any inertial system: $(\Delta s'_i)^2 = (\Delta s_i)^2 = (c\Delta t_i)^2 - (\Delta\vec{x}_i)^2$.

In summary, on the ith interval, the spaceship clock accumulates the time

$$\Delta\tau_i \equiv \frac{\Delta s_i}{c} = \Delta t_i' \tag{2.22a}$$

Note that $(\Delta s_i)^2 > 0$ since $\mathscr{P}$ is assumed to be *time-like*.

$$= \frac{\sqrt{(\Delta s_i')^2}}{c} \tag{2.22b}$$

$$\overset{1.84}{=} \frac{\sqrt{(\Delta s_i)^2}}{c} \tag{2.22c}$$

$$= \frac{\sqrt{(c\Delta t_i)^2 - (\Delta\vec{x}_i)^2}}{c} \tag{2.22d}$$

$$= \Delta t_i \sqrt{1 - \frac{(\Delta\vec{x}_i/\Delta t_i)^2}{c^2}} \tag{2.22e}$$

The above chain of arguments provided us with a *physical interpretation* for the Lorentz invariant spacetime interval $(\Delta s)^2 > 0$ of *time-like* separated events: It measures (up to a factor of c) the time accumulated by an inertial (= unaccelerated) clock that takes part in both events.

ii | Continuum limit $N \to \infty$ ($v(t) := |\vec{v}(t)| = |\dot{\vec{x}}(t)|$):

$$\mathrm{d}\tau = \frac{\mathrm{d}s}{c} = \mathrm{d}t\sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} \quad\Leftrightarrow\quad \frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} \quad\Leftrightarrow\quad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{v(t)} \tag{2.23}$$

Note that this is just an infinitesimal version of the time-dilation formula Eq. (2.14) with $\Delta t \to \mathrm{d}t$ and $\Delta t_0 \to \mathrm{d}\tau$.

Since $(\Delta s)^2 = (\Delta s')^2$ is *Lorentz invariant*:

$$K \overset{\Lambda}{\to} K': \quad \mathrm{d}t\sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} = \frac{\mathrm{d}s}{c} = \frac{\mathrm{d}s'}{c} = \mathrm{d}t'\sqrt{1 - \frac{\dot{\vec{x}}'(t')^2}{c^2}} \tag{2.24}$$

You can check this also explicitly using the Lorentz transformation Eq. (1.75).

iii | → ⁂ *Proper time* accumulated by the spaceship clock along the trajectory $\mathscr{P}$:

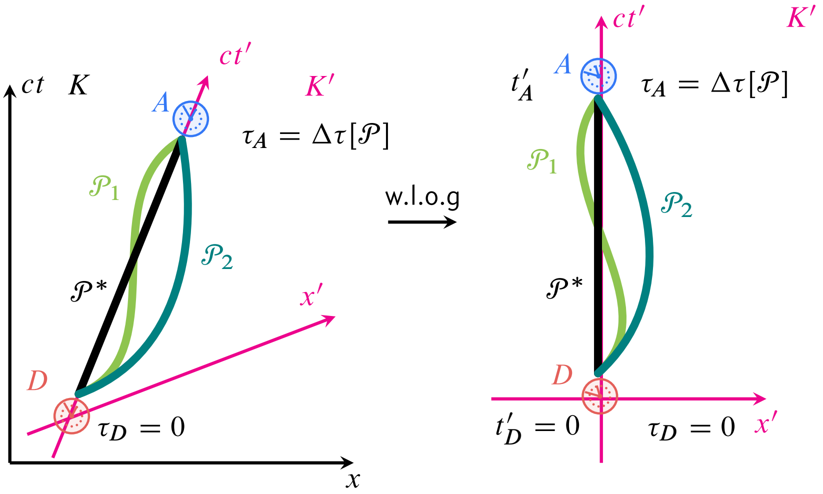
$$\Delta\tau[\mathscr{P}] = \lim_{N\to\infty} \sum_{\substack{\text{Segment}\\ i=1}}^{N} \Delta\tau_i = \int_{\mathscr{P}} \mathrm{d}\tau = \int_{\mathscr{P}} \frac{\mathrm{d}s}{c} := \int_{t_D}^{t_A} \mathrm{d}t\sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} \tag{2.25}$$

- As constructed, the proper time $\Delta\tau[\mathscr{P}]$ of a time-like trajectory $\mathscr{P}$, parametrized by $\vec{x}(t)$ for $t \in [t_0, t_1]$, is the time elapsed by a clock that follows this trajectory in spacetime.
- ¡! This result is valid for *accelerated* clocks.

  In general, SPECIAL RELATIVITY *can* described the physics of accelerated objects as long as the description of the process is given in an inertial coordinate system (as is the case here). Even the latter is not strictly necessary if one formulates SPECIAL RELATIVITY in a → *generally covariant form* (→ Section 9.2). However, then the equations are more complicated than in the Cartesian coordinates provided by inertial systems.

- ¡! The right-most expression in Eq. (2.25) yields the same result *in all inertial systems* K [recall Eq. (2.24)]. This is why $\tau[\mathcal{P}]$ is a function of the *event trajectory* $\mathcal{P}$ and not its coordinate parametrization $\vec{x}(t)$. This is important: It tells us that all inertial observers will agree on the reading of the spaceship clock τ_A at arrival A (although their *parametrization* $\vec{x}(t)$ may look different).
- Note that since $\vec{x}(t)$ is assumed to be time-like, it is $\forall_t : |\dot{\vec{x}}(t)| < c$ such that the radicand is always non-negative.
- $\tau[\bullet]$ is a *functional* of the trajectory $\mathcal{P}$; this is why we use square-brackets.

4 | Which trajectory $\mathcal{P}^*$ between the two events D and A *maximizes* the proper time $\Delta\tau$?



i | D and A are *time-like* separated $\rightarrow \exists$ Inertial system $K' = K(D,A)$ with

$$[D]_{K'} = (t'_D = 0, \vec{x}'_D = \vec{0}) \quad \text{and} \quad [A]_{K'} = (t'_A, \vec{x}'_A = \vec{0}) \tag{2.26}$$

That is, without loss of generality, we can Lorentz transform into an inertial system where the two events happen at the *same* location (and by translations we can assume that this location is the origin $\vec{0}$ and that the coordinate time is $t'_D = 0$ at D). We label the time and space coordinate in K' by t' and $\vec{x}'$. Because of the relativity principle **SR**, K' is as good as any system to describe events.

ii | Time of an arbitrary path $\mathcal{P} \ni D, A$ with $[\mathcal{P}]_{K'} = (t', \vec{x}'(t'))$:

$$\Delta\tau[\mathcal{P}] = \int_{t'_D}^{t'_A} \mathrm{d}t' \sqrt{1 - \frac{\dot{\vec{x}}'(t')^2}{c^2}} \overset{\vec{x}'(t')\equiv\vec{0}}{\leq} \int_{t'_D}^{t'_A} \mathrm{d}t' = t'_A - t'_D = \Delta\tau[\mathcal{P}^*] \tag{2.27}$$

Here $\mathcal{P}^*$ is the trajectory between D and A that is parametrized by the constant function $\vec{x}'(t') \equiv \vec{0}$ in K'. In other inertial systems, this trajectory will not be constant; however, it is inertial, i.e., $\mathcal{P}^*$ is described by a trajectory between D and A with uniform velocity.

Check this by applying a Lorentz transformation to the coordinates $(t', \vec{0})_{K'}$!

$\rightarrow$ Clocks that travel along the *inertial trajectory* $\mathcal{P}^*$ between D and A collect the largest proper time $\tau^* = \Delta\tau[\mathcal{P}^*]$.

Collecting the "largest time" means that the these clocks run the *fastest*.

5 | <u>It is important to let this result sink in:</u>

Let K' be the rest frame of earth (which is located in the origin $\vec{0}$) and consider two twins of age τ_D:

- **Twin S** departs with a **Spaceship** at D, flies away from earth, turns around and returns to earth at A. **Twin S** therefore follows a trajectory similar to $\mathcal{P}_2$ in the sketches above.
- **Twin E** stays on **Earth**. He follows the inertial trajectory $\mathcal{P}^*$ in the sketches above.

We just proved above:

$$\langle \text{Age of } \textbf{Twin S} \text{ at } A \rangle = \Delta\tau[\mathcal{P}_2] + \tau_D < \Delta\tau[\mathcal{P}^*] + \tau_D = \langle \text{Age of } \textbf{Twin E} \text{ at } A \rangle$$

This is the famous ⁂ *Twin "paradox"*: **Twin S** aged less than **Twin E**.

6 | Why there is *no* paradox:

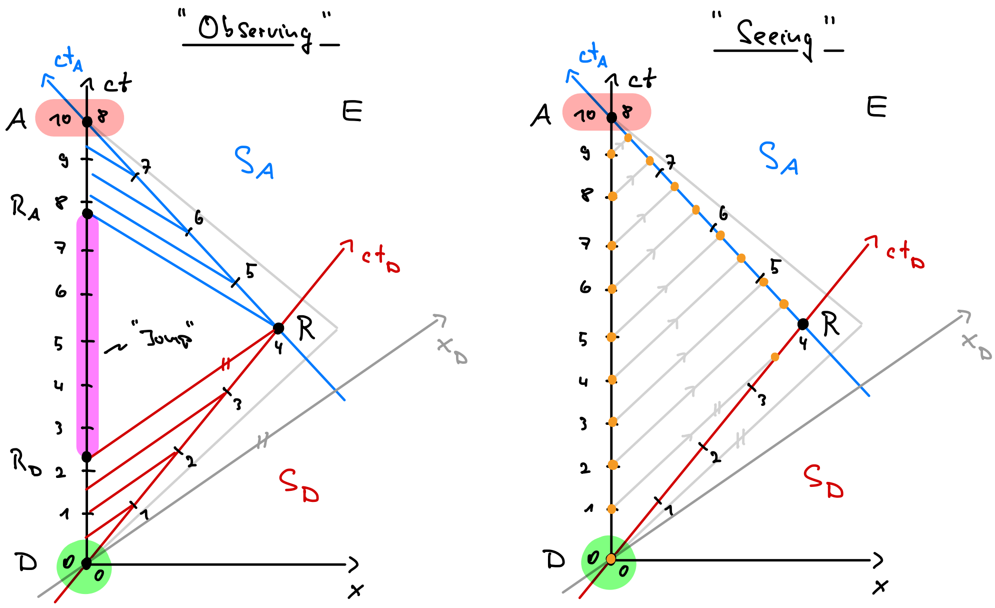
- If you don't see why the above result should be paradoxical:

  Good! Move along. Nothing to see here! ☺
- Why one *could* conclude that the above result is paradoxical (= logically inconsistent):
  - From the view of **Twin E**, **Twin S** speeds around quickly, thus time-dilation tells him that **Twin S** should age slower. And indeed, when **Twin S** returns, he actually didn't age as much.
  - Now, you conclude, due to the relativity principle SR, we could also take the perspective of **Twin S** (i.e., our system of reference is now attached to the spaceship). Then **Twin S** would conclude that time-dilation makes **Twin E** (who now, together with earth, speeds around quickly) age more slowly. But this does not match up with the above result that, when both twins meet again at A, **Twin S** is the younger one! *Paradox!*

  The resultion is quite straightforwad:

  The invocation of the relativity principle SR in the last point is not admissible! Remember that SR only makes claims about the equivalence of *inertial systems*. Now have a look at the trajectory $\mathcal{P}_2$ of the spaceship again: it is clearly accelerated and cannot be inertial. And that there *is* at least a period where the spaceship (and **Twin S**) is accelerating is a *neccessity* for **Twin S** to *return* to **Twin E** (at least in flat spacetimes, but not so in curved ones [54])! This implies that the reunion of both twins at A requires at least one of them to *not* stay in an inertial system. This breaks the symmetry between the two twins and explains why the result can be (and is) asymmetric.
- ¡! For historical (and anthropocentric) reasons, the "twin paradox" is called a "paradox." We stick to this term because we have to – and not because it is appropriate name. The term "paradox" suggests an intrinsic inconsistency of RELATIVITY. As we explained above: *This is not the case.* All "paradoxes" in RELATIVITY are a consequence of unjustified, seemingly "intuitive" reasoning. The root cause is almost always an inappropriate, vague notion of "absolute simultaneity" that cannot be operationalized.
- An overview on different geometric approaches to rationalize the phenomenon can be found in Ref. [55].

  Below are two widely used spacetime diagrams of an idealized version where **Twin S** changes inertial systems only once from S_D to S_A halfway through the journey at R. You can think of this as an instantenous acceleration at the kink. Note, however, that the acceleration itself is dynamically irrelevant for the arguments; it is only important that the inertial frames in which **Twin S** departs and returns are not the same:



- In the left diagram the *slices of simultaneity* in the two systems S_D and S_A are drawn. As predicted by time-dilation (and mandated by SR), **Twin S** observes the clocks of **Twin E** to run *slower* during his "inertial periods", i.e., while he stays in a single inertial system. However, the moment **Twin S** "jumps" from S_D to S_A at R, his notion of simultaneity changes instantaneously: In S_D, R and R_D are simultaneous; in S_A, however, R and R_A are simultaneous. Due to this jump, the record of **Twin S** contains now a temporal gap for events on earth (highlighted interval). It is this "missing" time interval that overcompensates the slower running clocks on earth (as observed from S_D and S_A) and makes **Twin S** conclude that **Twin E** ages faster (in agreement with the actual outcome of the experiment).

  If you wonder what happened to the (missing) observations of events in the triangle $R_A R R_D$: there is a nice explanation in Schutz [5]. (The bottom line is that **Twin S** constructs a bad coordinate system by stopping the recording of events in system S_D when he reaches R.)

- In the right diagram, we draw *light signals* ("pings") of an earth-bound clock next to **Twin E** sent to **Twin S**. **Twin S** receives these signals and measures their period. This idealizes how **Twin S** *sees* (not *observes*!) the clocks ticking on earth (and, by proxy, how fast **Twin E** ages). It is important to understand the difference between this "seeing" and our operational definition of *observing* (using the contraption called an ← *inertial system*, as used in the left diagram). As demonstrated by the diagram, **Twin S** first sees the clock on earth ticking *slower*; but when he turns around at R, the clocks on earth (apparently) speed up significantly. In the end, this speedup overcompensates for the slowdown during the first part of the journey so that **Twin S** again arrives at the (correct) conclusion that **Twin E** ages faster. Note that the speedup of the earth-bound clock *seen* by **Twin S** during the second half of his journey does not contradict time-dilation because *seeing* is not *observing*. This is similar to the ↑ *Penrose-Terrell effect* in that a genuine relativistic effect (here: time-dilation) is distorted by an additional "imaging effect" due to the finite speed of light.

• In our careful derivation above, we not only showed that **Twin S** ages less than **Twin E**; we also showed that this conclusion is *independent* of the inertial observer! Thus we know that there will be *no dispute* about the different ages between different inertial observers.

• The Hafele-Keating experiment [50,51] and the muon decay experiments [48], mentioned previously in the context of time-dilation, are experimental confirmations of the twin "paradox."

So our theoretical prediction above (that **Twin S** ages less than **Twin E**) is experimentally confirmed. End of discussion.

- Our derivation of the accumulated proper time along trajectories in spacetime is both mathematically sound and experimentally confirmed. This qualifies SPECIAL RELATIVITY as a successful theory of physics. *Operationally* there is nothing to complain about: the theory does its job to produce quantitative predictions of real phenomena. So why do so many people (physicists included) – despite the various efforts to visualize the phenomenon – have this nagging feeling of dissatisfaction that they cannot get rid of? The reason, so I would argue, is the human brain and its proclivity to inject concepts of absolute simultaneity into its model building. This qualifies the historical overemphasis of the twin "paradox" as a *meta problem*: The question to study is not how to "solve" the twin "paradox" (as we showed above, there is nothing to solve); the question to study is why so many people thought (and still think) that there is a problem in the first place. This *meta problem* is an actual problem to study; but it falls into the domain of cognitive science, and not physics!

7 | Two lessons to be learned from this:

> You can outlive your inertial-system-dwelling peers
> by changing inertial systems (= accelerating) at least once.

- ¡! You "live longer" when speeding around than your twin on earth, i.e., when you return, your twin might be 80 and have reached the end of his lifespan while you are still in your forties. This is a real, observable effect, not an illusion of sorts. However, "living longer" does *not* mean that you somehow have "more time to spend" than your twin because all physical phenomena in your spaceship experience the same effect. It is not your metabolism that slows down wrt. other physical phenomena around you, it is *time* itself. Put differently: If you and your twin both try to read as many books as possible during your lifetimes (say one per month), both of you will have read roughly the same amount of books when either of you dies (say at the age of 80).
- That the non-inertial space traveler can outlive his inertial, Earth-bound twin is nothing but *future-directed time travel*. Note that there is also no bound on how far into the future of Earth he can jump since the integrand of Eq. (2.25) can be made arbitrarily small for $|\dot{\vec{x}}(t)| \to c$. Note also that the fact that future-directed time travel is possible within the constraints of known physical laws is not surprising: There is no fundamental reason why the physical processes in your brain could not be artificially slowed down (or even halted) only to "reactivate" you far in the future. From your subjective experience, this *is* future-directed time travel.
- The mere fact that our universe *really* allows for this (at least in theory) makes it much more interesting than its boring alternative: a Galilean universe.

and

> Phenomena like length contraction and the twin "paradox" are physically *real*.
> Their "paradoxical" flavor is a phenomenon of human cognition, not physics.

This is why we put "paradox" always in quotes in the context of RELATIVITY.

↓ Lecture 8 [03.12.25]

3. Mathematical Tools I: Tensor Calculus

In this chapter we introduce tensor calculus (↑ *Ricci calculus*) for general coordinate transformations φ (which will be useful both in SPECIAL RELATIVITY and GENERAL RELATIVITY). The coordinate transformations φ relevant for SPECIAL RELATIVITY are Lorentz transformations (and therefore linear) which simplifies expressions often significantly (→ *Chapter 4*). However, this special feature of coordinate transformations in SPECIAL RELATIVITY is not crucial for the discussions in this chapter.

Goal: Construct Lorentz covariant (form invariant) equations
(for mechanics, electrodynamics, quantum mechanics)

Question: How to do this *systematically*?

Note that (we suspect that) Maxwell equations *are* Lorentz covariant. Clearly this is not obvious and requires some work to prove; we say that the Lorentz covariance is *not manifest*: it is there, but it is hard to see. Conversely, without additional tools that make Lorentz covariance more obvious, it is borderline impossible to *construct* Lorentz covariant equations from scratch (which we must do for mechanics and quantum mechanics!).

We are therefore looking for a "toolkit" that provides us with elementary "building blocks" and a set of rules that can be used to construct Lorentz covariant equations. This toolbox is known as *tensor calculus* or ↑ *Ricci calculus*; the "building blocks" are tensor fields and the rules for their combination are given by index contractions, covariant derivatives, etc. The rules are such that the expressions (equations) you can build with tensor fields are *guaranteed* to be Lorentz covariant. This implies in particular that if you can rewrite any given set of equations (like the Maxwell equations) in terms of these rules, you automatically show that the equations were Lorentz covariant all along. We then say that the Lorentz covariance is *manifest*: one glance at the equation is enough to check it.

Later, in GENERAL RELATIVITY, our goal will be to construct equations that are invariant under *arbitrary* (differentiable) coordinate transformations (not just global Lorentz transformations). Luckily, the formalism we introduce in this chapter is powerful enough to allow for the construction of such → *general covariant* equations as well. This is why we keep the formalism in this chapter as general as possible, and specialize it to SPECIAL RELATIVITY in the next Chapter 4. The discussion below is therefore already a preparation for GENERAL RELATIVITY; it is based on Schröder [1] and complemented by Carroll [56].

3.1. Manifolds, charts and coordinate transformations

1 | D-dimensional Manifold

= Topological space that *locally* "looks like" D-dimensional Euclidean space $\mathbb{R}^D$: