2. Kinematic Consequences

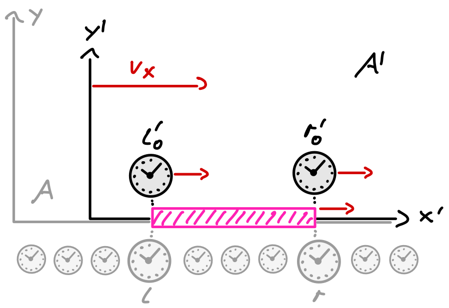
In this chapter we study implications of the special Lorentz transformations Eq. (1.75) and Eq. (1.77) that follow without imposing a model-specific dynamics (= equations of motion). We refer to these implications as *kinematic* because they follow from fundamental constraints on the degrees of freedom of all relativistic theories. The phenomena we will encounter are therefore *features of spacetime itself* – and not of some entities that live on/in (or couple to) spacetime.

¡! The phenomena we will encounter are *not* "illusions" (in the sense that we "see" things differently than they "really are"). Remember that we precisely defined what we mean by observers/reference frames; in particular, we emphasized that we do not "look" at anything, we *measure* events in a systematic way, using a well-defined structure called ← *inertial system*. All phenomena we will encounter are derived from and to be understood in this operational, physically meaningful context.

2.1. Length contraction and the Relativity of Simultaneity

1 | $\sphericalangle$ Inertial systems $A \xrightarrow{v_x} A'$ with rod on x'-axis and at rest in A':

Remember that $A \xrightarrow{v_x} A'$ denotes a boost in x-direction with v_x (as measured in A) where the spatial axes of both A and A' coincide at $t = 0$:



In such situations, we refer to A' as the ⁂ *rest frame* of the rod and A as the ⁂ *lab frame* (some call A the ⁂ *stationary frame*). In the following, coordinates of events in the inertial system A' are marked by primes.

↓ Lecture 6 [19.11.25]

2 | First, we have to define what we mean by the "lenght" of an object:

"Length" is an intrinsically non-local concept. It is not something you can measure or define at a single point in space. Consequently, there are no "length-events" in $\mathcal{E}$. Thus we need an algorithm (= operational definition) of what we mean by "length".

$\sphericalangle$ Two event *types*:

$$\{e_L\} = \{\langle\text{Left end of rod detected}\rangle\} \tag{2.1a}$$

$$\{e_R\} = \{\langle\text{Right end of rod detected}\rangle\} \tag{2.1b}$$

Think of an event *type* as a set (equivalence class) of all elementary events that you deem ↑ *type-identical* (but not ↑ *token-identical*). In the example given here, there will be many events e_L in spacetime that signify "Left end of rod detected" (if there is one rod, there will be one such event for each time t); these are *different* events of the *same type* $\{e_L\}$.

One could even declare that the event *type* $\{e_L\}$ *is* what we refer to as "the left end of the rod."

→ Algorithm LENGTH to compute *"Length of Rod"* in system K at time t:

LENGTH:
→ **Input:** Coincidences $\mathcal{E}$, Inertial system label K, Time t
← **Output:** Length l_K of rod at time t as measured in K

1. Find (unique) event $L \in \mathcal{E}$ with $\{e_L\} \in L$ and $(t, \vec{l})_K \in L$.
2. Find (unique) event $R \in \mathcal{E}$ with $\{e_R\} \in R$ and $(t, \vec{r})_K \in R$.
3. Return $l_K := |\vec{l} - \vec{r}|$.

Here, $\{e_L\} \in L$ is shorthand for $\{e_L\} \cap L \neq \emptyset$. In words: the coincidence class L contains an event of the *type* "Left end of rod detected".

Note that we define "length" as the spatial distance between the two ends of the rod *at the same time* t (as measured by the clocks in K). I hope you agree that this is what one typically means by "length."

3 | We now apply this algorithm twice, in the lab frame A and the rest frame A':

i | Rest frame A':

⁂ *Proper length* ≡ ⁂ *Rest length* := Length of rod in A':

$$l_0 := \mathrm{LENGTH}(\mathcal{E}, t'_0; A') = |\vec{l}'_0 - \vec{r}'_0| = |l'_0 - r'_0| \tag{2.2}$$

with simultaneous clock events $(t'_0, \vec{l}'_0)_{A'} \in L_0$ and $(t'_0, \vec{r}'_0)_{A'} \in R_0$.

The time t'_0 that we choose is irrelevant since the rod is (by definition) at rest in A'. Since the rod lies on the x'-axis, it is $\vec{l}'_0 = (l'_0, 0, 0)$ and $\vec{r}'_0 = (r'_0, 0, 0)$.

The subscript "0" in L_0 indicates that this is a specific event (coincidence class) we selected in A' to compute the length of the rod. It does *not* mean "as seen from the rest frame A'" or anything like that. Remember that coincidence classes in $\mathcal{E}$ are objective information!

ii | Lab frame A:

Length of moving rod in A:

$$l := \mathrm{LENGTH}(\mathcal{E}, t; A) = |\vec{l} - \vec{r}| \tag{2.3}$$

with simultaneous clock events $(t_l, \vec{l})_A \in L$ and $(t_r, \vec{r})_A \in R$ with $t_l = t_r = t$.

The time t that we choose might be irrelevant as well, but we do not know this yet.

¡! There is no reason to assume that the events L_0/R_0 chosen in A' to measure the length of the rod are identical to the events L/R used in A: $L_0 \neq L$ and $R_0 \neq R$ in general.

4 | How does l_0 relate to l?

i | In Section 1.5 we did a lot of hard work to compute the transformation φ which transforms the coordinates of an event in one inertial system into the coordinates of *the same event* in another inertial system. We identified the transformation as the Lorentz transformation:

$$\Lambda(A \xrightarrow{v_x} A') : \; [E]_A = (t, \vec{x}) = x \mapsto \Lambda_{v_x} x = x' = (t', \vec{x}') = [E]_{A'} \tag{2.4}$$

ii | So let us use this tool [namely Eq. (1.77)] to obtain the coordinates of the events L and R (used for the length measurement in A) in the rest frame A' of the rod:

$$[L]_{A'} = \begin{cases} ct'_l = \gamma\left(ct_l - \frac{v_x}{c} l_x\right) \\ l'_x = \gamma(l_x - v_x t_l) \\ l'_y = l_y \\ l'_z = l_z \end{cases} \quad \text{and} \quad [R]_{A'} = \begin{cases} ct'_r = \gamma\left(ct_r - \frac{v_x}{c} r_x\right) \\ r'_x = \gamma(r_x - v_x t_r) \\ r'_y = r_y \\ r'_z = r_z \end{cases} \tag{2.5}$$

Here we use $\vec{l} = (l_x, l_y, l_z)$ and $\vec{r} = (r_x, r_y, r_z)$. Since we declared that the rod is fixed on the x'-axis of A', and $\{e_L\} \in L$ and $\{e_R\} \in R$, it must be $l'_y = l'_z = r'_y = r'_z = 0$, and therefore $\vec{l} = (l_x, 0, 0)$ and $\vec{r} = (r_x, 0, 0)$. That is, the rod is not rotated by the boost and always lies on the x-axis of A as well. In particular: $l = |\vec{l} - \vec{r}| = |l_x - r_x|$.

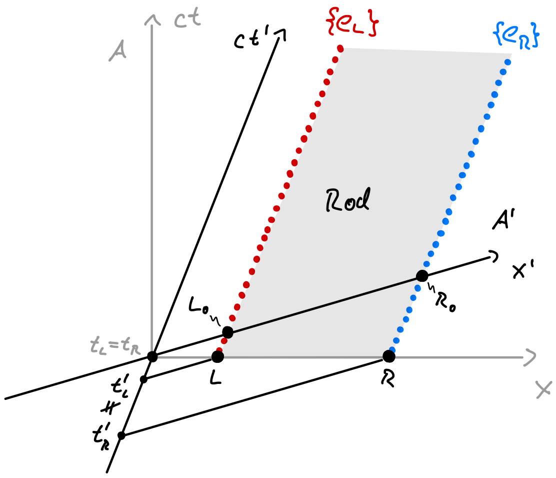
→ Two immediate conclusions:

a | In A' the two events L and R are *no longer simultaneous*:

$$t_l = t_r \text{ in } A \quad \text{but} \quad t'_l \neq t'_r \text{ in } A' \text{ (since } l_x \neq r_x\text{)}. \tag{2.6}$$

→ The simultaneity of events is *observer-dependent*.

This ambiguity of simultaneity can be graphically illustrated in a spacetime diagram (for details on how to draw the (t', x')-axes in A: ➔ Problemset 2):



- As a side note, this calculation implies that not only is it generally *not* true that $L_0 = L$ and $R_0 = R$, it is actually *impossible* (at least for both pairs).
- In the sketch above, the "interior of rod"-events are painted gray. One is tempted to ask: Which "line" of these events *is* the rod? The counterintuitive answer is that this depends on the observer: For A-observers, *horizontal* lines of gray events

make up "the rod", whereas for the A'-observer *tilted* lines are "the rod". It is actually more reasonable to think of the complete area of gray events as "the rod", just as the event type $\{e_L\}$ is "the left edge" of the rod. This suggests that our intuitive concept of the *instantaneous existence of extended objects* – which feels so natural to us – is, to some extend, misleading.

b | In A' the coordinate distance is different:

$$|l'_x - r'_x| \overset{t_l=t_r}{=} \gamma|l_x - r_x| \overset{v_x\neq 0}{\neq} |l_x - r_x| = l \tag{2.7}$$

¡! The time-dependence cancels so that the expressions are time-independent.

At this point, it is a bit premature to identify the left-hand side as the *rest length* l_0 of the rod because these are spatial coordinates of events that are *not simultaneous*! (Remember that the length of any object in any frame is defined as the coordinate distance of *simultaneous* events.)

However, since A' is (by definition) the *rest frame* of the rod, the position labels of the A'-clocks adjacent to the ends of the rod are the same for all events:

$$\left.\begin{array}{l} l'_x \overset{\{e_L\}\in L}{=} l'_0 \\ r'_x \overset{\{e_R\}\in R}{=} r'_0 \end{array}\right\} \Rightarrow |l'_x - r'_x| = |l'_0 - r'_0| = l_0 \tag{2.8}$$

→ ⁂ *Length contraction* ≡ ⁂ *Lorentz contraction:*

A rod of rest length l_0 is *shorter* if measured from an inertial system in relative motion:

$$l = l_0\sqrt{1-\frac{v^2}{c^2}} \overset{v\neq 0}{<} l_0 \tag{2.9}$$

- ¡! Due to isotropy, this result is true for any length of extended objects *in the direction of the boost*. A rod along the y'-axis, for example, is contracted according to Eq. (2.9) for a boost in y-direction, but not for a boost in x-direction.
- The rod is just a proxy for *any* physical object; the Lorentz contraction therefore affects all physical objects in the same way. The contraction is not a dynamical feature of the object itself (like a force that compresses the atomic lattice) but an intrinsic property of space(time).
- Note that we say above "if measured from ..." and not "as viewed from ...." This distinction is important: If you ask how you would *visually perceive* extended objects flying by (or how they look on a picture taken by a camera) you have to factor in that the photons bouncing of the object at different points take different times to reach your eye (our the camera sensor). If you do the math (➔ Problemset 3), this additional optical effect leads to the surprising result that 3D objects actually do *not look* "squeezed" but *rotated*. This implies in particular that a moving sphere still *looks* like a sphere and not like an ellipse (↑ *Penrose-Terrell effect* [43, 44], see also Ref. [45]).

  You can experience this effect (among others) in the educational game "A Slower Speed of Light," which has been developed by the MIT Game Lab for educational purposes, and can be downloaded here for Windows, Mac, and Linux (➔ Problemset 3):

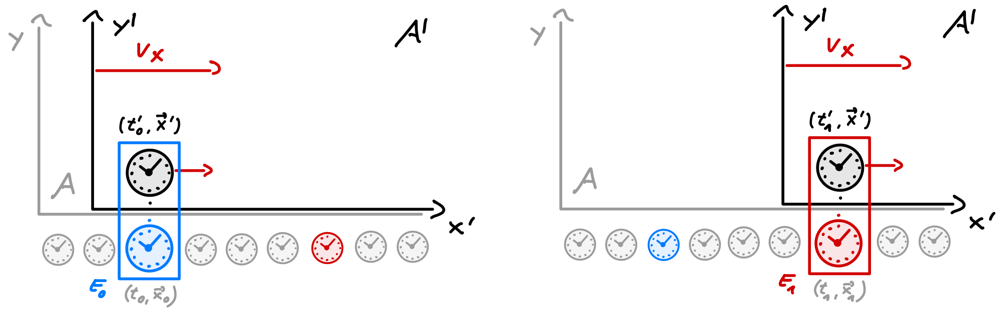
➔ Download "A Slower Speed of Light"

Recently, the Penrose-Terrell effect was visualized in a laboratory experiment where the speed of light was "virtually" reduced to less than $2\,\mathrm{m\,s^{-1}}$ [46].

You should always keep in mind, however, that this "looking" is *not* what we refer to as *observing* in RELATIVITY; the latter has been defined operationally as a measurement procedure at the beginning of this course.

2.2. Time dilation

1 | $\sphericalangle$ Inertial systems $A \xrightarrow{v_x} A'$ and a clock $\vec{x}'$ at rest in A':



2 | $\sphericalangle$ Two events:

$$A'\text{-Clock } \vec{x}' \text{ meets } A\text{-clock } \vec{x}_0\text{:} \quad (t'_0, \vec{x}')_{A'} \sim (t_0, \vec{x}_0)_A \in E_0 \tag{2.10a}$$

$$A'\text{-Clock } \vec{x}' \text{ meets } A\text{-clock } \vec{x}_1\text{:} \quad (t'_1, \vec{x}')_{A'} \sim (t_1, \vec{x}_1)_A \in E_1 \tag{2.10b}$$

¡! The two events E_0 and E_1 relate *three* different clocks: The single A'-clock $\vec{x}'$ and two *different* A-clocks $\vec{x}_0$ and $\vec{x}_1$.

3 | As for length, the concept of "duration" cannot be defined locally in spacetime. We therefore need an operational definition (algorithm) of "duration":

DURATION:
→ **Input:** Two events E_0 and E_1, Inertial system label K
← **Output:** Time interval Δt_K between events as measured in K

1. Find (unique) clock event $(t_0, \vec{x}_0)_K \in E_0$.
2. Find (unique) clock event $(t_1, \vec{x}_1)_K \in E_1$.
3. Return $\Delta t_K := t_1 - t_0$.

Hopefully you agree that this is a reasonable definition of the duration (or time interval) between two events.

4 | We can now apply this algorithm to determine the time elapsed between E_0 and E_1:

$$\text{In } A': \quad \Delta t' = \mathrm{DURATION}(E_0, E_1; A') = t'_1 - t'_0 \quad \text{Measured by a single clock!} \tag{2.11a}$$

$$\text{In } A: \quad \Delta t = \mathrm{DURATION}(E_0, E_1; A) = t_1 - t_0 \quad \text{Measured by two clocks!} \tag{2.11b}$$

5 | <u>How does Δt relate to $\Delta t'$?</u>

i | Since $(t_0', \vec{x}')_{A'} \sim (t_0, \vec{x}_0)_A$ and $(t_1', \vec{x}')_{A'} \sim (t_1, \vec{x}_1)_A$, we can use the Lorentz transformation to translate between the coordinates:

$\xrightarrow{\text{Inverse of Eq. (1.77)}}$

Remember that $\Lambda_{\vec{v}}^{-1} = \Lambda_{-\vec{v}}$ because of reciprocity; the inverse Lorentz transformation can then be obtained by substituting $v_x \mapsto -v_x$:

$$[E_0]_A = \begin{cases} ct_0 = \gamma\left(ct_0' + \frac{v_x}{c}x'\right) \\ x_0 = \gamma(x' + v_x t_0') \end{cases} \quad \text{and} \quad [E_1]_A = \begin{cases} ct_1 = \gamma\left(ct_1' + \frac{v_x}{c}x'\right) \\ x_1 = \gamma(x' + v_x t_1') \end{cases} \tag{2.12}$$

We omit the other two coordinates since they are invariant anyway; the transformation of the spatial coordinate is also not necessary for the following derivation.

ii | Subtracting the equations for the time coordinate of both events yields:

$$c(t_1 - t_0) = \gamma c(t_1' - t_0') \tag{2.13}$$

Note that in the inverse Lorentz transformation Eq. (2.12) the position coordinate in A' is x' for *both* events because the *same* A'-clock takes part in both coincidences.

iii | ⁂ *Time dilation:*

→ The moving clocks in A' run slower than the stationary clocks in A:

$$\boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1 - \frac{v^2}{c^2}}} \overset{v \neq 0}{>} \Delta t_0} \tag{2.14}$$

We renamed $\Delta t' \equiv \Delta t_0$ to emphasize the analogy to the *proper length* l_0:

Δt_0: ⁂ *Proper time* elapsed in A' between E_0 and E_1
Δt: Time elapsed in A between E_0 and E_1

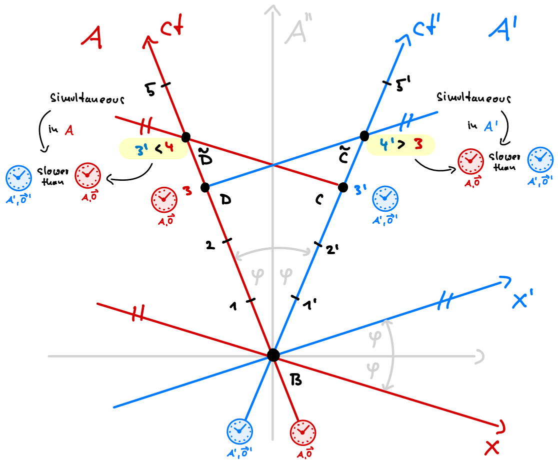
- The characteristic feature of the *proper time* Δt_0 between two (time-like separated) events E_0 and E_1 is that it can be measured by a *single* inertial clock that takes part in both events. All other time intervals must be measured by subtracting the reading of *two different* clocks. Eq. (2.14) tells you that these time intervals are always longer than the proper time Δt_0.
- ¡! Due to isotropy, our result above is true for boosts in any direction.

  Note that in the derivation above, we did *not* impose any special constraints on the positions of the clocks (except that they coincide pairwise at E_0 and E_1). In particular, we did not assume (despite the sketch suggesting this) that the clocks are located on the x/x'-axis. *All* clocks in A' are slowed down in the same way, irrespective of their location!
- This result does *not* contradict our assumption that all clocks are type-identical (= run with the same rate if put next to each other at rest) because the two events needed to compare the tick rate of moving clocks necessarily describe coincidences between *different* pairs of clocks.

6 | Relativity principle:

Because of the relativity principle SR time dilation must be completely *symmetrical*: The A'-clocks run slower compared to the A-clocks, and the A-clocks run slower compared to the A' clocks.

That this is indeed that case (without being a clock "paradox") is best illustrated in a symmetric spacetime diagram:



The existence of the "median frame" A'' between $A \xrightarrow{v_x} A'$ can be easily shown with the addition for collinear velocities Eq. (1.70). This symmetric form of a spacetime diagram is sometimes called ↑ *Loedel diagram* [47] and makes the symmetry between inertial frames manifest; in particular, the units on the axes of A and A' are identical (they are not identical to the units of A'', tough). In this symmetric form, the t'-axis is orthogonal to the x-axis and the t-axis to the x'-axis. Note that because of the relativistic addition of velocities, it is $A'' \xrightarrow{\tilde{v}_x} A'$ and $A'' \xrightarrow{-\tilde{v}_x} A$ with $\tilde{v}_x = v_x \frac{\gamma}{1+\gamma}$ and $\tan(\varphi) = \frac{\tilde{v}_x}{c}$ (➔ Problemset 3). Only in the non-relativistic limit $v_x/c \to 0$ one finds $\tilde{v}_x = \frac{v_x}{2}$ as naïvely expected.

Note that due to the relativity of simultaneity, the two observers use *different* pairs of clock-events to decide which of the two origin clocks runs slower:

- For A the two clock events $\tilde{D}$ and C are simultaneous such that one has to conclude that the (blue) A'-clock runs slower than the (red) A-clock.
- By contrast, for the observer A' the two events D and $\tilde{C}$ are simultaneous such that one has to conclude that the (red) A-clock runs slower than the (blue) A'-clock.

It is evident from the diagram that there is no disagreement about coincidences of events (or readings of clocks). It is just the observer-dependent concept of simultaneity that leads to the seemingly "paradoxical" reciprocity of time dilation.

7 | Experiments:

- *Muon decay [48]:*

  Muons quickly decay into electrons (and neutrinos):

  $$\mu^- \to e^- + \nu_\mu + \bar{\nu}_e \,. \tag{2.15}$$

  This decay can be readily observed in storage rings of particle colliders like CERN. The lifetime of muons *at rest* (measured by clocks in an inertial laboratory frame) is $\tau_\mu^0 \approx 2.1948(10)\,\mu\text{s}$. However, the lifetime of muons in flight (close to the speed of light) is measured to be $\tau_\mu \approx 64.368(29)\,\mu\text{s}$, i.e., much longer! If one carefully takes into account the speed of the muons and additional experimental imperfections, this result fits Eq. (2.14) with deviations of only $\sim 0.1\,\%$ [48].

  Notes:

  – In the rest frame of the flying *muons* one would measure the usual lifetime $\tau_\mu^0 \approx 2.1948(10)\,\mu\text{s}$. However, in this frame, the *laboratory* is *Lorentz contracted* such that the muon reaches exactly the same point in space where it decays in this "shorter" lifetime. Note how time-dilation and Lorentz contraction provide different explanations for the same experimental obervation.

- One can also use different particle species to study time dilation, for example *pions* (a sort of meson, i.e., a hadron with one quark and one antiquark) [49].

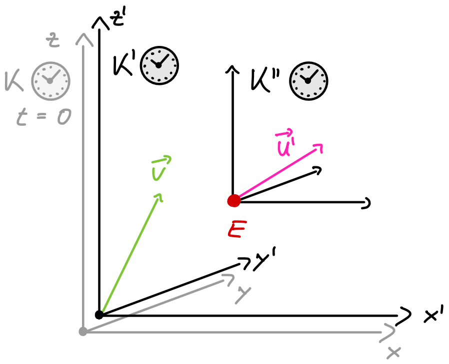
- *Hafele-Keating experiment [50, 51]:*

  In 1971, J.C. Hafele and R. E. Keating took four Cesium atomic clocks along commerical jet flights around the globe twice: once eastward and once westward. Compared to a reference clock on the ground, the clocks on the eastward flight lost on average $\sim$ 59 ns (= they ran slower) and the clocks on the westward flight gained $\sim$ 273 ns (= they ran faster). To understand this qualitatively, note that the reference clock on the ground is *rotating* (together with earth) and therefore is *not* an inertial clock. Therefore imagine an (approximately) inertial reference system flying along earth around the sun, and from this system look down on the north pole; earth is now slowly rotating beneath you. From this inertial system, the eastward flight has higher velocity than the reference clock, which, in turn, has higher velocity than the westward flight. Thus you find that the eastward clock runs slower than the reference clock which runs slower than the westward clock (this is also true if the clocks are accelerated, → *below*). These theoretical considerations are explained in [50].

2.3. Addition of velocities

Details: ➔ Problemset 2

1 | $\sphericalangle$ Particle moving with $\vec{u}' = \frac{d\vec{x}'}{dt'}$ in system K' and inertial system K with $K \xrightarrow{\vec{v}} K'$:



2 | Velocity $\vec{u}$ in K:

$$\vec{u} = \frac{d\vec{x}}{dt} \equiv \vec{v} \oplus \vec{u}' \stackrel{\circ}{=} \frac{1}{1+\frac{\vec{v}\cdot\vec{u}'}{c^2}}\left[\vec{v} + \frac{\vec{u}'}{\gamma_v} + \frac{\gamma_v}{c^2(1+\gamma_v)}(\vec{u}'\cdot\vec{v})\,\vec{v}\right] \tag{2.16}$$

Proof: Use Eq. (1.75) (➔ Problemset 2).

¡! The relativistic addition of velocities $\oplus$ is in general not commutative ($\vec{v} \oplus \vec{u} \neq \vec{u} \oplus \vec{v}$) nor associative [$\vec{v} \oplus (\vec{u} \oplus \vec{w}) \neq (\vec{v} \oplus \vec{u}) \oplus \vec{w}$]. As you can easily see from Eq. (2.16), it is also not linear: $(\lambda\vec{v}) \oplus (\lambda\vec{u}) \neq \lambda(\vec{v} \oplus \vec{u})$. Be careful: There are different notations (in particular: orderings) used in the literature.

3 | $\sphericalangle$ Non-relativistic limit ($c \to \infty \Rightarrow \gamma_v \to 1$):

$$\lim_{c\to\infty} \vec{v} \oplus \vec{u}' = \lim_{c\to\infty} \vec{u}' \oplus \vec{v} = \vec{v} + \vec{u}' \tag{2.17}$$

→ Galilean addition of velocities

4 | Special case: $\vec{v} = (v_x, 0, 0)$:

$$u_x \stackrel{\circ}{=} \frac{v_x + u'_x}{1 + \frac{v_x u'_x}{c^2}}, \quad u_y \stackrel{\circ}{=} \frac{u'_y/\gamma_v}{1 + \frac{v_x u'_x}{c^2}}, \quad u_z \stackrel{\circ}{=} \frac{u'_z/\gamma_v}{1 + \frac{v_x u'_x}{c^2}}. \tag{2.18}$$

¡! Note that also the transverse components of $\vec{u}'$ are modified, but in a different way than the collinear component u'_x. For $\vec{u}' = (u'_x, 0, 0)$ we get our previous result for collinear velocities Eq. (1.70) back.

5 | Thomas-Wigner rotation [52, 53]:

Remember that for *collinear* addition of velocities the concatenation of two boosts yields another boost: $\Lambda_{v_x}\Lambda_{u_x} = \Lambda_{w_x}$ [recall Eq. (1.57)].

As a straightforward (but tedious) calculation using two general boosts Eq. (1.75) shows, this is *not* true in general: $\Lambda_{\vec{v}}\Lambda_{\vec{u}} \neq \Lambda_{\vec{w}}$ with $\vec{w} = \vec{u} \oplus \vec{v}$. Rather one finds

$$\Lambda_{\vec{v}}\Lambda_{\vec{u}} = \Lambda_{\vec{u}\oplus\vec{v}}\Lambda_{R(\vec{u},\vec{v})} \tag{2.19}$$

with the ⁂ *Thomas-Wigner rotation* $R(\vec{u}, \vec{v}) \in \mathrm{SO}(3)$ (we omit the expliclit form of $R(\vec{u}, \vec{v})$ here).

This is not in contradiction with our general addition for velocities above because there we were only interested in the velocity of a moving particle (which you can identify with the origin of its rest frame K''); we completely ignored the *axes* of K''. The Thomas-Wigner rotation tells you that the concatenation of two *pure* boosts is *not* a pure boost in general.

2.4. Proper time and the twin "paradox"

1 | $\sphericalangle$ Time-like trajectory $\mathcal{P} \subseteq \mathcal{E}$ of a spaceship with departure $D \in \mathcal{P}$ and arrival $A \in \mathcal{P}$.

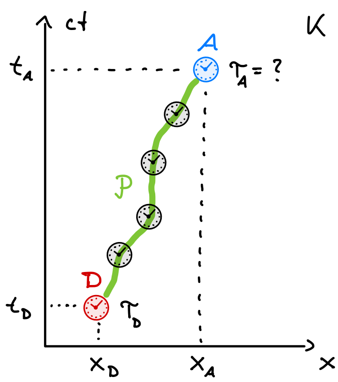
$\sphericalangle$ Coordinate parametrization $\vec{x}(t)$ of $\mathcal{P}$ in system K with

$$\text{departure } [D]_K = (t_D, \vec{x}_D) \quad \text{and} \quad \text{arrival } [A]_K = (t_A, \vec{x}_A): \tag{2.20}$$

Formally, $\mathcal{P}$ is a set of coincidence classes parametrized in K by the clock events $(t, \vec{x}(t))_K$:

$$\mathcal{P} = \{[(t, \vec{x}(t))_K] \,|\, t \in [t_D, t_A]\} \subseteq \mathcal{E}. \tag{2.21}$$

This suggests the formal notation $[\mathcal{P}]_K = (t, \vec{x}(t))$.



2 | Thought experiment:

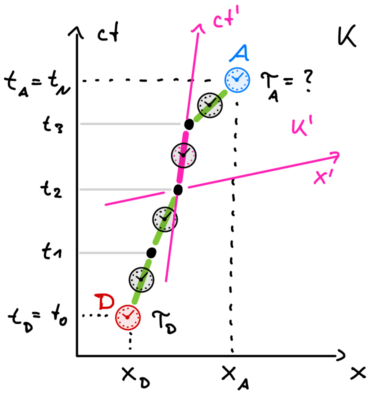
The spaceship takes a clock along and resets it to $\tau_D = \tau(t_D)$ at departure D.

What is the reading $\tau_A = \tau(t_A)$ of the clock at arrival A?

We assume that the clock in the spaceship is type-identical to the clocks used for inertial observers.

3 | *Idea:*

Approximate the trajectory by a *polygon* of N segments $i = 1, \dots, N$ separated by time steps t_i (with $t_0 := t_D$ and $t_N := t_A$):



i | Let $\Delta t_i := t_{i-1} - t_i$ and $\Delta \vec{x}_i := \vec{x}(t_{i-1}) - \vec{x}(t_i)$

For each segment, there is an inertial frame K' with a t'-axis that follows the spacetime segment (because all segments are time-like!). This is the instantaneous rest frame of the spaceship where the clock in the spaceship and the origin clock of K' are at the same place and at rest relative to each other. Since the clocks are type-identical, the time $\Delta\tau_i$ accumulated by the spaceship clock on this segment is identical to the time $\Delta t'_i$ elapsed for the origin clock of K' on this segment: $\Delta\tau_i = \Delta t'_i$. This time is equal to the spacetime interval $(\Delta s'_i)^2 = (c\Delta t'_i)^2 - 0$ because the origin clock is at rest in K' (so that $\Delta\vec{x}'_i = \vec{0}$). But remember that the spacetime interval $(\Delta s'_i)^2$ is Lorentz invariant so that we can calculate *the same number* in any inertial system: $(\Delta s'_i)^2 = (\Delta s_i)^2 = (c\Delta t_i)^2 - (\Delta\vec{x}_i)^2$.

In summary, on the ith interval, the spaceship clock accumulates the time

$$\Delta\tau_i = \frac{\Delta s_i}{c} := \frac{\sqrt{\Delta s_i^2}}{c} = \frac{\sqrt{(c\Delta t_i)^2 - (\Delta\vec{x}_i)^2}}{c} = \Delta t_i \sqrt{1 - \frac{(\Delta\vec{x}_i/\Delta t_i)^2}{c^2}} \tag{2.22}$$

The above chain of arguments provided us with a *physical interpretation* for the Lorentz invariant spacetime interval $(\Delta s)^2 > 0$ of time-like separated events: It measures (up to a factor of c) the time accumulated by an inertial (= unaccelerated) clock that takes part in both events.

ii | Continuum limit $N \to \infty$ ($v(t) := |\vec{v}(t)| = |\dot{\vec{x}}(t)|$):

$$\mathrm{d}\tau = \frac{\mathrm{d}s}{c} = \mathrm{d}t\sqrt{1 - \frac{\dot{\vec{x}}(t)^2}{c^2}} \quad\Leftrightarrow\quad \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{v(t)} \tag{2.23}$$

Note that this is just an infinitesimal version of the time-dilation formula Eq. (2.14) with $\Delta t \to \mathrm{d}t$ and $\Delta t_0 \to \mathrm{d}\tau$.

Since $(\Delta s)^2 = (\Delta s')^2$ is *Lorentz invariant*:

$$K \xrightarrow{\Lambda} K' : \quad \mathrm{d}t\sqrt{1-\frac{\dot{\vec{x}}(t)^2}{c^2}} = \frac{\mathrm{d}s}{c} = \frac{\mathrm{d}s'}{c} = \mathrm{d}t'\sqrt{1-\frac{\dot{\vec{x}}'(t')^2}{c^2}} \tag{2.24}$$

You can check this also explicitly using the Lorentz transformation Eq. (1.75).

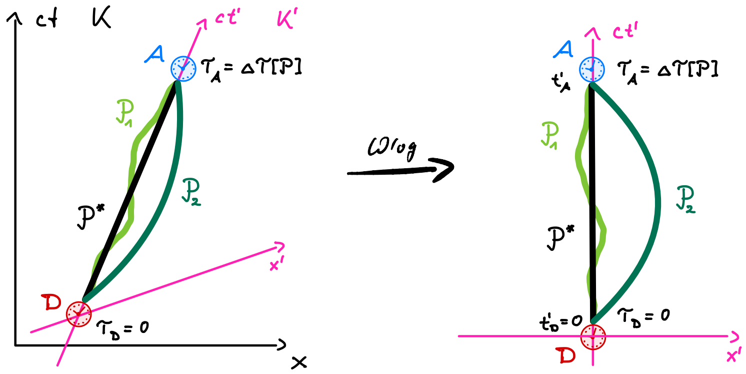
iii | → ⁂ *Proper time* accumulated by the spaceship clock along the trajectory $\mathcal{P}$:

$$\Delta\tau[\mathcal{P}] = \lim_{N\to\infty} \sum_{\substack{\text{Segment}\\ i=1}}^{N} \Delta\tau_i = \int_{\mathcal{P}} \mathrm{d}\tau = \int_{\mathcal{P}} \frac{\mathrm{d}s}{c} := \int_{t_D}^{t_A} \mathrm{d}t\sqrt{1-\frac{\dot{\vec{x}}(t)^2}{c^2}} \tag{2.25}$$

- As constructed, the proper time $\Delta\tau[\mathcal{P}]$ of a time-like trajectory $\mathcal{P}$, parametrized by $\vec{x}(t)$ for $t \in [t_0, t_1]$, is the time elapsed by a clock that follows this trajectory in spacetime.
- ¡! This result is valid for *accelerated* clocks.

  In general, SPECIAL RELATIVITY *can* described the physics of accelerated objects as long as the descpription of the process is given in an inertial coordinate system (as is the case here).
- ¡! The right-most expression in Eq. (2.25) yields the same result *in all inertial systems* K [recall Eq. (2.24)]. This is why $\tau[\mathcal{P}]$ is a function of the *event trajectory* $\mathcal{P}$ and not its coordinate parametrization $\vec{x}(t)$. This is important: It tells us that all inertial observers will agree on the reading of the spaceship clock τ_A at arrival A (although their *parametrization* $\vec{x}(t)$ may look different).
- Note that since $\vec{x}(t)$ is assumed to be time-like, it is $\forall_t : |\dot{\vec{x}}(t)| < c$ such that the radicand is always non-negative.
- $\tau[\bullet]$ is a *functional* of the trajectory $\mathcal{P}$; this is why we use square-brackets.

4 | Which trajectory $\mathcal{P}^*$ between the two events D and A *maximizes* the proper time $\Delta\tau$?



i | D and A are *time-like* separated → ∃ Inertial system $K' = K(D, A)$ with

$$[D]_{K'} = (t'_D = 0, \vec{x}'_D = \vec{0}) \quad \text{and} \quad [A]_{K'} = (t'_A, \vec{x}'_A = \vec{0}) \tag{2.26}$$

That is, without loss of generality, we can Lorentz transform into an inertial system where the two events happen at the *same* location (and by translations we can assume that this location is the origin $\vec{0}$ and that the coordinate time is $t'_D = 0$ at D). We label the time and space coordinate in K' by t' and $\vec{x}'$. Because of the relativity principle SR, K' is as good as any system to describe events.

ii | Time of an arbitrary path $\mathcal{P} \ni D, A$ with $[\mathcal{P}]_{K'} = (t', \vec{x}'(t'))$:

$$\Delta\tau[\mathcal{P}] = \int_{t'_D}^{t'_A} \mathrm{d}t' \sqrt{1 - \frac{\dot{\vec{x}}'(t')^2}{c^2}} \overset{\vec{x}'(t')\equiv\vec{0}}{\leq} \int_{t'_D}^{t'_A} \mathrm{d}t' = t'_A - t'_D = \Delta\tau[\mathcal{P}^*] \tag{2.27}$$

Here $\mathcal{P}^*$ is the trajectory between D and A that is parametrized by the constant function $\vec{x}'(t') \equiv \vec{0}$ in K'. In other inertial systems, this trajectory will not be constant; however, it is inertial, i.e., $\mathcal{P}^*$ is described by a trajectory between D and A with uniform velocity.

Check this by applying a Lorentz transformation to the coordinates $(t', \vec{0})_{K'}$!

→ Clocks that travel along the *inertial trajectory* $\mathcal{P}^*$ between D and A collect the largest proper time $\tau^* = \Delta\tau[\mathcal{P}^*]$.

Collecting the "largest time" means that the these clocks run the *fastest*.

5 | It is important to let this result sink in:

Let K' be the rest frame of earth (which is located in the origin $\vec{0}$) and consider two twins of age τ_D:

- **Twin S** departs with a **S**paceship at D, flies away from earth, turns around and returns to earth at A. **Twin S** therefore follows a trajectory similar to $\mathcal{P}_2$ in the sketches above.
- **Twin E** stays on **E**arth. He follows the inertial trajectory $\mathcal{P}^*$ in the sketches above.

We just proved above:

$$\langle\text{Age of **Twin S** at } A\rangle = \Delta\tau[\mathcal{P}_2] + \tau_D < \Delta\tau[\mathcal{P}^*] + \tau_D = \langle\text{Age of **Twin E** at } A\rangle$$

This is the famous ⁂ *Twin "paradox"*: **Twin S** aged less than **Twin E**.

6 | Why there is *no* paradox:

- If you don't see why the above result should be paradoxical:

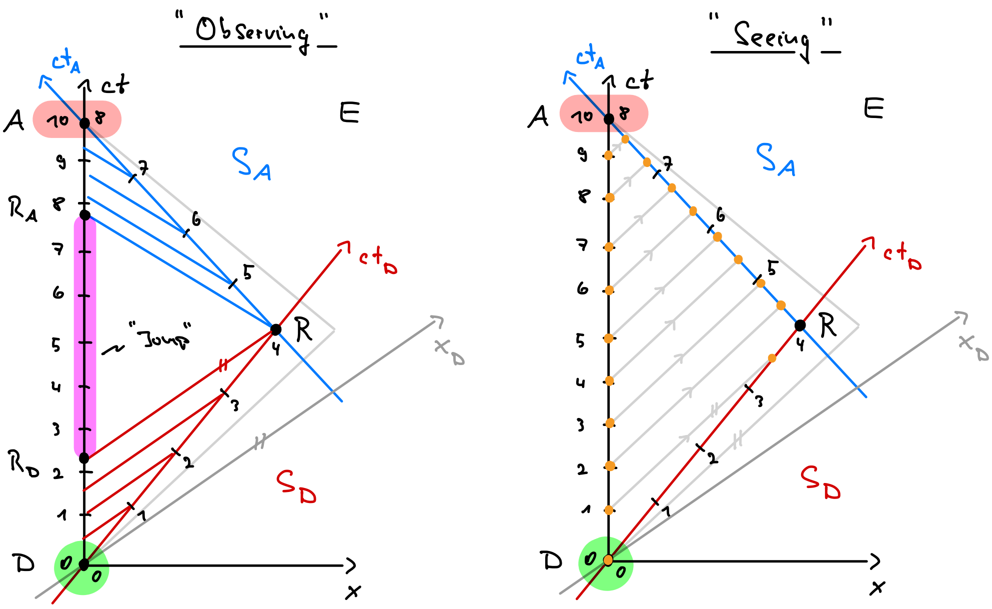
  Good! Move along. Nothing to see here! ☺
- Why one *could* conclude that the above result is paradoxical (= logically inconsistent):
  - From the view of **Twin E**, **Twin S** speeds around quickly, thus time-dilation tells him that **Twin S** should age slower. And indeed, when **Twin S** returns, he actually didn't age as much.
  - Now, you conclude, due to the relativity principle **SR**, we could also take the perspective of **Twin S** (i.e., our system of reference is now attached to the spaceship). Then **Twin S** would conclude that time-dilation makes **Twin E** (who now, together with earth, speeds around quickly) age more slowly. But this does not match up with the above result that, when both twins meet again at A, **Twin S** is the younger one! *Paradox!*

  The resultion is quite straightforwad:

  The invocation of the relativity principle **SR** in the last point is not admissible! Remember that **SR** only makes claims about the equivalence of *inertial systems*. Now have a look at the trajectory $\mathcal{P}_2$ of the spaceship again: it is clearly accelerated and cannot be inertial. And that there *is* at least a period where the spaceship (and **Twin S**) is accelerating is a *neccessity* for **Twin S** to *return* to **Twin E** (at least in flat spacetimes, but not so in curved ones [54])! This implies that the reunion of both twins at A requires at least one of them to *not* stay in an inertial system. This breaks the symmetry between the two twins and explains why the result can be (and is) asymmetric.

- ¡! For historical (and anthropocentric) reasons, the "twin paradox" is called a "paradox." We stick to this term because we have to – and not because it is appropriate name. The term "paradox" suggests an intrinsic inconsistency of RELATIVITY. As we explained above: *This is not the case.* All "paradoxes" in RELATIVITY are a consequence of unjustified, seemingly "intuitive" reasoning. The root cause is almost always an inappropriate, vague notion of "absolute simultaneity" that cannot be operationalized.
- An overview on different geometric approaches to rationalize the phenomenon can be found in Ref. [55].

  Below are two widely used spacetime diagrams of an idealized version where **Twin S** changes inertial systems only once from S_D to S_A halfway through the journey at R. You can think of this as an instantenous acceleration at the kink. Note, however, that the acceleration itself is dynamically irrelevant for the arguments; it is only important that the inertial frames in which **Twin S** departs and returns are not the same:



  - In the left diagram the *slices of simultaneity* in the two systems S_D and S_A are drawn. As predicted by time-dilation (and mandated by SR), **Twin S** observes the clocks of **Twin E** to run *slower* during his "inertial periods", i.e., while he stays in a single inertial system. However, the moment **Twin S** "jumps" from S_D to S_A at R, his notion of simultaneity changes instantaneously: In S_D, R and R_D are simultaneous; in S_A, however, R and R_A are simultaneous. Due to this jump, the record of **Twin S** contains now a temporal gap for events on earth (highlighted interval). It is this "missing" time interval that overcompensates the slower running clocks on earth (as observed from S_D and S_A) and makes **Twin S** conclude that **Twin E** ages faster (in agreement with the actual outcome of the experiment).

    If you wonder what happened to the (missing) observations of events in the triangle $R_A R R_D$: there is a nice explanation in Schutz [5]. (The bottom line is that **Twin S** constructs a bad coordinate system by stopping the recording of events in system S_D when he reaches R.)
  - In the right diagram, we draw *light signals* ("pings") of an earth-bound clock next to **Twin E** sent to **Twin S**. **Twin S** receives these signals and measures their period. This idealizes how **Twin S** *sees* (not *observes*!) the clocks ticking on earth (and, by proxy, how fast **Twin E** ages). It is important to understand the difference between this "seeing"

and our operational definition of *observing* (using the contraption called an ← *inertial system*, as used in the left diagram). As demonstrated by the diagram, **Twin S** first sees the clock on earth ticking *slower*; but when he turns around at R, the clocks on earth (apparantly) speed up significantly. In the end, this speedup overcompensates for the slowdown during the first part of the journey so that **Twin S** again arrives at the (correct) conclusion that **Twin E** ages faster. Note that the speedup of the earth-bound clock *seen* by **Twin S** during the second half of his journey does not contradict time-dilation because *seeing* is not *observing*. This is similar to the ↑ *Penrose-Terrell effect* in that a genuine relativistic effect (here: time-dilation) is distorted by an additional "imaging effect" due to the finite speed of light.

- In our careful derivation above, we not only showed that **Twin S** ages less than **Twin E**; we also showed that this conclusion is *independent* of the inertial observer! Thus we know that there will be *no dispute* about the different ages between different inertial observers.
- The Hafele-Keating experiment [50,51] and the muon decay experiments [48], mentioned previously in the context of time-dilation, are experimental confirmations of the twin "paradox." So our theoretical prediction above (that **Twin S** ages less than **Twin E**) is experimentally confirmed. End of discussion.
- Our derivation of the accumulated proper time along trajectories in spacetime is both mathematically sound and experimentally confirmed. This qualifies SPECIAL RELATIVITY as a successfull theory of physics. *Operationally* there is nothing to complain about: the theory does its job to produce quantiative predictions of real phenomena. So why do so many people (physicists included) – despite the various efforts to visualize the phenomenon – have this nagging feeling of dissatisfaction that they cannot get rid of? The reason, so I would argue, is the human brain and its proclivity to inject concepts of absolute simulateneity into its model building. This qualifies the historical overemphasis of the twin "paradox" as a *meta problem*: The question to study is not how to "solve" the twin "paradox" (as we showed above, there is nothing to solve); the question to study is why so many peoply thought (and still think) that there is a problem in the first place. This *meta problem* is an actual problem to study; but it falls into the domain of cognitive science, and not physics!

7 | Two lessons to be learned from this:

You can outlive your inertial-system-dwelling peers
by changing inertial systems (= accelerating) at least once.

- ¡! You "live longer" when speeding around than your twin on earth, i.e., when you return, your twin might be 80 and have reached the end of his lifespan while you are still in your fourties. This is a real, observable effect, not an illusion of sorts. However, "living longer" does *not* mean that you somehow have "more time to spend" than your twin because all physical phenomena in your spaceship experience the same effect. It is not your metaboslism that slows down wrt. other physical phenomena around you, it is *time* itself. Put differently: If you and your twin both try to read as many books as possible during your lifetimes (say one per month), both of you will have read roughly the same amount of books when either of you dies (say at the age of 80).
- The mere fact that our universe *really* allows for this (at least in theory) makes it much more interesting than its boring alternative: a Galilean universe.

and

Phenomena like length contraction and the twin "paradox" are physically *real*. Their "paradoxical" flavor is a phenomenon of human cognition, not physics.

This is why we put "paradox" always in quotes in the context of RELATIVITY.

↓ Lecture 7 [26.11.25]

3. Mathematical Tools I: Tensor Calculus

In this chapter we introduce tensor calculus (↑ *Ricci calculus*) for general coordinate transformations φ (which will be useful both in SPECIAL RELATIVITY and GENERAL RELATIVITY). The coordinate transformations φ relevant for SPECIAL RELATIVITY are Lorentz transformations (and therefore linear) which simplifies expressions often significantly (→ *Chapter 4*). However, this special feature of coordinate transformations in SPECIAL RELATIVITY is not crucial for the discussions in this chapter.

Goal: Construct Lorentz covariant (form invariant) equations
(for mechanics, electrodynamics, quantum mechanics)

Question: How to do this *systematically*?

Note that (we suspect that) Maxwell equations *are* Lorentz covariant. Clearly this is not obvious and requires some work to prove; we say that the Lorentz covariance is *not manifest*: it is there, but it is hard to see. Conversely, without additional tools that make Lorentz covariance more obvious, it is borderline impossible to *construct* Lorentz covariant equations from scratch (which we must do for mechanics and quantum mechanics!).

We are therefore looking for a "toolkit" that provides us with elementary "building blocks" and a set of rules that can be used to construct Lorentz covariant equations. This toolbox is known as *tensor calculus* or ↑ *Ricci calculus*; the "building blocks" are tensor fields and the rules for their combination are given by index contractions, covariant derivatives, etc. The rules are such that the expressions (equations) you can build with tensor fields are *guaranteed* to be Lorentz covariant. This implies in particular that if you can rewrite any given set of equations (like the Maxwell equations) in terms of these rules, you automatically show that the equations were Lorentz covariant all along. We then say that the Lorentz covariance is *manifest*: one glance at the equation is enough to check it.

Later, in GENERAL RELATIVITY, our goal will be to construct equations that are invariant under *arbitrary* (differentiable) coordinate transformations (not just global Lorentz transformations). Luckily, the formalism we introduce in this chapter is powerful enough to allow for the construction of such → *general covariant* equations as well. This is why we keep the formalism in this chapter as general as possible, and specialize it to SPECIAL RELATIVITY in the next Chapter 4. The discussion below is therefore already a preparation for GENERAL RELATIVITY; it is based on Schröder [1] and complemented by Carroll [56].

3.1. Manifolds, charts and coordinate transformations

1 | D-dimensional Manifold

= Topological space that *locally* "looks like" D-dimensional Euclidean space $\mathbb{R}^D$: