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Information

Since we accidentally made too many exercises, enjoy this additional bonus sheet (all exercises on this sheet are optional). If there is demand for discussing the solutions, we will offer an additional tutorial at the start of the lecture-free period. Please let us know if you are interested in this extra tutorial during the last tutorial session.

Problem 7.1: The Lense-Thirring effect* [8 bonuspt(s)]

ID: ex_lense_thirring:rt24

Learning objective

The *Lense-Thirring effect* describes the dragging of inertial frames in the vicinity of a rotating mass. For example, in this exercise you show that the inertial systems on the north pole of Earth slowly rotate with respect to the fixed stars due to the rotation of Earth. The Lense-Thirring effect has no Newtonian counterpart and thus serves as a good test for general relativity. As the effect is a consequence of the angular momentum of the central body, it cannot be derived from the *Schwarzschild metric* (which is only valid for non-rotating masses) but requires the *Kerr metric* instead.

To avoid the (complicated) Kerr metric, here you work in its weak-field approximation were one can use the linearized Einstein equations [derived in Problem 6.1]. You show that in this limit, mass sources can be treated analogously to charge distributions in electrodynamics, and lead to *gravitoelectromagnetic effects*, one of which is the Lense-Thirring effect.

In Problem 6.1 you derived the linearized Einstein tensor $G_{\mu\nu}$, which simplifies in the Hilbert gauge ${\phi^{\mu\alpha}}_{,\alpha} = 0$ to

$$G_{\mu\nu} = \frac{1}{2}\Box\phi_{\mu\nu} \quad \text{with} \quad \phi_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \quad \text{and} \quad h_{\mu\nu}(x) := g_{\mu\nu}(x) - \eta_{\mu\nu}\,. \tag{1}$$

Here, $h_{\mu\nu}$ denotes the (small) deviation of the metric $g_{\mu\nu}$ from flat Minkowski space (in which the theory is linearized). With this, the linearized Einstein equations read

$$\Box\phi^{\mu\nu} = -2\kappa T^{\mu\nu} \quad \text{with} \quad \kappa = \frac{8\pi G}{c^4}\,. \tag{2}$$

For our purpose, the energy-momentum tensor of the source is given by $T^{\mu\nu}(x) = \rho(x)u^\mu(x)u^\nu(x)$, where $\rho(x)$ denotes the mass density and $u^\mu(x) = \frac{dx^\mu}{d\tau}$ is the four-velocity field (i.e., the four-velocity of a small chunk of matter at x).

Here we want to focus on the case of a time-independent energy-momentum tensor, $\partial_t T^{\mu\nu} = 0$ [so that $T^{\mu\nu}(x) = T^{\mu\nu}(\boldsymbol{x})$]. In this case, the linearized Einstein equations (2) simplify to

$$\Delta\phi^{\mu\nu} = 2\kappa T^{\mu\nu}\,, \tag{3}$$

where we used that the metric of a time-independent source should be time-independent as well: $\partial_t \phi^{\mu\nu} = 0$ (the metric should be *stationary*).

As you know (e.g., from electrodynamics), the general solutions of the Poisson Eq. (3) are given by

$$\phi^{\mu\nu} = -\frac{4G}{c^4} \int d^3x' \, \frac{T^{\mu\nu}(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, . \tag{4}$$

a) Show that for non-relativistic velocities $\frac{v}{c} \ll 1$, the line element up to first order is given by **1**pt(s)

$$ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right) \delta_{ij} \, dx^i dx^j - 2A_i \, dt dx^i \, , \tag{5}$$

where we introduced $\Phi := \frac{c^2}{4}\phi^{00}$ and $A_i := c\,\phi^{0i}$.

Note: The source term of Φ is the mass *density*, whereas A_i is related to the mass *current*.

b) Let us now consider a non-relativistic, free falling test particle that follows a geodesic $x^\mu(s)$ in this spacetime. First, show that the spatial part of the geodesic equation **2**pt(s)

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \tag{6}$$

can be approximately written as

$$\ddot{x}^i + \Gamma^i{}_{00} c^2 + 2c\,\Gamma^i{}_{0j}\, \dot{x}^j - c^2\, \Gamma^0{}_{00}\, \dot{x}^i = 0 \, , \quad \text{where } \dot{x}^i = \frac{dx^i}{dt} \text{ and } \ddot{x}^i = \frac{d^2x^i}{dt^2} \, , \tag{7}$$

if one parametrizes the trajectory in coordinate time t (instead of the affine parameter s), and expands the result in first order of v/c.

Now use the metric (5) to calculate the Christoffel symbols, and show that the geodesic equation Eq. (7) can be written in the form

$$\ddot{\boldsymbol{x}} = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \, , \tag{8}$$

where (in analogy to electrodynamics) we introduced the *gravito-electric field* $\boldsymbol{E} := -\nabla\Phi$ and the *gravito-magnetic field* $\boldsymbol{B} := \nabla \times \boldsymbol{A}$.

Interpret this result.

We now consider the special case of a rotating sphere of total mass M and radius R made from uniformly distributed matter; we assume the sphere is rotating with angular velocity $\boldsymbol{\omega}$. This is of course a model for celestial bodies like Earth.

In this case, the scalar and vector potential outside the sphere ($r > R$) are given by

$$\Phi(\boldsymbol{r}) = -\frac{GM}{r} \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{r}) = -\frac{4GMR^2}{5c^2} \frac{\boldsymbol{\omega} \times \boldsymbol{r}}{r^3} \, . \tag{9}$$

Note: The gravito-electric and gravito-magnetic fields can be derived completely analogous to the electric and magnetic fields of a rotating charged sphere in electrodynamics.

c) Calculate the corresponding gravito-electric and gravito-magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$. **2**pt(s)

As a sanity check, show that a test particle – initially located at rest on the rotation axis of the sphere (e.g., above the north pole) – is accelerated towards the center of the sphere (and thus remains on the rotation axis).

Finally, we want to explore the phenomenon of *frame dragging*.

To this end, we consider a test particle with *spin* s^μ (= internal angular momentum, think of a gyroscope). We use this spin as a reference vector that indicates an inertial frame (in an inertial frame, the angular momentum of a gyroscope remains constant and does not precess).

The idea is that the effects of the rotating mass (Earth) on the spin (a gyroscope) tells us how the local inertial frames are affected by the rotation of Earth.

The four-spin vector s^μ is defined in the rest frame of the spin as $s^\mu_{\text{rf}} = (0, \boldsymbol{s}_{\text{rf}})$, where $\boldsymbol{s}_{\text{rf}}$ denotes the conventional (internal) angular momentum of the gyroscope. The evolution of the spin s^μ along a geodesic $x^\alpha(\tau)$ is given by *parallel transport* of spin vector:

$$\frac{ds^\mu}{d\tau} = -\Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}s^\beta \,. \tag{10}$$

With this machinery, you can determine the effects of any spacetime on free-falling gyroscopes:

d) Calculate the evolution of the spin s^μ of a gyroscope that is freely falling towards the north pole of Earth (i.e, along its rotation axis). 3pt(s)

Hint: It is sufficient to consider the evolution in 0-th order of $\frac{v}{c}$, i.e., $u^\mu = \frac{dx^\mu}{d\tau} \approx (c, \mathbf{0})$.

The precession of the spin is called *Lense-Thirring effect* and indicates the dragging of local inertial frames by rotating masses (called *frame dragging*). This means that on the poles of Earth, the local inertial frames slowly rotate with respect to the fixed stars (which are assumed to be at rest in the coordinates used in this exercise).

Coming back to Newton's bucket experiment (discussed in the first lecture), this result shows that for the water surface to stay perfectly flat, one must rotate the bucket slowly with respect to the fixed stars *because* Earth is rotating beneath the bucket. This particular consequence of frame dragging is unfortunately too small to be measured.

However, the frame dragging predicted by the Lense-Thirring effect has been measured by the space-borne experiment `Gravity Probe B` (in combination with a much stronger generally relativistic effect called *geodetic precession*, which has nothing to do with the rotation of Earth). For details see the "NASA Fact Sheet"

https://doi.itp3.info/ff8d7cf9cca72031dbd39af2a952808a

and the original publication

https://doi.itp3.info/10.1103/physrevlett.106.221101 .

Problem 7.2: Kruskal-Szekeres coordinates* [7 bonuspt(s)]

ID: ex_kruskal_szekeres_coordinates:rt24

Learning objective

Different coordinate systems can offer different perspectives on a physical problem. This is especially true in General Relativity. In the lecture, it was claimed that the singularity of the Schwarzschild metric at the Schwarzschild radius $r = r_s$ is a *coordinate singularity* – an artifact of Schwarzschild coordinates.

The goal of this exercise is to construct and motivate a new coordinate system, called *Kruskal-Szekeres coordinates*, that removes the coordinate singularity at $r = r_s$. A first benefit of these new coordinates is that one can describe the trajectory of probes falling into a black hole without singularities at the event horizon [compare Problem 6.2]. Furthermore, you will see that one can *extend* the original Schwarzschild spacetime (consisting of the interior and the exterior of the black hole) quite naturally by two additional regions: a *white hole* and another asymptotically flat region that cannot be reached from our universe.

The Schwarzschild metric is given in Schwarzschild coordinates (ct, r, θ, φ) as

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \,, \tag{11}$$

with Schwarzschild radius r_s and $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$.

In these coordinates, we have the problem that (some of) the metric components $g_{\mu\nu}$ are singular at the horizon $r = r_s$. If one computes the null cones at every point of the $t - r$ diagram, one finds that they close up when approaching the horizon ($r \searrow r_s$) and flip in the interior where r becomes a time-like coordinate.

This suggest that if we somehow could construct coordinates such that the null cones look the same everywhere, the coordinate singularity might disappear. We can find such a coordinate system by using the null geodesics followed by light rays as a new coordinate grid.

For now we focus only on the region outside of the horizon were $r > r_s$:

a) First, show that the equations **2**pt(s)

$$u = ct - r_s \ln\left(\frac{r}{r_s} - 1\right) - r \quad \text{and} \quad v = ct + r_s \ln\left(\frac{r}{r_s} - 1\right) + r \tag{12}$$

describe the geodesics of radially in- ($v = \text{const.}$) and outgoing ($u = \text{const.}$) light rays in the Schwarzschild metric.

Then show that if we use u and v as new coordinates (replacing t and r), the Schwarzschild metric takes the form

$$ds^2 = \left(1 - \frac{r_s}{r}\right) du\, dv - r^2 d\Omega^2 \,, \tag{13}$$

with $r = r(u, v)$ defined via Eq. (12) implicitly.

b) In the (u, v)-coordinates, we have now the problem that the event horizon is infinitely far away: **2**pt(s)
Show that the event horizon at $r = r_s$ is mapped to $(u = \infty, v)$ and $(u, v = -\infty)$.

We can fix this via another coordinate transformation:

$$U = -e^{-\frac{u}{2r_s}} \quad \text{and} \quad V = e^{\frac{v}{2r_s}} \,. \tag{14}$$

This moves the event horizon to the finite coordinates $(U = 0, V)$ and $(U, V = 0)$.

Show that in these coordinates the Schwarzschild metric takes the new form

$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_e} dU dV - r^2 d\Omega^2 \tag{15}$$

where $r = r(U, V)$ is defined implicitly via Eq. (14) and Eq. (12).

At this point, we reached our goal to find coordinates which do not diverge at the event horizon: it is now located at finite values of U and V.

Unfortunately, both U and V are *light-like* (*null*) coordinates (why?). This means that neither U nor V deserves the label "(coordinate) time"!

The support intuition, it would be nice if we had instead one *time-like* and three *space-like* coordinates:

c) To this end, introduce the new coordinates **1**pt(s)

$$T = \frac{1}{2}(V + U) \quad \text{and} \quad R = \frac{1}{2}(V - U)\,, \tag{16}$$

and show that the Schwarzschild metric reads now

$$ds^2 = \frac{4r_s^3}{r} e^{-r/r_s} \left(dT^2 - dR^2\right) - r^2 d\Omega^2\,. \tag{17}$$

Use this result to argue that T is time-like and R space-like, respectively.

This is the Schwarzschild metric in *Kruskal-Szekeres coordinates* (T, R, θ, φ).

Note: Here $r = r(T, R)$ is a function of T and R and implicitly defined via the previous transformations.

Note: Note that the metric components in Eq. (17) still satisfy the Einstein field equations in vacuum because you transformed the original tensor components $g_{\mu\nu}$ in Eq. (11) (for which we have explicitly shown in the lecture that they satisfy the EFEs) according to the rules of a rank-2 tensor. This is the general covariance of the Einstein field equations in action!

The full transformation from Schwarzschild coordinates (ct, r) to Kruskal-Szekeres coordinates (T, R) can be constructed by combining all coordinate transformations that we did so far.

One finds the explicit expressions valid outside of the black hole ($r > r_s$):

$$T = \sqrt{\frac{r}{r_s} - 1}\, e^{\frac{r}{2r_s}} \sinh\left(\frac{ct}{2r_s}\right) \quad \text{and} \quad R = \sqrt{\frac{r}{r_s} - 1}\, e^{\frac{r}{2r_s}} \cosh\left(\frac{ct}{2r_s}\right)\,. \tag{18}$$

Almost identical expressions (some sign changes) are valid *inside* the black hole ($r < r_s$):

$$T = \sqrt{1 - \frac{r}{r_s}}\, e^{\frac{r}{2r_s}} \cosh\left(\frac{ct}{2r_s}\right) \quad \text{and} \quad R = \sqrt{1 - \frac{r}{r_s}}\, e^{\frac{r}{2r_s}} \sinh\left(\frac{ct}{2r_s}\right)\,. \tag{19}$$

With these explicit expressions at hand, we can now study the domains for which the coordinates (T, R) are defined, and how these map to the original Schwarzschild coordinates (ct, r) (for which we have physical interpretations):

d) Draw a $T - R$ diagram (*Kruskal diagram*) and sketch ... **2**pt(s)

- lines of constant t (= constant $\frac{T}{R}$).

- lines of constant r (= constant $T^2 - R^2$); highlight the lines for $r = 0$ and $r = r_s$.
- some null cones at points of your choosing.

Mark the region that corresponds to the exterior of the black hole ($r > r_s$) and its interior ($r < r_s$), respectively.

You might be surprised that these two regions do not cover the full domain of the coordinates (T, R). This means that we not only got rid of the singularity on the event horizon; we actually *extended* the solution of the Einstein field equations into new regions of spacetime!

More specifically, you can identify two new regions in the Kruskal diagram that describe the interior of a *white hole* and the asymptotically flat spacetime of a *mirror universe*. Which is which and why?

Note: The solution of the Einstein field equation that you constructed in this exercise is called the *maximally extended Schwarzschild metric*. Since it is a static solution (in the exterior regions) in a universe devoid of energy and matter (except for the singularity), it does *not* describe the spacetime of real black holes in our universe because (1) these form *dynamically* via the collapse of stars and (2) our universe it not empty. The additional regions that we found here do not exist for such more realistic solutions! There is no evidence for the existence of white holes, and no known physical mechanism that could produce them.