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July 4^{th}, 2024
SS 2024

Problem 6.1: Gravitational waves [12 pt(s)]

ID: ex_gravitational_waves:rt24

Learning objective

The Einstein field equations are non-linear second-order partial differential equations for the metric tensor. If one considers scenarios where spacetime only weakly deviates from flat Minkowski space (as is the case in interstellar space, far away from large masses), one can linearize the equations to find a *wave equation* for the metric.

In this exercise, you study the wave-like excitations (*gravitational waves*) of the metric field described by this equation. You show – by exploiting gauge degrees of freedom of the linearized theory – that there are only two possible polarizations of gravitational waves.

To study gravitational waves, we consider a metric linearized around Minkowski spacetime:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)\,, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}(x)$ a small perturbation, i.e., all entries of $h_{\mu\nu}$ (and its derivatives) are assumed to be much smaller than 1.

The inverse metric is then given (up to first order) by $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ (Show this!).

Note: Up to first order you can pull indices of $h_{\mu\nu}$ (and tensors of the same order) up and down with the *Minkowski* metric $\eta_{\mu\nu}$ instead of the full metric $g_{\mu\nu}$.

a) As a first step, we want to express all interesting quantities in first order of $h_{\mu\nu}$. **4**pt(s)

Therefore, calculate ...

- the Christoffel symbols $\Gamma^{\alpha}{}_{\mu\nu}$,
- the Riemann curvature tensor $R_{\alpha\mu\nu\beta}$,
- the Ricci tensor $R_{\mu\nu} = R^{\alpha}{}_{\mu\nu\alpha}$,
- and the Ricci scalar $R = R^{\mu}{}_{\mu}$

... up to first order in $h_{\mu\nu}$.

b) Show that the linearized Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is given by **2**pt(s)

$$G_{\mu\nu} = -\frac{1}{2}\partial^{\alpha}\partial^{\beta}H_{\mu\alpha\nu\beta}\,, \tag{2}$$

where

$$H_{\mu\alpha\nu\beta} := -\phi_{\mu\nu}\eta_{\alpha\beta} - \phi_{\alpha\beta}\eta_{\mu\nu} + \phi_{\mu\beta}\eta_{\nu\alpha} + \phi_{\nu\alpha}\eta_{\mu\beta}\,, \tag{3}$$

with $\phi_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$ and $h := h^{\mu}{}_{\mu}$.

Thus, the linearized Einstein field equations in vacuum (without cosmological constant) read

$$\partial^\alpha \partial^\beta H_{\mu\alpha\nu\beta} = 0\,. \tag{4}$$

Note: These are exactly the same equations we already encountered in Problem 1.3, where we studied a linear tensor field as gravitational field.

c) Consider a local transformation of the small perturbation $h_{\mu\nu}$ to **1**pt(s)

$$\tilde{h}_{\mu\nu}(x) := h_{\mu\nu}(x) - \xi_{\mu,\nu}(x) - \xi_{\nu,\mu}(x)\,, \tag{5}$$

where $\xi_\mu(x)$ (and its derivatives) are arbitrary but small and of the same order as $h_{\mu\nu}$.

Show explicitly that the Riemann curvature tensor $R_{\alpha\mu\nu\beta}$ is invariant under this transformation. Use this to argue that $\tilde{h}(x)$ is also a solution of the linearized Einstein field equations and therefore describes a gauge transformation.

Note: This gauge transformation can be motivated by the coordinate transformation $\tilde{x}^\mu = x^\mu + \xi^\mu(x)$. But be careful: The gauge transformation (5) does not change the coordinates x^μ!

In the following, we choose a gauge such that $\tilde{\phi}^{\mu\alpha}{}_{,\alpha} = 0$. (How do you have to choose ξ^μ to achieve this?). This gauge is called the *Hilbert gauge*; it is analogous to the Lorenz gauge ($\partial A = 0$) in electrodynamics. To simplify our notation, we omit the tilde henceforth.

d) Show that in the gauge $\phi^{\mu\alpha}{}_{,\alpha} = 0$ the linearized Einstein tensor $G_{\mu\nu}$ simplifies to **2**pt(s)

$$G_{\mu\nu} = \frac{1}{2}\Box\phi_{\mu\nu} \quad \text{with} \quad \Box := \eta^{\alpha\beta}\partial_\alpha\partial_\beta\,. \tag{6}$$

Use this to show that Einstein field equations in the Hilbert gauge read

$$\Box h_{\mu\nu} = 0\,. \tag{7}$$

These are simple wave equations for the components of $h_{\mu\nu}$.

The solutions of the wave equations for $h_{\mu\nu}$ are (linear combinations of) plane waves:

$$h_{\mu\nu}(x) = A_{\mu\nu} e^{ik_\alpha x^\alpha}\,. \tag{8}$$

Convince yourself that $h_{\mu\nu}(x)$ solves Eq. (7) and satisfies the Hilbert gauge if and only if ...

- the wave vector k_α satisfies the dispersion relation $k_\alpha k^\alpha = 0$, and ...
- the symmetric tensor $A_{\mu\nu}$ satisfies $A_{\mu\nu}k^\nu = \frac{1}{2}Ak_\mu$ with $A := A^\mu{}_\mu$.

To simplify our discussion, we consider only waves propagating in z-direction: $k^\mu = (k, 0, 0, k)$.

Note: Of course the metric $h_{\mu\nu}$ should be *real-valued.* As usual, one can achieve this taking only the real part of the complex solution above. For simplicity, we ignore this in the following.

e) Write down the conditions $A_{\mu\nu}k^\nu = \frac{1}{2}Ak_\mu$ explicitly for $k^\mu = (k, 0, 0, k)$. **1**pt(s)

How many independent components does the symmetric tensor $A_{\mu\nu}$ have?

f) By fixing the Hilbert gauge above we have not yet used up all gauge degrees of freedom: **1**pt(s)

Consider gauge transformations $\xi^\mu(x)$ of the form $\xi^\mu = -i\epsilon^\mu e^{ik_\alpha x^\alpha}$ and show that the Hilbert gauge condition $\phi^{\mu\alpha}{}_{,\alpha} = 0$ is preserved under this transformation.

How do the components of $A_{\mu\nu}$ transform under this gauge transformation?

Use the freedom to choose the vector ϵ^μ to show that there are only *two* degrees of freedom left, and one can write

$$A^{\mu\nu} = a_{\mathrm{I}} E_{\mathrm{I}}^{\mu\nu} + a_{\mathrm{II}} E_{\mathrm{II}}^{\mu\nu} \tag{9}$$

with the two *polarization tensors*

$$E_{\mathrm{I}}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad E_{\mathrm{II}}^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{10}$$

g) Similar to photons, we can define two *circular polarizations* as $E_\pm^{\mu\nu} := E_{\mathrm{I}}^{\mu\nu} \pm i E_{\mathrm{II}}^{\mu\nu}$. **1**pt(s)

How do the circular polarization tensors $E_\pm^{\mu\nu}$ transform under a rotation

$$R^\mu{}_\nu(\varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

around the propagation direction (z-axis)?

What does this tell you about the helicity (and therefore the spin) of gravitational waves?

Problem 6.2: Falling into a Black Hole [6 pt(s)]

ID: ex_dropping_into_a_black_hole:rt24

Learning objective

In the lecture, we derived the Schwarzschild metric that describes the gravitational field outside a spherically symmetric (non-rotating and uncharged) mass. In Schwarzschild coordinates, this solution has a singularity at a finite radius $r = r_s$, called the *Schwarzschild radius*. An object so dense that this radius is not hidden beneath the surface (were the solution is invalid) is called a *black hole*.

In this exercise you calculate the trajectory of a probe that is dropped into a (non-rotating) Schwarzschild black hole from far away. This is a particularly interesting thought experiment as it is a priori unclear how the singularity at the Schwarzschild radius (event horizon) affects the probe.

Consider a probe of mass m that is located at radius $r_0 \gg r_s$ above the black hole. At coordinate time t_0, the probe is let go and begins to fall freely towards the black hole. General relativity tells us that the probe follows a time-like geodesic in the Schwarzschild metric

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \,, \tag{12}$$

where r_s is the Schwarzschild radius of the black hole.

Because the probe starts at rest, the initial conditions are $\frac{dr}{d\tau} = 0$, $\frac{d\vartheta}{d\tau} = 0$ and $\frac{d\varphi}{d\tau} = 0$, together with the initial position $x_0^\mu = (ct_0, r_0, \theta = \frac{\pi}{2}, \varphi = 0)$. Due to symmetry, φ and ϑ remain constant throughout the drop and can be completely ignored in the following.

We could tackle this problem by writing down the Geodesic equation for the Schwarzschild metric and solve it for the given initial conditions. However, here we use constants of motion (conserved quantities) to derive the trajectory of the probe in a less technical way:

a) Show that the quantity **1** pt(s)

$$A = \left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau} \tag{13}$$

is conserved along a geodesics in the Schwarzschild spacetime (here t is the Schwarzschild coordinate time and τ the proper time of the probe).

Hint: Use a Lagrangian that defines geodesic motion and specialize it to the Schwarzschild metric. Use that the Euler-Lagrange equations yield constants of motion for cyclic variables and that we parametrize the trajectory with proper time τ.

b) Calculate the proper time τ that elapses for the probe as a function of its radial position r. **3** pt(s)

Hint: Use the conserved quantity A and $c\,d\tau = \mathrm{d}s$ to derive an expression for $\frac{dr}{d\tau}$ that you can integrate.

Hint: The integral

$$\int dx \sqrt{\frac{x}{a-x}} = a \arctan\left(\sqrt{\frac{x}{a-x}}\right) - \sqrt{x(a-x)}\,, \tag{14}$$

may be useful.

From this calculation you can already conclude that the probe reaches (and passes through) the event horizon at *finite* proper time without any issues.

c) Now find and expression for the *coordinate* time t of the probe in terms of its radial position. **1** pt(s)

Hint: The integral

$$\int dx \frac{x^{3/2}}{(x-b)\sqrt{a-x}} = -b\sqrt{\frac{b}{a-b}} \ln\left|\frac{\sqrt{\frac{b}{a-b}} + \sqrt{\frac{x}{a-x}}}{\sqrt{\frac{b}{a-b}} - \sqrt{\frac{x}{a-x}}}\right| + (a+2b) \arctan\left(\sqrt{\frac{x}{a-x}}\right) - \sqrt{x(a-x)} \tag{15}$$

may be useful.

Note: As explained in the lecture, the coordinate time t can be interpreted as the (approximate) proper time of an observer far away from the black hole (say at $r_0 \gg r_s$ from which he dropped the probe).

d) Sketch the trajectory in both a $\tau - r$ and a $t - r$ diagram. **1** pt(s)

Discuss the differences and interpret your results.