↓ Lecture 14 [28.01.26]

7 | Transformation of the electromagnetic field:

The field strength tensor Eq. (6.30) has the useful properties that (1) we know how it transforms under Lorentz transformations, and (2) we know how it relates to the observable fields $\vec{E}$ and $\vec{B}$. Hence we can use it to derive the transformation of the electromagnetic field when transitioning from one inertial system to another.

i | The (contravariant) FST transforms under a Lorentz transformation Λ as follows:

$$\underbrace{\bar{F}^{\mu\nu}(\bar{x})}_{\{\bar{E}_i(\bar{x}),\bar{B}_i(\bar{x})\}} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\underbrace{F^{\alpha\beta}(x)}_{\{E_i(x),B_i(x)\}} \tag{6.33}$$

Here it is $F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}$ as usual.

ii | $\sphericalangle$ Boost $\Lambda_{\vec{v}}$ [Eq. (4.9)]:

$$\vec{\bar{E}}(\bar{x}) \triangleq \gamma\left[\vec{E}(x) + \tfrac{1}{c}\vec{v}\times\vec{B}(x)\right] - (\gamma-1)\frac{\vec{v}\cdot\vec{E}(x)}{v^2}\vec{v} \tag{6.34a}$$

$$\vec{\bar{B}}(\bar{x}) \triangleq \gamma\left[\vec{B}(x) - \tfrac{1}{c}\vec{v}\times\vec{E}(x)\right] - (\gamma-1)\frac{\vec{v}\cdot\vec{B}(x)}{v^2}\vec{v} \tag{6.34b}$$

with $x^\mu = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu}\,\bar{x}^\nu$.

¡! Note that on the left-hand side the arguments are $\bar{x}$ and on the right-hand side x!

→ Electric and magnetic fields "mix" under boosts!

- Please appreciate what we showed: If you start from Maxwell's Eq. (6.10) and perform an arbitrary Lorentz boost $\bar{x}^\mu = \Lambda^{\mu}{}_{\nu}x^\nu$, transforming the derivatives as $\bar{\partial}_\mu = \Lambda_{\mu}{}^{\nu}\partial_\nu$, you obtain a set of horribly complicated PDEs. But if you recombine the equations appropriately, group the terms according to Eq. (6.34) and define the new fields $\vec{\bar{E}}(\bar{x})$, $\vec{\bar{B}}(\bar{x})$, the equations look again like Eq. (6.10), only with bars over coordinates and fields.

  You *could* show this directly, without ever introducing the gauge field A^μ and without using the machinery of tensor calculus (this is what Einstein did for a boost in z-direction in his 1905 paper "Zur Elektrodynamik bewegter Körper" [10], see also ➔ Problemset 7); but hopefully you agree that our more advanced route (using the gauge field and tensor calculus) is a more elegant approach.

- Because of our motivation from Einstein's principle of Special Relativity SR, we frame our discussion in the terminology of *passive* transformations (= coordinate transformation): The same electromagnetic field that looks like $\vec{E}(x)$, $\vec{B}(x)$ in an inertial system K looks like $\vec{\bar{E}}(\bar{x})$, $\vec{\bar{B}}(\bar{x})$ in another system $\bar{K}$.

  Because we showed that the Maxwell equations satisfy SR, they have exactly the same form in $\bar{K}$ as in K. This, however, allows you to interpret the transformation *actively*: If you are given a solution of Maxwell equations $\vec{E}(x)$, $\vec{B}(x)$, then, for any $\vec{v}$, the *new* functions $\vec{\bar{E}}(\bar{x})$, $\vec{\bar{B}}(\bar{x})$ defined by Eq. (6.34) and $x^\mu = (\Lambda_{-\vec{v}})^{\mu}{}_{\nu}\,\bar{x}^\nu$ are again solutions (in the same coordinates). This shows that the Lorentz group is (part of) the invariance group of the PDE system Eq. (6.10) we call Maxwell equations (just like the Galilei group was an invariance group of Newton's equation, recall Section 1.2).

iii | ⊲ Non-relativistic limit:

$$\text{Eq. (6.34)} \xrightarrow{\gamma\approx 1} \begin{cases} \bar{\vec{E}}(\bar{x}) \approx \vec{E}(x) + \frac{1}{c}\vec{v}\times\vec{B}(x) \\ \bar{\vec{B}}(\bar{x}) \approx \vec{B}(x) - \frac{1}{c}\vec{v}\times\vec{E}(x) \end{cases} \tag{6.35}$$

- The interconversion between magnetic and electric fields happens already in linear order of v/c.
- The separation of the electromagnetic field into "electric" and "magnetic" components is observer dependent!
- Example: A charge at rest has a non-zero electric field, but a vanishing magnetic field. The *same* charge as seen from an inertial system in relative motion gives rise to a current that is accompanied by a non-vanishing magnetic field perpendicular to the direction of motion and the electric field. This is a direct consequence of Eq. (6.35): $\bar{\vec{B}}(\bar{x}) \approx -\frac{1}{c}\vec{v}\times\vec{E}(x) \neq \vec{0}$.

iv | ⊲ Special case: Boost Λ_{v_x} in x-direction: Eq. (6.34) $\xrightarrow{\vec{v}=(v_x,0,0)}$

$$\bar{E}_x = E_x\,,\quad \bar{E}_y = \gamma\left[E_y - \tfrac{v}{c}B_z\right],\quad \bar{E}_z = \gamma\left[E_z + \tfrac{v}{c}B_y\right], \tag{6.36a}$$

$$\bar{B}_x = B_x\,,\quad \bar{B}_y = \gamma\left[B_y + \tfrac{v}{c}E_z\right],\quad \bar{B}_z = \gamma\left[B_z - \tfrac{v}{c}E_y\right], \tag{6.36b}$$

(Here the fields in $\bar{K}$ on the left-hand side are functions of $\bar{x}$ whereas the fields in K on the right-hand side are functions of x.)

→

- The field components *parallel* to the boost remain unchanged.
- The *perpendicular* components mix and get enhanced by a Lorentz factor $\gamma > 1$.
- Einstein derived this transformation directly (without using gauge fields and tensor notation) in his 1905 paper "Zur Elektrodynamik bewegter Körper" [10]; you follow this path in ➲ Problemset 7.

v | <u>Lorentz scalars:</u>

The electric and magnetic field components transform in a complicated way under Lorentz transformations. Is it possible to combine them into scalar quantities? Thanks to our knowledge of tensor calculus and the field strength tensor, this question is easy to answer:

a | We can construct a scalar by contracting the FST with itself:

$$\boxed{F^{\mu\nu}F_{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta}F_{\mu\nu} \mathrel{\hat{=}} 2(\vec{B}^2 - \vec{E}^2)} \tag{6.37}$$

→ If $|\vec{E}| \gtreqless |\vec{B}|$ is true in one IS, it is true in all IS.

b | We can construct a *pseudo* scalar by contracting the FST with the DFST:

$$\boxed{\tilde{F}^{\mu\nu}F_{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}F_{\mu\nu} \mathrel{\hat{=}} -4(\vec{E}\cdot\vec{B})} \tag{6.38}$$

→ If $\vec{E} \perp \vec{B}$ is true in one IS, it is true in all IS.

Some comments:

- Note that $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} \stackrel{\circ}{=} -F^{\mu\nu}F_{\mu\nu}$ (use contraction identities for Levi-Civita symbols to show this, ➔ Problemset 7); i.e., the two quantities above exhaust all elementary gauge-invariant scalar fields that we can construct ($A^\mu A_\mu$ is of course also a scalar, but not a gauge-invariant one).
- The combination of Eq. (6.37) and Eq. (6.38) can be used to infer whether inertial systems exist in which either the electric or magnetic field vanishes. For example: If $\tilde{F}^{\mu\nu}F_{\mu\nu} = 0$ and $F^{\mu\nu}F_{\mu\nu} > 0$, it is possible to find an inertial system where $\vec{E} = 0$ and $\vec{B} \neq 0$ (but not the other way around). If $\tilde{F}^{\mu\nu}F_{\mu\nu} \neq 0$ there is no inertial system in which one of the fields vanishes.

8 | Manifest covariant form of the Maxwell equations:

Using the FST and the DFST, we can write the Maxwell equations manifestly covariant without using the gauge field and/or fixing a gauge (cf. Eq. (6.19)):

i | The equations we look for must be …

- …manifestly covariant ($\to$ tensor equations).
- …linear in the FST or the DFST (the ME are linear in $\vec{E}$ and $\vec{B}$).
- …use one 4-divergence ∂_μ (the ME are first-order PDEs).

$\to \triangleleft$

$$\partial_\nu \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \varepsilon^{\mu\nu\rho\sigma}\partial_\nu\partial_\rho A_\sigma = 0 \tag{6.39a}$$

$$\partial_\nu F^{\mu\nu} = \partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial^\mu(\partial A) - \partial^2 A^\mu \tag{6.39b}$$

ii | The homogeneous ME Eqs. (6.10a) and (6.10b) must be *identically* true if the fields are given in terms of gauge fields. Eq. (6.39a) then suggests that the homogeneous ME are:

$$\partial_\nu \tilde{F}^{\mu\nu} = 0 \qquad \text{Homogeneous ME (?)} \tag{6.40}$$

To check this evaluate:

$$\partial_\nu \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} \tag{6.41a}$$

$$= \frac{1}{6}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\nu F_{\rho\sigma} + \partial_\sigma F_{\nu\rho} + \partial_\rho F_{\sigma\nu}\right) \tag{6.41b}$$

$$= \frac{1}{2}\sum_{\nu<\rho<\sigma}\varepsilon^{\mu\nu\rho\sigma}\left(\partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho}\right) \tag{6.41c}$$

Here we used that the Levi-Civita symbol is invariant under cyclic permutations of (subsets) of indices and that the FST (and the Levi-Civita symbol) is antisymmetric in its indices. Note that for every fixed μ there are $3! = 6$ non-vanishing assignments of indices $\nu\rho\sigma$. However, pairs of terms like $\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = \varepsilon^{\mu\nu\sigma\rho}\partial_\nu F_{\sigma\rho}$ are identical, so that only 3 distinct terms remain. These can be w.l.o.g. written like cyclic permutations as in Eq. (6.41c). Note that for a fixed index μ, the sum contains only one non-vanishing summand.

$\to$

$$\underbrace{\forall_{\nu<\rho<\sigma}:\ \partial_\nu F_{\rho\sigma} + \partial_\rho F_{\sigma\nu} + \partial_\sigma F_{\nu\rho} = 0}_{\uparrow\ \textit{Bianchi identity}\ \text{(4 equations)}} \quad \Leftrightarrow \quad \underbrace{\forall_\mu:\ \partial_\nu \tilde{F}^{\mu\nu} = 0}_{\text{(4 equations)}} \tag{6.42}$$

It is straightforward to check by hand, using Eq. (6.30), that the four Bianchi identities correspond to the four homogeneous Maxwell Eqs. (6.10a) and (6.10b). For example:

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} \overset{\circ}{=} -\nabla \cdot \vec{B} = 0 \quad \Leftrightarrow \quad \text{Eq. (6.10a)} \tag{6.43}$$

Details: Problemset 7

- As shown in Eq. (6.39a), the homogeneous ME are *identities* if the FST is expressed in terms of gauge fields.
- By contrast, if the FST is expressed in terms of physical fields $\vec{E}$ and $\vec{B}$ [as given in Eq. (6.30)], the equation $\partial_\nu \tilde{F}^{\mu\nu} = 0$ becomes a non-trivial constraint on the field configurations.

iii | $\triangleleft$ Lorenz gauge Eq. (6.23) →

$$\text{Eq. (6.39b)} \quad \Rightarrow \quad \partial_\nu F^{\mu\nu} = -\partial^2 A^\mu \tag{6.44}$$

Compare Eq. (6.19) (inhomogeneous ME in Lorenz gauge):

$$-\partial^2 A^\mu = -\frac{4\pi}{c} j^\mu \tag{6.45}$$

This suggests that the inhomogeneous ME are:

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \qquad \text{Inhomogeneous ME (?)} \tag{6.46}$$

It is straightforward to check by hand that these four equations are equivalent to the four inhomogeneous ME Eqs. (6.10c) and (6.10d) using Eq. (6.30). For example for $\mu = 0$:

$$\partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} \overset{\circ}{=} -\nabla \cdot \vec{E} = -\frac{4\pi}{c} j^0 = -4\pi\rho \quad \Leftrightarrow \quad \text{Eq. (6.10c)} \tag{6.47}$$

Details: Problemset 7

- In this form, the continuity equation Eq. (6.24) follows trivially from the antisymmetry of the FST:

$$\partial_\mu j^\mu = -\frac{c}{4\pi} \partial_\nu \partial_\mu F^{\mu\nu} = 0 \tag{6.48}$$

- If you express the FST in terms of the gauge field, the inhomogeneous ME read (without fixing a gauge!):

$$\partial^2 A^\mu - \partial^\mu (\partial A) = \frac{4\pi}{c} j^\mu \tag{6.49}$$

  This equation becomes Eq. (6.19) in the Lorenz gauge Eq. (6.23). It is easy to check that this equation is still gauge symmetric under the transformation Eq. (6.25).

iv | In summary, the 8 (=1+3+1+3=4+4) Maxwell equations can be written in the covariant form:

$$\text{Homogeneous ME:} \quad \partial_\nu \tilde{F}^{\mu\nu} = 0 \tag{6.50a}$$

$$\text{Inhomogeneous ME:} \quad \partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \tag{6.50b}$$

- ¡! Using Eqs. (6.30) and (6.32), these equations make sense without introducing the gauge field.
- Note that these equations show that under Lorentz transformations the four homogeneous (inhomogeneous) Maxwell equations mix among each other. You show this explicitly in ➔ Problemset 7 for a boost in z-direction.
- In particular, this means that the Maxwell equations written in their conventional form Eq. (6.10) (i.e., as two scalar and two vector equations) remain *not* invariant under Lorentz transformations for each equation separately, rather the magnetic Gauss law mixes with the Maxwell-Faraday law, and the electric Gauss law mixes with Ampère's law. This explains why showing the Lorentz covariance of the PDE system Eq. (6.10) is quite messy and complicated without using the tensor formalism. This is why we say that its Lorentz covariance *is not manifest*. By contrast, the Lorentz covariance of the formulation Eq. (6.50) *is manifest* as these are tensor equations.

9 | Lagrangian formulation:

Our final goal is to make a connection to the formalism introduced in Section 6.1 and obtain the Lorentz covariant Maxwell equations as the Euler-Lagrange equations of some action/Lagrangian:

i | It is convenient to construct the Lagrangian as a function of the gauge fields A^μ because in this formulation the HME are identically satisfied:

$$\partial_\nu \tilde{F}^{\mu\nu} \equiv 0 \quad \Rightarrow \quad \mathcal{L} = \mathcal{L}(A, \partial A) \tag{6.51}$$

→ Only the inhomogeneous ME must follow as Euler-Lagrange equations

Note that the counting matches: We have four fields A^μ and thus four Euler-Lagrange equations – just as we have four IME: $\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu$.

ii | We have the following hints to construct a reasonable Lagrangian density:

- The IME are Lorentz covariant. This can be ensured by a Lagrangian density that is a Lorentz (pseudo) scalar.
- The Maxwell equations are linear (superposition principle!); thus the Lagrangian must be quadratic in the fields.
- The IME are gauge invariant. This can be ensured by a Lagrangian density that is gauge invariant up to a total derivative (here: surface term) which does not affect the equations of motion.

→ Most general form:

$$\mathcal{L}(A, \partial A) = a_1\, F^{\mu\nu} F_{\mu\nu} + a_2 \underbrace{\tilde{F}^{\mu\nu} F_{\mu\nu}}_{\text{Surface term}} + a_3 \underbrace{\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu}}_{\propto F^{\mu\nu} F_{\mu\nu}} + a_4 \underbrace{A_\mu j^\mu}_{\text{Gauge inv. up to surface term}} \tag{6.52}$$

Details: ➔ Problemset 7

- It is straightforward to check that

$$\tilde{F}^{\mu\nu} \tilde{F}_{\mu\nu} \stackrel{\circ}{=} -F^{\mu\nu} F_{\mu\nu} \tag{6.53}$$

so that we can drop the a_3-term without loss of generality.

- One can also check that

$$\tilde{F}^{\mu\nu}F_{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho) \tag{6.54a}$$

$$= \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu \partial_\rho A_\sigma + \partial_\nu A_\mu \partial_\sigma A_\rho - \partial_\nu A_\mu \partial_\rho A_\sigma - \partial_\mu A_\nu \partial_\sigma A_\rho) \tag{6.54b}$$

$$= 2\varepsilon^{\mu\nu\rho\sigma}(\partial_\mu A_\nu)(\partial_\rho A_\sigma) \tag{6.54c}$$

$$= \underbrace{2\,\partial_\mu \overbrace{\varepsilon^{\mu\nu\rho\sigma}(A_\nu \partial_\rho A_\sigma)}^{\uparrow \text{ Chern-Simons 3-form}}}_{\text{Surface term}} \tag{6.54d}$$

so that the a_2-term has no effect on the equations of motion and we can drop it as well.

Note: The a_2-term is known as the ↑ *θ-term* and is special because it is *topological* (it does not "feel" the *geometry* of spacetime). This is easy to see: One does not need a metric tensor to construct it because the contravariant indices of the DFST stem from the Levi-Civita symbol! Despite being a surface term, such terms are important when one quantizes the theory and/or when the gauge theory is non-Abelian (like the SU(3) gauge theory of the strong interaction). Note also that this term is a *pseudo* scalar, i.e., it breaks parity symmetry (which we know electrodynamics does not).

- The a_4-term is *not* gauge invariant. However, the continuity equation ensures that it modifies the Lagrangian only by a surface term under gauge transformations:

$$\tilde{A}_\mu j^\mu = (A_\mu - \partial_\mu \lambda) j^\mu = A_\mu j^\mu - (\partial_\mu \lambda) j^\mu = A_\mu j^\mu - \underbrace{\partial_\mu(\lambda j^\mu)}_{\text{Surface term}} \tag{6.55}$$

(Here we used the continuity equation $\partial_\mu j^\mu = 0$.)

Consequently, the equations of motion must be gauge invariant despite the a_4-term.

- It is easy to check that the quadratic Lorentz scalar $A_\mu A^\mu$ is not gauge invariant (not even up to a surface term); thus it is forbidden.

Note: Coincidentally, it is this term that would give the quantized excitations of the A-field a mass. Thus if you want massive gauge excitations (like the $W^\pm$- and Z-bosons of the weak interaction), you must find a way to smuggle the term $A_\mu A^\mu$ into your Lagrangian. This is what the ↑ *Higgs mechanism* achieves.

iii | Thus we propose the

Lagrangian density for Maxwell theory:

$$\boxed{\mathcal{L} \equiv \mathcal{L}_{\text{Maxwell}}(A, \partial A) = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}A_\mu j^\mu} \tag{6.56}$$

The prefactors have been chosen such that the Euler-Lagrange equations match the IME (→ *next step*).

iv | Euler-Lagrange equations:

Details: ➔ Problemset 7

There are four ($\mu = 0, 1, 2, 3$) Euler-Lagrange equations:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)}\right) = 0 \tag{6.57}$$

Straightforward calculations yield:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = -\frac{1}{c} j^\mu \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \stackrel{\circ}{=} \frac{1}{4\pi} F^{\mu\nu} \tag{6.58}$$

Hence the Euler-Lagrange equations are exactly the inhomogeneous Maxwell equations:

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \tag{6.59}$$

→ Eq. (6.56) is the correct Lagrangian density for Maxwell theory.

10 | Coordinate-free notation:

Remember the coordinate-free concepts introduced in Chapter 3: All tensor fields ${T^I}_J$ are the chart-dependent components of chart-independent objects T (the actual tensor fields). This formalism allows us to reformulate the Maxwell equations in the language of differential geometry, without using coordinates altogether:

i | First, write the gauge field

$$A := A_\mu \mathrm{d}x^\mu \tag{6.60}$$

and the field strength coordinate-free:

$$F := F_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \frac{1}{2} F_{\mu\nu} \underbrace{[\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu]}_{=:\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \text{ (“wedge product”)}} . \tag{6.61}$$

We say that A is a 1-form and F is a 2-form.

ii | We can evaluate ↑ *exterior derivative* of the gauge field:

$$\mathrm{d}A \stackrel{\text{def}}{=} \mathrm{d}A_\nu \wedge \mathrm{d}x^\nu = \partial_\mu A_\nu\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = F \tag{6.62}$$

The exterior derivative d maps k-forms onto $k+1$-forms.

iii | Now evaluate the exterior derivative of the field strength:

$$\mathrm{d}F \stackrel{\text{def}}{=} \frac{1}{2} \partial_\sigma F_{\mu\nu}\, \mathrm{d}x^\sigma \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \tag{6.63a}$$

$$= \frac{1}{6} \left(\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma}\right) \mathrm{d}x^\sigma \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \tag{6.63b}$$

$$= \frac{1}{2} \sum_{\sigma<\nu<\mu} \left(\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma}\right) \mathrm{d}x^\sigma \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \tag{6.63c}$$

(Here we used the antisymmetry of the wedge product in all factors.)

Thus we find:

$$\mathrm{d}F = 0 \quad \Leftrightarrow \quad \partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0 \quad \stackrel{6.42}{\Leftrightarrow} \quad \partial_\nu \tilde{F}^{\mu\nu} = 0 \tag{6.64}$$

If the field strength is expressed in terms of the gauge field, the homogeneous Maxwell equations $\partial_\nu \tilde{F}^{\mu\nu} = 0$ are identities. In the coordinate-free notation of differential geometry, this identity follows from the fact that applying an exterior derivative twice produces the zero field:

$$\mathrm{d}F = \mathrm{d}\mathrm{d}A = 0 \quad \text{since} \quad \mathrm{d}^2 = 0 \tag{6.65}$$

The relation $\mathrm{d}F = 0$ is known as a ↑ *Bianchi identity*.

iv | Define the linear ↑ *Hodge star operator* (here for a 4-dimensional Minkowski manifold):

$$\star(\mathrm{d}x^\mu) := \frac{1}{3!}\varepsilon^\mu{}_{\nu\rho\sigma}\,(\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma) \tag{6.66a}$$

$$\star(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu) := \frac{1}{2!}\varepsilon^{\mu\nu}{}_{\rho\sigma}\,(\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma) \tag{6.66b}$$

$$\star(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho) := \frac{1}{1!}\varepsilon^{\mu\nu\rho}{}_\sigma\,(\mathrm{d}x^\sigma) \tag{6.66c}$$

Note that the definition makes use of the metric tensor via pulling up/down indices of the Levi-Civita symbols. This implies in particular that any equation that uses the Hodge star depends on the geometry of spacetime (here flat Minkowski space).

v | The dual field-strength tensor (DFST) is the Hodge dual of the field-strength tensor (FST):

$$\star F = \frac{1}{2}F_{\mu\nu}\star(\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu) \tag{6.67a}$$

$$= \frac{1}{4}F_{\mu\nu}\varepsilon^{\mu\nu}{}_{\rho\sigma}\,(\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma) \tag{6.67b}$$

$$= \frac{1}{2}\tilde{F}_{\rho\sigma}\,(\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma) \equiv \tilde{F} \tag{6.67c}$$

Beware: The Hodge star $\star$ is not a multiplication symbol (as the notation on the right-hand side might suggest) but a linear operator that acts on the differential form to the right.

vi | The Hodge dual of the exterior derivative of the DFST yields:

$$\star\mathrm{d}(\star F) = \frac{1}{4}\varepsilon^{\mu\nu}{}_{\rho\sigma}\partial_\pi F_{\mu\nu}\star(\mathrm{d}x^\pi\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma) \tag{6.68a}$$

$$= \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}\varepsilon^{\pi\rho\sigma}{}_\alpha\partial_\pi F^{\mu\nu}\,(\mathrm{d}x^\alpha) \tag{6.68b}$$

$$= \frac{1}{2}(\delta^\pi_\mu\eta_{\nu\alpha}-\delta^\pi_\nu\eta_{\mu\alpha})\partial_\pi F^{\mu\nu}\,(\mathrm{d}x^\alpha) \tag{6.68c}$$

$$= \eta_{\nu\alpha}\partial_\mu F^{\mu\nu}\,(\mathrm{d}x^\alpha) \tag{6.68d}$$

$$\overset{6.50b}{=} \frac{4\pi}{c}j_\alpha\,(\mathrm{d}x^\alpha) \tag{6.68e}$$

Here we used a contraction identity for Levi-Civita symbols (over the two red pairs of indices).

vii | This motivates the definition of the coordinate-free current:

$$J := \frac{4\pi}{c}j_\mu\,\mathrm{d}x^\mu \tag{6.69}$$

viii | In conclusion, the Maxwell equations can be written without using a coordinate system as:

$$\text{Homogeneous ME:}\quad \mathrm{d}F = 0 \tag{6.70a}$$

$$\text{Inhomogeneous ME:}\quad \star\mathrm{d}(\star F) = J \tag{6.70b}$$

- If one uses that $(\star)^2 = +1\cdot\mathbb{1}$ on odd differential forms ($\mathrm{d}(\star F)$ is a 3-form), Eq. (6.70b) can alternatively be written as $d(\star F) = \star J$. If one then defines the current not as a 1-form but as the dual 3-form, $J := \frac{4\pi}{c}j_\mu \star \mathrm{d}x^\mu$, the inhomogeneous Maxwell equations take their simplest form: $d(\star F) = J$.
- Eq. (6.70) is the most general and elegant formulation of the Maxwell equations. In this form, the equations remain valid even in GENERAL RELATIVITY on curved space times. Then the Minkowski metric used in the definition of the Hodge star $\star$ (to pull the indices of the Levi-Civita symbols up/down) must be replaced by the dynamic, potentially curved metric of GENERAL RELATIVITY.

6.3. Noether theorem and the energy-momentum tensor

In the following, we consider first a generic (classical, relativistic) field theory, and specialize to electrodynamics later. This is to emphasize that most of the results in this chapter are not specific to electrodynamics.

Details: Chapter 1 of my QFT script [21]

1 | ⊲ General transformation of field $\phi \mapsto \phi'$:

$$x \mapsto x' = x'(x) \quad \text{and} \quad \phi(x) \mapsto \phi'(x') = \mathcal{F}(\phi(x)) \tag{6.71}$$

Two effects: coordinates and (values of the) field transformed

These are *active transformations* that change physics. $x' = x'(x)$ is *not* a (passive) coordinate transformation; the frame of reference remains fixed in the following!

> **Example 1: Homogeneous Lorentz transformations**
>
> The (active) homogeneous Lorentz transformation of a *vector field* A^μ reads
>
> $$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \text{and} \quad A_\nu(x) \mapsto A'_\mu(x') = \underbrace{\Lambda_\mu{}^\nu A_\nu(x)}_{\mathcal{F}(A_\mu(x))} \tag{6.72}$$
>
> whereas the Lorentz transformation of a *scalar field* ϕ reads
>
> $$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad \text{and} \quad \phi(x) \mapsto \phi'(x') = \underbrace{\phi(x)}_{\mathcal{F}(\phi(x))} . \tag{6.73}$$

2 | ⊲ Infinitesimal transformations ($|w_a| \ll 1$):

$$x'^\mu = x^\mu + w_a\,\delta^a x^\mu(x) \quad \text{and} \quad \phi'(x') = \phi(x) + w_a\,\delta^a\phi(x) \tag{6.74}$$

- Here, $w_a \in \mathbb{R}$ denotes infinitesimal parameters of the transformation (sum over a implied!) and we label different transformations by the labels a.
- Note that the symbolic expressions $\delta^a x^\mu(x)$ and $\delta^a\phi(x)$ denote *functions* of x, indexed by labels a and μ.
- ¡! The indices a are generically *not* Lorentz indices; hence it is irrelevant and purely conventional whether we write them as sub- or superscripts.

> **Example 2: Homogeneous Lorentz transformations**
>
> Infinitesimal homogeneous Lorentz transformations take the form (→ Problemset 4)
>
> $$\Lambda_w = \exp\left(-\frac{i}{2} w_{\alpha\beta}\,\mathcal{J}^{\alpha\beta}\right) \overset{|w_{\alpha\beta}|\ll 1}{\approx} \mathbb{1} - \frac{i}{2} w_{\alpha\beta}\,\mathcal{J}^{\alpha\beta} \tag{6.75}$$
>
> (note that the $a = \alpha\beta$ are labels of generators that are not required to be tensor indices)
>
> with generators
>
> $$(\mathcal{J}^{\alpha\beta})^\mu{}_\nu = i\left(\eta^{\alpha\mu}\delta^\beta_\nu - \delta^\alpha_\nu\eta^{\beta\mu}\right) . \tag{6.76}$$

With this it follows for the coordinates

$$w_{\alpha\beta}\,\delta^{\alpha\beta}x^\mu = x'^\mu - x^\mu = -\frac{i}{2}w_{\alpha\beta}(\mathcal{J}^{\alpha\beta})^\mu{}_\nu x^\nu = w_{\alpha\beta}\,\underbrace{\frac{1}{2}\left(\eta^{\alpha\mu}\delta^\beta_\nu - \delta^\alpha_\nu\eta^{\beta\mu}\right)x^\nu}_{\delta^{\alpha\beta}x^\mu} \tag{6.77}$$

so that

$$\delta^{\alpha\beta}x^\mu = \frac{1}{2}\left(\eta^{\alpha\mu}x^\beta - \eta^{\beta\mu}x^\alpha\right)\,. \tag{6.78}$$

Similar arguments yield $\delta^{\alpha\beta}A^\mu = \frac{1}{2}\left(\eta^{\alpha\mu}A^\beta - \eta^{\beta\mu}A^\alpha\right)$ for a vector field and $\delta^{\alpha\beta}\phi = 0$ for a scalar field.

3 | ‡ Generators of infinitesimal transformations:

$$\delta_w\phi(x) := \phi'(x) - \phi(x) \equiv -i\,w_a\,G^a\phi(x) \tag{6.79}$$

With (omit first line and refer to previous equation)

$$\phi'(x') = \phi(x) + w_a\,\delta^a\phi(x) \tag{6.80a}$$

$$= \phi(x') - w_a(\delta^a x^\mu)\partial_\mu\phi(x') + w_a\delta^a\phi(x') + \mathcal{O}(w^2) \tag{6.80b}$$

(Here we replaced x by x' in the last term because this is a $\mathcal{O}(w^2)$ modification. We used $x^\mu = x'^\mu - w_a\delta^a x^\mu$ and a Taylor expansion of ϕ to obtain the first two terms.)

…it follows (replace x' by x; these are just labels!)

$$\boxed{i\,G^a\phi = (\delta^a x^\mu)\partial_\mu\phi - \delta^a\phi} \tag{6.81}$$

This function describes the infinitesimal change of the field at the *same* point.

➔ Example 3: Translations

i | $x'^\mu := x^\mu + w^\mu \equiv x^\mu + w^\nu\delta_\nu x^\mu$ with $\delta_\nu x^\mu = \delta^\mu_\nu$

ii | $\delta_\nu\phi = 0$ (This is true for scalar and vector fields.)

iii | $i\,G_\mu\phi = \delta^\nu_\mu\partial_\nu\phi - 0$ and therefore

$$G_\mu = -i\,\partial_\mu \equiv P_\mu \tag{6.82}$$

→ The "momentum operator" generates translations.

Note that here we switched the position of the indices $a \equiv \mu$ from sub- to superscripts (and vice versa for the generator) because here they correspond to a Lorentz index and we want remain consistent with tensor calculus.

4 | So far the continuous transformations $\phi \mapsto \phi'$ were arbitrary.

⊲ Continuous transformation [with infinitesimal form Eq. (6.74)] which is a

$$\boxed{\text{Symmetry of the action} \quad :\Leftrightarrow \quad S[\phi] = S[\phi']} \tag{6.83}$$

In principle, the action can vary by a surface term – equivalently, the Lagrangian density $\mathcal{L}$ can vary by a 4-divergence $\partial_\mu K^\mu(\phi, x)$ – under the symmetry transformation (because such modifications do not affect the equations of motion). Here we consider for simplicity only the case where no such terms exist and the action is strictly invariant.

Then one can prove (see Chapter 1 of my QFT script [21] or Refs. [1, 81]): $\overset{*}{\rightarrow}$

5 | ⁂ *Noether's (first) theorem*:

For solutions ϕ of the equations of motion, the ⁂ *(canonical) (Noether) currents*

$$j_a^\mu \stackrel{*}{=} \left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\nu \phi - \delta_\nu^\mu \mathcal{L}\right\} \delta_a x^\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_a \phi \tag{6.84}$$

(associated to the infinitesimal transformations of coordinates $\delta_a x^\nu$ and fields $\delta_a \phi$)

satisfy the continuity equations

$$\forall_a : \qquad \partial_\mu j_a^\mu = 0 . \tag{6.85}$$

This means there is one conserved current j_a^μ for each generator a of the continuous symmetry.

6 | Conserved charge:

The currents Eq. (6.84) are called "conserved" because they describe the flow of a conserved ...

$$Q_a := \int_{\text{Space}} \mathrm{d}^{D-1}x\, j_a^0 \qquad \text{⁂ (Noether) charge} \tag{6.86}$$

There is one conserved charge Q_a for each generator a of the continuous symmetry.

Indeed:

$$\frac{1}{c}\frac{\mathrm{d}Q_a}{\mathrm{d}t} = \int_{\text{Space}} \mathrm{d}^{D-1}x\, \partial_0 j_a^0 \stackrel{6.85}{=} -\int_{\text{Space}} \mathrm{d}^{D-1}x\, \partial_k j_a^k \stackrel{\text{Gauss}}{=} -\int_{\text{Surface}} \mathrm{d}\sigma_k\, j_a^k = 0 \tag{6.87}$$

Here we assume that $j_a^k \equiv 0$ on the spatial boundaries—typically at infinity, i.e., the universe is closed. $k = 1, 2, 3$ denotes the spatial coordinates.

Note 6.1

The current Eq. (6.84) is called canonical current because it is not unique:

$$\tilde{j}_a^\mu := j_a^\mu + \partial_\nu B_a^{\mu\nu} \quad \text{with} \quad B_a^{\mu\nu} = -B_a^{\nu\mu} \quad \text{arbitrary} \quad \Rightarrow \quad \partial_\mu \tilde{j}_a^\mu = 0 \tag{6.88}$$

This is particularly important for the energy-momentum tensor (→ *below*).

↓ Lecture 15 [04.02.26]

6.3.1. Application: The Energy-Momentum Tensor (EMT)

Details: Problemset 7

7 | $\lessdot$ Infinitesimal spacetime translations:

$$x'^\mu = x^\mu + w^\mu \quad \Rightarrow \quad \delta_\nu x^\mu = \delta^\mu_\nu \quad \text{and} \quad \delta_\nu \phi = 0 \tag{6.89}$$

& Translation-invariant action: $S' = S$ (This includes translations in time!)

8 | Conserved currents: Eq. (6.84) →

$$\Theta^\mu{}_\nu := \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\rho \phi - \delta^\mu_\rho \mathcal{L} \right\} \underbrace{\delta_\nu x^\rho}_{\delta^\rho_\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \tag{6.90}$$

Note that the generator index a is in this case a proper Lorentz index ν so that we can pull it up, $\Theta^{\mu\nu} = \eta^{\nu\rho} \Theta^\mu{}_\rho$, and obtain:

⁂ *(Canonical) Energy-Momentum Tensor:*

$$\Theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - \eta^{\mu\nu} \mathcal{L} \tag{6.91}$$

with

$$\partial_\mu \Theta^{\mu\nu} = 0 \quad \text{and four conserved charges} \quad P^\nu := \frac{1}{c} \int \mathrm{d}^3x \, \Theta^{0\nu} \, . \tag{6.92}$$

- Note that these quantities are only conserved for *solutions* of the Euler-Lagrange equations.
- P^ν is a 4-vector (show this!). Note that this is a non-trivial statement because d^3x is not a Lorentz scalar and $\Theta^{0\nu}$ not a 4-vector.
- The prefactor $1/c$ ensures that P^0 has the same dimension as a conventional 4-momentum with $p^0 = E/c$; note that Θ^{00} has the dimension of an *energy density* because $\mathcal{L}$ has this dimension.

9 | Interpretation:

i | *Energy* ($\nu = 0$):

$$cP^0 = \int \mathrm{d}^3x \, \Theta^{00} = \int \mathrm{d}^3x \, \underbrace{\left\{ \frac{\partial \mathcal{L}}{\partial \dot\phi} \dot\phi - \mathcal{L} \right\}}_{\text{Hamiltonian } density} = \underbrace{\int \mathrm{d}^3x \, \mathcal{H}(\phi, \pi)}_{\text{Hamiltonian}} = H \tag{6.93}$$

→ The Hamiltonian is the component of a 4-vector and not Lorentz invariant!
By contrast, the Lagrangian *is* Lorentz invariant (for relativistic field theories).

ii | *Kinetic momentum* ($\nu = i$):

$$P^i = \int \mathrm{d}^3x \, \Theta^{0i} = \int \mathrm{d}^3x \, \frac{\partial \mathcal{L}}{\partial \dot\phi} (-\partial_i \phi) = -\int \mathrm{d}^3x \, \pi \, \partial_i \phi \tag{6.94}$$

π is the *canonical* momentum conjugate to the field ϕ.