- The additional term that makes the transformation of the connection coefficients non-tensorial is needed to compensate for a corresponding non-tensorial term from the total (non-covariant) derivative $\frac{\mathrm{d}A^i}{\mathrm{d}\lambda}$.
- Every set of fields $\Gamma^i{}_{kl}$ that transforms according to Eq. (10.39) can be used to define a connection (and therefore a notion of what "parallel" means on a manifold). This definition allows for more solutions than the specific type of connection that we used for our motivation, namely connections derived from declaring a given vector field as "constant." Interestingly, not all connections can be constructed in this way (the ones that can are actually quite boring because they do not have → *curvature*), and in Section 10.3 we will find a recipe to construct a special connection from every Riemannian metric.

↓ **Lecture 20** [05.05.26]

8 | Torsion:

In general, the connection coefficients are *not* symmetric in their lower two indices. →

$$\Gamma^i{}_{kl} = \underbrace{\frac{1}{2}\left(\Gamma^i{}_{kl} + \Gamma^i{}_{lk}\right)}_{\Gamma^i{}_{(kl)}} + \underbrace{\frac{1}{2}\overbrace{\left(\Gamma^i{}_{kl} - \Gamma^i{}_{lk}\right)}^{S^i{}_{kl}}}_{\Gamma^i{}_{[kl]}} \tag{10.40}$$

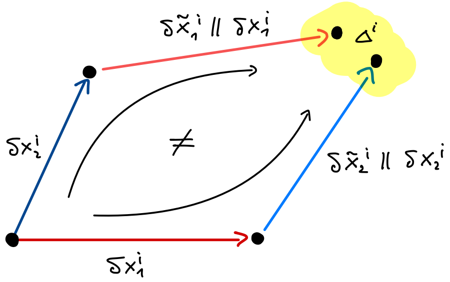
$\xrightarrow{\text{Eq. (10.39)}}$ (Note that the non-tensorial part in Eq. (10.39) is symmetric in k and l!)

$$\bar{S}^i{}_{kl} = \frac{\partial\bar{x}^i}{\partial x^m}\frac{\partial x^n}{\partial\bar{x}^k}\frac{\partial x^o}{\partial\bar{x}^l}S^m{}_{no} \tag{10.41}$$

→ *Antisymmetric* part $S^i{}_{kl}$ of connection is a *tensor*: ⁂ *Torsion tensor*

- ¡! This is not true for the *symmetric* part.
- GENERAL RELATIVITY is based on the assumption that the affine connection of spacetime is *torsion-free*. Hence it is sufficient to focus on symmetric, torsion-free connections to formulate the theory.
- Interpretation:

  On a manifold with torsion, infinitesimal parallelograms do not close:



To see this, consider two infinitesimal vectors δx_1^i and δx_2^i at some point $p \in M$. Then parallel transport δx_1^i along δx_2^i to produce $\delta\tilde{x}_1^i$ and vice versa:

$$\delta\tilde{x}_1^i = \delta x_1^i - \Gamma^i{}_{kl}(\delta x_1^k)(\delta x_2^l)\,, \tag{10.42a}$$

$$\delta\tilde{x}_2^i = \delta x_2^i - \Gamma^i{}_{kl}(\delta x_2^k)(\delta x_1^l)\,. \tag{10.42b}$$

The amount by which this infinitesimal parallelogram does not close is:

$$\begin{aligned}\Delta^i &:= (\delta x_1^i + \delta\tilde{x}_2^i) - (\delta x_2^i + \delta\tilde{x}_1^i) = (\delta x_1^i - \delta\tilde{x}_1^i) - (\delta x_2^i - \delta\tilde{x}_2^i) \\ &\overset{10.42}{=} (\Gamma^i{}_{kl} - \Gamma^i{}_{lk})\,(\delta x_1^k)(\delta x_2^l) \overset{\text{def}}{=} S^i{}_{kl}\,(\delta x_1^k)(\delta x_2^l)\,. \end{aligned} \tag{10.43}$$

Non-vanishing torsion therefore implies:

$$\Delta^i = S^i{}_{kl}\,(\delta x_1^k)\,(\delta x_2^l) \neq 0 \quad \Leftrightarrow \quad S^i{}_{kl}\,(\delta x_1^k)\,(\delta x_2^l) \neq S^i{}_{kl}\,(\delta x_2^k)\,(\delta x_1^l) \tag{10.44}$$

→ The direction of paths matters: First going along δx_1^k and then parallel to δx_2^l leads to a different point than doing the opposite. (Similar to the motion of a screw, which is different for clockwise and counterclockwise rotation.)

- It is possible to extend GENERAL RELATIVITY by allowing the torsion of spacetime to be non-zero (and dynamic as well) [134, 135]. In such theories, the ↓ *spin* of particles becomes the source of torsion, just as their *mass* is the source of → *curvature*. Such theories can predict additional forces between spinful particles, see Ref. [136] for a review.
- Since torsion is "just another tensor field" (which is not true for the symmetric part of the connection), it is reasonable to keep a geometric theory of gravity slim and assume torsion to vanish. If the theory matches observations, we didn't produce unnecessary clutter by dragging torsion along (↓ *Occam's razor*); however, if there happen to be phenomena that cannot be explained, we can still "patch" the theory by adding new (tensor) fields (that might play the role of torsion). In any case, there is no experimental evidence to date that makes a torsion field necessary.

→ Henceforth we consider only torsion-free connections:

$$\boxed{\Gamma^i{}_{kl} = \Gamma^i{}_{lk}}$$

9 | Locally geodesic coordinate systems:

Since we know how the coefficients of a connection transform, we can ask whether there are special coordinate systems in which the connection looks particularly simple:

Details: ➔ Problemset 2

i | Goal:

Show that for every point $p \in M$ there is a coordinate system in which the connection coefficients *in this point* vanish:

$$\forall p \in M\ \exists\ \text{Chart}\ u\ \text{with}\ u(p) = x_0 : \ \Gamma^i{}_{kl}(x_0) = 0 \quad \forall_{ikl} \tag{10.45}$$

u: ⁂ *Locally geodesic coordinate system*

ii | First, show the alternative form of the transformation: [recall Eq. (3.75)]

$$\bar{\Gamma}^i{}_{kl} \overset{\circ}{=} \frac{\partial\bar{x}^i}{\partial x^m}\frac{\partial x^n}{\partial\bar{x}^k}\frac{\partial x^o}{\partial\bar{x}^l}\Gamma^m{}_{no} - \frac{\partial x^m}{\partial\bar{x}^l}\frac{\partial x^p}{\partial\bar{x}^k}\left(\frac{\partial^2\bar{x}^i}{\partial x^p\partial x^m}\right) \tag{10.46}$$

This follows from Eq. (10.39) by differentiating $\frac{\partial\bar{x}^i}{\partial x^k}\frac{\partial x^k}{\partial\bar{x}^j} = \delta^i_j$.

iii | $\sphericalangle$ Coordinates v with $v(p) = 0 \in \mathbb{R}^D$ (in general it is $\Gamma^i{}_{kl}(0) \neq 0$ in this chart)
$\rightarrow$ Coordinate transformation $\bar{x} = \varphi(x) = u \circ v^{-1}(x)$ in vicinity of $p \in M$:

$$\bar{x}^i = x^i + \tfrac{1}{2} C^i{}_{kl}(0)\, x^k x^l + \dots \tag{10.47}$$

with (wlog) *symmetric* coefficients $C^i{}_{kl} = C^i{}_{lk}$.

iv | $\rightarrow$ Partial derivatives at $u(p) = 0 = v(p)$:

$$\left.\frac{\partial \bar{x}^i}{\partial x^m}\right|_{x=0} = \delta^i_m \quad \text{and} \quad \left.\frac{\partial^2 \bar{x}^i}{\partial x^p \partial x^m}\right|_{x=0} = C^i{}_{pm}(0) \tag{10.48}$$

$$\xrightarrow{\text{Eq. (10.46)}} \bar{\Gamma}^i{}_{kl} \overset{\circ}{=} \Gamma^i{}_{kl} - C^i{}_{kl}$$

v | $\bar{\Gamma}^i{}_{kl}(0) \overset{!}{=} 0$ and $\bar{\Gamma}^i{}_{kl} = \bar{\Gamma}^i{}_{lk}$ (torsion-free!) $\rightarrow C^i{}_{kl}(0) := \Gamma^i{}_{kl}(0)$ ■

Notes:

- ¡! Note that we only showed that the connection coefficients can be made zero *in a single point*; in general one cannot find a coordinate system where the coefficients vanish everywhere. This also implies that in general the *derivatives* $\partial_m \Gamma^i{}_{kl}(0)$ do *not* vanish in p.
- In locally geodesic coordinates, the absolute derivative Eq. (10.37) is simply the "normal" total derivative. As a consequence, in the context of Riemannian manifolds, the coordinate lines are local geodesics ("shortest paths", → *later*) – hence the name.
- The above argument fails for connections with non-vanishing torsion $S^i{}_{kl} \neq 0$ since the latter transforms as a tensor and cannot be zeroed by a coordinate transformation (unless it vanishes in all coordinates).
- The fact that locally geodesic coordinates exist at every point will be the foundation for the implementation of Einstein's equivalence principle **EEP** in the mathematical framework of GENERAL RELATIVITY. Physically, these coordinates will be identified with the free falling, local inertial frames.

10.2.1. Covariant derivatives

10 | The definition Eq. (10.37) of the → *absolute derivative* did not require $A^i(\lambda)$ to be defined in a neighborhood of the curve $\gamma(\lambda)$. However, if $A^i(\lambda) \equiv A^i(\gamma(\lambda))$ *is* defined on the whole manifold (or at least in a neighborhood of the curve), we can define a more useful derivative:

$$\frac{\mathrm{d}A^i}{\mathrm{d}\lambda} = \frac{\partial A^i}{\partial x^k}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \quad\Rightarrow\quad \frac{\mathrm{D}A^i}{\mathrm{D}\lambda} \overset{10.37}{=} \left(\frac{\partial A^i}{\partial x^k} + \Gamma^i{}_{mk} A^m\right)\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \equiv A^i{}_{;k}\frac{\mathrm{d}x^k}{\mathrm{d}\lambda} \tag{10.49}$$

→ ⁂ *Covariant derivative* of a *contravariant* vector:

$$\boxed{\underbrace{\begin{Bmatrix} D_k A^i \\ \nabla_k A^i \\ A^i{}_{;k} \end{Bmatrix}}_{\text{Alternative notations}} := \underbrace{\begin{Bmatrix} \frac{\partial A^i}{\partial x^k} \\ \partial_k A^i \\ A^i{}_{,k} \end{Bmatrix}}_{\text{Alternative notations}} + \Gamma^i{}_{mk} A^m} \tag{10.50}$$

$\overset{\circ}{\rightarrow}$ $A^i{}_{;k}$ is $(1,1)$-tensor

Proof: Via the ← *quotient theorem* or by straightforward calculation using Eq. (10.39) (← *Section 3.6*).

11 | Covariant derivative of a scalar:

$$\boxed{\Phi_{;k} := \Phi_{,k}} \tag{10.51}$$

$\overset{\circ}{\rightarrow}$ $\Phi_{;k}$ is $(0,1)$-tensor [Proof: Eq. (3.19)]

That the partial derivatives of scalar fields encode geometric objects, and there is no need to use the additional structure of a connection, is a consequence of the fact that scalar fields map to $\mathbb{R}$ and not T_pM. Note that it makes sense to talk about a *constant* scalar field $\phi(p) = \phi(q)$ for all $p, q \in M$ without referring to a particular coordinate system or specifying an additional structure!

12 | One demands that the ↓ *Leibniz product rule* is valid for covariant derivatives:

$$\boxed{(A^i B_i)_{;k} \overset{!}{=} A^i{}_{;k} B_i + A^i B_{i;k}} \tag{10.52}$$

→ Covariant derivative of *covariant* vector:

$$\boxed{B_{i;k} := B_{i,k} - \Gamma^m{}_{ik} B_m} \tag{10.53}$$

Cf. Eq. (10.50): Different summation indices and different sign!

$\overset{\circ}{\rightarrow}$ $B_{i;k}$ is $(0,2)$-tensor

Proof. First we note that

$$A^i{}_{;k} B_i + A^i B_{i;k} \overset{10.52}{=} (A^i B_i)_{;k} \overset{10.51}{=} (A^i B_i)_{,k} = A^i{}_{,k} B_i + A^i B_{i,k} \tag{10.54}$$

since $A^i B_i$ is a scalar. With the definition Eq. (10.50) it follows

$$A^i B_{i;k} \overset{\circ}{=} A^i \left(B_{i,k} - \Gamma^m{}_{ik} B_m\right) . \tag{10.55}$$

Since this must be true for arbitrary A^i, Eq. (10.53) follows. ■

13 | Covariant derivatives of higher-rank tensors:

The above structure can be generalized to tensors of arbitrary rank:

$$\boxed{T^{ik\dots}{}_{rs\dots;l} := T^{ik\dots}{}_{rs\dots,l} \underbrace{+\Gamma^i{}_{ml} T^{mk\dots}{}_{rs\dots} + \dots}_{\forall \text{upper indices}} \underbrace{-\Gamma^m{}_{rl} T^{ik\dots}{}_{ms\dots} - \dots}_{\forall \text{lower indices}}} \tag{10.56}$$

Example:

Covariant derivatives of *rank-2 tensors*:

$$T^{ik}{}_{;l} = T^{ik}{}_{,l} + \Gamma^i{}_{ml} T^{mk} + \Gamma^k{}_{ml} T^{im} \quad \rightarrow (2,1)\text{-tensor} \tag{10.57a}$$

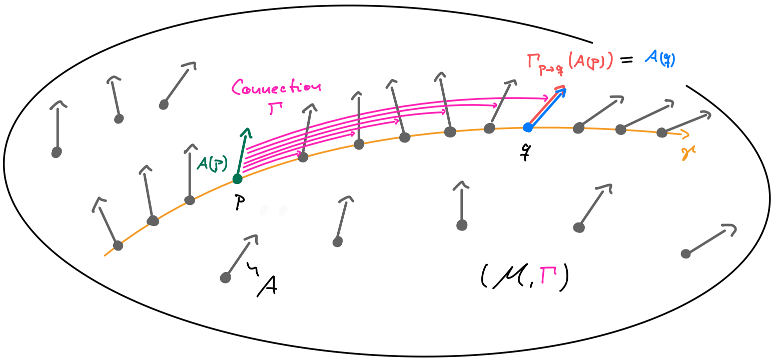
$$T_{ik;l} = T_{ik,l} - \Gamma^m{}_{il} T_{mk} - \Gamma^m{}_{kl} T_{im} \quad \rightarrow (0,3)\text{-tensor} \tag{10.57b}$$

$$T^i{}_{k;l} = T^i{}_{k,l} + \Gamma^i{}_{ml} T^m{}_k - \Gamma^m{}_{kl} T^i{}_m \quad \rightarrow (1,2)\text{-tensor} \tag{10.57c}$$

For a proof, see SCHRÖDER [2] (p. 53).

10.2.2. Parallel vector fields and autoparallel curves

14 | ⊲ Vector field $A = A^i \partial_i$ & curve γ:



$$\boxed{\begin{array}{l} A \text{ is a ⁂ } \textit{parallel (vector field)} \text{ along } \gamma \\ :\Leftrightarrow \quad \frac{\mathrm{D}A^i}{\mathrm{D}\lambda} = \frac{\mathrm{d}A^i}{\mathrm{d}\lambda} + \Gamma^i{}_{kl} A^k \frac{\mathrm{d}x^l}{\mathrm{d}\lambda} \overset{!}{=} 0 \end{array}} \tag{10.58}$$

- ¡! The notion of "parallel vectors" does *not* allow for local rescaling of vectors! It is therefore stricter than the conventional notion of "parallel"; a parallel vector field along a curve γ is "constant" on the curve (as its covariant derivative vanishes). If you start with a parallel vector field and multiply it by a (non-constant) scalar field, $A^i \mapsto \tilde{A}^i \equiv \phi A^i$, the new field is in general no longer parallel: $\frac{\mathrm{D}\tilde{A}^i}{\mathrm{D}\lambda} \neq 0$.
- Given a connection Γ, Eq. (10.58) is a first-order differential equation for A^i. By solving it for a given initial value of $A^i(\lambda = 0)$, one can reconstruct a parallel vector field on the curve γ.
- For higher-rank tensors, one defines parallelism along a curve analogously:

$$\frac{\mathrm{D}T^{ik\dots}{}_{mn\dots}}{\mathrm{D}\lambda} \overset{!}{=} 0 \tag{10.59}$$

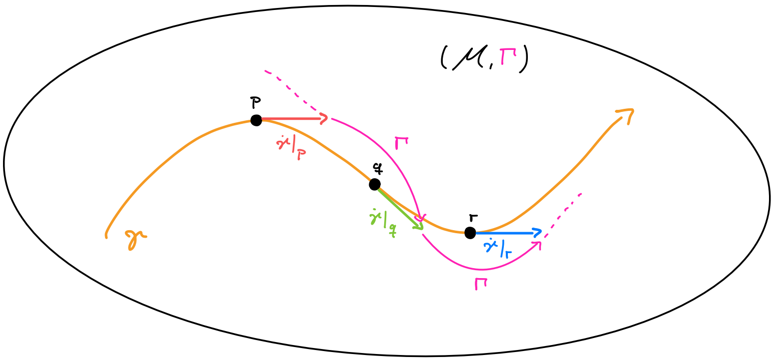
15 | ⁂ *Autoparallel curve*: Generalization of a *straight line* in $\mathbb{R}^D$:

Straight line: Curve that "keeps its direction constant."

We cannot characterize a straight line as "the shortest curve between two points" because we do not have a metric, only a connection!

⊲ Curve γ with parametrization $\gamma^i(\lambda)$ (in some chart)

γ is ⁂ *autoparallel* $:\Leftrightarrow$ Tangent field $A = A^i \partial_i := \frac{\mathrm{d}\gamma^i}{\mathrm{d}\lambda}\partial_i$ is ← *parallel* along γ:



¡! The tangent “field” is not a vector field as it is not defined on (an open set of the) the manifold but only on the curve γ. As we do not rely on the covariant derivative but on the absolute derivative (which is well-defined for such vector-valued functions), this is not a problem.

$\xrightarrow{\text{Eq. (10.58)}}$

$$\boxed{\frac{\mathrm{d}^2\gamma^i}{\mathrm{d}\lambda^2}+\Gamma^i{}_{kl}\frac{\mathrm{d}\gamma^k}{\mathrm{d}\lambda}\frac{\mathrm{d}\gamma^l}{\mathrm{d}\lambda}=0 \quad\Rightarrow\quad \gamma \text{ is ⁂ } \textit{autoparallel}} \tag{10.60}$$

- ¡! If a parametrization of a curve satisfies the DE Eq. (10.60), the curve is autoparallel and the given parametrization is called ⁂ *affine*. Since Eq. (10.60) is *not* reparametrization invariant (→ *below*), there are other (non-affine) parametrizations of the same autoparallel curve that do *not* satisfy Eq. (10.60). Every autoparallel curve has such an affine parametrization (which is unique up to affine transformations).
- Once we have a metric and a compatible connection (→ *Section 10.3*), the autoparallel curves will be identical to the curves of *shortest length* (→ *geodesics*).
- Let us assume that an affine parametrization of an autoparallel curve satisfies Eq. (10.60). Now consider a reparametrization $\mu = f(\lambda)$ given by some strictly monotone function f. The new parametrization is then $\tilde\gamma^i(\mu) = \tilde\gamma^i(f(\lambda)) := \gamma^i(\lambda)$ and satisfies the DE

$$\frac{\mathrm{d}^2\tilde\gamma^i}{\mathrm{d}\mu^2}+\Gamma^i{}_{kl}\frac{\mathrm{d}\tilde\gamma^k}{\mathrm{d}\mu}\frac{\mathrm{d}\tilde\gamma^l}{\mathrm{d}\mu} \stackrel{\circ}{=} h(\mu)\frac{\mathrm{d}\tilde\gamma^i}{\mathrm{d}\mu} \quad \text{with} \quad h(\mu) = -\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}\left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)^{-2}. \tag{10.61}$$

  The definition of h is equivalent to the DE

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}+h(\mu)\left(\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\right)^{2}=0\,. \tag{10.62}$$

  If λ is an affine parameter, the transformation f yields *another* affine parameter μ if and only if $h(\mu)\equiv 0$, *i.e.*,

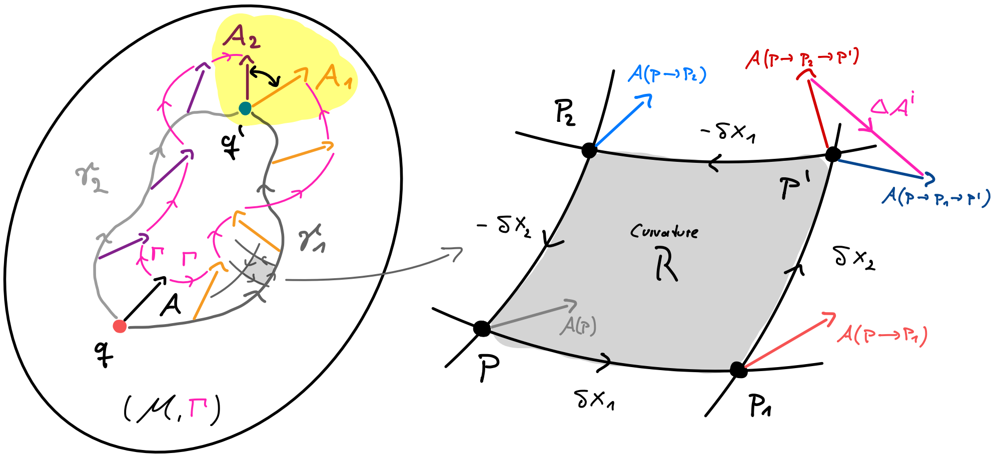
$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\lambda^2}=0\,, \tag{10.63}$$

  which is solved by reparametrizations of the *affine* form $\mu = f(\lambda) = a\lambda + b$. That is, affine parametrizations are unique up to affine *re*parametrizations.
- This problem does not affect the definition of a parallel vector field because Eq. (10.58) *is* reparametrization invariant.

10.2.3. The curvature tensor

Now that we have a formal concept of the parallel transport of vectors from one tangent space to another, we can ask whether the result of such a transport depends only on the final destination, or whether the path of the transport also plays a role. The answer will be that, for a generic connection, parallel transport indeed is path dependent, and that this path dependence is a manifestation of the intrinsic *curvature* of the manifold (more precisely: its connection).

16 | ⊲ Parallel transport of vector $A = A^i\partial_i$ from q to q' via different paths γ_1 and γ_2:



→ It is easier (and sufficient) to study an infinitesimal parallelogram.

17 | ⊲ Path $p \xrightarrow{p_1} p'$:

The first parallel transport along δx_1 yields:

$$A^i(p \xrightarrow{\delta x_1} p_1) \overset{10.34}{=} \underbrace{A^i(p) + \delta_1 A^i(p)}_{\equiv A^i + \delta_1 A^i} = A^i(p) - \Gamma^i{}_{kl}(p) A^k \delta x_1^l \tag{10.64}$$

The subsequent parallel transport along δx_2 yields:

$$A^i(p \xrightarrow{\delta x_1} p_1 \xrightarrow{\delta x_2} p') = A^i(p \xrightarrow{\delta x_1} p_1) + \delta_2 A^i(p \xrightarrow{\delta x_1} p_1) \tag{10.65a}$$

$$= A^i + \delta_1 A^i - \Gamma^i{}_{nm}(p_1)\left[A^n + \delta_1 A^n\right]\delta x_2^m \tag{10.65b}$$

Our goal is to express everything in the initial point p. →

$$\Gamma^i{}_{nm}(p_1) \approx \Gamma^i{}_{nm}(p) + \partial_l \Gamma^i{}_{nm}(p)\,\delta x_1^l \tag{10.66}$$

(Since we consider an *infinitesimal* parallelogram, we only need linear variations of all quantities.)

With this expansion, we find for the parallel vector in p':

$$\begin{aligned} A^i(p \xrightarrow{\delta x_1} p_1 \xrightarrow{\delta x_2} p') \hat{=}\; & A^i \underbrace{-\Gamma^i{}_{kl} A^k \delta x_1^l}_{\delta_1 A^i(p)} \underbrace{-\Gamma^i{}_{nm} A^n \delta x_2^m}_{\delta_2 A^i(p)} \\ & + \Gamma^i{}_{nm}\Gamma^n{}_{kl} A^k\,\delta x_1^l\,\delta x_2^m - \partial_l \Gamma^i{}_{nm} A^n\,\delta x_1^l\,\delta x_2^m \\ & + \mathcal{O}\left((\delta x)^3\right) \end{aligned} \tag{10.67}$$

In this expression, all connection coefficients and fields are evaluated in p!

18 | ⊲ Path $p \xrightarrow{p_2} p'$: Same expression with $\delta x_1 \leftrightarrow \delta x_2$:

$$A^i(p \xrightarrow{\delta x_2} p_2 \xrightarrow{\delta x_1} p') \triangleq A^i \underbrace{-\Gamma^i{}_{kl}\, A^k \delta x_2^l}_{\delta_2 A^i(p)} \underbrace{-\Gamma^i{}_{nm}\, A^n \delta x_1^m}_{\delta_1 A^i(p)} + \Gamma^i{}_{nm}\,\Gamma^n{}_{kl}\, A^k\, \delta x_2^l\, \delta x_1^m - \partial_l \Gamma^i{}_{nm}\, A^n\, \delta x_2^l\, \delta x_1^m + \mathcal{O}\left((\delta x)^3\right) \tag{10.68}$$

19 | → Path dependence:

$$\Delta A^i := A^i(p \xrightarrow{\delta x_1} p_1 \xrightarrow{\delta x_2} p') - A^i(p \xrightarrow{\delta x_2} p_2 \xrightarrow{\delta x_1} p')$$

$$= \left\{ \begin{array}{l} \text{Change of } A^i \text{ after parallel transport along} \\ \textit{closed path } p \to p_1 \to p' \to p_2 \to p. \end{array} \right\}$$

Drop $\mathcal{O}\left((\delta x)^3\right)$ terms.

$$\equiv R^i{}_{klm} A^k\, \delta x_1^m\, \delta x_2^l \tag{10.69}$$

with the ⁂ *curvature tensor*

$$\boxed{R^i{}_{klm} \triangleq \partial_l \Gamma^i{}_{km} - \partial_m \Gamma^i{}_{kl} + \Gamma^i{}_{nl}\, \Gamma^n{}_{km} - \Gamma^i{}_{nm}\, \Gamma^n{}_{kl}\,.} \tag{10.70}$$

Although $\Gamma^i{}_{kl}$ is not a tensor, this particular combination is a $(1,3)$-tensor (Proof: → *next*).

20 | Covariant derivatives are defined by an infinitesimal parallel transport. As parallel transport is path dependent, the subsequent application of two covariant derivatives in different directions cannot be commutative. Indeed:

$$\boxed{A_{k[;l;m]} \equiv A_{k;l;m} - A_{k;m;l} \triangleq R^i{}_{klm} A_i \qquad \text{⁂ \textit{Ricci identity}}} \tag{10.71}$$

→ Covariant derivatives of tensors are *not* commutative (in general)!

[Eq. (10.71) is valid in this form only for torsion-free connections.]

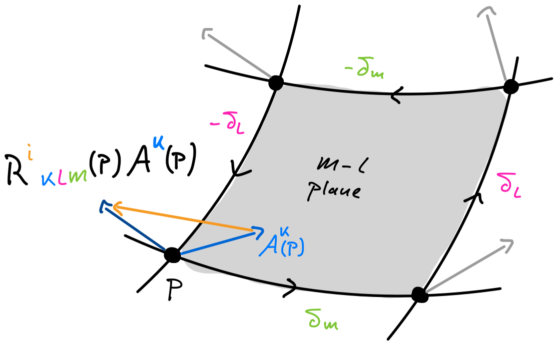
$A_{k;l;m}$ is $(0,3)$-tensor $\xrightarrow{\leftarrow \textit{Quotient theorem}}$ $R^i{}_{klm}$ is $(1,3)$-tensor ✓

- Alternatively, you can prove the tensorial transformation of $R^i{}_{klm}$ manually using the expression Eq. (10.70) and the transformation of the connection coefficients Eq. (10.39) and partial derivatives Eq. (3.5).
- Compare the non-commutativity of the covariant derivative of tensors with the commutativity of conventional partial derivatives:

$$A_{k[,l,m]} \equiv A_{k,l,m} - A_{k,m,l} = \partial_m \partial_l A_k - \partial_l \partial_m A_k = 0\,. \tag{10.72}$$

21 | Notes:

- The curvature tensor can be interpreted geometrically as follows:



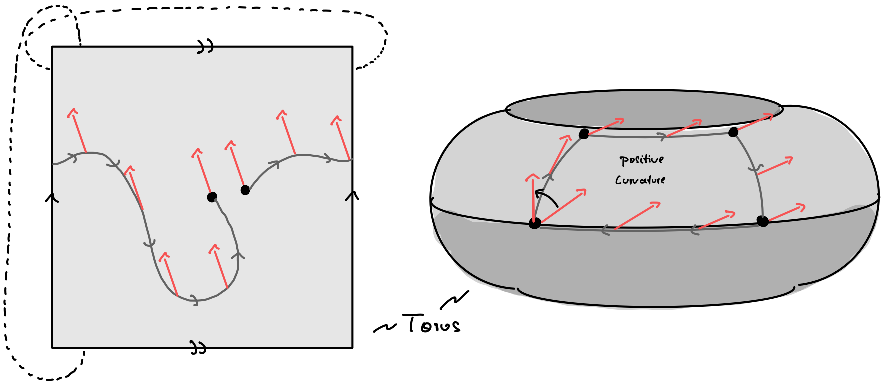
  Since curvature is the property that vectors parallel transported around infinitesimal loops change their direction, one can encode all features of curvature in an object that tells you how an arbitrary vector is transformed if transported around any infinitesimal parallelogram in the ml-plane. This object is the curvature tensor, and from this perspective it is clear that it must be of rank four (two indices to specify the plane, two for the transformation of the vector).

- (A manifold with) a connection is called *flat* iff the curvature tensor is identically zero everywhere: $R^i{}_{klm}(p) \equiv 0$. In particular, this means (for a torsion-free connection) that in a *neighborhood* of every point on the manifold (and not just the point itself!) you can find a coordinate system in which the connection coefficients vanish identically (*i.e.*, these neighborhoods behave like flat Euclidean space).

  In summary, the following statements are equivalent:

  - The curvature tensor vanishes identically.
  - The manifold is flat.
  - Parallel transport is path-independent.
  - Covariant derivatives are commutative.

- Whether a space is curved or not is a property of its *connection* and not of its *topology*! For example, here are two topologically equivalent (↑ *homeomorphic*) tori ("donuts"):



  The left one is defined by identifying opposite edges with each other and inherits the connection of the Euclidean plane. The right torus is embedded in 3D Euclidean space and inherits the metric of $\mathbb{R}^3$ and its induced connection. Both spaces are topological tori, but the left one is flat whereas the right one is not [as illustrated by the path(in)dependence of parallel transport].

  So if someone asks you whether a torus is flat or curved, the correct answer is that this is an undefined question unless a particular connection is specified! (Interestingly, this is not true for the two-dimensional sphere S^2. While there are many connections you can assign to a 2D sphere, none of them is flat! This is a corollary of the ↑ *Gauss-Bonnet theorem* or, alternatively, the ↑ *hairy ball theorem*.)

↓ **Lecture 21** [12.05.26]

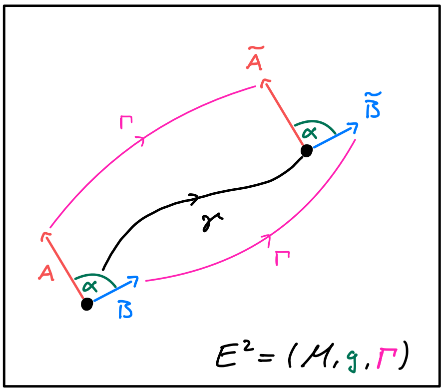
10.3. Affine connections on Riemannian manifolds

We already know the benefits of a Riemannian manifold (M, g), *i.e.*, a manifold equipped with a (pseudo-)Riemannian metric g. In the previous section, we studied another type of structure that lives on a manifold: a connection Γ. In this section we bring both (a priori independent) concepts together by asking whether, among all possible connections, there are *distinguished* ones on a Riemannian manifold. This will lead us to a connection that can be constructed directly from the metric and plays a central role in GENERAL RELATIVITY.

10.3.1. The LEVI-CIVITA connection

1 | Motivation:

In Euclidean space, the parallel transport of two vectors does not change their inner product (in particular, their norm/length remains constant):



→ It makes sense to generalize this property to general Riemannian manifolds with a connection.

2 | ⊲ Riemannian manifold (M, g) with (pseudo-)Riemannian metric $g_{ij}(x)$

A connection Γ is called a ⁂ *metric-compatible*

$$:\Leftrightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\lambda}\langle A, B\rangle \overset{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}\lambda}(g_{ik}A^i B^k) \overset{10.51}{=} \frac{\mathrm{D}}{\mathrm{D}\lambda}(g_{ik}A^i B^k) \overset{!}{=} 0 \tag{10.73}$$

along any curve $\gamma(\lambda)$ for all *parallel* vector fields A and B along γ.

Recall that for a *scalar* the total and absolute derivative are identical.

A and B arbitrary parallel vector fields: $\frac{\mathrm{D}A^i}{\mathrm{D}\lambda} = 0 = \frac{\mathrm{D}B^k}{\mathrm{D}\lambda} \rightarrow$

$$\text{Eq. (10.73)} \quad \Leftrightarrow \quad \forall_{i,k,\gamma(\lambda)} : \frac{\mathrm{D}g_{ik}}{\mathrm{D}\lambda} \overset{10.49}{=} g_{ik;l}\frac{\mathrm{d}\gamma^l}{\mathrm{d}\lambda} \overset{!}{=} 0 \quad \Leftrightarrow \quad \forall_{i,k,l} : g_{ik;l} \overset{!}{=} 0 \tag{10.74}$$