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Problem 3.1: Properties of the Riemann Curvature Tensor

[**Written** | 8 pt(s)]

ID: ex_properties_riemann_tensor:rt24

> **Learning objective**
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> The purpose of this task is to become familiar with the Riemann curvature tensor, our mathematical tool to describe curvature in general relativity. You study the symmetries of this tensor (which can simplify calculations significantly), and show that there are not as many independent components as one naïvely expects.

As a reminder, the *Christoffel symbols* are given by

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{im} \left(g_{mj,k} + g_{km,j} - g_{jk,m} \right) . \tag{1}$$

These are the connection coefficients of the unique, torsion-free, metric-compatible *Levi-Civita connection* on a Riemannian manifold.

In the lecture, the curvature tensor was derived as function of the connection coefficients:

$$R_{iklm} = g_{is} \left(\partial_l \Gamma^s{}_{km} - \partial_m \Gamma^s{}_{kl} + \Gamma^s{}_{nl} \Gamma^n{}_{km} - \Gamma^s{}_{nm} \Gamma^n{}_{kl} \right) . \tag{2}$$

R_{iklm} is called *Riemann* curvature tensor if the connection is the Levi-Civita connection Eq. (1).

a) Show that the Riemann curvature tensor takes the form **2**pt(s)

$$R_{iklm} = \frac{1}{2} \left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l} \right) + g_{ab} \left(\Gamma^a{}_{kl} \Gamma^b{}_{im} - \Gamma^a{}_{km} \Gamma^b{}_{il} \right) . \tag{3}$$

b) The Riemann curvature tensor features several symmetries and identities. **3**pt(s)

It is antisymmetric (A) in the first two and last two indices and symmetric (S) under a swap of the first two with the last two indices. It also fulfills the first Bianchi identity (B):

$$\text{(A)} \quad R_{iklm} = -R_{kilm} = -R_{ikml} \tag{4a}$$

$$\text{(S)} \quad R_{iklm} = R_{lmik} \tag{4b}$$

$$\text{(B)} \quad R_{iklm} + R_{ilmk} + R_{imkl} = 0 \tag{4c}$$

Prove these identities.

We are now prepared to discuss the independent components of the Riemann curvature tensor. Simple counting tells us that the tensor consists of D^4 components (fields), where D denotes the dimension of the manifold. For example, in the usual $D = 4$-dimensional spacetime of general relativity, we have 256 components. However, symmetries reduce this number considerably:

c) Show that the symmetries (A), (S) and (B) reduce the number of (algebraically) independent components to 3pt(s)

$$\frac{D^2(D^2-1)}{12}\,. \tag{5}$$

Hint: First, infer the number of independent components due to (A) and (S), and then subtract the number of unique equations that (B) implies.

In conclusion, you have shown that on the $D = 4$-dimensional spacetime of general relativity, the 256 fields that make up the Riemann curvature tensor reduce to only 20 algebraically independent fields.

Note: This does not mean that all of these 20 fields are *physical* degrees of freedom in general relativity! As we will discuss in the lecture, there are actually far fewer since general relativity has a gauge symmetry (which is related to its background independence and invariance under arbitrary coordinate transformations).

Problem 3.2: Geodesic Deviation

[**Oral** | 5 pt(s)]

ID: ex_geodesic_deviation:rt24

Learning objective

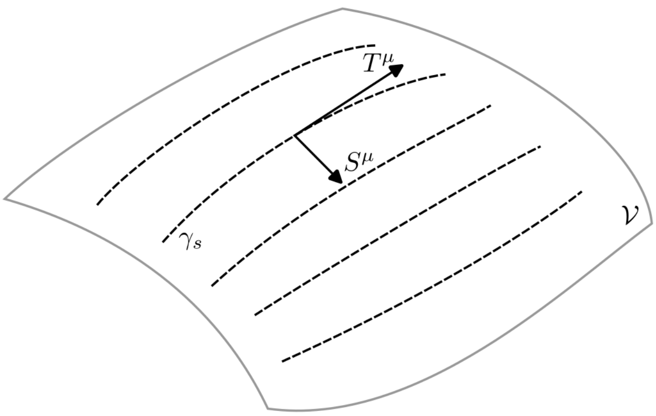
In our everyday experience of (nearly) flat space, we expect parallel straight lines to keep the same distance from each other, no matter how far they extend. In curved spaces, this is not necessarily the case: Initially parallel Geodesics – the straight lines on Riemannian manifolds – may cross or diverge.

In this exercise, you show that this behavior can be directly related to the *curvature* of the manifold. The resulting *geodesic deviation equation* serves as a motivation for a geometric theory of gravity, in that it suggests a purely geometric description of gravitational tidal effects, and a relation between gravity and curvature.

A geodesic is a curve $\gamma(\lambda)$ on a manifold which (locally) minimizes the distance between two points. Because we only consider the torsion-free Levi-Civita connection, geodesics are equivalent to autoparallel curves.

The defining equation for (affinely parametrized) geodesics is then

$$\frac{d^2\gamma^\mu}{d\lambda^2} + \Gamma^\mu{}_{\alpha\beta}\frac{d\gamma^\alpha}{d\lambda}\frac{d\gamma^\beta}{d\lambda} = 0\,. \tag{6}$$



We now consider a small, two-dimensional patch $\mathcal{V}$ embedded in a larger, higher dimensional manifold $\mathcal{M}$, and a continuous family of geodesics $\gamma_s(\lambda)$ parametrized by $s \in \mathbb{R}$. We assume that every point in $\mathcal{V}$ belongs to exactly one geodesic and can therefore be labeled by the coordinates (s, λ) (we assume that the geodesics do not cross in $\mathcal{V}$!).

It is then natural to define the two vector fields

$$S^\mu := \frac{\partial\gamma_s^\mu(\lambda)}{\partial s} \quad \text{(“Deviation”)} \quad \text{and} \quad T^\mu := \frac{\partial\gamma_s^\mu(\lambda)}{\partial\lambda} \quad \text{(“Velocity”)}\,. \tag{7}$$

This setup is illustrated in the sketch above.

In the following we want to show that, when traveling along a geodesic, adjacent geodesics can be attracted or repelled with a rate determined by the local curvature.

a) As a first preliminary step, show that **1**pt(s)

$$\frac{DS^\mu}{D\lambda} = \frac{DT^\mu}{Ds} \quad \text{or equivalently} \quad T^\alpha S^\mu{}_{;\alpha} = S^\alpha T^\mu{}_{;\alpha} \,, \tag{8}$$

since the Levi-Civita connection is torsion free (= symmetric).

b) As a second preliminary step, convince yourself that T^μ is a parallel vector field, i.e., show that **1**pt(s)

$$T^\alpha T^\beta{}_{;\alpha} = 0 \,. \tag{9}$$

Because of the relation

$$\gamma^\mu_{s+ds} - \gamma^\mu_s = S^\mu ds \,, \tag{10}$$

we can interpret S^μ as the distance to another infinitesimally close geodesic. A change of this vector (more precisely: its "acceleration") then tells us whether nearby geodesics are being attracted or repelled from each other.

The *geodesic deviation* A^μ is therefore defined as the second derivative of S^μ along the geodesic:

$$A^\mu := \frac{DS^\mu}{D\lambda^2} \,. \tag{11}$$

c) Use your preliminary results to derive the *geodesic deviation equation* **3**pt(s)

$$A^\mu = T^\alpha \left(T^\beta S^\mu{}_{;\beta} \right)_{;\alpha} = R^\mu{}_{\alpha\beta\nu} T^\alpha T^\beta S^\nu \,. \tag{12}$$

Hint: Remember the Ricci identity from the lecture, which relates the commutator of covariant derivatives to the curvature tensor:

$$B_{\nu;\alpha;\beta} - B_{\nu;\beta;\alpha} = R^\sigma{}_{\nu\alpha\beta} B_\sigma \,. \tag{13}$$

Problem 3.3: Charged particle in a gravitational field

[**Oral** | 9 (+6 bonus) pt(s)]

ID: ex_charge_in_gravitational_field:rt24

Learning objective

In this exercise you explore and analyze the apparent "paradox" of a charge at rest in a gravitational field.

Einstein's equivalence principle (EEP) states that local effects of gravity are indistinguishable from acceleration: a small laboratory at rest on Earth is indistinguishable from an accelerated rocket far away from Earth. It is well known from electrodynamics that accelerated charges emit radiation (synchrotron radiation, bremsstrahlung, ...). In combination with the EEP, this seems to suggest that a charge *at rest in a gravitational field* (say, on a table in an earth-bound laboratory) should radiate as if it were accelerated[a]! But our daily experience clearly tells us that this is not the case!

This "paradox" has a long-standing history. It has first been studied by Max Born and Wolfgang Pauli at the beginning of the 20th century, but wasn't conclusively explained until much later in 1960 by Fulton and Rohrlich. In this exercise, we follow the pedagogic exposition by Almeida and Saa:

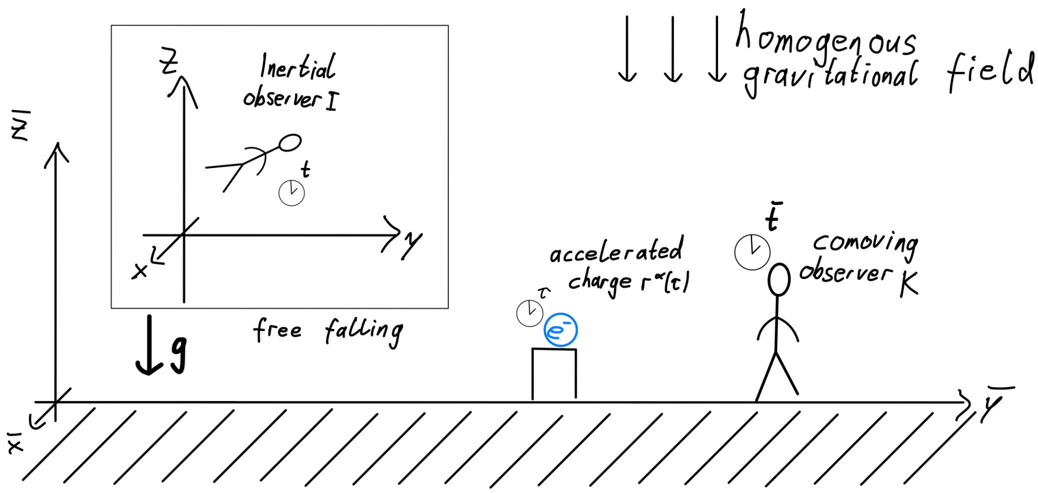
`https://doi.itp3.info/10.1119/1.2162548`

[a]According to general relativity, it *is* accelerated, as it does not follow a geodesic in spacetime.

Let us consider a charged particle with charge e at rest in a (non-inertial) laboratory K on the surface of Earth with coordinates $(c\bar{t}, \bar{x}, \bar{y}, \bar{z})$, henceforth called the *comoving observer* (see sketch below). The proper time displayed by a clock attached to the charge is denoted τ.

For our purpose, we can consider the gravitational field of Earth as homogeneous, so that – in accordance with Einstein's equivalence principle – the situation is equivalent to a charge in a laboratory with constant *proper acceleration* $g \equiv c^2/\alpha$ in z-direction.

For comparison (and to simplify the problem), we also consider an inertial (free-falling) observer I with inertial coordinates (ct, x, y, z) observing the same charge. Note that the charge is accelerated upwards with respect to this inertial observer:



The idea is now to use the inertial frame I to solve the problem within the framework of *special* relativity (i.e., by using the Maxwell equations in their Lorentz covariant form). To resolve the "paradox," we then transform these solutions into the comoving frame K, exploiting the general covariance of Maxwell's equations, and the tensorial transformation of the field strength tensor $F_{\mu\nu}$.

In Problem 6.1 (last semester) you already calculated the world line of a relativistic rocket with constant proper acceleration in an inertial frame. Thus, we already know that the world line of the charged particle has the form

$$r^\mu(\tau) = (\alpha \sinh(c\tau/\alpha), 0, 0, \alpha \cosh(c\tau/\alpha))_I = \left(ct_e, 0, 0, \alpha\sqrt{1 + \frac{c^2 t_e^2}{\alpha^2}}\right)_I \tag{14}$$

in the inertial coordinates of the free-falling observer I. Here, τ is the proper time of the particle and t_e is the coordinate time measured by clocks at rest in I.

a) To get an intuition for the hyperbolic motion of the charged particle Eq. (14), sketch the world line in the z-t-plane. Split the spacetime diagram of I into four regions, separated by the light cone of the origin, and explain the *Rindler horizon* of the accelerated charged particle. **1pt(s)**

In Problem 7.2 (last semester) you learned that the Maxwell equations can be written in their manifestly Lorentz covariant form as

$$F^{\mu\nu}{}_{,\nu} = \partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} j^\mu \,, \tag{15}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ is the electromagnetic field strength tensor, A^μ is the four-potential, and j^μ is the four-current induced by the charged particle. For a point particle, it is given by

$$j^\mu(x) = ec \int d\tau\, \delta^{(4)}(x - r(\tau))\, u^\mu \,, \tag{16}$$

where $u^\mu = \frac{dr^\mu}{d\tau}$ is the four-velocity of the charged particle.

b) Use the Lorenz gauge $\partial_\mu A^\mu = 0$ to derive the wave equation for the four-potential and solve it. **2pt(s)**

Hint: The solution of the wave equation $\Box A^\mu(x) = \frac{4\pi}{c} j^\mu(x)$, which also respects the event horizons discussed in a), can be written as

$$A^\mu(x) = \int d^4x'\, G_{\text{ret}}(x - x') \frac{4\pi}{c} j^\mu(x') \,, \tag{17}$$

where the retarded Green's function is given by

$$G_{\text{ret}}(x) = \frac{1}{2\pi} \theta(x^0) \delta\left(x^\nu x_\nu\right) \tag{18}$$

and $\theta(x^0)$ is the Heaviside step function.

*c) Use your solution from b) to calculate the electromagnetic field strength tensor $F^{\mu\nu}$. **+1pt(s)**

Hint: Start from the four-potential in the form

$$A^\mu = 2e \int d\tau\, \theta(ct - r^0(\tau)) \delta\big((x - r(\tau))^2\big)\, u^\mu(\tau) \,, \tag{19}$$

and calculate the gradient $\partial^\nu A^\mu$.

The following chain rule might be useful: $\partial^\nu \delta(f) = \frac{\partial f / \partial x_\nu}{df/d\tau} \frac{d}{d\tau} \delta(f)$.

The final result should read

$$\partial^\nu A^\mu = \frac{e}{|uR|} \frac{d}{d\tau} \left[\frac{u^\mu R^\nu}{uR}\right]\Bigg|_{\tau = \tau_{\text{ret}}} \,, \tag{20}$$

where all expressions are to be evaluated at the retarded proper time τ_{ret}, which is implicitly defined by $R^\mu R_\mu = (x^\mu - r^\mu(\tau))^2 \overset{!}{=} 0$. For simplicity, we introduced the relative vector $R^\mu := x^\mu - r^\mu(\tau)$ and omit indices in scalar contractions, i.e., $uR \equiv u^\mu R_\mu$.

Recall that the electromagnetic field strength tensor relates to the electric and magnetic fields as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} . \tag{21}$$

d) Calculate the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ of the accelerated charged particle in the inertial frame I using Eq. (20) and (21), and the world line $r^\mu(\tau)$ in Eq. (14). **2pt(s)**

The result reads

$$\boldsymbol{E} = \frac{c^3 e}{\alpha\, |uR|^3} \begin{pmatrix} xz \\ yz \\ \frac{(x^\mu x_\mu)^2}{4\alpha^2} - \frac{\alpha^2}{4} - \frac{(uR)^2}{c^2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \frac{c^3 e}{\alpha\, |uR|^3} \begin{pmatrix} -cty \\ ctx \\ 0 \end{pmatrix} , \tag{22}$$

where the uR is always to be evaluated at the retarded proper time τ_{ret}.

*e) Show that the prefactor $uR = u^\mu R_\mu$ evaluates to **+3pt(s)**

$$u^\mu R_\mu\big|_{\tau=\tau_{\text{ret}}} = \pm\frac{c}{2\alpha}\sqrt{(\alpha^2 + x^\mu x_\mu)^2 + 4\alpha^2(x^2+y^2)} \,, \tag{23}$$

where the right-hand side has no τ-dependence anymore.

Hint: Rewrite the expression in terms of $r^0 = ct_e$ instead of τ and use the conditions for the retarded time.

You now know the fields of a charged particle with constant proper acceleration in an inertial frame: Eq. (22), together with Eq. (23), which show that there is a *non-vanishing magnetic field*, and both fields are *time dependent*. This suggests that the constantly accelerated charge indeed radiates in the free-falling frame I, consistent with what you have been told in electrodynamics!

(If you are in doubt: Compute the Poynting vector and verify that the charge emits radiation.)

So far, so unexciting. But what does the comoving observer K see? The coordinates of this non-inertial/accelerated observer are given by *Rindler coordinates* $\bar{x}^\mu = (c\bar{t}, \bar{x}, \bar{y}, \bar{z})$, defined by the (non-linear) coordinate transformation

$$x^\mu = (ct, x, y, z)_I \equiv \left(\sqrt{2\bar{z}\alpha}\,\sinh\left(c\bar{t}/\alpha\right),\, \bar{x},\, \bar{y},\, \sqrt{2\bar{z}\alpha}\,\cosh\left(c\bar{t}/\alpha\right)\right)_I . \tag{24}$$

If you compare this transformation to the trajectory of the accelerated charge Eq. (14), you find

$$\bar{r}^\mu(\tau) = (c\tau, 0, 0, \alpha/2)_K \tag{25}$$

i.e., the charge is at rest in the Rindler coordinate system [at the position $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \alpha/2)$], and the Rindler time $\bar{t}$ coincides with the proper time τ. This shows that the Rindler coordinates are a reasonable coordinate system for the earth-bound laboratory in which the charge is at rest.

Note: The Rindler coordinates only cover the spacetime region $|ct| \leq z$ in I (the so called *Rindler wedge*). You can use your sketch from a) to visualize this region.

f) Express the Rindler coordinates $\bar{x}^\mu$ of K in terms of the inertial coordinates x^μ of I. **3pt(s)**

Then calculate the transformation matrix $\frac{\partial x^\mu}{\partial \bar{x}^\nu}$ and its inverse $\frac{\partial \bar{x}^\mu}{\partial x^\nu}$.

Since the electromagnetic field strength $F^{\mu\nu}$ is a contravariant tensor, it transforms under arbitrary coordinate transformations as

$$\bar{F}^{\mu\nu} = \frac{\partial \bar{x}^\mu}{\partial x^\alpha}\frac{\partial \bar{x}^\nu}{\partial x^\beta} F^{\alpha\beta} \,. \tag{26}$$

Note that also in Rindler coordinates the components of the electromagnetic field strength tensor can be identified as the electric and magnetic fields $\bar{\boldsymbol{B}}$ and $\bar{\boldsymbol{E}}$:

$$\bar{F}^{\mu\nu} = \begin{pmatrix} 0 & -\bar{E}_x & -\bar{E}_y & -\bar{E}_z \\ \bar{E}_x & 0 & -\bar{B}_z & \bar{B}_y \\ \bar{E}_y & \bar{B}_z & 0 & -\bar{B}_x \\ \bar{E}_z & -\bar{B}_y & \bar{B}_x & 0 \end{pmatrix} \,. \tag{27}$$

Note: This is a subtle claim. To show this, you must first operationally define what the electric and magnetic fields *are*, for example, by using the Lorentz force. Then you must show that the fields $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{B}}$ in Eq. (27) play the proper role in the Lorentz force law in K.

g) Calculate the magnetic field $\bar{\boldsymbol{B}}$ in K by transforming the relevant components of the electromagnetic field tensor [using Eq. (22) for the fields]. **1**pt(s)

Use your result to argue that the Poynting vector $\bar{\boldsymbol{S}}$ vanishes for the comoving observer. With this you have shown that – consistent with observations – the charge does *not* radiate in the earth-bound laboratory!

*h) For completeness, calculate also the transformed electric field $\bar{\boldsymbol{E}}$ by using the explicit form of $u^\mu R_\mu$ in Eq. (23). **+2**pt(s)

Show that the electric field in this frame is no longer time dependent – consistent with the result from g) that the charge does not radiate!

In conclusion, you have shown that the inertial observer I measures an accelerating and radiating charge, whereas the non-inertial comoving observer measures a charge at rest without magnetic field and only a static electric field.

More generally, you have shown that the concept of *radiation* is observer dependent! This means that statements like "object X emits radiation" or "object X emits N photons" are not observer-independent, an insight that has ramifications far beyond classical electrodynamics. (For example, in quantum field theory, the notion of "particles" becomes observer dependent, which leads to phenomena like *Hawking radiation* or the *Unruh effect.*)

Note: Instead of solving the Maxwell equations in an inertial frame, and subsequently transforming the solutions into the Rindler frame, one can also solve the Maxwell equations directly in the Rindler frame. These equations follow from the generally covariant form of the Maxwell equations:

$$F^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{g}}\partial_\nu(\sqrt{g}F^{\mu\nu}) = \frac{4\pi}{c}J^\mu \,. \tag{28}$$

Here, the semicolon ; denotes the covariant derivative, g is given by the determinant of the metric tensor $g = -\det(g_{\mu\nu})$, and $J^\mu = \frac{1}{\sqrt{g}} j^\mu$ is the covariant four-current.

In particular, this implies that the conventional form of Maxwell's equations you learn in electrodynamics is, strictly speaking, not valid in earth-bound laboratories as these are *not* inertial (and the usual [not generally covariant] Maxwell equations are only valid in inertial coordinates). However, the deviations due to the (homogeneous) gravitational field in a typical lab are so small that they can be ignored. Nonetheless, one can show (using your results from above), that the electric field of a point charge at rest on the surface of earth is *not exactly* a spherically symmetric Coulomb field!