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Problem 4.1: The Hilbert energy-momentum tensor

[**Written** | 6 pt(s)]

ID: ex_hilbert_energy_momentum_tensor:rt24

Learning objective

In this exercise, you familiarize yourself with the *Hilbert energy-momentum tensor* which is the source of gravity in general relativity. It is named after the mathematician David Hilbert, who introduced it in his derivation of the field equations

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as variation of the Lagrangian density with respect to the metric of spacetime. (Can you identify the relevant equation in the paper?)

Here you calculate the Hilbert energy-momentum tensor explicitly for the electromagnetic field in vacuum; you will recover the symmetric *Belinfante-Rosenfeld energy-momentum tensor* discussed in Problem 7.3 (last semester). As a further example, you study the real *Klein-Gordon field theory*, derive its generally covariant equation of motion, and its Hilbert energy-momentum tensor.

In the lecture, the Hilbert energy-momentum tensor was defined as

$$T_{\mu\nu} := \frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}L)}{\delta g^{\mu\nu}} \equiv \frac{2}{\sqrt{g}}\left[\frac{\partial(\sqrt{g}L)}{\partial g^{\mu\nu}} - \partial_\lambda \frac{\partial(\sqrt{g}L)}{\partial({g^{\mu\nu}}_{,\lambda})}\right] , \tag{1}$$

where $\mathcal{L} = \sqrt{g}L$ is the Lagrangian density, with the scalar Lagrange function L, the metric tensor $g_{\mu\nu}$ and its determinant $g = -\det(g_{\mu\nu})$.

The general covariant action of electrodynamics in vacuum is given by

$$S_{\text{EM}}[A_\alpha, g^{\mu\nu}] = \int d^4x\,\sqrt{g}L = \int d^4x\,\sqrt{g}\left(-\frac{1}{16\pi}g^{\alpha\mu}g^{\beta\nu}F_{\alpha\beta}F_{\mu\nu}\right) , \tag{2}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor and A_μ the four-potential.

a) Calculate the Hilbert energy-momentum tensor $T_{\mu\nu}$ of electromagnetic fields in vacuum. **2pt(s)**

Compare it to the symmetric Belinfante-Rosenfeld energy-momentum tensor from last semester.

Hint: From Problem 2.1 you know that $\frac{\partial g}{\partial g_{\mu\nu}} = g g^{\mu\nu}$.

Use $g_{\mu\lambda}g^{\lambda\nu} = \delta^\nu_\mu$ to show that $\frac{\partial g}{\partial g^{\mu\nu}} = -g_{\mu\alpha}g_{\nu\beta}\frac{\partial g}{\partial g_{\alpha\beta}}$.

As a second example, we want to study the *real Klein-Gordon field*, the simplest relativistic field theory. On *flat Minkowski space*, the *Lorentz covariant* action of the Klein-Gordon field is given by

$$S^{\text{SRT}}_{\text{KG}}[\phi] = \int d^4x\,\frac{1}{2}\left(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2\right) \tag{3}$$

with real scalar field ϕ and mass parameter m; $\eta^{\mu\nu}$ is the Minkowski metric in inertial coordinates.

b) Use the minimal coupling principle (MCP) to transform the action (3) into its generally covariant form $S_{\mathrm{KG}}^{\mathrm{ART}}[\phi, g^{\mu\nu}]$ for an arbitrary metric $g^{\mu\nu}$ in arbitrary coordinates. **1**pt(s)

c) Use the Euler-Lagrange equations (i.e., vary the action with respect to ϕ) to derive the generally covariant Klein-Gordon equation in curved spacetime. **1**pt(s)

Hint: Use the Laplace-Beltrami operator $\Delta\phi := \phi^{;\mu}{}_{;\mu} = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\phi_{,\nu}\right)$ from Problem 2.1 to simplify your result.

d) Now calculate the Hilbert energy-momentum tensor $T_{\mu\nu}$ for the Klein-Gordon field. **1**pt(s)

e) Finally, show explicitly that the covariant divergence of the energy-momentum tensor vanishes for solutions of the equations of motion: $T^{\mu\nu}{}_{;\nu} = 0$. **1**pt(s)

As demonstrated in the lecture, this is a consequence of the diffeomorphism invariance of $S_{\mathrm{KG}}^{\mathrm{ART}}[\phi, g^{\mu\nu}]$. It can be interpreted as energy-momentum conservation in *local* inertial systems, but does not correspond to an integral conservation law (= conserved charge) on generic spacetimes (= spacetimes without Killing vectors).

Problem 4.2: General Relativity in 2+1 Dimensions

[**Oral** | 6 pt(s)]

ID: ex_gravity_in_two_dimensions:rt24

Learning objective

In general relativity, the effects of gravity are locally described by the curvature of a four-dimensional Lorentzian spacetime manifold. But the Einstein field equations make sense in *arbitrary* spacetime dimensions (that is, on $D = d + 1$-dimensional Lorentzian manifolds). Since the degrees of freedom encoded in the curvature depend on the dimension of the manifold, this raises the question whether higher or lower-dimensional twins of general relativity behave differently than our “normal” 3+1-dimensional theory.

In this exercise, you study the special case of $2 + 1$-dimensional general relativity, i.e., the theory that a hypothetical “flatland Einstein” – an inhabitant of a world with two spatial dimensions– would have written down. Specifically, you show that there is no gravitational force in such a world; in this sense, general relativity in $2 + 1$-dimensions is “trivial”.

In Problem 3.1 you have shown that the *Riemann curvature tensor* $R^\rho{}_{\mu\nu\sigma}$ has $D^2(D^2 - 1)/12$ independent components due to its symmetries. The *Ricci tensor* $R_{\mu\nu} = R^\sigma{}_{\mu\nu\sigma}$, on the other hand, is a symmetric matrix, and therefore has $D(D + 1)/2$ independent components.

Interestingly, for $D = 2 + 1$ these two numbers are equal, suggesting that one can express the Riemann curvature tensor in terms of the Ricci tensor. We show now that this is true:

a) We denote by *Curv* the space of rank-4 tensors $A_{\alpha\beta\mu\nu}$ with the same symmetries as the Riemann curvature tensor: antisymmetric in the first and last two indices (A), symmetric under exchanging the first and last two indices (S), and it fulfills the first Bianchi identity (B): **3**pt(s)

$$\text{(A)} \quad A_{\alpha\beta\mu\nu} = -A_{\beta\alpha\mu\nu} = -A_{\alpha\beta\nu\mu}\,, \tag{4a}$$

$$\text{(S)} \quad A_{\alpha\beta\mu\nu} = A_{\mu\nu\alpha\beta}\,, \tag{4b}$$

$$\text{(B)} \quad A_{\alpha\beta\mu\nu} + A_{\mu\alpha\beta\nu} + A_{\beta\mu\alpha\nu} = 0\,. \tag{4c}$$

Show that for any symmetric rank-2 tensor $A_{\alpha\beta}$ the rank-4 tensor defined via

$$A_{\alpha\beta\mu\nu} := \frac{1}{D-2}\left[A_{\alpha\nu}g_{\beta\mu} - A_{\beta\nu}g_{\alpha\mu} - A_{\alpha\mu}g_{\beta\nu} + A_{\beta\mu}g_{\alpha\nu} - \frac{g^{\sigma\rho}A_{\sigma\rho}}{D-1}\left(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\nu}\right)\right], \quad (5)$$

is an element of *Curv*.

Furthermore, show that any symmetric 2-tensor $A_{\beta\mu}$ can be written as the trace of a tensor in *Curv*, e.g., $A_{\beta\mu} = g^{\alpha\nu}A_{\alpha\beta\mu\nu}$.

b) Use your results from a), in combination with a dimensionality argument, to show that there is exactly one rank-4 tensor which traces to the Ricci tensor in $D = 2+1$ dimensions. Conclude that this must be the curvature tensor. **2**pt(s)

In the lecture it was shown that, for a matter distribution given by $T^{\mu\nu}$, the Ricci tensor is constrained by the Einstein field equations (here without cosmological constant)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\kappa T_{\mu\nu}\,. \quad (6)$$

Here $R = R^{\mu}{}_{\mu}$ is the Ricci scalar. Note that these equations are well-defined in arbitrary spacetime dimensions D.

c) Use your result from b) to show that in vacuum ($T_{\mu\nu} = 0$), the curvature tensor is identically zero in $D = 2+1$ spacetime dimensions. That is: Spacetime is locally flat! Convince yourself that this conclusion cannot be drawn in our real $D = 3+1$-dimensional spacetime. **1**pt(s)

You have shown that in $D = 2+1$ dimensions the Riemann curvature tensor vanishes identically in vacuum. Intuitively, this tells us that there are no gravitational effects away from matter. For example, the tidal effects described by the geodesic deviation equation (Problem 3.2) are absent.

One can show that if the curvature tensor vanishes in an open region of spacetime, then one can construct coordinates such that the metric simplifies to the Minkowski metric $\eta_{\mu\nu}$ in that region. Thus, in $D = 2+1$ dimensions, empty space becomes Minkowski space, and no gravitational attraction is possible (also other gravitational effects are absent, such as gravitational waves).

Note: Note that *Newtonian* gravity in two spatial dimensions is *not* trivial. (What is the Green's function of the Laplace operator in 2D?) Thus $2+1$-dimensional general relativity and 2-dimensional Newtonian gravity are not equivalent in the non-relativistic limit!

Note: The fact that there are no (local) dynamical degrees of freedom in pure 2+1 gravity does not mean that the theory is completely "trivial". As Edward Witten showed, it is equivalent to a *topological* Chern-Simons field theory (a particular class of gauge theories):

https://doi.itp3.info/10.1016/0550-3213(88)90143-5

The paper is – as usual for Witten – nice to read, but requires mathematics that is beyond this course. (But not by much. We are mostly missing the *tetrad formalism*.)

Problem 4.3: Einstein's original formulation of the field equations [**Oral** | 3 (+2 bonus) pt(s)]

ID: ex_einsteins_original_field_equations:rt24

Learning objective

In 1915 Einstein published his paper *"Die Feldgleichungen der Gravitation"*

https://doi.itp3.info/650aebc80e4c295a6c3d40cd0df42889

in which he spelled out the field equations of general relativity in their final form.

In this exercise, your task is to read this (short!) paper and translate Einstein's original formulation into the modern form of the field equations.

a) Read the 1915 paper and identify the equation that corresponds to the modern form of the Einstein field equations (without cosmological constant Λ) **3**pt(s)

$$G_{\mu\nu} = -\kappa T_{\mu\nu}\,, \tag{7}$$

where the Einstein tensor is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, with Ricci tensor $R_{\mu\nu}$, Ricci scalar R, and the metric tensor $g_{\mu\nu}$. $T_{\mu\nu}$ is the energy-momentum tensor and κ is Einsteins gravitational constant.

Which of Einstein's symbols correspond to which of our modern notation?

Finally, identify and discuss equations and concepts in Einstein's paper that do *not* have a direct equivalent in our modern treatment of the field equations.

*b) In the lecture, we introduced the more general form of the field equations **+2**pt(s)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu}\,, \tag{8}$$

with *cosmological constant* $\Lambda \in \mathbb{R}$.

This modification was not yet included in the 1915 paper, but introduced by Einstein in 1917 in his paper *"Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie"*

https://doi.itp3.info/45606292de2af71198a29a3045ecbb05

in which he studied the consequences of his field equations for the universe as a whole.

Read the paper and identify the complete field equations (8) in Einstein's notation.

For what purpose did Einstein introduce the cosmological constant?