ii | The variational derivative Eq. (11.101) simplifies because L is again independent of derivatives of the metric:

$$\frac{\delta(\sqrt{g}L)}{\delta g^{\mu\nu}} \overset{11.101}{=} \frac{\partial(\sqrt{g}L)}{\partial g^{\mu\nu}} \overset{11.113}{=} -\frac{1}{2}\sqrt{g}g_{\mu\nu}L + \frac{1}{2}\sqrt{g}(\partial_\mu\phi)(\partial_\nu\phi) \tag{11.118}$$

iii | Eq. (11.107) $\overset{\circ}{\rightarrow}$

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\left(\phi^{,\alpha}\phi_{,\alpha} - m^2\phi^2\right) \tag{11.119}$$

This is the HEMT of the real Klein-Gordon field.

iv | Let us verify that it satisfies Eq. (11.110):

$$T^{\mu\nu}{}_{;\nu} = (\phi^{,\mu}\phi^{,\nu})_{;\nu} - \frac{1}{2}g^{\mu\nu}\left(\phi^{,\alpha}\phi_{,\alpha} - m^2\phi^2\right)_{;\nu} \tag{11.120a}$$

$$= \phi^{,\mu}\phi^{,\nu}{}_{;\nu} \underbrace{+\phi^{,\mu}{}_{;\nu}\phi^{,\nu} - g^{\mu\nu}\phi^{,\alpha}\phi_{,\alpha;\nu}}_{\overset{10.53}{=}0} + m^2\phi^{,\mu}\phi \tag{11.120b}$$

$$= \phi^{,\mu}\underbrace{\left(\phi^{,\nu}{}_{;\nu} + m^2\phi\right)}_{\overset{11.38}{=}0} \doteq 0\,. \tag{11.120c}$$

Here we used $g^{\mu\nu}{}_{;\rho} = 0$ since the connection is metric-compatible. Note that we had to invoke the equation of motion in the last step to show that the divergence vanishes.

↓ Lecture 25 [16.06.26]

11.5. ‡ Killing vector fields and conservation laws

As noted previously, Eq. (11.110) cannot be used to define conserved quantities in general. Here we discuss this in more detail and characterize the conditions under which conserved quantities actually do exist.

Killing (vector) fields are named after German mathematician ↑ *Wilhelm Karl Joseph Killing* (1847–1923). Note that this is a situation were case sensitivity is of paramount importance: *Killing fields* and *the* ↑ *Killing Fields* are not related whatsoever; luckily, we're only concerned with the former.

Details: ➔ Problemset 6

1 | Problem: [Compare this to Eq. (6.85) which implies Eq. (6.87).]

$$T^{\mu\nu}{}_{;\nu} \overset{10.57a}{=} T^{\mu\nu}{}_{,\nu} + \Gamma^{\nu}{}_{\rho\nu}T^{\mu\rho} + \Gamma^{\mu}{}_{\rho\nu}T^{\rho\nu} \overset{11.110}{\doteq} 0 \tag{11.121a}$$

$$\overset{10.93}{\Longleftrightarrow} \quad \frac{1}{\sqrt{g}}\left(\sqrt{g}T^{\mu\nu}\right)_{,\nu} + \underbrace{\Gamma^{\mu}{}_{\rho\nu}T^{\rho\nu}}_{\text{No surface term ☹}} \doteq 0 \tag{11.121b}$$

→ Cannot be integrated by Gauss to define conserved charge!

→ *Question:* Are there conditions under which conserved quantities can be defined?

2 | Let $\varepsilon^\mu(x) \equiv \varepsilon\,\xi^\mu(x)$ with $\varepsilon \ll 1$; ◁ Eq. (11.105) and demand

$$\delta_\varepsilon g^{\mu\nu} = \varepsilon\left(\xi^{\mu}{}_{;\lambda}g^{\lambda\nu} + \xi^{\nu}{}_{;\lambda}g^{\lambda\mu}\right) \overset{!}{=} 0 \tag{11.122}$$

For a given metric, this is a differential equation to be solved for vector fields $\xi^\mu(x)$.

Pull indices down →

$$\boxed{\begin{aligned} \xi_{\mu;\nu} + \xi_{\nu;\mu} &= 0 \quad &&\text{⁂ Killing equation} \\ \Leftrightarrow \ \xi^\mu(x) & &&\text{⁂ Killing (vector) field} \end{aligned}} \tag{11.123}$$

- Whether the Killing Eq. (11.123) has solutions depends on the metric $g_{\mu\nu}$; it can have none / one / multiple solutions. One then says that the metric has no / one / multiple Killing fields.
- At this point it is unclear how Killing fields can help us to find conserved quantities. However, it is intuitively clear what Killing fields are: They generate a continuous group of diffeomorphisms that do not change the metric! Such special diffeomorphisms are called ↑ *isometries* of the Riemannian manifold; *i.e.*, the infinitesimal generators of the isometry group of a Riemannian manifold are the Killing fields. This group obviously depends on the geometry of the manifold, and thus its metric. Killing fields therefore characterize the *symmetries* of a spacetime manifold. A generic spacetime will have "bumps" and "twists", and therefore no symmetries/Killing fields.
- Example: Minkowski space:

  On Minkowski space we can use global inertial coordinates, so that the covariant derivatives become partial derivatives everywhere:

  $$\xi_{\mu,\nu} + \xi_{\nu,\mu} = 0\,. \tag{11.124}$$

  It is straightforward to check that the most general solution reads

  $$\xi_\mu(x) = a_\mu + b_{\mu\nu}x^\nu \quad \text{with} \quad b_{\mu\nu} = -b_{\nu\mu} \tag{11.125}$$

  with a_μ and $b_{\mu\nu}$ arbitrary integration constants. There are 4 linearly independent solutions parametrized by a^μ, and 6 solutions parametrized by the antisymmetric $b_{\mu\nu}$; so in total Minkowski space has 10 Killing vector fields. These correspond to 4 *translations* (the a_μ-solutions), 3 spatial rotations and 3 boosts (the $b_{\mu\nu}$-solutions). The isometric diffeomorphisms of Minkowski space are the ← *Poincaré transformations*! Recall our discussion of the Lorentz group in Section 4.3.
- One can show that a $3+1$-dimensional spacetime can have at most 10 Killing fields. In general, a D-dimensional Riemannian manifold can have at most $\frac{1}{2}D(D+1)$ Killing fields; such manifolds are called ↑ *maximally symmetric spaces*. Minkowski space is an example of a maximally symmetric space. For more details: ↑ Carroll [4] (§3.8 & §3.9, pp. 133–144).

3 | Conserved quantities:

We show now that Killing fields can be used to construct conserved quantities:

i | ◁ Spacetime with Killing vector field ξ^μ → Define the 4-current

$$\boxed{J^\mu_\xi := T^{\mu\nu}\xi_\nu\,.} \tag{11.126}$$

Then it follows for the covariant divergence of this current:

$$\frac{1}{\sqrt{g}}\left(\sqrt{g}J^\mu_\xi\right)_{,\mu} \overset{10.96}{=} J^\mu_{\xi;\mu} = (T^{\mu\nu}\xi_\nu)_{;\mu} \overset{11.123}{=} \xi_\nu T^{\mu\nu}{}_{;\mu} \underset{11.110}{\doteq} 0 \tag{11.127}$$

Here we used that the contraction of a symmetric with an antisymmetric tensor vanishes.

ii | Then the charge

$$\boxed{Q_\xi := \int \mathrm{d}^3x \sqrt{g} J^0_\xi \quad \text{is conserved:} \quad \frac{\mathrm{d}Q_\xi}{\mathrm{d}x^0} \doteq 0\,.} \tag{11.128}$$

This is true if $J^i_\xi = 0$ on the boundary of space; *i.e.*, if one considers closed systems.

Proof. From Eq. (11.127) we have

$$0 \doteq \int \mathrm{d}^3x \left(\sqrt{g} J^\mu_\xi\right)_{,\mu} = \int \mathrm{d}^3x \left(\sqrt{g} J^0_\xi\right)_{,0} + \int \mathrm{d}^3x \left(\sqrt{g} J^i_\xi\right)_{,i} \tag{11.129}$$

and therefore

$$\frac{\mathrm{d}Q_\xi}{\mathrm{d}x^0} = \int \mathrm{d}^3x \left(\sqrt{g} J^0_\xi\right)_{,0} \doteq -\int \mathrm{d}^3x \left(\sqrt{g} J^i_\xi\right)_{,i} \overset{\text{Gauss}}{=} -\int_\partial \mathrm{d}\sigma_i \sqrt{g} J^i_\xi = 0\,. \tag{11.130}$$

Here we used the assumption that the current vanishes on the spatial surface ∂: $J^i_\xi = 0$. ■

iii | For the special case of point mechanics, one finds a more explicit expression:

⊲ Free particle described by the equation of motion:

$$\frac{\mathrm{D}u^\mu}{\mathrm{D}\tau} = 0 \quad \text{with 4-velocity } u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{11.131}$$

→ With the Killing vector ξ^μ it follows:

$$\frac{\mathrm{d}(\xi_\mu u^\mu)}{\mathrm{d}\tau} = \frac{\mathrm{D}(\xi_\mu u^\mu)}{\mathrm{D}\tau} = \underbrace{u^\mu \frac{\mathrm{D}\xi_\mu}{\mathrm{D}\tau}}_{=0} + \underbrace{\xi_\mu \frac{\mathrm{D}u^\mu}{\mathrm{D}\tau}}_{=0} \overset{\circ}{=} 0 \tag{11.132}$$

The second summand vanishes because of Eq. (11.131) and the first one because of the Killing Eq. (11.123):

$$u^\mu \frac{\mathrm{D}\xi_\mu}{\mathrm{D}\tau} \overset{10.49}{=} u^\mu \xi_{\mu;\nu} u^\nu = 0\,. \tag{11.133}$$

Here we used that the Killing equation tells us that $\xi_{\mu;\nu}$ is an antisymmetric tensor and that the contraction of a symmetric with an antisymmetric tensor vanishes.

→ Conserved quantity:

$$\boxed{\xi_\mu u^\mu = \text{const}} \tag{11.134}$$

This can be useful to integrate the geodesic equation on symmetric spacetimes.

4 | Stationary & Static spacetimes:

Using Killing fields, we can define two special classes of spacetimes with useful properties:

i | We can define the following special class of metrics g:

$$\boxed{g \text{ is ⁂ } \textit{stationary} \quad :\Leftrightarrow \quad g \text{ has (asymptotically) } \textit{time-like} \text{ Killing vector}}$$

"Asymptotically time-like" means that there is a Killing vector field that becomes time-like at infinity. It is possible that the Killing field becomes space-like in some finite region of space.

ii | $\triangleleft$ Time-like Killing field ξ^μ (*i.e.*, $\xi^\mu \xi_\mu > 0$)

→ Choose wlog coordinates where $\xi^\mu = (1, 0, 0, 0)$

This means that we choose the x^0-axis such that it points along the Killing field.

→

$$\text{Killing Eq. (11.123)} \quad \xleftrightarrow{\text{Eq. (11.104)}} \quad \frac{\partial g^{\mu\nu}}{\partial x^0} = 0 \tag{11.135}$$

→ $g_{\mu\nu}(x) = g_{\mu\nu}(\vec{x})$ independent of the time-like coordinate $x^0 \equiv t$!

→ In these coordinates, a stationary metric has the general form:

$$\boxed{\mathrm{d}s^2 = g_{00}(\vec{x})\mathrm{d}t^2 + g_{0i}(\vec{x})\mathrm{d}t\,\mathrm{d}x^i + g_{i0}(\vec{x})\mathrm{d}x^i\,\mathrm{d}t + g_{ij}(\vec{x})\mathrm{d}x^i\,\mathrm{d}x^j} \tag{11.136}$$

- *Interpretation:* A *stationary* spacetime "does exactly the same thing at every time" [4].
- *Example:* The ↑ *Kerr metric* of a rotating black hole is stationary.

iii | There is an interesting subclass of stationary spacetimes:

$\triangleleft$ Stationary metric g. *Iff* there exists a coordinate system such that $g_{0i}(\vec{x}) = 0$,

$$\boxed{\mathrm{d}s^2 = g_{00}(\vec{x})\mathrm{d}t^2 + g_{ij}(\vec{x})\mathrm{d}x^i\,\mathrm{d}x^j} \tag{11.137}$$

the metric is called ⁂ *static*.

- *Interpretation:* A *static* spacetime "doesn't do anything at all" [4].
- *Example:* The (exterior) → *Schwarzschild metric* of a non-rotating black hole is static.
- The feature that distinguishes a stationary (non-static) metric Eq. (11.136) from a static metric Eq. (11.137) is that the latter is invariant under reversal $t \mapsto -t$ of the time-like coordinate.

  This makes sense if you compare the static Schwarzschild metric (non-rotating black hole) with the stationary Kerr metric (rotating black hole): If you invert time, a non-rotating black hole looks exactly the same, so also its metric should not change; hence it has the form Eq. (11.137) and is static. By contrast, the angular momentum of a rotating Kerr black hole changes sign under $t \mapsto -t$, so that it looks different in a time-reversed world; hence its metric should change as well, which necessitates the off-diagonal terms of a stationary (non-static) metric Eq. (11.136). (Note that the Kerr black hole still describes a *stationary* system in that its angular momentum doesn't change in time, and consequently the metric "does the same thing at every time".)

iv | In summary:

$$\boxed{\textit{Static}\text{ spacetime} \quad \substack{\Rightarrow \\ \not\Leftarrow} \quad \textit{Stationary}\text{ spacetime}}$$

- *Counterexample:* The ↑ *Kerr metric* is stationary but not static.

12. The Einstein Field Equations

We are only one step away from completing the theoretical framework of GENERAL RELATIVITY.

In the previous Chapter 11 we studied how matter fields are affected by the metric of spacetime. What we are missing is the converse: How is the metric of spacetime determined in the first place? This is the question we will answer in this chapter, and it will lead us to the most important result of this course: The Einstein field equations.

12.1. Derivation of the Einstein field equations

In the following, we make the following general (and rather weak) assumptions:

§ Assumptions

- **3P1** Spacetime is a $3+1$-dimensional Lorentzian manifold.
- **MTR** There exists *one* dynamical metric field g.
- **FLD** All other degrees of freedom ("matter") are described by fields ϕ.

  Note that we write ϕ as placeholder for a family of (not necessarily scalar) fields.
- **VAR** The classical dynamics of all fields can be described by a variational principle.
- **LOC** All actions are given by integrals over local Lagrangians.
- **COV** All theories are generally covariant (**GR**).

There will be only *one* additional assumption → *below*.

Follow the arguments below carefully; each step is quite simple, so that the derivation borders on magic:

1 | The action of Everything:

MTR + **FLD** + **VAR** $\to \sphericalangle$ "Action of Everything":

$$S[g,\phi] = \underbrace{S[g]}_{\text{Only metric}} + \underbrace{S_g[\phi]}_{\text{Rest}} \tag{12.1}$$

Without loss of generality, we can divide the action into a purely metric part and a "rest", all terms of which contain at least one matter field.

Combining GENERAL RELATIVITY with the Standard Model of particle physics tells us what this action actually looks like (at least in the infrared limit), recall the ← *Core Theory* mentioned in Section 0.4. These details are not relevant for what follows, though.

2 | Equations of motion of Everything:

As usual, the physical solutions extremize the action (variational principle):

$$\text{VAR} \rightarrow \quad \delta S[g,\phi] \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \begin{cases} \delta_g S[g,\phi] = \delta_g S[g] + \delta_g S_g[\phi] \stackrel{!}{=} 0 \\ \delta_\phi S[g,\phi] = \delta_\phi S_g[\phi] \stackrel{!}{=} 0 \end{cases} \tag{12.2}$$

To extremize the action, both equations on the right must be satisfied simultaneously:

- EOMs for *matter fields*:

$$\delta_\phi S_g[\phi] \stackrel{!}{=} 0 \tag{12.3}$$

These equations describe the dynamics of *matter fields* on a given "background" metric g.

→ Already known & understood! (← Chapter 11)

An example is the Maxwell action (11.78), the variation of which leads to the Maxwell Eq. (11.79).

- EOMs for *metric field*:

$$\delta_g S[g] \stackrel{!}{=} -\delta_g S_g[\phi] \tag{12.4}$$

These equations describe the dynamics of the *metric* and its interaction with matter.

→ New! What can we say about this equation of motion?

3 | LOC →

Because of locality, we can write both parts of the action as integrals of Lagrangian(densities):

$$S[g] = \int d^4x\,\sqrt{g}\,L_{\text{Metric}}(g,\partial g) \tag{12.5a}$$

$$S_g[\phi] = \int d^4x\,\sqrt{g}\,L_{\text{Matter}}(\phi,\partial\phi,g,\partial g) \tag{12.5b}$$

These expressions actually require an additional prefactor $\frac{1}{c}$ for dimensional reasons because we measure time coordinates in units of length ($x^0 = ct$); we omit these prefactors because they are irrelevant in the following and drop out in the next step anyway.

Eq. (12.4) is then equivalent to →

$$\int \underbrace{d^4x\,\sqrt{g}}_{\text{Scalar}}\,\underbrace{\frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}L_{\text{Metric}})}{\delta g^{\mu\nu}}}_{=:\,\Box_{\mu\nu}\,\text{(unknown)}}\,\underbrace{\delta g^{\mu\nu}}_{\text{Variation}} \stackrel{!}{=} -\int \underbrace{d^4x\,\sqrt{g}}_{\text{Scalar}}\,\underbrace{\frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}L_{\text{Matter}})}{\delta g^{\mu\nu}}}_{\stackrel{11.107}{=}\,T_{\mu\nu}}\,\underbrace{\delta g^{\mu\nu}}_{\text{Variation}} \tag{12.6}$$

Here we used the variational derivative Eq. (11.101), multiplied the equation by 2 and inserted $\sqrt{g}$ to identify the Hilbert energy-momentum tensor Eq. (11.107) on the right-hand side.

Eq. (12.6) valid for all variations $\delta g^{\mu\nu}(x)$ →

$$\delta_g S[g,\phi] \stackrel{!}{=} 0 \quad \Leftrightarrow \quad \Box_{\mu\nu} \stackrel{!}{=} -T_{\mu\nu} \tag{12.7}$$

4 | What do we know about $\Box_{\mu\nu}$?

- $\Box_{\mu\nu}$ depends on metric g and its derivatives ∂g, $\partial^2 g$, …

  Note that $\Box_{\mu\nu}$ does not contain matter fields ϕ by construction.

- COV → $\Box_{\mu\nu}$ is $(0,2)$-tensor

  This follows from the definition in Eq. (12.6) with the same arguments as for $T_{\mu\nu}$ in Section 11.4.

- $\Box_{\mu\nu}$ is symmetric

  This follows from the definition in Eq. (12.6) with the same arguments as for $T_{\mu\nu}$ in Section 11.4.

- COV → $\Box_{\mu\nu}$ is identically divergence-free: $\Box^{\mu\nu}{}_{;\nu} \equiv 0$

  An often heard argument for this condition is the following: Since $T_{\mu\nu}$ satisfies $T^{\mu\nu}{}_{;\nu} \doteq 0$ [recall Eq. (11.110)], Eq. (12.7) implies that $\Box^{\mu\nu}{}_{;\nu} = 0$. This argument is sloppy at best because of the little dot over the equal sign in $T^{\mu\nu}{}_{;\nu} \doteq 0$; recall that this indicates that the equation is only true for *special* matter fields, namely those that satisfy the equation of motion Eq. (12.3). Thus, from this line of argument, one can only conclude that $\Box^{\mu\nu}{}_{;\nu} \doteq 0$, *i.e.*, $\Box_{\mu\nu}$ is divergence-free *for solutions* (g,ϕ) *of the equations of motion*.

  But our claim – which is crucial for the next step – is much stronger: $\Box^{\mu\nu}{}_{;\nu} \equiv 0$ is an *identity* (that's why we use $\equiv$ and not $=$), *i.e.*, it is valid for *arbitrary* metric fields. That this must be true follows from our derivation in Section 11.4: Note that, because of Eq. (12.6), $\Box_{\mu\nu}$ plays formally (not physically, → *later*) the role of $T_{\mu\nu}$ for a purely gravitational theory $S[g] = S_g[\bullet]$ without matter fields. One can then retrace our derivation in Section 11.4 with the simplification that $\delta_\phi S_g[\phi] \equiv 0$ is trivially satisfied (because there are no fields ϕ). Instead of $\Box^{\mu\nu}{}_{;\nu} \doteq 0$, one finds the identity $\Box^{\mu\nu}{}_{;\nu} \equiv 0$. This unconditional identity is therefore a consequence of the general covariance (= diffeomorphism invariance) of the gravitational action $S[g]$ and the fact that it does not depend on any other fields. $\Box^{\mu\nu}{}_{;\nu} \equiv 0$ is an example of a so called ↑ *Noether identity* that follows, via Noether's *second* theorem, from a group of local (gauge) symmetries (here: diffeomorphisms) [159].

These are all necessary properties of $\Box_{\mu\nu}$; no discussions!

5 | We now make one (and the only) simplifying assumption, namely:

> **§ Assumptions**
>
> 2ND The tensor $\Box_{\mu\nu}$ depends on g, ∂g, $\partial^2 g$ (but not on higher-order derivatives).

- This is the only simplicity assumption we use in our derivation. If you drop it, you can construct (more complicated) *modifications* of GENERAL RELATIVITY (→ *later*).
- Can you think of any equation of motion (classical or quantum, doesn't matter) that contains third- or even higher-order derivatives? No? Nothing? So our assumption isn't that outlandish after all …

6 | <u>Lovelock's theorem:</u>

We already know *two* tensors that satisfy all these properties:

$$\text{Metric } g_{\mu\nu}: \quad g^{\mu\nu}{}_{;\nu} \overset{10.74}{=} 0 \quad \text{(Metric-compatibility)} \tag{12.8a}$$

$$\text{Einstein tensor } G_{\mu\nu}: \quad G^{\mu\nu}{}_{;\nu} \overset{10.123}{=} 0 \quad \text{(Bianchi identity)} \tag{12.8b}$$

Recall that the Einstein tensor depends linearly on the curvature tensor which, in turn, depends on second (and first) derivatives of the metric.

3P1 + 2ND $\xrightarrow{\uparrow\,\textit{Lovelock's theorem}}$ This list is exhaustive!

Lovelock's theorem states that, in $D = 4$ spacetime dimensions, the only divergence-free rank-2 tensors that can be constructed from the metric and its first and second derivatives are the Einstein tensor $G_{\mu\nu}$ and the metric $g_{\mu\nu}$ itself (for details see notes → *below*).

- Since Lovelock's theorem is just a mathematical fact with a technical proof [142, 143], we take it at face value. Note that this does not open a conceptual gap in our derivation. We do not push any assumptions under the rug! See also Misner *et al.* [3] (§17.1, pp. 407–408).
- Consider rank-2 tensors A^{ij} that are …
  (a) … functions of the metric and its first two derivatives: $A^{ij} = A^{ij}(g, \partial g, \partial^2 g)$.
  (b) … divergence-free: $A^{ij}{}_{;j} = 0$.
  (c) … symmetric: $A^{ij} = A^{ji}$.
  (d) … linear in $\partial^2 g$.

  The statement of Lovelock's theorem is the following:

  The only tensors with the properties (a)-(d) are G_{ij} and g_{ij}.

  (This result is actually not due to Lovelock but Cartan, Weyl and Vermeil, see references in [143].)

  Note that this statement is independent of the spacetime dimension D!

  However, if one presumes that spacetime is $D = 4$-dimensional (which we did anyway starting from Chapter 11), Lovelock showed [143] that the assumptions of symmetry (c) and linearity in the second derivative (d) are *superfluous* and can be dropped!

  Thus we are left with the only *non-trivial assumption that the EOM of the metric field does not contain higher than second derivatives of the metric.*

→ Most general form of Eq. (12.7):

$$\Box_{\mu\nu} = \alpha G_{\mu\nu} + \beta g_{\mu\nu} \stackrel{!}{=} -T_{\mu\nu} \tag{12.9}$$

Note all conditions above are preserved by linear combinations.

7 | ⁂ *Einstein field equations (EFE)*:

Let us reshuffle and rename the unknown constants α and β a bit:

$$\boxed{\underbrace{\overbrace{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}}^{G_{\mu\nu}} + \Lambda g_{\mu\nu}}_{\text{“Geometry”}} = - \underbrace{\kappa T_{\mu\nu}}_{\text{“Matter”}}} \tag{12.10}$$

Two unknown parameters:

- ⁂ *Einstein gravitational constant* κ
- ⁂ *Cosmological constant* Λ

We will discuss these two parameters → *below*.

Notes:

- The minus on the right-hand side of Eq. (12.10) depends on the convention; here we follow Schröder [2] (who follows the original convention by Einstein). There is a plethora of sign conventions in the literature, in some of which the minus in Eq. (12.10) is not present [↑ Misner *et al.* [3] (first page)].
- If one removes the inconsistency of the linearized tensor gravity discussed in Section 8.2 (*cf.* Eq. (8.10), see also ➔ Problemset 1), one inevitably ends up with the Einstein field Eq. (12.10) [105]. Recall that we identified the *linearity* of Eq. (8.10) as the root cause for its inconsistency; in Section 8.2 we then argued on very general grounds that a relativistic theory of gravity must be *non-linear* in the gravitational field. Eq. (12.10) satisfies this: the Einstein tensor is *non-linear* in the metric (and its derivatives).

  →

  The superposition principle is *not* valid for the EFE!

  This makes solving the EFE extremely hard in general.
- The above derivation of the EFEs used surprisingly few (and simple) assumptions. This makes the EFEs very "generic," and one shouldn't be surprised that there are many different routes to derive them. An overview over alternative derivations (or axiomatizations) of the Einstein field equations can be found in Misner *et al.* [3] (pp. 417–428).
- The EFEs are the Euler-Lagrange equations that come from the variation of an action (which we don't know yet); *i.e.*, we formulate general relativity in the ↓ *Lagrangian formalism*. There is also a ↓ *Hamiltonian formulation* of general relativity, the so called ↑ *ADM formalism* [160], which plays an important role as a starting point for some theories of quantum gravity.
- The Einstein field equations are both very simple and very complicated:

  They are simple in the sense that their derivation doesn't need much physical input; as we have seen above, under very general assumptions (like general covariance), the EFEs are *inevitable*. In that sense, general relativity is a very "cheap" theory (we don't have to "pay" with a lot of assumptions about reality).

  On the other hand, because of their non-linearity, the EFEs are mathematically extremely complicated and hard to solve (→ *below*). This sounds bad, but is actually their greatest strength: because of their complexity, they predict and describe a plethora of non-trivial, unanticipated phenomena [black holes, gravitational waves, gravitational lenses, an expanding universe, ...; all of this is hidden in the innocuous-looking Eq. (12.10)].

  Good physical theories have a high "compression ratio" of input vs. output: they describe a variety of phenomena with little input. This makes the EFEs (and thereby general relativity) one of the most successful physical theories of all time.

  It is almost *too* good, at least as a starting point for a "theory of everything" (presumably a theory of quantum gravity). To find such a theory, we need *input*: features and phenomena of reality that we can use as starting points to "bootstrap" a more fundamental theory. The problem is that general relativity tells us that a big chunk of the crazy stuff happening in our world (black holes *etc.*) can be traced back to Eq. (12.10), which, as we have seen, is implied by rather generic assumptions about reality. Thus, while every viable theory of quantum gravity must necessarily lead to Eq. (12.10) in a classical regime, this might not be such a distinguishing feature as one might hope. Put differently: it might turn out to be hard to write down reasonable theories of quantum gravity that *do not* lead to Eq. (12.10).

- Some historical notes:
  - A precursor to GENERAL RELATIVITY was developed by Einstein and his friend and colleague Marcel Grossmann (a mathematician who introduced Einstein to differential geometry) already in 1913, the so called *"Entwurftheorie"* [129]; it contained essentially all parts needed to formulate GENERAL RELATIVITY, but not yet the correct field Eq. (12.10).
  - Einstein developed GENERAL RELATIVITY, culminating the EFEs Eq. (12.10), in a sequence of papers between October and November 1915 in the *"Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin"*:
    - On 4. November 1915, Einstein publishes *"Zur allgemeinen Relativitätstheorie"* [12] (extended by an addendum), where he proposed the (not yet quite correct) field equations $R_{\mu\nu} = -\kappa T_{\mu\nu}$. That is, he still missed the term $-\frac{1}{2}g_{\mu\nu}R$ that converts $R_{\mu\nu}$ into the Einstein tensor $G_{\mu\nu}$ (which satisfies the necessary condition $G^{\mu\nu}{}_{;\nu} \equiv 0$). (Beware: Einstein denoted the Ricci tensor by $G_{\mu\nu}$ ☺☹.)
    - On 25. November 1915, Einstein published in *"Die Feldgleichungen der Gravitation"* [13] finally the correct field equations (without cosmological constant).

      (Beware: Einstein's notation differs from the modern notation, so be careful when comparing Ref. [13] with Eq. (12.10); ➔ Problemset 6.)
    - On 8. Februar 1917, Einstein introduces the cosmological term $\Lambda g_{\mu\nu}$ in *"Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie"* [15] [Eq. (13a) on p. 151].
  - The German mathematician David Hilbert arrived at the Einstein field equations almost at the same time as Einstein (↑ p. 8 in Ref. [161]). Hilbert introduced the HEMT Eq. (11.107) and obtained Eq. (12.10) (without cosmological constant) directly via the variation of an action (the → *Einstein-Hilbert action*), derived from a Lagrangian (which Hilbert called *"Weltfunktion"*, essentially our "Action of Everything").
  - A first comprehensive account of GENERAL RELATIVITY, summarizing all his previous results that had appeared in many different papers, was provided by Einstein in *"Die Grundlage der allgemeinen Relativitätstheorie"* in 1916 [23].
  - Details on the historical genesis of the Einstein field equations can be found in Ref. [162].

8 | Trace-inverted form:

Let $T := T^\mu{}_\mu$ be the trace of the energy-momentum tensor $\overset{\circ}{\rightarrow}$

$$\boxed{\text{Eq. (12.10)} \quad \Leftrightarrow \quad R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu}} \tag{12.11}$$

This is the (completely equivalent) *trace-inverted* form of the Einstein field equations.

Proof. Taking the trace on both sides of Eq. (12.10) yields

$$R^\mu{}_\mu - \frac{1}{2}R\delta^\mu_\mu + \Lambda\delta^\mu_\mu = -\kappa T^\mu{}_\mu \quad \Leftrightarrow \quad R = \kappa T + 4\Lambda \tag{12.12}$$

where we used $\delta^\mu_\mu = 4$. We can now apply Eq. (12.12) to replace R in Eq. (12.10),

$$R_{\mu\nu} - \frac{1}{2}(\kappa T + 4\Lambda)g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}\,, \tag{12.13}$$

which can be reshuffled to Eq. (12.11). ■

9 | Vacuum field equations:

The cosmological constant has the interpretation of a vacuum energy (→ *below*). If we assume this contribution to be absent, "vacuum" means "no energy & momentum":

$$\boxed{\text{Eq. (12.11)} \xrightarrow{\Lambda=0,\, T_{\mu\nu}=0} R_{\mu\nu} = 0} \tag{12.14}$$

→ Vacuum solutions = ⁂ *Ricci-flat* spacetime manifolds

- ¡! Note that $R_{\mu\nu\rho\sigma} = 0$ implies $R_{\mu\nu} = 0$ *but not the other way around*! Ricci-flat spacetimes are therefore not necessarily flat (= Minkowskian).
- Note that $R_{\mu\nu} = 0$ is equivalent to $G_{\mu\nu} = 0$.
- Simplest solution: Minkowski space $g_{\mu\nu} = \eta_{\mu\nu}$

  There are also more complicated, non-trivial solutions; *e.g.*, the → *Schwarzschild solution*, which describes the exterior geometry of a spherically symmetric mass, or gravitational wave solutions (note that these waves propagate through vacuum: $T_{\mu\nu} = 0$).
- Eq. (12.14) looks simple, right? Well, not so much:

$$0 \stackrel{!}{=} R_{\mu\nu} \tag{12.15a}$$

$$\stackrel{\substack{10.106\\10.107\\10.115}}{=} \tfrac{1}{2} g^{\lambda\pi} \left(g_{\lambda\pi,\mu,\nu} + g_{\mu\nu,\lambda,\pi} - g_{\lambda\nu,\mu,\pi} - g_{\mu\pi,\lambda,\nu}\right) + g^{\lambda\pi}\left(\Gamma_{\rho\mu\nu}\Gamma^{\rho}{}_{\lambda\pi} - \Gamma_{\rho\mu\pi}\Gamma^{\rho}{}_{\lambda\nu}\right) \tag{12.15b}$$

$$\begin{aligned}\stackrel{10.80}{=} & \tfrac{1}{2} g^{\lambda\pi} \left(g_{\lambda\pi,\mu,\nu} + g_{\mu\nu,\lambda,\pi} - g_{\lambda\nu,\mu,\pi} - g_{\mu\pi,\lambda,\nu}\right) \\ & + \tfrac{1}{4} g^{\lambda\pi} g^{\rho\eta} \left(g_{\rho\mu,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}\right)\left(g_{\eta\lambda,\pi} + g_{\pi\eta,\lambda} - g_{\lambda\pi,\eta}\right) \\ & - \tfrac{1}{4} g^{\lambda\pi} g^{\rho\eta} \left(g_{\rho\mu,\pi} + g_{\pi\rho,\mu} - g_{\mu\pi,\rho}\right)\left(g_{\eta\lambda,\nu} + g_{\nu\eta,\lambda} - g_{\lambda\nu,\eta}\right)\end{aligned} \tag{12.15c}$$

$$\begin{aligned}= & \tfrac{1}{2} g^{\lambda\pi} \left(g_{\lambda\pi,\mu,\nu} + g_{\mu\nu,\lambda,\pi} - g_{\lambda\nu,\mu,\pi} - g_{\mu\pi,\lambda,\nu}\right) \\ & + \tfrac{1}{4} g^{\lambda\pi} g^{\rho\eta} \begin{bmatrix} g_{\rho\mu,\nu} g_{\eta\lambda,\pi} + g_{\nu\rho,\mu} g_{\eta\lambda,\pi} - g_{\mu\nu,\rho} g_{\eta\lambda,\pi} \\ + g_{\rho\mu,\nu} g_{\pi\eta,\lambda} + g_{\nu\rho,\mu} g_{\pi\eta,\lambda} - g_{\mu\nu,\rho} g_{\pi\eta,\lambda} \\ - g_{\rho\mu,\nu} g_{\lambda\pi,\eta} - g_{\nu\rho,\mu} g_{\lambda\pi,\eta} + g_{\mu\nu,\rho} g_{\lambda\pi,\eta} \\ - g_{\rho\mu,\pi} g_{\eta\lambda,\nu} - g_{\pi\rho,\mu} g_{\eta\lambda,\nu} + g_{\mu\pi,\rho} g_{\eta\lambda,\nu} \\ - g_{\rho\mu,\pi} g_{\nu\eta,\lambda} - g_{\pi\rho,\mu} g_{\nu\eta,\lambda} + g_{\mu\pi,\rho} g_{\nu\eta,\lambda} \\ + g_{\rho\mu,\pi} g_{\lambda\nu,\eta} + g_{\pi\rho,\mu} g_{\lambda\nu,\eta} - g_{\mu\pi,\rho} g_{\lambda\nu,\eta} \end{bmatrix}\end{aligned} \tag{12.15d}$$

  Happy solving! ☺☹ →

  Even the vacuum EFEs are extremely complicated and only a few exact solutions are known.
- If one allows for a finite cosmological constant $\Lambda \neq 0$, and considers an otherwise empty universe ($T_{\mu\nu} = 0$), one finds the more general vacuum EFE

$$R_{\mu\nu} = \Lambda g_{\mu\nu}\,. \tag{12.16}$$

  Solutions of this equation are called ⁂ *Einstein manifolds*.

  Note that flat Minkowski space does *not* solve this equation for $\Lambda \neq 0$; since interstellar space (= vacuum) is very close to flat Minkowski space (SPECIAL RELATIVITY is valid to good approximation), this already tells us that the cosmological constant, if nonzero, cannot be very large in our universe. This is why the cosmological constant Λ is often set to zero for non-cosmological calculations (*e.g.*, for tests in the solar system).

10 | Properties:

- How many *independent* EFEs are there?

  The Einstein field Eq. (12.10) in vacuum (without cosmological constant)

  $$G_{\mu\nu} = 0 \tag{12.17}$$

  is a set of *second-order* partial differential equations (PDEs) that determine the evolution of the metric tensor field $g_{\mu\nu}(x)$. Thus, for a *three*-dimensional spatial slice at time coordinate x_*^0, you can provide initial data $g_* \equiv g_{\mu\nu}(x_*^0, \vec{x})$ and $\dot{g}_* \equiv g_{\mu\nu,0}(x_*^0, \vec{x})$, and the EFE should provide you with a solution $g_{\mu\nu}(x)$ defined on the full spacetime (ignoring issues with singularities). Since $G_{\mu\nu}$ is symmetric, the EFEs correspond to 10 PDEs, which matches the 10 independent components of the metric $g_{\mu\nu}$ (which is also symmetric).

  There is a catch, though: The four Bianchi *identities* $G^{\mu\nu}{}_{;\nu} \equiv 0$ $(\mu = 0, \cdots, 3)$ tell us that not all of these 10 PDEs are independent. Due to these constraints, we actually loose four of the 10 equations, which makes the EFEs *underconstrained*. That is, we should expect that solutions $g_{\mu\nu}$ of the EFEs retain four unconstrained degrees of freedom that can be changed arbitrarily. This reflects of course our freedom to change coordinates! Viewed as an active transformation, this freedom corresponds to the diffeomorphism invariance of the Einstein-Hilbert action, which must be interpreted as a *gauge symmetry* (with four generators): different solutions $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ that are related by a coordinate transformation / diffeomorphism describe the *same* physics! The Bianchi identities $G^{\mu\nu}{}_{;\nu} \equiv 0$ can then be interpreted as ↑ *Noether identities*, following from Noether's *second* theorem.

- Degrees of freedom:

  So how many *physical* degrees of freedom do the EFEs then actually describe? Subtracting the 4 gauge DOF from the 10 DOF of the metric yields 6 DOF; but, as we will later see in our discussion of → *gravitational waves*, this cannot be the end of the story because gravitational waves have only *two* polarizations and not 6 (just like photons)! What is going on?

  To solve this puzzle, we must first recognize that all dynamical degrees of freedom of a *deterministic* theory, described by a second-order PDE, are encoded in the initial data $(g_*, \dot{g}_*)$. Since the EFEs describe a gauge theory, they are only deterministic if we throw all gauge-equivalent solutions into a common "gauge equivalence class"; thus let $[g_*, \dot{g}_*]$ denote the class of field configurations on the spatial slice at x_*^0 that are equivalent modulo coordinate transformations (diffeomorphisms). The physical degrees of freedom are then the DOF that parametrize these gauge classes; and – according to our argument above – there should be 6 such degrees of freedom (in configuration space, not in phase space).

  The problem is that not all initial field configurations $[g_*, \dot{g}_*]$ are allowed (= yield solutions) because the initial data must satisfy *four* constraint equations:

  $$G^{\mu 0} = f(g, \dot{g}, \partial_i g, \partial_i^2 g) \overset{!}{=} 0 \quad \text{for} \quad \mu = 0, \dots, 3\,. \tag{12.18}$$

  These equations are just part of the EFEs Eq. (12.17); the point is that the $G^{\mu 0}$ are functions of only *first* time derivatives of the metric. Hence they are not evolution equations at all – they are *constraint* equations that must be satisfied by the initial data $[g_*, \dot{g}_*]$. Put differently: You cannot hand in an arbitrary initial configuration $[g_*, \dot{g}_*]$ and expect the EFEs to spit out a solution. Only the special subclass of initial configurations that satisfy Eq. (12.18) yield solutions. As Eq. (12.18) provides four constraints, this cuts down the physical DOF by another 4. So in summary there are only $10 - 4 - 4 = 2$ physical DOF described by the EFEs per point of *space*, which matches the two polarizations of gravitational waves.

  How to see that Eq. (12.18) is correct? We want to avoid an expansion of the Einstein tensor in terms of the metric (because it is ugly). To this end, expand the Bianchi identity:

  $$G^{\mu\nu}{}_{;\nu} \overset{10.57a}{=} G^{\mu 0}{}_{,0} + G^{\mu i}{}_{,i} + \ldots \equiv 0 \tag{12.19}$$

where the ... part does not contain derivatives of G. So we have

$$G^{\mu 0}{}_{,0} \equiv -G^{\mu i}{}_{,i} - \dots . \tag{12.20}$$

But the right-hand side contains at most *second* time derivatives of the metric. Since this is an identity, $G^{\mu 0}$ can only contain at most *first* time derivatives of the metric. This is the statement of Eq. (12.18).

- Quite surprisingly, the equations of motion for the *matter* fields Eq. (12.3) are already contained in the integrability constraint $T^{\mu\nu}{}_{;\nu} \stackrel{!}{=} 0$ that follows from the identity $G^{\mu\nu}{}_{;\nu} \equiv 0$ (↑ Ref. [163]). Put differently:

  The Einstein field equations (12.10) *are not only the differential equations that determine the geometry of spacetime in response to the energy and momentum of the matter fields, but, at the same time, determine the evolution of the matter fields themselves!*

  This is possible because the EFEs are non-linear [163].

  To understand how strange this is, recall our theory in Section 6.4 that described the joint evolution of an electromagnetic field coupled to charged, massive particles. There we derived *two* equations of motion: the "matter EOM" Eq. (6.130) describes the motion of particles in response to the EM field, and the "field EOM" Eq. (6.125) describes the evolution of the EM field in response to the current produced by the charged particles. To describe the evolution of the full system, one needs *both* EOMs – one cannot derive the Lorentz force law Eq. (6.132) from the inhomogeneous Maxwell Eq. (6.125) (at least not without assuming conservation of total energy and momentum).

  Naïvely, the Einstein field Eq. (12.10) parallel the inhomogeneous Maxwell Eq. (6.125) in that they describe the response of a field (the metric) to a source (energy & momentum). The difference is that the EFEs are so restrictive (due to their non-linearity), that they already contain (local) conservation of energy and momentum, and thereby the matter EOMs! Thus, in GENERAL RELATIVITY, the geometry of spacetime and the evolution of matter are so tightly interwoven, that one can only solve them together (which makes solving the EFEs in general extremely hard, if not impossible).

  As an example, consider ↑ *Einstein-Maxwell theory* that describes a universe filled with Maxwell's EM field but nothing else (no charges). The source of the gravitational field is then given by the HEMT Eq. (11.116) of Maxwell theory,

$$T_{\mu\nu} = \frac{1}{4\pi}\left[F_{\mu\lambda}F^{\lambda}{}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right], \tag{12.21}$$

  and the equations of motion of the coupled system read

$$\text{Eq. (12.3)} \;\Leftrightarrow\; F^{\mu\nu}{}_{;\nu} = 0 \qquad \text{(Inhomogeneous Maxwell eqs.)}, \tag{12.22a}$$
$$\text{Eq. (12.4)} \;\Leftrightarrow\; G_{\mu\nu} = -\kappa T_{\mu\nu} \qquad \text{(Einstein field eqs.)}. \tag{12.22b}$$

  We assume that $F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}$ with the gauge field A_μ, so that the homogeneous Maxwell equations Eq. (11.69) are identically satisfied.

  If we combine Eq. (12.21) with Eq. (12.22b), we obtain the ⁂ *Einstein-Maxwell equations*

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{\kappa}{4\pi}\left[F_{\mu\lambda}F^{\lambda}{}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right]. \tag{12.23}$$

  The crucial (and surprising) insight is that Eq. (12.23) [equivalently: Eq. (12.22b) and Eq. (12.21)] already *contains the inhomogeneous Maxwell* Eq. (12.22a):

$$G_{\mu\nu} = -\kappa T_{\mu\nu} \quad \stackrel{*}{\Rightarrow} \quad F^{\mu\nu}{}_{;\nu} = 0\,. \tag{12.24}$$

So writing down the Einstein field equations – with an explicit expression of the energy-momentum tensor in terms of the matter fields on the right – is tantamount to writing down *all* equations of motion!

For more details [and a proof of Eq. (12.24)] see MISNER *et al.* [3] (§20.6, pp. 471–483).

For a fully "geometrized" formulation of Einstein-Maxwell theory see Ref. [164].

↓ Lecture 26 [23.06.26]

12.1.1. Newtonian limit

We now want to study the relation between the EFEs and Newtonian mechanics to determine the Einstein gravitational constant κ via a correspondence principle.

11 | Non-relativistic limit:

- Slowly varying, weak gravitational fields → Metric almost Minkowskian:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \quad \text{with small perturbation } |h_{\mu\nu}(x)| \ll 1 \tag{12.25}$$

  In the following, we keep only the lowest order terms in $h_{\mu\nu}$.

- Slow bodies ($v \ll c$) → Source of gravity = Mass density $\rho(x)$ (= rest energy)

$$T_{\mu\nu} = \begin{cases} \rho c^2 & \mu\nu = 00 \\ 0 & \text{otherwise} \end{cases} \quad \Rightarrow \quad T = T^{\mu}{}_{\mu} = \rho c^2 \tag{12.26}$$

  The energy-momentum tensor of a ↑ *perfect fluid* of mass-energy density $\rho(x)$, pressure $p(x)$, and 4-velocity field $u^\mu(x)$ is given by

$$T^{\mu\nu}(x) = \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu - p g^{\mu\nu} \,. \tag{12.27}$$

  In a comoving frame (where the fluid is at rest), it is $u^\mu = (c, 0, 0, 0)$ and $g^{\mu\nu} = \eta^{\mu\nu}$ so that

$$T^{\mu\nu} = \operatorname{diag}\left(\rho c^2, p, p, p\right) \overset{p\approx 0}{\approx} \operatorname{diag}\left(\rho c^2, 0, 0, 0\right) \tag{12.28}$$

  if we assume the pressure to be negligible wrt the rest energy.

12 | $\sphericalangle\ \mu\nu = 00$ in Eq. (12.11) $\xrightarrow{\Lambda=0}$

We are interested in the $\mu\nu = 00$ component because in Eq. (11.65) we related this component of the metric with the Newtonian gravitational potential in the non-relativistic limit.

$$R_{00} = -\frac{\kappa}{2}\rho c^2 + \cancel{\mathcal{O}(h)} \tag{12.29}$$

We drop $h_{\mu\nu}(x)$ on the right-hand side because this is a higher-order perturbation that can be neglected for $|h_{\mu\nu}| \ll 1$ (in the following, we always keep only the lowest non-trivial order).

i | Because of $h_{\mu\nu}(x)$ there is *no* global inertial coordinate system; but there is a coordinate system where $\mathcal{O}(\Gamma) = \mathcal{O}(h)$ with connection coefficients $\Gamma \sim |\Gamma^\rho{}_{\mu\nu}|$.

Eqs. (10.105) and (10.107) →

$$R_{00} = \partial_0 \Gamma^\mu{}_{0\mu} - \partial_\mu \Gamma^\mu{}_{00} + \cancel{\mathcal{O}(h^2)} \tag{12.30}$$