true for the two-dimensional sphere S^2. While there are many connections you can assign to a 2D sphere, none of them is flat! This is a corollary of the ↑ *Gauss-Bonnet theorem* or, alternatively, the ↑ *hairy ball theorem*.)

↓ Lecture 21 [12.05.26]

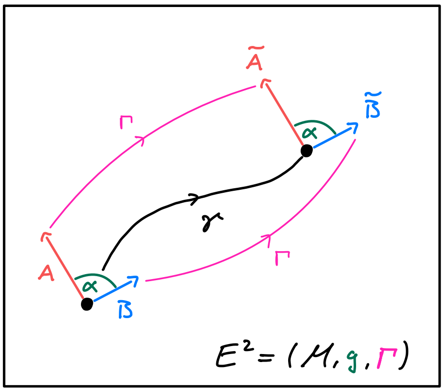
10.3. Affine connections on Riemannian manifolds

We already know the benefits of a Riemannian manifold (M, g), *i.e.*, a manifold equipped with a (pseudo-)Riemannian metric g. In the previous section, we studied another type of structure that lives on a manifold: a connection Γ. In this section we bring both (a priori independent) concepts together by asking whether, among all possible connections, there are *distinguished* ones on a Riemannian manifold. This will lead us to a connection that can be constructed directly from the metric and plays a central role in GENERAL RELATIVITY.

10.3.1. The LEVI-CIVITA connection

1 | Motivation:

In Euclidean space, the parallel transport of two vectors does not change their inner product (in particular, their norm/length remains constant):



→ It makes sense to generalize this property to general Riemannian manifolds with a connection.

2 | ∢ Riemannian manifold (M, g) with (pseudo-)Riemannian metric $g_{ij}(x)$

A connection Γ is called a ⁂ *metric-compatible*

$$:\Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\langle A, B\rangle \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{ik}A^iB^k) \overset{10.51}{=} \frac{\mathrm{D}}{\mathrm{D}\lambda}(g_{ik}A^iB^k) \overset{!}{=} 0 \tag{10.73}$$

along any curve $\gamma(\lambda)$ for all *parallel* vector fields A and B along γ.

Recall that for a *scalar* the total and absolute derivative are identical.

A and B arbitrary parallel vector fields: $\frac{\mathrm{D}A^i}{\mathrm{D}\lambda} = 0 = \frac{\mathrm{D}B^k}{\mathrm{D}\lambda} \rightarrow$

$$\text{Eq. (10.73)} \quad \Leftrightarrow \quad \forall_{i,k,\gamma(\lambda)} : \frac{\mathrm{D}g_{ik}}{\mathrm{D}\lambda} \overset{10.49}{=} g_{ik;l}\frac{\mathrm{d}\gamma^l}{\mathrm{d}\lambda} \overset{!}{=} 0 \quad \Leftrightarrow \quad \forall_{i,k,l} : g_{ik;l} \overset{!}{=} 0 \tag{10.74}$$

Use the Leibniz product rule Eq. (10.52) to show this.

→ $g_{ij}(x)$ is *covariantly constant:*

$$\Gamma \text{ is } \textit{metric-compatible} \quad \Leftrightarrow \quad \forall_{i,k,l}: \; g_{ik;l} = 0 \tag{10.75}$$

3 | $\xrightarrow{\text{Eq. (10.57b)}}$

$$\partial_l g_{ik} - \Gamma^m{}_{il}\, g_{mk} - \Gamma^m{}_{kl}\, g_{im} \overset{!}{=} 0 \tag{10.76}$$

Since Eq. (10.74) holds for arbitrary indices, we also have equations with cyclic permutations:

$$\partial_k g_{li} - \Gamma^m{}_{lk}\, g_{mi} - \Gamma^m{}_{ik}\, g_{lm} \overset{!}{=} 0\,, \tag{10.77a}$$

$$-\partial_i g_{kl} + \Gamma^m{}_{ki}\, g_{ml} + \Gamma^m{}_{li}\, g_{km} \overset{!}{=} 0\,. \tag{10.77b}$$

Adding up the three equations yields

$$\begin{aligned} \Gamma_{i(kl)} \equiv \Gamma^m{}_{(kl)}\, g_{mi} &\overset{!}{=} \frac{1}{2}\left(\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl}\right) + \frac{1}{2}\left(S^m{}_{li}\, g_{mk} + S^m{}_{ki}\, g_{ml}\right) \\ &= \frac{1}{2}\left(\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl}\right) + S_{(kl)i} \end{aligned} \tag{10.78}$$

with torsion $S^m{}_{li} = \Gamma^m{}_{li} - \Gamma^m{}_{il}$ and the symmetrized coefficient $\Gamma^m{}_{(kl)} := \frac{1}{2}\left(\Gamma^m{}_{kl} + \Gamma^m{}_{lk}\right)$ and torsion tensor $S_{(kl)i} := \frac{1}{2}\left(S_{kli} + S_{lki}\right)$.

If we assume a torsion-free connection, it is $\Gamma_{i(kl)} = \Gamma_{ikl}$ and $S_{(kl)i} = 0$ so that

$$\Gamma_{ikl} = \frac{1}{2}\left(\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl}\right)\,. \tag{10.79}$$

These are the connection coefficients of the unique Levi-Civita connection.

4 | Use symmetry $\Gamma^i{}_{kl} = \Gamma^i{}_{lk}$ (torsion-free!) and definition $\Gamma_{ikl} := g_{im}\Gamma^m{}_{kl}$

$\xrightarrow{\text{Eqs. (10.76) and (10.77)}}$

⁂ *Christoffel symbols (of the first kind)*

$$\Gamma_{ikl} \overset{\circ}{=} \frac{1}{2}\left(\partial_l g_{ik} + \partial_k g_{li} - \partial_i g_{kl}\right) \tag{10.80a}$$

⁂ *Christoffel symbols (of the second kind)*

$$\Gamma^i{}_{kl} = \frac{1}{2} g^{im}\left(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl}\right) \tag{10.80b}$$

¡! You *cannot* pull indices up/down inside partial derivatives because the metric itself depends on the coordinates. For example: $g^{im}\partial_l g_{mk} \neq \partial_l (g^{im} g_{mk}) = \partial_l \delta^i_k = 0$.

This torsion-free, metric-compatible connection is *unique* and called the *Levi-Civita connection*:

Christoffel symbols $\Gamma^i{}_{kl}$ = Connection coefficients of the ⁂ *Levi-Civita connection*

- In GENERAL RELATIVITY, we only work with the Levi-Civita connection; *i.e.*, when we use the symbols $\Gamma^i{}_{kl}$, we always refer to the Christoffel symbols Eq. (10.80) (and not to generic coefficients of a [metric-compatible] connection, → *below*).

- For a given metric, there are many compatible connections (→ *next* and ➲ Problemset 4). However, if we demand *in addition* that the connection is symmetric (= torsion-free), there is only one possible choice: the Levi-Civita connection (↑ *Fundamental theorem of Riemannian geometry*).
- The Christoffel symbols are sometimes written as [135, 136] (Details: ➲ Problemset 4)

$$\left\{ {i \atop kl} \right\} = \frac{1}{2} g^{im} \left(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl} \right) . \tag{10.81}$$

(Einstein used an "upside down" version of this notation in his original work on GENERAL RELATIVITY, *e.g.*, in Ref. [12].)

Then it follows from Eq. (10.78) that a *general* metric-compatible connection can be written as

$$\Gamma^i{}_{kl} = \Gamma^i{}_{(kl)} + \Gamma^i{}_{[kl]} = \left\{ {i \atop kl} \right\} + \underbrace{\frac{1}{2} \left(S^i{}_{kl} - S_l{}^i{}_k + S_{kl}{}^i \right)}_{=: -K^i{}_{kl}} , \tag{10.82}$$

with $\Gamma^i{}_{[kl]} = \frac{1}{2} S^i{}_{kl}$; the tensor $K^i{}_{kl}$ is known as ↑ *contorsion tensor* ("Verdrehungstensor").

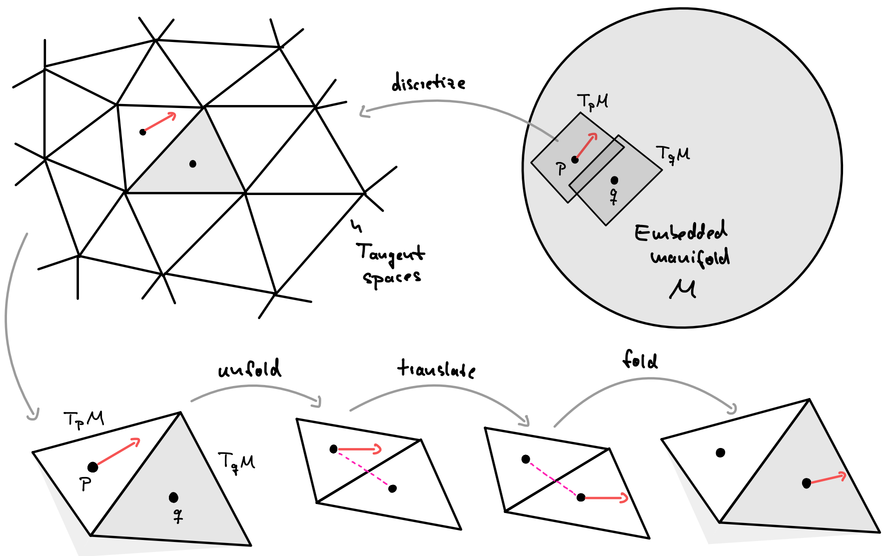
The torsion-free Levi-Civita connection is the special case where

$$\Gamma^i{}_{kl} = \left\{ {i \atop kl} \right\} . \tag{10.83}$$

Because we use only the torsion-free Levi-Civita connection in GENERAL RELATIVITY, we don't make use of this notation and only write $\Gamma^i{}_{kl}$.

5 | Interpretation:

For the special case of a 2D manifold embedded in 3D Euclidean space, the Levi-Civita connection can be geometrically interpreted as follows:



¡! This illustration is based on an embedding of the manifold into an ambient Euclidean space (which induces a metric on the manifold). Note, however, that the Levi-Civita connection is *intrinsically* defined and does not require such an embedding.

6 | Corollaries:

- Working with a metric-compatible connection has the benefit that one can pull indices up and down *within* a covariant derivative:

$$T_{i;k} = (g_{im}T^m)_{;k} = \underbrace{g_{im;k}}_{=0} T^m + g_{im}T^m{}_{;k} \overset{10.74}{=} g_{im}T^m{}_{;k} \tag{10.84}$$

- The inverse metric is also covariantly constant:

$$g^{ik}{}_{;l} = 0 \tag{10.85}$$

To show this, note that $\delta^i_{j;l} = 0$ [Eq. (10.57b)] and use the Leibniz product rule:

$$0 = \delta^i_{j;l} = (g^{ik}g_{kj})_{;l} = g^{ik}{}_{;l}g_{kj} + g^{ik}g_{kj;l} \overset{10.74}{=} g^{ik}{}_{;l}g_{kj}\,. \tag{10.86}$$

7 | Local inertial coordinates: (Details: ➔ Problemset 2)

i | ⊲ Levi-Civita connection in ← *locally geodesic coordinates* at $p \in M$:

(For simplicity, we assume that the point p has the coordinates $u(p) = 0$.)

$$\partial_l g_{ik}(0) \overset{10.76}{=} \underbrace{\Gamma^m{}_{il}(0)}_{=0}\, g_{mk} + \underbrace{\Gamma^m{}_{kl}(0)}_{=0}\, g_{im} = 0 \tag{10.87}$$

→ In these coordinates, the metric tensor is constant in linear order:

$$g_{ij}(x) = g_{ij}(0) + \frac{1}{2}\partial_\alpha\partial_\beta g_{ij}(0)\, x^\alpha x^\beta + \mathcal{O}(x^3) \tag{10.88}$$

ii | ⊲ *Affine* coordinate transformation: $\bar{x}^i = M^i{}_j\, x^j + b^i \xrightarrow{\text{Eq. (10.39)}} \bar{\Gamma}^i{}_{kl} = 0$

Note that under affine/linear coordinate transformations, the connection coefficients transform like tensors! In particular, if the connection coefficients vanish in one (geodesic) coordinate system, they vanish in all coordinates that can be reached by affine transformations; *i.e.*, geodesic coordinates are not unique!

→ Use linear transformation to bring metric of signature (r, s) into the form

$$\bar{g}_{ij}(0) = \text{diag}(\underbrace{+1, \dots, +1}_{\times r}, \underbrace{-1, \dots, -1}_{\times s})\,. \tag{10.89}$$

That this is possible follows from ↑ *Sylvester's law of inertia*: First, use the symmetry of the metric to diagonalize the matrix $g_{ij}(0)$ by an orthogonal transformation, then use another non-singular transformation to normalize the eigenvalues to ± 1.

iii | ⊲ Special case $(r = 1, s = 3)$ = Lorentzian manifold →

Metric in ⁂ *locally inertial coordinates*:

$$\bar{g}_{\mu\nu}(\bar{x}) \overset{\bar{x}\to 0}{\approx} \eta_{\mu\nu} + \frac{1}{2}\bar{\partial}_\alpha\bar{\partial}_\beta \bar{g}_{\mu\nu}(0)\, \bar{x}^\alpha \bar{x}^\beta \tag{10.90}$$

- In words: For every point of a Lorentzian manifold there exist coordinate systems such that the metric in this point takes the Minkowski form $\eta_{\mu\nu}$ and is constant in linear order; we call such charts *locally inertial coordinates*.

- Recall that Lorentz transformations are linear and leave the Minkowski metric invariant [← *Eq.* (4.20)]. This implies that locally inertial coordinates are also not unique: You can use arbitrary Lorentz transformations without changing the structure of Eq. (10.90).

8 | Useful relations:

Here we list a few identities that will be useful for many calculations in GENERAL RELATIVITY. You prove these relations in ➔ Problemset 2.

- The trace of the Christoffel symbols simplifies to

$$\Gamma^i{}_{ki} \stackrel{\circ}{=} \frac{1}{2} g^{im} g_{im,k} \,. \tag{10.91}$$

- With the determinant of the metric $g = \det(g_{im})$, the *inverse* metric can be written as

$$g^{im} \stackrel{\circ}{=} \frac{1}{g} \frac{\partial g}{\partial g_{im}} \,. \tag{10.92}$$

- With Eqs. (10.91) and (10.92), the trace of the Christoffel symbols takes the simple form

$$\Gamma^i{}_{ki} = \frac{1}{2g} g_{,k} = \left(\ln \sqrt{\pm g}\right)_{,k} \,, \tag{10.93}$$

  such that $\pm g > 0$.

  Note: In GENERAL RELATIVITY it is $\det(g_{\mu\nu}) < 0$ (because of the Lorentzian signature) and we redefine $g := -\det(g_{\mu\nu}) > 0$ to simplify expressions.

- The other trace of the Christoffel symbols can also be written in a compact form:

$$g^{kl} \Gamma^i{}_{kl} \stackrel{\circ}{=} -\frac{1}{\sqrt{g}} \left(\sqrt{g} g^{im}\right)_{,m} \,. \tag{10.94}$$

- It is straightforward to show the following useful identity:

$$g_{ik} (g^{kl})_{,m} \stackrel{\circ}{=} -(g_{ik})_{,m} \, g^{kl} \,. \tag{10.95}$$

- The ⁂ *covariant divergence* of a contravariant vector field is defined as one would expect:

$$A^i{}_{;i} \stackrel{10.93}{=} A^i{}_{,i} + A^l (\ln \sqrt{g})_{,l} = \frac{1}{\sqrt{g}} \left(\sqrt{g} A^i\right)_{,i} \tag{10.96}$$

- For the covariant divergence of an *antisymmetric* $(2,0)$-tensor there is a similar expression:

$$A^{ik}{}_{;k} \stackrel{\circ}{=} \frac{1}{\sqrt{g}} \left(\sqrt{g} A^{ik}\right)_{,k} \quad \text{with} \quad A^{ij} = -A^{ji} \,. \tag{10.97}$$

- Eq. (10.96) can be used to rewrite the covariant Laplacian (divergence of a gradient) of a scalar:

$$\Delta\phi \equiv \phi^{;i}{}_{;i} = \frac{1}{\sqrt{g}} \left(\sqrt{g} g^{ik} \phi_{,k}\right)_{,i} \,. \tag{10.98}$$

  The differential operator Δ maps scalar functions onto scalar functions and is known as ↑ *Laplace-Beltrami operator*.

- Generalized divergence theorem:

i | $\triangleleft$ Coordinate transformation $\bar{x} = \varphi(x)$
→ D-dimensional (oriented) volume element (more precisely: volume form) transforms as (← *Eq.* (3.39))

$$\mathrm{d}^D\bar{x} = \det\left(\frac{\partial\bar{x}}{\partial x}\right)\mathrm{d}^D x \tag{10.99}$$

with ↓ *Jacobian determinant* $\det\left(\frac{\partial\bar{x}}{\partial x}\right)$.

ii | The determinant of the metric transforms in the opposite way [← *Eq.* (3.54)]:

$$\sqrt{\bar{g}} = \left|\det\left(\frac{\partial x}{\partial\bar{x}}\right)\right|\sqrt{g} \tag{10.100}$$

(Note the absolute value of the Jacobian determinant!)

iii | Hence the product of metric determinant and (oriented) volume element transforms like a pseudo scalar:

$$\sqrt{\bar{g}}\,\mathrm{d}^D\bar{x} = \mathrm{sign}\left[\det\left(\frac{\partial\bar{x}}{\partial x}\right)\right]\sqrt{g}\,\mathrm{d}^D x\,. \tag{10.101}$$

Here $\mathrm{sign}\left[\det\left(\frac{\partial\bar{x}}{\partial x}\right)\right]$ denotes the sign of the Jacobian determinant, which encodes whether the coordinate transformation is orientation preserving $(+1)$ or not (-1). This makes $\sqrt{g}\,\mathrm{d}^D x$ transform like a *pseudo* scalar.

If we are only interested in non-oriented volume elements, or restrict ourselves to orientation-preserving coordinate transformations, Eq. (10.101) simplifies to a true scalar transformation:

$$\sqrt{\bar{g}}\,\mathrm{d}^D\bar{x} = \sqrt{g}\,\mathrm{d}^D x\,. \tag{10.102}$$

This subtlety will not be important in the following and we use Eq. (10.102) henceforth.

iv | Eq. (10.102) is the reason why integrals over scalar quantities $\bar{\phi}(\bar{x}) = \phi(x)$ are form-invariant under arbitrary coordinate transformations if we use the "modified" volume element $\sqrt{g}\,\mathrm{d}^D x$ for integration:

$$\int \underbrace{\underbrace{\mathrm{d}^N x\sqrt{g(x)}}_{\text{Scalar}}\ \underbrace{\phi(x)}_{\text{Scalar}}}_{\text{Scalar}} \overset{\bar{x}=\varphi(x)}{=} \int \mathrm{d}^N\bar{x}\sqrt{\bar{g}(\bar{x})}\,\bar{\phi}(\bar{x}) \tag{10.103}$$

v | Using the covariant divergence Eq. (10.96) and the modified volume element Eq. (10.102), we find the generalized form of the divergence theorem

$$\int_V \mathrm{d}^D x\sqrt{g}\,A^i{}_{;i} \overset{10.96}{=} \int_V \mathrm{d}^D x\,\partial_i\left(\sqrt{g}A^i\right) \overset{\text{Gauss}}{=} \oint_{\partial V}\mathrm{d}\sigma_i\sqrt{g}\,A^i\,, \tag{10.104}$$

where ∂V is the surface of V and $\mathrm{d}\sigma_i$ denotes the $D-1$-dimensional surface element.

10.3.2. The RIEMANN curvature tensor

Now that we identified the special Levi-Civita connection (which can be computed from the metric), we can also express its curvature tensor (then called *Riemann* curvature tensor) in terms of the metric as well:

Detailed calculations: ➲ Problemset 3

9 | ⊲ Locally geodesic coordinates LG:

$$\{R_{iklm}\}^{\mathrm{LG}} = \{g_{ia}R^a{}_{klm}\}^{\mathrm{LG}} \overset{10.70}{=} g_{ia}\left(\partial_l\Gamma^a{}_{km} - \partial_m\Gamma^a{}_{kl}\right) \tag{10.105}$$

Recall that the connection coefficients – but not their derivatives – vanish in these coordinates!

10 | Now use the explicit form of the Levi-Civita connection to find an expression in terms of the metric:

$$\{R_{iklm}\}^{\mathrm{LG}} \overset{10.80}{=} \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) \tag{10.106}$$

- Recall that $g_{ij,k} = 0$ in locally geodesic coordinates [← *Eq.* (10.87)].
- This expression tells us that curvature prevents us from finding coordinates in which the *second* derivatives of the metric vanish.

11 | In general coordinates, the expression becomes more complicated:

$$R_{iklm} \overset{\circ}{=} \{R_{iklm}\}^{\mathrm{LG}} + g_{ab}\left(\Gamma^a{}_{kl}\Gamma^b{}_{im} - \Gamma^a{}_{km}\Gamma^b{}_{il}\right) \tag{10.107}$$

To show this, start from Eqs. (10.70) and (10.80) and use Eqs. (10.76) and (10.95).

12 | Algebraic identities: (➔ Problemset 3)

- Eqs. (10.106) and (10.107) →

$$R_{iklm} = -R_{kilm}\,, \quad R_{iklm} = -R_{ikml}\,, \quad R_{iklm} = R_{lmik} \tag{10.108}$$

In words: the Riemann tensor is *antisymmetric* in the first two and last two indices, but *symmetric* if both pairs of indices are swapped.

- ⁂ *First/Algebraic* BIANCHI *identity*:

The *cyclic sums* of Riemann tensors vanish identically:

$$R_{i\langle klm\rangle} \equiv R_{iklm} + R_{ilmk} + R_{imkl} \overset{\circ}{=} 0 \tag{10.109}$$

The same is true for the cyclic sums of arbitrary triples of indices.

The relations Eqs. (10.108) and (10.109) are *identities*, *i.e.*, their validity follows directly from the definition of the Riemann curvature tensor, independent of the specific metric. This means that a Riemann tensor in D-dimensions has less independent components as the naïve count D^4 suggests.

For example, on the $D = 4$-dimensional spacetime of GENERAL RELATIVITY, at most 20 (and not $4^4 = 256$) numbers are needed to specify R_{iklm} in every point of the spacetime manifold (➔ Problemset 3). [Beware: This does not mean that there are 20 *physical* degrees of freedom in GENERAL RELATIVITY! R_{iklm} is still a tensor and can be modified by arbitrary coordinate transformations without changing its physical content. We will see → *later* that GENERAL RELATIVITY has a large gauge group (→ *diffeomorphism invariance*) so that there are way less physical degrees of freedom than the 20 alluded to above.]

13 | ⁂ *Second/Differential* Bianchi *identity*:

The cyclic sums of covariant derivatives of the Riemann tensor vanish identically:

$$R^a{}_{k\langle lm;n\rangle} \equiv R^a{}_{klm;n} + R^a{}_{kmn;l} + R^a{}_{knl;m} \stackrel{\circ}{=} 0 \tag{10.110}$$

Proof. A neat trick to prove tensor relations is to choose a coordinate system in which their derivation is simple, and then use the tensor character of the involved objects to infer the validity of the relation in general coordinates.

Both the Riemann tensor and covariant derivatives are particularly simple in locally geodesic coordinates:

$$\left\{R^a{}_{klm;n}\right\}^{\text{LG}} \overset{10.70}{\underset{10.56}{=}} \Gamma^a{}_{km,l,n} - \Gamma^a{}_{kl,m,n}\,. \tag{10.111}$$

Adding up the cyclic permutations of this expression yields:

$$\left\{R^a{}_{k\langle lm;n\rangle}\right\}^{\text{LG}} = \left\{R^a{}_{klm;n}\right\}^{\text{LG}} + \left\{R^a{}_{kmn;l}\right\}^{\text{LG}} + \left\{R^a{}_{knl;m}\right\}^{\text{LG}} \tag{10.112a}$$

$$= \Gamma^a{}_{km,l,n} - \Gamma^a{}_{kl,m,n} + \Gamma^a{}_{kn,m,l} - \Gamma^a{}_{km,n,l} + \Gamma^a{}_{kl,n,m} - \Gamma^a{}_{kn,l,m} \tag{10.112b}$$

$$= 0 \tag{10.112c}$$

Now, since $R^a{}_{k\langle lm;n\rangle}$ is a tensor and vanishes in one coordinate system, it vanishes in *all* coordinate systems (because tensor components transform linearly under coordinate transformations); thus $R^a{}_{k\langle lm;n\rangle} = 0$ and we are done. ■

Notes:

- Remember that commutators $[A, B] = AB - BA$ satisfy the ↓ *Jacobi identity*:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0\,. \tag{10.113}$$

But the ← *Ricci identity* Eq. (10.71) relates the curvature tensor (not necessarily a Riemannian one, but the connection must be torsion-free) to the commutator of covariant derivatives:

$$A_{k[;l;m]} = A_a R^a{}_{klm}\,. \tag{10.114}$$

Using this, one can derive the second (and also the first) Bianchi identity from the Jacobi identity; see Nakahara [137] (p. 269).

14 | Derived tensors:

The following tensors can be derived from the Riemann tensor and will play an important role in the formulation of General Relativity:

i | The only non-trivial contraction of the Riemann tensor sums one index of the first pair with one index of the second pair (all other contractions vanish due to symmetries):

$$\text{⁂ Ricci } \textit{tensor:} \quad R_{kl} := R^a{}_{kla} = -R^a{}_{kal} \tag{10.115}$$

ii | The Ricci tensor is symmetric:

$$R_{kl} = R_{lk} \tag{10.116}$$

To show this, contract the first Bianchi identity Eq. (10.109),

$$R^a{}_{kla} + R^a{}_{lak} + R^a{}_{akl} = 0\,, \tag{10.117}$$

and use $R^a{}_{akl} = 0$ due to the antisymmetry of the Riemann tensor.

→ In $D = 4$ dimensions, the Ricci tensor has 10 algebraically independent components.

iii | We can contract the Ricci tensor to obtain a curvature scalar:

$$⁂\ \text{Ricci scalar:} \quad R := g^{ab} R_{ab} = R^a{}_a \tag{10.118}$$

iv | ⁂ *Contracted* BIANCHI *identity*:

Ricci tensor and -scalar obey an identity that derives from the second Bianchi identity:

$$R^a{}_{n;a} = \frac{1}{2} R_{;n} \tag{10.119}$$

Proof. To show this, contract the differential Bianchi identity Eq. (10.110) over a and m:

$$R_{kl;n} - R_{kn;l} - R_k{}^a{}_{nl;a} = 0\,. \tag{10.120}$$

Tracing out k and l (recall that our connection is metric-compatible, *i.e.*, we are allowed to pull indices up/down inside covariant derivatives) yields:

$$0 = g^{kl} R_{kl;n} - g^{kl} R_{kn;l} - g^{kl} R_k{}^a{}_{nl;a} \tag{10.121a}$$

$$\overset{10.118}{=} R_{;n} - R^l{}_{n;l} - R^{la}{}_{nl;a} \tag{10.121b}$$

$$\overset{10.115}{=} R_{;n} - R^l{}_{n;l} - R^a{}_{n;a} \tag{10.121c}$$

$$= R_{;n} - 2R^a{}_{n;a}\,. \tag{10.121d}$$

■

v | As preparation for GENERAL RELATIVITY, we define another tensor using the Ricci tensor, Ricci scalar, and metric:

$$⁂\ \text{Einstein tensor:} \quad G_{ij} := R_{ij} - \frac{1}{2} g_{ij} R \tag{10.122}$$

For $D = 4$ on a Lorentzian manifold, this tensor will be used as the left-hand side of the → *Einstein field equations*.

vi | The form of Eq. (10.122) is structurally similar to the contracted Bianchi identity. Indeed, Eq. (10.119) immediately implies:

$$\text{Eq. (10.119)} \quad \Rightarrow \quad G^a{}_{i;a} = 0 \tag{10.123}$$

- Eq. (10.123) will be crucial for the consistency of the → *Einstein field equations* with energy momentum conservation.

- For $D = 4$ one can show that the Einstein tensor $G_{\mu\nu}$ (besides the metric tensor $g_{\mu\nu}$) is the *only* rank-2 tensor with vanishing (covariant) divergence that one can construct from the metric and its first and second derivatives [138, 139]. This result is known as ↑ *Lovelock's theorem* and states under which conditions the field equations of GENERAL RELATIVITY (including the cosmological constant) are *unique* (→ *later*). The uniqueness of $G_{\mu\nu}$ and Lovelock's theorem impose important constraints on possible extensions (or modifications) of GENERAL RELATIVITY.

10.3.3. Geodesics

In Section 10.2 we defined "straight lines" as curves that keep their direction constant, and formalized this notion as ← *autoparallel curves*. Now that we have a metric at hand, we can also define "straight lines" as the *shortest* curves connecting two points. We will show now that these two concepts coincide for the metric-compatible, torsion-free Levi-Civita connection induced by the metric:

15 | ◁ Length of curve γ connecting two points P_1 and P_2 [← *Eq.* (3.55)]:

$$L[\gamma] = \int_\gamma \mathrm{d}s = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{g_{ij}\dot{x}^i\dot{x}^j} \tag{10.124}$$

Here, $x^i(\lambda_{1/2})$ are the coordinates of $P_{1/2}$ in some chart. The right expression is independent of both the parametrization $x^i(\lambda)$ of the curve and the coordinate system.

To see the latter, recall that for a coordinate transformation $\bar{x} = \varphi(x)$ it is

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}\lambda} = \frac{\partial \bar{x}^i}{\partial x^m}\frac{\mathrm{d}x^m}{\mathrm{d}\lambda} \quad \text{and} \quad \bar{g}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j} g_{kl}\,. \tag{10.125}$$

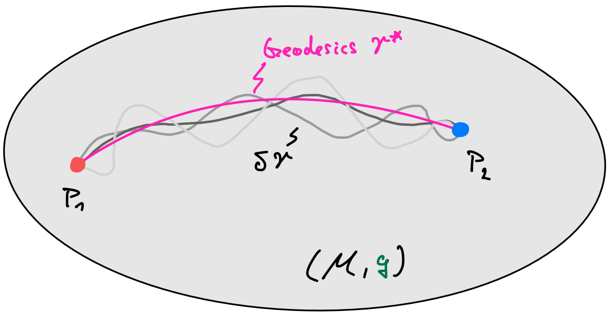
Remember that the directional derivatives $\dot{x}^i\partial_i$ along a curve are vectors in the tangent space T_pM and transform accordingly. Thus, in the expression Eq. (10.124), the total derivative wrt λ is important! In contrast to the special coordinate transformations of SPECIAL RELATIVITY (Lorentz transformations), the coordinates x^i themselves do *not* transform as tensors (they transform like $\bar{x} = \varphi(x)$, which is non-linear in general).

16 | "Straight line" from P_1 to P_2 ≡ Shortest curve γ^* (⁂ *Geodesics*) from P_1 to P_2

¡! Strictly speaking, we will not study *globally shortest* curves, but curves that *locally extremize* the length functional Eq. (10.124). For now, you can think of geodesics as "shortest curve" connecting two points, but keep in mind that this is not necessarily true (→ *comments below*).

→ Extremize length over curves starting at P_1 and terminating at P_2:

$$\delta L = \delta \int_{P_1}^{P_2} \mathrm{d}s \stackrel{!}{=} 0 \tag{10.126}$$



17 | $\sphericalangle$ Strictly monotonic, differentiable function χ & Class of "Lagrangians"

$$\mathfrak{L}_\chi(x,\dot{x}) := \chi\big(\underbrace{g_{kl}(x)\dot{x}^k\dot{x}^l}_{=:y}\big) \tag{10.127}$$

- For example: $\chi(x) = \sqrt{x}$ yields the integrand of Eq. (10.124) as Lagrangian.
- ¡! We choose this more generic "Lagrangian" – as compared to the obvious choice in Eq. (10.124) – to show that it has no effect on which curves extremize the length functional (χ will drop out); this freedom to choose χ to describe geodesics can be used to simplify calculations.

→ More general variation principle:

$$\delta \int_{P_1}^{P_2} \mathrm{d}\lambda\, \mathfrak{L}_\chi(x,\dot{x}) = 0 \tag{10.128}$$

Depending on χ, this "action" is no longer reparametrization invariant in general.

18 | → Euler-Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\partial \mathfrak{L}_\chi}{\partial \dot{x}^i}\right) - \frac{\partial \mathfrak{L}_\chi}{\partial x^i} = 0 \quad\Leftrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\chi'(y)2g_{ik}\dot{x}^k\right) - \chi'(y)\frac{\partial g_{kl}}{\partial x^i}\dot{x}^k\dot{x}^l = 0 \tag{10.129}$$

19 | $\sphericalangle$ Parametrization with $y = g_{ij}(x)\dot{x}^i\dot{x}^j \equiv \|\dot{x}\|_x^2 \stackrel{!}{=} 1 = \text{const}$

This choice fixes an *affine parametrization* $\lambda = s$ of the curve γ where the "velocity" $\|\dot{x}\|_x$ is constant. Since we require $\|\dot{x}\|_x = 1$, the "time" λ is equal to the length s of the curve from the start to $x^i(\lambda)$ (up to a constant offset).

Later, on the (pseudo-Riemannian) Lorentzian manifolds of GENERAL RELATIVITY, we will also consider *space-like* geodesics with $y < 0$; for such curves, you must add an additional minus in the square root of Eq. (10.124) and choose $y = -1 = \text{const}$ instead. The rest of the derivation is then completely analogous.

→ $\chi'(y) = \text{const} \neq 0$ (strict monotonic!) →

$$\text{Eq. (10.129)} \quad\Leftrightarrow\quad g_{ik}\ddot{x}^k + g_{ik,l}\dot{x}^k\dot{x}^l - \frac{1}{2}g_{kl,i}\,\dot{x}^k\dot{x}^l = 0 \tag{10.130}$$

Note that this differential equation is independent of χ!

$$\text{Eq. (10.130)} \quad\Leftrightarrow\quad g_{ik}\ddot{x}^k + \underbrace{\frac{1}{2}\left(g_{il,k} + g_{ik,l} - g_{kl,i}\right)}_{\Gamma_{ikl}}\dot{x}^k\dot{x}^l = 0\,. \tag{10.131}$$

20 | Identify Christoffel symbols Eq. (10.80) $\stackrel{\circ}{\rightarrow}$

$$\boxed{\frac{\mathrm{d}^2x^i}{\mathrm{d}\lambda^2} + \Gamma^i{}_{kl}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda}\frac{\mathrm{d}x^l}{\mathrm{d}\lambda} = 0 \qquad ⁂\ \textit{Geodesic equation}} \tag{10.132}$$

Solutions of this DE are called ⁂ *(affinely parametrized) Geodesics*.

↓ Lecture 22 [19.05.26]