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Problem 5.1: The Palatini action

[**Written** | 4 (+3 bonus) pt(s)]

ID: ex_palatini_formalism:rt24

> **Learning objective**
>
> In the lecture, it was claimed that the Einstein field equations follow by variation of the Einstein-Hilbert action. In this exercise you prove this. However, instead of the straightforward (but cumbersome) calculation presented in the script, here you employ the so called *Palatini formalism*, where one treats the metric and the connection as *independent* degrees of freedom. A benefit of this approach is that the Levi-Civita connection (Christoffel symbols) does not have to be postulated – it follows naturally from the variational principle.

In the lecture, you learned that the metric $g_{\mu\nu}$ and the connection $\Gamma^{\alpha}{}_{\mu\nu}$ of a differentiable manifold are a priori unrelated concepts. Remember that the curvature tensor $R^{\alpha}{}_{\mu\nu\beta}$ is defined via the *connection* (and not the metric); only after one introduces the metric-compatible and torsion-free *Levi-Civita connection*, it becomes the *Riemann* curvature tensor that can be expressed in terms of the metric.

Here we take half a step back and treat metric and the connection as *independent* entities (we restrict the connection to be torsion-free [symmetric] but not necessarily metric-compatible). The Einstein-Hilbert action (without cosmological constant) is then a functional of both the metric and the connection:

$$S[g,\Gamma] = \frac{c^3}{16\pi G}\int \mathrm{d}x^4 \sqrt{g} g^{\mu\nu} R_{\mu\nu}(\Gamma)\,. \tag{1}$$

Here the Ricci tensor is computed from the curvature tensor as usual, but the latter is computed from the connection coefficients and does not depend on the metric.

Eq. (1) must now be varied with respect to both $g_{\mu\nu}$ and $\Gamma^{\alpha}{}_{\mu\nu}$; this is called the *Palatini formalism*.

a) Vary the action with respect to the metric $g^{\mu\nu}$, **1**pt(s)

$$\delta_g S[g,\Gamma] \stackrel{!}{=} 0\,, \tag{2}$$

and show that this yields the Einstein field equations in vacuum.

Since the connection coefficients are independent degrees of freedom, we should expect another set of equations of motion from varying the action wrt. the connection:

$$\delta_\Gamma S[g,\Gamma] \stackrel{!}{=} 0\,. \tag{3}$$

To this end, it is convenient to write the general (torsion-free) connection $\Gamma^{\alpha}{}_{\mu\nu}$ as a sum of the metric-compatible Levi-Civita connection $\hat{\Gamma}^{\alpha}{}_{\mu\nu}$ (Christoffel symbols) and a symmetric tensor $C^{\alpha}{}_{\mu\nu}$:

$$\Gamma^{\alpha}{}_{\mu\nu} = \hat{\Gamma}^{\alpha}{}_{\mu\nu} + C^{\alpha}{}_{\mu\nu}\,. \tag{4}$$

This is always possible because the difference of two connections is a tensor. (Show this!)

b) Compute the equations of motion for $C^{\alpha}{}_{\mu\nu}$ by varying the action wrt. $C^{\alpha}{}_{\mu\nu}$ (instead of $\Gamma^{\alpha}{}_{\mu\nu}$). Show that $C^{\alpha}{}_{\mu\nu} = 0$ satisfies them. 3pt(s)

Hint: Before performing the variation, you can identify the terms $\hat{\nabla}_{\nu} C^{\alpha}{}_{\mu\alpha} - \hat{\nabla}_{\alpha} C^{\alpha}{}_{\mu\nu}$ in the action ($\hat{\nabla}$ is the usual covariant derivative with the Christoffel Symbols $\hat{\Gamma}^{\alpha}{}_{\mu\nu}$). They do not contribute to the variation. Why is that?

With your result in b) you have shown that any metric that solves the Einstein Field equations with the Levi-Civita connection also extremizes the action in the Palatini formalism. What remains to be shown is that the Levi-Civita connection is the *unique* solution that extremizes the Palatini action under the constraint of vanishing torsion.

*c) Show that the equations of motions obtained in part b) *imply* that $C^{\alpha}{}_{\mu\nu} = 0$ for a *torsion-free* connection ($C^{\alpha}{}_{\mu\nu} = C^{\alpha}{}_{\nu\mu}$). That is, show that the unique, torsion-free solution of Eq. (3) is the Levi-Civita connection. +3pt(s)

Hint: It is easier to work with equations where all indices are contravariant, i.e., $C^{\alpha\beta\gamma}$.

Hint: Use the EOMs to establish relations between the traces of $C^{\alpha\beta\gamma}$ and show that they must vanish.

Problem 5.2: Hafele-Keating experiment — Part 2 [**Oral** | 7 pt(s)]

ID: ex_hafele-keating_experiment_gr:rt24

Learning objective

In this exercise, we want to revisit the *Hafele-Keating experiment*, for which you already calculated the contributions from *special relativistic time dilation* in the last semester (Problem 5.1). Here you use our newly developed machinery of general relativity to re-derive your previous results, but now including an additional contribution due to *gravitational time dilation* (which is needed to explain the measured data of the experiment!).

As a reminder, the experimental results were reported in

https://doi.itp3.info/10.1126/science.177.4044.168

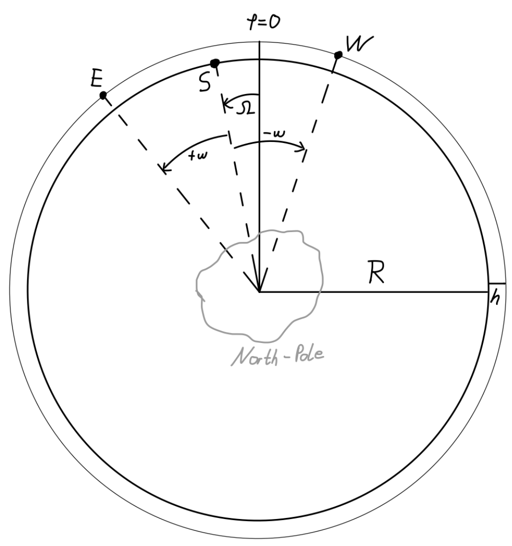
and the theory was developed in

https://doi.itp3.info/10.1126/science.177.4044.166.

We consider a coordinate system with origin in the center of Earth that is not rotating (see sketch below). In the time scales we are interested in, one can ignore the motion of Earth along its orbit (we only need to consider its rotation); one can also neglect effects of the sun and other celestial bodies.

The experiment makes use of *three* clocks (recall the description in Problem 5.1 of last semester):

- The **stationary clock** S is located on the surface of Earth ($R = 6.4 \times 10^6$ m) at the equator with an initial longitude $\varphi = 0$. Since our coordinate system is not rotating with the Earth, the longitude of this clock changes with Earth's angular velocity $\Omega = 2\pi/24\,\mathrm{h}$.
- The **eastward flying clock** E is initially located at a height h above the stationary clock S (in an airplane), and then flies eastward with the total angular velocity $\Omega + \omega$.
- The **westward flying clock** W starts at the same point as E (in another airplane), but then flies westward with the total angular velocity $\Omega - \omega$.



Here, $\omega = v/(R+h)$ corresponds to the angular velocity of the planes carrying the clocks with respect to Earth; v denotes their speed (wrt. ground), h is their altitude above ground, and R is the radius of Earth. We assume that all clocks are *ideal*, i.e., they measure *proper time*.

a) Parametrize the world lines of the clocks S, E, and W in spherical coordinates (ct, r, φ, θ). After which (coordinate) time T, and at which longitude Φ will the three clocks meet again? (Ignore that the airplanes are then still h above the stationary clock.) **1pt(s)**

The proper time $\Delta\tau$ measured by an ideal clock along its world line γ is given by

$$\Delta\tau = \frac{1}{c}\int_\gamma ds = \frac{1}{c}\int_\gamma \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,d\lambda\,, \tag{5}$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$ is the tangent vector to the world line and λ is some parametrization.

As shown in the lecture, the metric of the curved spacetime outside of Earth is well approximated by the radially symmetric *Schwarzschild metric*

$$ds^2 = \left(1 - \frac{r_S}{r}\right)c^2dt^2 - \left(1 - \frac{r_S}{r}\right)^{-1}dr^2 - r^2\sin^2(\theta)\,d\varphi^2 - r^2\,d\theta^2\,, \tag{6}$$

where $r_S = 2GM_\oplus/c^2 \approx 8.9\,\mathrm{mm}$ is the Schwarzschild radius of Earth.

b) Assume that all three clocks are set to $\tau = 0$ at their departure. Derive an expression for the proper time τ_i displayed by the three clocks $i \in \{S, E, W\}$ when they meet again at S. **1pt(s)**

c) Calculate the proper time differences $\tau_E - \tau_S$ and $\tau_W - \tau_S$ and expand your results for small values of $\frac{r_S}{R} \ll 1$, $\frac{R\Omega}{c} \ll 1$ and $\frac{h}{R} \ll 1$. **3pt(s)**

Compare your result to the equations (2) and (3) of the theory paper

https://doi.itp3.info/10.1126/science.177.4044.166

by Hafele and Keating.

Hint: For the expansion, check the order of magnitudes of the three different ratios to decide up to which order you need to expand the result.

d) Finally, use a velocity of $v = 200\,\mathrm{m/s}$ and a height of $h = 10\,\mathrm{km}$ for the east and westward flying clocks. Calculate the proper time differences and compare your numbers with the results in the paper. **2**pt(s)

Compare the contribution from the special relativistic time dilation with the contribution from the "new" gravitational time dilation. Can one neglect the effects of general relativity to explain the experimental data?

Problem 5.3: Pendulum clocks as ideal clocks

[**Oral** | 9 pt(s)]

ID: ex_ideal_clocks:rt24

> **Learning objective**
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> *Ideal* clocks are defined as any periodic process that measures proper time τ. Thus, whether a clock is ideal or not depends on the specific process used to build the clock and the situation in which it is used.
>
> The purpose of this exercise is to demonstrate that whether a clock is ideal can be calculated using a theory describing the clock. To illustrate this, you show that a simple pendulum clock measures proper time if it is at rest on the surface of Earth. To this end, you show that Newtonian mechanics in a homogeneous gravitational field follows from the Schwarzschild metric under reasonable approximations.

We consider a simple pendulum clock at rest on the surface of Earth. Our goal is to show that (within reasonable approximations) the clock oscillates periodically in the proper time that elapses at the position of the clock.

As shown in the lecture, the spacetime outside of Earth is approximately the *Schwarzschild metric*

$$ds^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2(\theta)\, d\varphi^2)\,, \tag{7}$$

here given in Schwarzschild coordinates (ct, r, φ, θ), with the Schwarzschild radius of Earth $r_s = 2GM_\oplus/c^2 \approx 8.9\,\mathrm{mm}$.

a) As a fist step, calculate the proper time τ, which elapses at the position of the pendulum at $r = r_e$, as a function of the coordinate time t. (r_e is the radial coordinate of the surface of Earth.) **1**pt(s)

Now that we know the relationship between τ and t, we want to show that the pendulum is a periodic process in τ.

Since the equations of motion for classical mechanics on curved spaces are quite complicated, we use some (justified) approximations. The following scales are relevant for our problem:

The Earth radius r_e, the Schwarzschild radius r_s, the length of our pendulum l, and the typical velocity of the pendulum v. We know that $\frac{r_s}{r_e} \ll 1$, and it is reasonable to assume that $\frac{l}{r_e} \ll 1$ and $\frac{v}{c} \ll 1$ (the pendulum is small and non-relativistic).

b) Introduce the new spatial coordinates (the time coordinate remains the same) **3**pt(s)

$$x := r\sin\theta\cos\varphi, \quad y := r\sin\theta\sin\varphi, \quad z := r\cos\theta - r_e\,, \tag{8}$$

with the clock located at $\boldsymbol{x} = (x, y, z) = 0$.

Show that the metric in the vicinity of the clock ($|\boldsymbol{x}| \lesssim l$) takes the approximate form

$$ds^2 \approx \left(1 - \frac{r_s}{r_e} + \frac{zr_s}{r_e^2}\right) c^2dt^2 - dx^2 - dy^2 \\ - \left(1 + \frac{r_s}{r_e} + \frac{r_s^2}{r_e^2} - \frac{zr_s}{r_e^2}\right) dz^2 - \frac{2xr_s}{r_e^2} dxdz - \frac{2yr_s}{r_e^2} dydz\,, \tag{9}$$

where the z-direction points away from the center of Earth.

Hint: Rewrite the Schwarzschild metric in the new coordinates and expand to second order in $\frac{r_s}{r_e}$ and $\frac{l}{r_e}$.

c) The position x^μ of the mass of the pendulum follows the generally covariant equation of motion **4pt(s)**

$$K^\mu = m\frac{Du^\mu}{D\tau} = m\left(\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right) \quad \text{with } u^\mu = \frac{dx^\mu}{d\tau}. \tag{10}$$

Here, K^μ denotes the (non-gravitational) force exerted by the string or rod of the pendulum on the mass m.

Use leading-order approximations in $\frac{r_s}{r_e} \ll 1$, $\frac{l}{r_e} \ll 1$ and $\frac{v}{c} \ll 1$ to show that the right-hand side (= gravitational part) of the equations of motion can be approximated as

$$K^x = m\frac{d^2x}{dt^2}\,, \quad K^y = m\frac{d^2y}{dt^2} \quad K^z = m\frac{d^2z}{dt^2} + mg \tag{11}$$

with the gravitational acceleration $g = \frac{GM_\oplus}{r_e^2} = \frac{r_sc^2}{2r_e^2}$.

Hint: Approximate the two terms in Eq. (10) separately to leading order: First, use Eq. (9) to evaluate the leading order of $\frac{d^2x}{d\tau^2}$ (you can assume that $\frac{d^2t}{d\tau^2}$ is of higher order). Then show that the second term can be approximated by $c^2\Gamma^\mu{}_{00}$. Since the pendulum is driven by gravitational effects, the two terms are of the same order of magnitude, e.g. $\frac{d^2x^i}{dt^2} \sim g$.

Note: The τ in this subtask is not the proper time from subtask (a) that elapses for someone at rest next to the pendulum, but the proper time that elapses for someone swinging along with the pendulum!

You have now shown that one can recover the equations of motion of classical Newtonian mechanics in a homogeneous gravitational field from the Schwarzschild metric. In this approximation, we can easily solve the equations of motion of the pendulum and, using our result from subtask a), relate the periodicity of the pendulum to the proper time:

d) Assume that the pendulum swings in the xz-plane. The force $\boldsymbol{K} = (K^x, K^y, K^z)$ must then be chosen such that the motion of the mass is constrained to **1pt(s)**

$$x = l\sin\phi \quad \text{and} \quad z = -l\cos\phi\,, \tag{12}$$

where ϕ denotes the deflection of the pendulum from the vertical.

Derive the solution to the equations of motion for the pendulum in the small angle approximation ($\sin\phi \approx \phi$) and use your results from a) to show that it is periodic in τ.