

### 6.3.1. Application: The Energy-Momentum Tensor (EMT)

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7 | < Infinitesimal spacetime translations:

$$x'^{\mu} = x^{\mu} + w^{\mu} \Rightarrow \delta_{\nu} x^{\mu} = \delta_{\nu}^{\mu} \quad \text{and} \quad \delta_{\nu} \phi = 0 \quad (6.89)$$

& Translation-invariant action:  $S' = S$  (This includes translations in time!)

8 | Conserved currents: Eq. (6.84) →

$$\Theta^{\mu}_{\nu} := \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\rho}\phi - \delta_{\rho}^{\mu} \mathcal{L} \right\} \underbrace{\delta_{\nu} x^{\rho}}_{\delta_{\nu}^{\rho}} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\nu}\phi - \delta_{\nu}^{\mu} \mathcal{L} \quad (6.90)$$

Note that the generator index  $\alpha$  is in this case a proper Lorentz index  $\nu$  so that we can pull it up,  $\Theta^{\mu\nu} = \eta^{\nu\rho} \Theta^{\mu}_{\rho}$ , and obtain:

\* (Canonical) Energy-Momentum Tensor:

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial^{\nu}\phi - \eta^{\mu\nu} \mathcal{L} \quad (6.91)$$

with

$$\partial_{\mu} \Theta^{\mu\nu} = 0 \quad \text{and four conserved charges} \quad P^{\nu} := \frac{1}{c} \int d^3x \Theta^{0\nu}. \quad (6.92)$$

- Note that these quantities are only conserved for *solutions* of the Euler-Lagrange equations.
- $P^{\nu}$  is a 4-vector (show this!). Note that this is a non-trivial statement because  $d^3x$  is not a Lorentz scalar and  $\Theta^{0\nu}$  not a 4-vector.
- The prefactor  $1/c$  ensures that  $P^0$  has the same dimension as a conventional 4-momentum with  $p^0 = E/c$ ; note that  $\Theta^{00}$  has the dimension of an *energy density* because  $\mathcal{L}$  has this dimension.

9 | Interpretation:

i | *Energy* ( $\nu = 0$ ):

$$cP^0 = \int d^3x \Theta^{00} = \int d^3x \underbrace{\left\{ \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L} \right\}}_{\text{Hamiltonian density}} = \int d^3x \underbrace{\mathcal{H}(\phi, \pi)}_{\text{Hamiltonian}} = H \quad (6.93)$$

→ The Hamiltonian is the component of a 4-vector and not Lorentz invariant!  
By contrast, the Lagrangian *is* Lorentz invariant (for relativistic field theories).

ii | *Kinetic momentum* ( $\nu = i$ ):

$$P^i = \int d^3x \Theta^{0i} = \int d^3x \frac{\partial \mathcal{L}}{\partial \dot{\phi}} (-\partial_i \phi) = - \int d^3x \pi \partial_i \phi \quad (6.94)$$

$\pi$  is the *canonical* momentum conjugate to the field  $\phi$ .

10 | The canonical EMT of electrodynamics:

i | < Free ( $j^\mu = 0$ ) electromagnetic field:  $\mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$

→ Invariant under spacetime translations

Indeed, with  $x'^\mu = x^\mu + w^\mu$  and the field transformation  $A'_\mu(x) := A_\mu(x - w)$  it is

$$S_{\text{em}}[A'] = \int d^4x \mathcal{L}_{\text{em}}(A'(x), \partial A'(x)) = \int d^4x \mathcal{L}_{\text{em}}(A(x - w), \partial A(x - w)) \quad (6.95a)$$

$$= \int d^4y \mathcal{L}_{\text{em}}(A(y), \partial A(y)) = S_{\text{em}}[A] \quad (6.95b)$$

where we integrate over the full Minkowski spacetime  $\mathbb{R}^{1,3}$ , substituted  $y^\mu = x^\mu - w^\mu$  and used  $d^4x = d^4y$ .

ii | → Canonical EMT conserved:  $\partial_\mu \Theta_{\text{em}}^{\mu\nu} = 0$  with

$$\Theta_{\text{em}}^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{em}}}{\partial(\partial_\mu A_\sigma)} \partial^\nu A_\sigma - \eta^{\mu\nu} \mathcal{L}_{\text{em}} \stackrel{6.58}{=} \frac{1}{4\pi} F^{\sigma\mu} \partial^\nu A_\sigma + \frac{\eta^{\mu\nu}}{16\pi} F_{\sigma\rho} F^{\sigma\rho} \quad (6.96)$$

Note that because the gauge field has multiple components  $A_\mu$ , there is now an additional summation in the first term over these components (marked indices). This follows directly from a generalization of the proof of Noether's theorem for fields with multiple components.

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iii | Problems:

The canonical EMT  $\Theta_{\text{em}}^{\mu\nu}$  has two problematic properties:

- Because of the term  $\partial^\nu A_\sigma$ ,  $\Theta_{\text{em}}^{\mu\nu}$  is gauge-dependent!

This is problematic because it means that we cannot hope to identify physical quantities like the energy density or the momentum density of the electromagnetic field with (the components) of this tensor.

- The canonical EMT is non-symmetric:  $\Theta_{\text{em}}^{\mu\nu} \neq \Theta_{\text{em}}^{\nu\mu}$ !

In GENERAL RELATIVITY, we will find that the right-hand side of the → *Einstein field equations* (which determine how spacetime curves and evolves)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \quad (6.97)$$

is given by the → *Hilbert energy-momentum tensor*

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta(\mathcal{L}_{\text{Matter}})}{\delta g_{\mu\nu}} \quad (6.98)$$

where  $\mathcal{L}_{\text{Matter}}$  describes the Lagrangian density of all fields in the universe (except the metric tensor field). For example,  $\mathcal{L}_{\text{Matter}}$  contains the Maxwell Lagrangian  $\mathcal{L}_{\text{em}}$  (“matter” here includes every degree of freedom that has energy & momentum, i.e., also electromagnetic radiation).

Note that  $T^{\mu\nu}$  is *symmetric* because the metric  $g_{\mu\nu}$  is. Hence it cannot be identified with the canonical EMT  $\Theta^{\mu\nu}$  in general (here for the example of Maxwell theory).

;! These problems are not specific to electrodynamics but typically affect all theories that are gauge theories and/or include non-scalar fields.

→ How to solve these issues?

### 6.3.2. The Belinfante-Rosenfeld energy-momentum tensor (BRT)

We consider again first a generic field theory, and specialize to electrodynamics later.

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- 11 | Remember (Note 6.1) that the canonical EMT is not the only conserved EMT because

$$\tilde{\Theta}^{\mu\nu} := \Theta^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} \quad \text{with} \quad K^{\rho\mu\nu} = -K^{\mu\rho\nu} \quad (6.99)$$

yields another EMT  $\tilde{\Theta}^{\mu\nu}$  for any suitable tensor  $K^{\rho\mu\nu}$ .

→ *Idea*: Find  $K^{\rho\mu\nu}$  such that  $\tilde{\Theta}^{\mu\nu} = \tilde{\Theta}^{\nu\mu}$  is symmetric (and hopefully gauge-invariant).

- 12 | Let us assume that our theory is also invariant under homogeneous Lorentz transformations (in addition to the spacetime translations needed for the conservation of the EMT).

◁ Generators of homogeneous LTs for coordinates:

$$\text{Eq. (6.78)} \rightarrow \delta^{\alpha\beta} x^\mu = \frac{1}{2} \left( \eta^{\alpha\mu} x^\beta - \eta^{\beta\mu} x^\alpha \right). \quad (6.100)$$

Assume that fields transform as  $\delta^{\alpha\beta} \phi$ . (Remember: Here it is  $a \equiv \alpha\beta$ .)

For the following arguments, we do not need to fix whether our fields transform as scalar, vector, or even → *spinor fields*.

Eq. (6.84) & Eq. (6.91) & Eq. (6.100) →

Noether currents for homogeneous LTs:

$$L^{\mu\alpha\beta} \doteq \frac{1}{2} \left( \Theta^{\mu\alpha} x^\beta - \Theta^{\mu\beta} x^\alpha \right) + \frac{1}{2} S^{\mu\alpha\beta} \quad (6.101)$$

with

$$\text{** Spin current: } S^{\mu\alpha\beta} := -2 \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta^{\alpha\beta} \phi \quad (6.102)$$

which satisfies  $S^{\mu\alpha\beta} = -S^{\mu\beta\alpha}$ .

(This follows because  $\delta^{\alpha\beta} \phi = -\delta^{\beta\alpha} \phi$  as the generators of homogeneous LTs are antisymmetric.)

The continuity equation reads

$$\partial_\mu L^{\mu\alpha\beta} = 0. \quad (6.103)$$

Because homogeneous LTs describe *rotations* in space and time, the conserved current  $L^{\mu\alpha\beta}$  can be identified as ↑ (*canonical*) *angular momentum current*. The first part in Eq. (6.101) corresponds to the (canonical) *orbital angular momentum* while the second part  $S^{\mu\alpha\beta}$  encodes the *intrinsic angular momentum* of the field (= its ↓ *spin*). This immediately explains why for a scalar field with  $\delta^{\alpha\beta} \phi = 0$ , the spin current vanishes:  $S^{\mu\alpha\beta} = 0$ .

- 13 | Eq. (6.92) & Eq. (6.103) →

$$\partial_\mu S^{\mu\alpha\beta} \doteq \Theta^{\alpha\beta} - \Theta^{\beta\alpha} \quad (6.104)$$

This means that a non-vanishing divergence in the spin current is responsible for the “non-symmetry” of the canonical EMT!

14 | Now define

$$K^{\rho\mu\nu} := -\frac{1}{2} (S^{\mu\nu\rho} + S^{\nu\mu\rho} - S^{\rho\nu\mu}) \quad (6.105)$$

→  $K^{\rho\mu\nu} = -K^{\mu\rho\nu}$  (This follows from  $S^{\mu\alpha\beta} = -S^{\mu\beta\alpha}$ .)

If you wonder where this rather arbitrary looking sum comes from: → Eq. (6.107) below.

With this we can finally define the ...

\*\* *Belinfante-Rosenfeld energy-momentum tensor (BRT):*

$$T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\rho K^{\rho\mu\nu} := \Theta^{\mu\nu} - \frac{1}{2} \partial_\rho (S^{\mu\nu\rho} + S^{\nu\mu\rho} - S^{\rho\nu\mu}) \quad (6.106)$$

15 | It remains to be shown that  $T^{\mu\nu}$  is always symmetric:

$$T^{\mu\nu} - T^{\nu\mu} \stackrel{6.104}{=} 0 \quad \odot \quad (6.107)$$

- You can use this calculation to bootstrap the arbitrary looking definition of  $K^{\rho\mu\nu}$  in Eq. (6.105). Start from the most general linear combination  $K^{\rho\mu\nu} = \alpha(S^{\mu\nu\rho} - S^{\rho\nu\mu}) + \beta S^{\nu\mu\rho}$  which satisfies  $K^{\rho\mu\nu} = -K^{\mu\rho\nu}$  and determine  $\alpha$  and  $\beta$ .
- It can be rigorously shown that the BRT is identical to the Hilbert EMT that shows up in GENERAL RELATIVITY as the source of gravity [82]. This is why the BRT gets its own symbol  $T^{\mu\nu}$ .

16 | The BRT of electrodynamics:

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- i | Using  $\mathcal{L}_{\text{em}} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$  and the transformation of a vector field (= spin-1)

$$\delta^{\alpha\beta} A_\mu = \frac{1}{2} (\delta_\mu^\alpha A^\beta - \delta_\mu^\beta A^\alpha) \quad (6.108)$$

in Eq. (6.102) yields the spin current:

$$S_{\text{em}}^{\mu\alpha\beta} \doteq \frac{1}{4\pi} (F^{\mu\alpha} A^\beta - F^{\mu\beta} A^\alpha) \quad (6.109)$$

- ii | Eq. (6.96) & Eq. (6.106) & Eq. (6.109) →

$$T_{\text{em}}^{\mu\nu} \doteq \frac{1}{4\pi} F^\mu_\rho F^{\rho\nu} - \eta^{\mu\nu} \mathcal{L}_{\text{em}} \quad (6.110a)$$

$$= \frac{1}{4\pi} \left[ F^\mu_\rho F^{\rho\nu} + \frac{\eta^{\mu\nu}}{4} F^{\rho\sigma} F_{\rho\sigma} \right] \quad (6.110b)$$

$$\doteq \left( \begin{array}{c|c} \mathcal{E} & c \vec{\Pi}^T \\ \hline c \vec{\Pi} & \Sigma \end{array} \right)_{\mu\nu} \quad (6.110c)$$

To show this you have to use the Maxwell equations in vacuum:  $\partial_\nu F^\nu_\mu = 0$ .

Components:

Energy density:  $\mathcal{E} = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$  (6.111a)

Momentum density:  $\vec{\Pi} = \frac{1}{4\pi c}(\vec{E} \times \vec{B})$  (6.111b)

\**Maxwell stress tensor*:  $\Sigma_{ij} = \frac{1}{4\pi} \left[ \frac{\delta_{ij}}{2}(\vec{E}^2 + \vec{B}^2) - E_i E_j - B_i B_j \right]$  (6.111c)

! Convince yourself that  $T_{em}^{\mu\nu}$  is *symmetric* and *gauge invariant*. Note that we did not construct it to be gauge invariant, only to be symmetric! We got this as a bonus.

iii | The conservation  $\partial_\mu T^{\mu\nu} = 0$  of the BRT implies the following physical interpretations:

- $\nu = 0$ :

$$\partial_\mu T^{\mu 0} = \frac{1}{c} \frac{\partial \mathcal{E}}{\partial t} + c \nabla \cdot \vec{\Pi} = 0 \quad (6.112)$$

→ ↓ *Poynting's theorem* (in vacuum)

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \vec{S} = 0 \quad (6.113)$$

with

$$\downarrow \text{Poynting vector: } \vec{S} = c^2 \vec{\Pi} = \frac{c}{4\pi}(\vec{E} \times \vec{B}) \quad (6.114)$$

Eq. (6.113) → Poynting vector = Energy current density

This is simply the formal statement of *energy conservation* for the free electromagnetic field. As energy is the Noether charge for translations in time, it is of course no coincidence that the Poynting theorem follows from the time-component  $\nu = 0$ .

- $\nu = i$ :

$$\partial_\mu T^{\mu i} = \frac{\partial \Pi_i}{\partial t} + \partial_k \Sigma_{ki} = 0 \quad (6.115)$$

→ Conservation of momentum *with ...*

- $\Pi_i$ : *i*-momentum density
- $\Sigma_{ki}$ : *i*-momentum current density

→ Maxwell stress tensor = Momentum current density

Note that there are three momentum densities and corresponding current densities because there are three spatial momenta:  $i = x, y, z$ .

iv | Some final remarks:

- With the symmetric BRT one can define a gauge-invariant and conserved *angular momentum tensor*

$$M^{\rho\mu\nu} := T^{\rho\mu} x^\nu - T^{\rho\nu} x^\mu \quad (6.116)$$

with  $\partial_\rho M^{\rho\mu\nu} = 0$  (show this!). The conserved Noether charges are

$$J^{\mu\nu} := \frac{1}{c} \int d^3x M^{0\mu\nu} = \frac{1}{c} \int d^3x (T^{0\mu} x^\nu - T^{0\nu} x^\mu) \quad (6.117)$$

which encodes the *total angular momentum* of the field. Indeed, for the spatial components one finds

$$J_{ij} := \int d^3x (\Pi_i x_j - \Pi_j x_i) . \quad (6.118)$$

Since  $\Pi_i$  is the momentum density, the three components  $J_x \equiv J_{32}$ ,  $J_y \equiv J_{13}$  and  $J_z \equiv J_{21}$  can be identified as the total angular momentum  $\vec{J}$  of the field.

- If the electric current  $j^\mu$  does *not* vanish (i.e., the field is not in vacuum), the BRT derived above is no longer conserved. Rather one finds

$$\partial_\mu T_{\text{em}}^{\mu\nu} = -\frac{1}{c} F^{\nu\rho} j_\rho \quad (6.119)$$

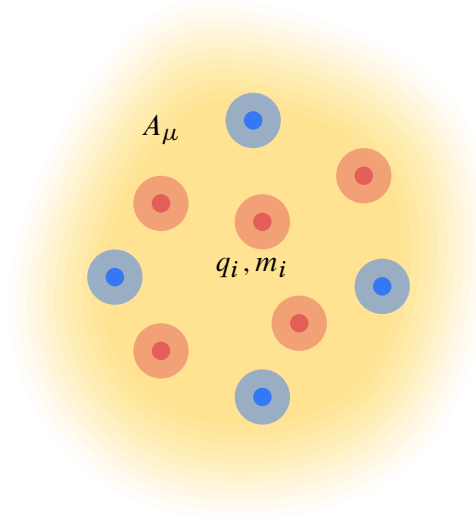
which can be identified as the *Lorentz force density*. This is perfectly reasonable as an external (non-dynamic) current  $j^\mu$  breaks the translation symmetry of the system in space and time on which the conservation of the BRT relies. Physically, the electromagnetic field is no longer a closed system because it can exchange momentum and energy with the charges described by  $j^\mu$ . Only if one describes the charges as dynamic degrees of freedom (→ *next section*) and considers the total BRT

$$T^{\mu\nu} = T_{\text{em}}^{\mu\nu} + T_{\text{charges}}^{\mu\nu} \quad (6.120)$$

one would recover the conservation  $\partial_\mu T^{\mu\nu} = 0$ ; this is then a statement about total energy and momentum conservation, including the energy and momentum of the charges.

## 6.4. Charged point particles in an electromagnetic field

1 |  $\ll N$  charged point particles with charge  $q_i$  and mass  $m_i$  in an EM field  $A_\mu$ :



Eq. (6.56) & Eq. (5.40) → Relativistic action of the complete system:

$$S[\{x_k\}, A] = \int d^4x \left[ \underbrace{-\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu}}_{\substack{S_{em}[A] \\ \text{EM field}}} - \underbrace{\frac{1}{c^2} A_\mu j^\mu}_{\substack{S_c[\{x_k\}, A] \\ \text{Coupling}}} \right] - \sum_{i=1}^N \underbrace{m_i c \int ds_i}_{\substack{S_p[x_i] \\ \text{Particle } i}} \quad (6.121)$$

$S_A[\{x_k\}]$ :  $N$  particles in static field  $A_\mu$   
 $S_j[A]$ : EM field with static current  $j^\mu$

Note that the Lagrangian is a Lorentz scalar!  $S[\{x_k\}, A]$  is short for  $S[x_1, \dots, x_N, A]$ .  
with current density

$$j^\mu(x) \stackrel{6.18}{=} \sum_i \rho_i(x) \frac{dx_i^\mu}{dt} = \sum_i \underbrace{q_i \delta(\vec{x} - \vec{x}_i)}_{\text{Point particle}} \frac{dx_i^\mu}{dt}. \quad (6.122)$$

2 |  $\ll$  Coupling:

$$S_c[\{x_k\}, A] = -\frac{1}{c^2} \int d^4x A_\mu(x) j^\mu(x) \stackrel{6.18}{=} \sum_i \underbrace{\left\{ -\frac{q_i}{c} \int A_\mu(ct, \vec{x}_i) dx_i^\mu \right\}}_{S_c[x_i, A]} \quad (6.123)$$

Here we used  $\frac{dx_i^\mu}{dt} dt = dx_i^\mu$ ; the last integral is therefore a four-dimensional  $\downarrow$  *line integral* of the 4-vector field  $A_\mu$  along the trajectory of particle  $i$ .

3 | Hamilton's principle:

$$\delta S[x_k, A] = 0 \Leftrightarrow \begin{cases} \frac{\delta S_{\text{em}}[A]}{\delta A} + \frac{\delta S_{\text{c}}[\{x_k\}, A]}{\delta A} = \frac{\delta S_j[A]}{\delta A} = 0 \\ \forall_i : \frac{\delta S_{\text{c}}[x_i, A]}{\delta x_i} + \frac{\delta S_{\text{p}}[x_i]}{\delta x_i} = \frac{\delta S_A[\{x_k\}]}{\delta x_i} = 0 \end{cases} \quad (6.124)$$

 4 | < Gauge field variations  $\delta A$ :

Here we don't have to do anything because we already computed the Euler-Lagrange equations:

$$\frac{\delta S_j[A]}{\delta A} = 0 \xleftrightarrow{6.56 \& 6.58} \partial_\nu F^{\nu\mu} \stackrel{6.122}{=} \frac{4\pi}{c} \sum_i q_i \delta(\vec{x} - \vec{x}_i) \frac{dx_i^\mu}{dt} \quad (6.125)$$

These are the inhomogeneous Maxwell equations with the  $N$  point particles as sources of the EM field. Note that this PDE system couples the particle coordinates  $\{x_k^\mu\}$  to the EM field  $A^\mu$ .

 5 | < Particle trajectory variations  $\delta x_i$ :

i | Eqs. (6.121) and (6.123) →

$$S_A[\{x_k\}] = - \sum_i \int \left[ m_i c \sqrt{\dot{x}_{i\mu} \dot{x}_i^\mu} + \frac{q_i}{c} A_\mu(x_i) \dot{x}_i^\mu \right] d\lambda \quad (6.126)$$

Note that this action is again reparametrization invariant.

→ Euler-Lagrange equation for particle  $i$ :

$$\frac{\delta S_A[\{x_k\}]}{\delta x_i} = 0 \xleftrightarrow{x^\mu \equiv x_i^\mu} \frac{d}{d\lambda} \left[ \frac{m_i c \dot{x}_\mu}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} \right] + \frac{q_i}{c} \left[ \dot{A}_\mu(x) - \dot{x}^\nu \frac{\partial A_\nu(x)}{\partial x^\mu} \right] = 0 \quad (6.127)$$

ii | Choose proper-time parametrization  $\lambda = \tau$ :

$$m_i \frac{du_\mu}{d\tau} + \frac{q_i}{c} \left[ \underbrace{\frac{dA_\mu}{d\tau}}_{\frac{\partial A_\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau}} - \frac{\partial A_\nu}{\partial x^\mu} \frac{dx^\nu}{d\tau} \right] = 0 \quad (6.128)$$

Thus we find as the EOM for particle  $i$ :

$$m_i \frac{du_\mu}{d\tau} = \frac{q_i}{c} \underbrace{\left( \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right)}_{F_{\mu\nu}} u^\nu \quad (6.129)$$

Or in the form discussed previously in Chapter 5 (we restore the particle index  $i$ ):

$$\frac{dp_i^\mu}{d\tau} = \frac{q_i}{c} F^\mu{}_\nu(x_i) u_i^\nu \quad (6.130)$$

with 4-momentum  $p_i^\mu = m_i u_i^\mu$ .

! The field strength tensor is evaluated at the position of the particle at a given time.

iii | Compare Eqs. (5.5) and (6.130) → 4-force:

$$K^\mu = \begin{pmatrix} \gamma_v \frac{\vec{F} \cdot \vec{v}}{c} \\ \gamma_v \vec{F} \end{pmatrix} = \frac{q_i}{c} F^\mu{}_\nu \gamma_v \frac{dx^\nu}{dt} \quad (6.131)$$

→ 3-force (we restore the particle index  $i$ ):

$$\vec{F}_i \doteq q_i \vec{E}_i + \frac{q_i}{c} (\vec{v}_i \times \vec{B}_i) \quad \downarrow \text{Lorentz force} \quad (6.132)$$

with  $\vec{E}_i = \vec{E}(x_i)$ ,  $\vec{B}_i = \vec{B}(x_i)$  and  $\vec{v}_i = \frac{d\vec{x}_i}{dt}$ .

- This result demonstrates that our concept of the relativistic 3-force introduced in Eq. (5.10) was reasonable: for a force due to an electromagnetic field, it exactly matches the Lorentz force.
- It also demonstrates that the common expression for the Lorentz force is already fully relativistic. However, note that the 3-force determines the change rate of the *relativistic* 3-momentum  $\vec{p} = \gamma_v m \vec{v}$ , recall Eq. (5.15).

## 6 | Comments:

- Eqs. (6.125) and (6.130) together are the equations of motion of the composite system, i.e., the EM field and the  $N$  particles. Note that the system of differential equations is coupled: The dynamical positions of the particles determine the evolution of the EM field via Eq. (6.125), and the dynamical EM field determines the trajectories of the charged particles via Eq. (6.130).
- This model of  $N$  charged particles interacting with and via an electromagnetic field is the culmination of our discussion of *relativistic mechanics* in Chapter 5 and *electrodynamics* in Chapter 6.
- The theory Eqs. (6.125) and (6.130) is fully relativistic as the EOMs are manifestly Lorentz covariant (they are tensor equations).
- Note that this model describes interactions between the  $N$  particles not directly via forces (as one would in Newtonian mechanics), but via coupling to the dynamic EM field. Thus a particle can *locally* affect the EM field due to its motion, the EM field then can propagate with the speed of light through space and affect the trajectory of any other particle within the lightcone of the first. There is no instantaneous interaction between the particles!
- One can also consider the  $\mu = 0$  component of Eq. (6.130). Then one finds with  $p_i^0 = E_i/c$ :

$$\frac{dE_i}{dt} \doteq q_i \vec{E}_i \cdot \vec{v}_i. \quad (6.133)$$

This is just the statement that the change of energy for particle  $i$  is given by the distance it travels collinear with the electric field per time. This is no surprise: The Lorentz force Eq. (6.132) tells us that the force due to the magnetic field is always perpendicular to the direction of motion and therefore cannot perform work on the particle.

## 7 | Corollary: Single particle in a static electromagnetic field:

i | The action follows from Eq. (6.126) with  $N = 1$  as:

$$S_A[x] = \int d\lambda L(x^\mu, \dot{x}^\mu) = - \int \left[ mc \sqrt{\dot{x}_\mu \dot{x}^\mu} + \frac{q}{c} A_\mu(x_i) \dot{x}^\mu \right] d\lambda \quad (6.134)$$

where  $A_\mu$  is a fixed parameter (the static field configuration).

ii | < Parametrization in coordinate time  $\lambda = t$ :

$$L(\vec{x}, \vec{v}) = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} + \frac{q}{c} \vec{A} \cdot \vec{v} - q\varphi \quad (6.135)$$

with  $A_\mu = (\varphi, -\vec{A})$  (covariant!) and  $\dot{\vec{x}} = \vec{v}$ .

iii | Canonical momentum:

$$\vec{\pi} := \frac{\partial L}{\partial \vec{v}} \doteq m\gamma_v \vec{v} + \frac{q}{c} \vec{A} \quad (6.136)$$

with *mechanical momentum*  $\vec{p} = m\gamma_v \vec{v} \rightarrow$

$$\vec{p} = \vec{\pi} - \frac{q}{c} \vec{A} \quad (6.137)$$

$\vec{p}$ : Measurable momentum

→ Mechanical momentum  $\vec{p}$  gauge-invariant

→ Canonical momentum  $\vec{\pi}$  *not* gauge-invariant

iv | Hamiltonian:

$$H = \vec{\pi} \cdot \vec{v} - L \doteq \underbrace{\frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}}_{\text{Mechanical energy } E} + q\varphi \stackrel{5.25}{=} c \sqrt{\left(\pi - \frac{q}{c} \vec{A}\right)^2 + m^2 c^2} + q\varphi \quad (6.138)$$

so that

$$E = H - q\varphi \quad (6.139)$$

$E$  is gauge invariant →  $H$  is *not* gauge invariant

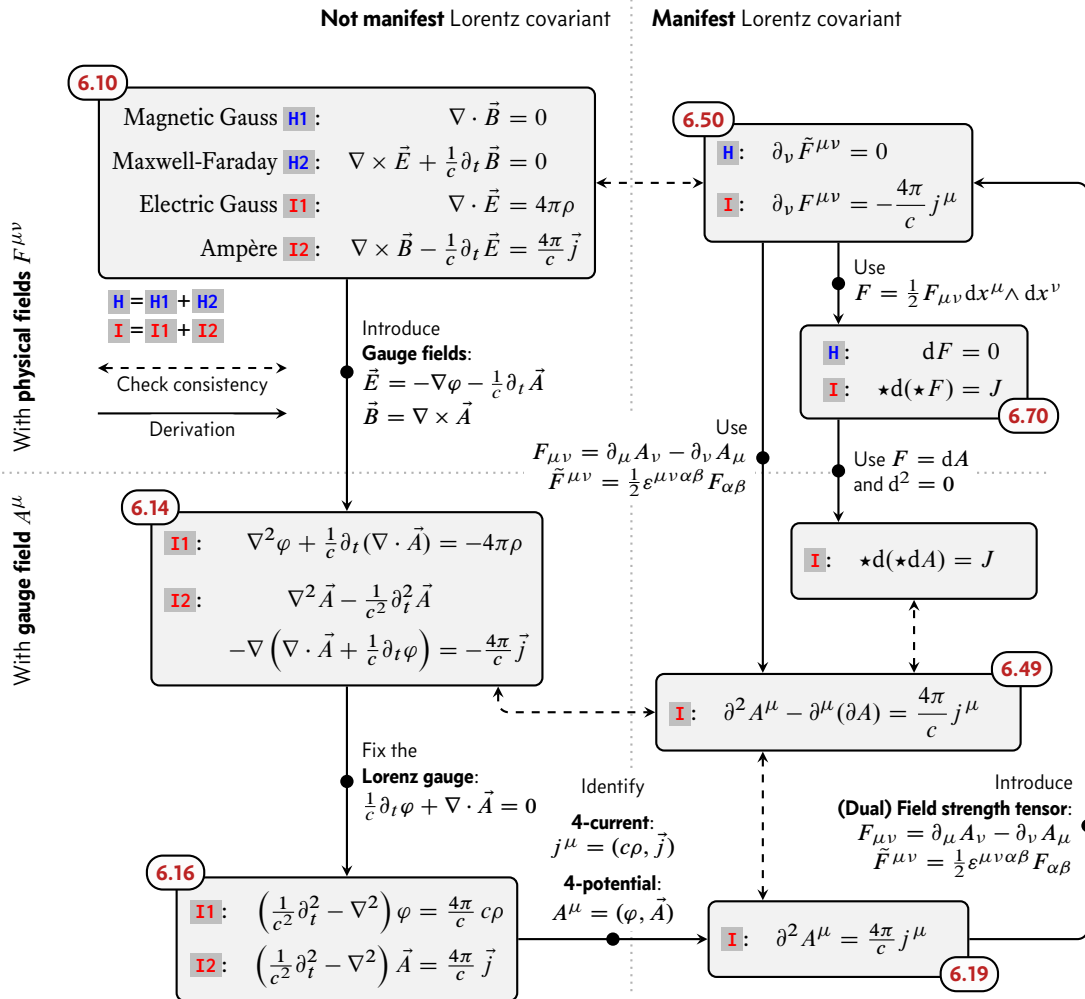
v | Summary:

$$\text{Gauge invariant} \left\{ \begin{array}{l} E = H - q\varphi \\ \vec{p} = \vec{\pi} - \frac{q}{c} \vec{A} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} E + q\varphi = H \\ \vec{p} + \frac{q}{c} \vec{A} = \vec{\pi} \end{array} \right\} \text{Gauge dependent} \quad (6.140)$$

For more details on the aspect of the gauge-(in)variance of certain quantities, see Ref. [83]. Note that these subtleties are not specific to a relativistic treatment, they already appear in Newtonian mechanics (only the specific dependency of the Hamiltonian on the mechanical/canonical momentum and the functional form of the Lagrangian are relativistic).

### 6.5. Summary: The many faces of Maxwell's equations

Here is a compact overview of the many (physically equivalent) forms of Maxwell's equations that we encountered in this chapter:



## 7. Relativistic Field Theories II: Relativistic Quantum Mechanics

### Reminder

- 1 | The ↓ *Schrödinger equation* (SE)

$$i\hbar\partial_t\psi(t, \vec{x}) = \hat{H}\psi(t, \vec{x}) \tag{7.1}$$

is a linear field equation with ↓ *Hamilton operator*

$$\hat{H} = \frac{\vec{p}^2}{2m} + V(\vec{x}) = -\frac{\hbar^2}{2m}\Delta + V(\vec{x}) \tag{7.2}$$

and the complex-valued field  $\psi : \mathbb{R}^{1,3} \rightarrow \mathbb{C}$ .

It describes the time evolution of a single quantum particle with mass  $m$  in a potential  $V(\vec{x})$  that is initially described by the wavefunction  $\psi_0(\vec{x}) = \psi(0, \vec{x})$  at  $t = 0$ .

- 2 | The wavefunction has the interpretation

$$|\psi(t, \vec{x})|^2 = \langle \text{Probability to find particle at time } t \text{ at position } \vec{x} \rangle \tag{7.3}$$

which necessitates the normalization condition

$$\forall_t : \|\psi(t)\|_2 := \int d^3x |\psi(t, \vec{x})|^2 = 1. \tag{7.4}$$

Thus the wavefunction is an element of the Hilbert space  $\psi \in L^2 \equiv L^2(\mathbb{R}^3, \mathbb{C})$  of square-integrable functions.

The Hermiticity  $\hat{H} = \hat{H}^\dagger$  of the Hamiltonian implies a unitary time evolution and thereby guarantees a conserved norm:

$$\frac{d}{dt}\|\psi(t)\|_2 = \int d^3x [\psi^* \partial_t \psi + \psi \partial_t \psi^*] \stackrel{7.1}{=} \frac{1}{i\hbar} \int d^3x [\psi^* (\hat{H}\psi) - \psi (\hat{H}\psi)^*] \stackrel{7.6}{=} 0, \tag{7.5}$$

where we used that for  $\psi, \phi \in L^2$  and a Hermitian Hamiltonian

$$\int d^3x \phi^* (\hat{H}\psi) \stackrel{\text{def}}{=} \langle \phi | \hat{H}\psi \rangle \stackrel{\text{def}}{=} \langle \hat{H}^\dagger \phi | \psi \rangle \stackrel{\text{def}}{=} \int d^3x (\hat{H}^\dagger \phi)^* \psi \stackrel{\hat{H} = \hat{H}^\dagger}{=} \int d^3x \psi (\hat{H}\phi)^*. \tag{7.6}$$

- 3 | Problem: The SE is Galilei covariant but *not* Lorentz covariant! (recall ↻ Problemset 1)

- The SE is of first order in time but of second order in the spatial derivatives. This asymmetry already suggests that the equation cannot be Lorentz covariant: Time is treated differently than space in (non-relativistic) quantum mechanics.
- We would like quantum mechanics to be described by a Lorentz covariant equation because we subscribed to ← *Einstein's principle of special relativity* SR at the beginning of this course: All laws of physics must take the same form in all inertial systems (which are related by Lorentz transformations). This certainly includes quantum mechanics.