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April 9th, 2024
SS 2024

Information on lecture and tutorials

Here a few infos on the modalities of the course ***"Relativitätstheorie 2"***:

- The C@MPUS-ID of this course is 043700002.
- You can find detailed information on lecture and tutorials on the website of our institute:
  `https://itp3.info/rt24`
- You can also find detailed information on lecture and tutorials on ILIAS:
  `https://ilias3.uni-stuttgart.de/goto_Uni_Stuttgart_crs_3644328.html`
- **Written** problems have to be handed in via ILIAS and will be corrected by the tutors. You must earn at least **66 %** of the written points to be admitted to the exam.
- **Oral** problems have to be prepared for the exercise session and will be presented by a student at the blackboard. You must earn at least **66 %** of the oral points to be admitted to the exam.
- Every student is required to **present** at least **1** of the oral problems at the blackboard to be admitted to the exam.
- Problems marked with an asterisk ($*$) are optional and can earn you bonus points.
- If you have questions regarding the problem sets, feel free to contact your tutor at any time.

Problem 1.1: Equivalence principle - The Eötvös experiment

[**Oral** | 3 pt(s)]

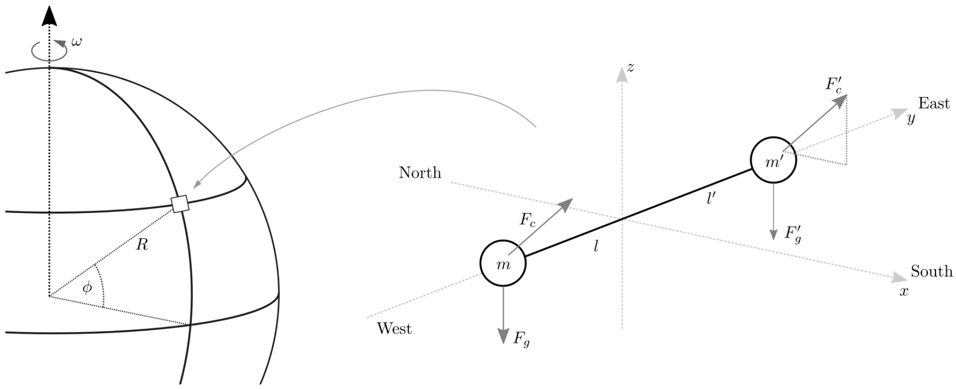
ID: ex_equivalence_principle:rt24

Learning objective

In this exercise we want to understand the experimental setup used by Roland Eötvös to tighten the bounds of the equivalence of inertial and gravitational mass. This fact was already known to Galileo and Newton, but never confirmed to the precision achieved by the Eötvös experiment. The fundamental nature of this equivalence was then recognized by Einstein and captured in his equivalence principle, which is the foundation of general relativity.

The original papers on the experiment(s) can be downloaded here:
`https://doi.itp3.info/34f140e314a0ea3b4b9a98cd58e61b9c`
`https://doi.itp3.info/10.1002/andp.19223730903`

The centerpiece of the experiment is a torsion balance that is located in a laboratory at the latitude ϕ, as depicted in the sketch below. The balance is suspended from a thin wire which is attached to the balance at the origin of the coordinate system. There are two two test masses (made from different materials) attached at distances l and l'. The two test masses experience the gravitational force $\boldsymbol{F}_g$ and the centrifugal force $\boldsymbol{F}_c$ (due to the rotation of earth). As the experiment is very small compared to the radius of the earth, we can assume homogeneous gravitational and centrifugal forces:



a) We choose our local coordinate system such that positive x-axis points south, the positive y-axis east, and the positive z-axis up. In this coordinate system, the forces have the form **1**pt(s)

$$\boldsymbol{F}_g = -m_G g \boldsymbol{e}_z \,, \quad \boldsymbol{F}_c = m_I \begin{pmatrix} a_x \\ 0 \\ a_z \end{pmatrix} \,, \tag{1}$$

where m_G denotes the gravitational and m_I the inertial mass of one of the two test masses (the masses of the other test mass are labeled by primes and experience the same accelerations).

Write down the balance condition for the torsion balance. Find expressions for a_x and a_z.

b) The torsion balance experiences a torque τ along the z-axis due to the centrifugal force. **1**pt(s)

Show that this torque is proportional to $\left[\frac{m_G}{m_I} - \frac{m'_G}{m'_I}\right]$.

Hint: Use the balance condition to eliminate l'.

By demonstrating that there is *no* torque, one can therefore experimentally verify the equivalence principle. The way this is done in practice is to determine the equilibrium position of the torsion balance (in the x-y plane); then one turns the complete setup by 180 degrees and determines the equilibrium position again.

c) Explain why this experimental protocol can be used to show that there is no torque. **1**pt(s)

Problem 1.2: Scalar gravity

[**Written** | 10 pt(s)]

ID: ex_scalar_gravity:rt24

Learning objective

The most straightforward generalization of Newtonian gravity to the relativistic regime is to consider a *scalar* gravitational field $\phi(t, \boldsymbol{x})$ with a Lorentz invariant action. While in principle such theories are a possible option to treat gravity relativistically, you will show in this exercise (for a specific scalar theory) that it *cannot* accurately describe phenomena like the perihelion precession of Mercury and the bending of light; both well-established phenomena that are confirmed by observations.

For a historical account on the role (and failure) of scalar theories of gravity, see
`https://doi.itp3.info/10.1007/BF00375886`

For a detailed analyis of a specific (particularly simple) scalar theory of gravity, see
`https://doi.itp3.info/10.1016/j.shpsb.2007.09.001`

We consider a particle with (inertial = gravitational) mass m with parametrized world line $z^\mu(\lambda)$ and a scalar gravitational field $\phi(t, \boldsymbol{x}) = \phi(x)$. Their dynamics and the interaction between particle and gravitational field are given by the following Lorentz invariant action:

$$S[z, \phi] = S_{\text{particle}}[z] + S_{\text{field}}[\phi] + S_{\text{int}}[z, \phi] \tag{2}$$

(This is structurally similar to our discussion in the lecture of charged, relativistic particles that interact with and via an electromagnetic field.)

- The action of the relativistic particle is the same that we discussed in special relativity:

$$S_{\text{particle}}[z] = -mc \int d\lambda\, \sqrt{\eta_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta} \equiv \int d\lambda\, L_{\text{particle}}(z, \dot{z}) \tag{3}$$

  Here we abbreviate $\dot{z}^\alpha = \frac{dz^\alpha}{d\lambda}$.

- The action of the gravitational field is given by the scalar that can be constructed from the 4-gradient of the field:

$$S_{\text{field}}[\phi] = \frac{c^3}{8\pi G} \int d^4x\, \eta^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi) \equiv \int d^4x\, \mathcal{L}_{\text{field}}(\phi, \partial\phi) \tag{4}$$

- The interaction between field and particle is chosen as follows:

$$\begin{aligned} S_{\text{int}}[z, \phi] &= -mc \int d^4x\, \left(e^{\phi(x)} - 1\right) \int d\lambda\, \sqrt{\eta_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta}\, \delta^{(4)}(x - z(\lambda)) \\ &\equiv \int d^4x\, \mathcal{L}_{\text{int}}(\phi, \partial\phi) \equiv \int d\lambda\, L_{\text{int}}(z, \dot{z}) \end{aligned} \tag{5}$$

a) Use the Lagrangian density $\mathcal{L} = \mathcal{L}_{\text{field}} + \mathcal{L}_{\text{int}}$ to derive the Euler-Lagrange equations for the gravitational field $\phi(x)$. **3**pt(s)

Hint: You can use the proper time τ as parametrization of the world line, i.e., $cd\tau = \sqrt{\eta_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta} d\lambda$.

Solve the resulting field equation for a point particle with mass M at rest in the origin, $z^\alpha(\tau) = (c\tau, \mathbf{0})$. Make the assumption that the gravitational field is weak ($e^{\phi(x)} \approx 1$) and confirm that this approximation is reasonable for the gravitational field of the sun at Mercury's orbit.

Hint: The solution of the field equation, which is of the form $\Box\phi = f(x)$, is given by

$$\phi(x) = \int d^4x'\, G(x-x')f(x') \tag{6}$$

with the Green's function $G(x) = \delta\big(x^0 - r\big)/(4\pi r)$.

b) Now use the Lagrangian $L = L_{\text{particle}} + L_{\text{int}}$ to derive the Euler-Lagrange equations for the world line of the particle in a given scalar field $\phi(x)$. 1pt(s)

Compare those equations for small velocities to Newton's law $\boldsymbol{F} = m\boldsymbol{a}$.

Hint: Use again the proper time parametrization $cd\tau = \sqrt{\eta_{\alpha\beta}\dot{z}^\alpha\dot{z}^\beta}d\lambda$ to simplify the result.

We now want to solve the equations of motion for a small body (like Mercury) in the gravitational field of the sun to determine the perihelion precession. Instead of solving the equations of motion from b) directly, it is more convenient to study the Lagrangian and its conserved quantities first.

c) Write down the Lagrangian $L = L_{\text{particle}} + L_{\text{int}}$ for a particle of mass m (Mercury) in the gravitational field of the sun [use your result from a) with M the mass of the sun]. In the following we consider the gravitational field as static. 3pt(s)

Transform the Lagrangian to spherical coordinates and use that the motion is restricted to a plane (Why?); choose the $x-y$ plane with $\theta = \pi/2$.

Hint: The Lagrangian should read $L = -mc\, e^{-\frac{MG}{c^2 r}}\sqrt{c^2\dot{t}^2 - (\dot{r}^2 + r^2\dot{\varphi}^2)}$.

Use this result to calculate the conserved quantities (Why are they conserved?)

$$m\alpha = \frac{\partial L}{\partial\dot{\varphi}} \quad , \quad m\gamma = -\frac{\partial L}{\partial\dot{t}} \quad \text{and} \quad c^2 = \eta_{\alpha\beta}\frac{dz^\alpha}{d\tau}\frac{dz^\beta}{d\tau}\,. \tag{7}$$

Finally, introduce the new coordinate $u = 1/r$ and use $\frac{du}{d\varphi} = \frac{du/d\tau}{d\varphi/d\tau}$ to derive a differential equation of the form $u' \equiv \frac{du}{d\varphi} = f(u)$. Taylor-expand the right-hand side for $MGu/c^2 \ll 1$ to bring it into the form

$$(u')^2 = a_0 + a_1 u - a_2 u^2\,. \tag{8}$$

d) Solve the differential equation Eq. (8) for $u(\varphi)$ and calculate the perihelion precession $(\Delta\varphi)_{\text{Scalar}}$ between two maxima of $u(\varphi)$ (i.e., per revolution). Compare your result with the prediction 2pt(s)

$$(\Delta\varphi)_{\text{GR}} = +\frac{6\pi MG}{c^2 r_{\text{min}}(1+\epsilon)} \tag{9}$$

of general relativity (which matches observations).

Hint: By differentiating the equation with respect to φ you can rewrite it as a simpler second-order differential equation. The solution of the differential equation should be

$$u(\varphi) = \frac{a_1}{2}\left[1 + \epsilon\cos\Big(\sqrt{a_2}(\varphi - \varphi_0)\Big)\right]\,. \tag{10}$$

e) Last but not least we want to examine whether scalar gravity can deflect massless particles like photons. To do so, rewrite the Euler-Lagrange equations from part b) using the parametrization $\xi = \tau/m$ and the 4-momentum $k^\mu = \frac{dz^\mu}{d\xi}$. 1pt(s)

Take the limit $m \to 0$ to show that the quantity $q^\mu = e^\phi k^\mu$ is conserved. Use this to argue that there is no bending of light in scalar gravity.

In summary, you have shown that the scalar gravity theory Eq. (2) contradicts both the observed perihelion precession, and the bending of light, and can therefore be ruled out as a relativistic theory of gravity.

Problem 1.3: Linear symmetric tensor gravity

[**Written** | 3 (+2 bonus) pt(s)]

ID: ex_tensor_gravity:rt24

> **Learning objective**
>
> In Problem 1.2 you studied a *scalar* theory of gravity that failed to reproduce crucial observations. As mentioned in the lecture, *vector* gravitational fields suffer from the same issues (plus gravitational waves with negative energy!). It is thus reasonable to consider a tensor-valued gravitational field next.
>
> Since the procedure to study this theory is quite similar to Problem 1.2, you have to do few calculations. The purpose of this exercise is then to work out the differences between the scalar theory of Problem 1.2 and a theory on flat Minkowski space with a symmetric tensor field.

We consider a particle with (inertial = gravitational) mass m and parametrized world line $z^\mu(\lambda)$. In contrast to Problem 1.2, the gravitational field is now described by symmetric tensor field: $h^{\mu\nu}(x) = h^{\nu\mu}(x)$. Note that the theory is still defined on flat Minkowski space, $h^{\mu\nu}$ is not the metric!

The dynamics and interaction between particle and gravitational field are given by the following Lorentz invariant action:

$$S[z,h] = S_{\text{particle}}[z] + S_{\text{field}}[h] + S_{\text{int}}[z,h] \tag{11}$$

The three parts of the action are defined as follows:

- The relativistic particle action is:

$$S_{\text{particle}}[z] = -\frac{m}{2}\int d\tau\, \eta_{\alpha\beta}\dot{z}^\alpha \dot{z}^\beta \equiv \int d\tau\, L_{\text{particle}}(z,\dot{z}) \tag{12}$$

  This action is more convenient as it drops the usual square root. While it is not reparametrization invariant, it is physically equivalent and leads to the same equations of motion.

  Here the *dot* derivative is defined with respect to the *proper time*: $\dot{z}^\mu \equiv \frac{dz^\mu}{d\tau}$.

- The action for the tensor field is chosen quadratic in the field:

$$S_{\text{field}}[h] = \frac{c^3}{32\pi G}\int d^4x \left[\frac{1}{2}h_{\nu\beta,\alpha}\bar{h}^{\nu\beta,\alpha} - \bar{h}_{\mu\alpha}{}^{,\alpha}\bar{h}^{\mu\beta}{}_{,\beta}\right] \equiv \int d^4x\, \mathcal{L}_{\text{field}}(h,\partial h) \tag{13}$$

  Here we defined $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^\alpha{}_\alpha$.

  Recall that a *comma* denotes a partial derivative: $h_{\mu\nu,\alpha} \equiv \partial_\alpha h_{\mu\nu}$.

- The interaction is simply the contraction of the gravitational field $h_{\mu\nu}$ with the energy-momentum tensor

$$T^{\mu\nu} = mc\int d\tau\, \dot{z}^\mu \dot{z}^\nu \delta^{(4)}(x - z(\tau)) \tag{14}$$

of the massive particle, i.e.,

$$
\begin{aligned}
S_{\text{int}}[z,h] &= -\frac{m}{2}\int d^4x\, h_{\mu\nu}(x)\int d\tau\, \dot{z}^\mu \dot{z}^\nu \delta^{(4)}(x-z(\tau)) \\
&\equiv \int d^4x\, \mathcal{L}_{\text{int}}(h,\partial h) \equiv \int d\tau\, L_{\text{int}}(z,\dot{z})\,.
\end{aligned} \tag{15}
$$

a) Use the Lagrangian density $\mathcal{L} = \mathcal{L}_{\text{field}} + \mathcal{L}_{\text{int}}$ to derive the Euler-Lagrange equations for the tensor field $h^{\mu\nu}(x)$. **1**pt(s)

Similarly to the four-potential A^μ of the electromagnetic field, the gravitational tensor field $h^{\mu\nu}$ is not fully determined by the equations of motion as it has a gauge degree of freedom. One can use this freedom to fix the gauge $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$ (similar to the Lorenz gauge $A^\alpha{}_{,\alpha} = 0$ in electrodynamics). In this gauge, the field equations become quite simple:

$$
\Box \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}\,. \tag{16}
$$

Solving this field equation for a point particle at rest (similar to Problem 1.2) one finds

$$
h_{\mu\nu} = \begin{cases} \frac{-2MG}{c^2 r} & \text{for } \mu=\nu=0 \\ 0 & \text{otherwise} \end{cases}\,. \tag{17}
$$

b) Show that the Lagrangian $L = L_{\text{particle}} + L_{\text{int}}$ for a particle in the static gravitational field (17) is given in spherical coordinates (and $\theta = \pi/2$) by **2**pt(s)

$$
L = -\frac{mc^2}{2}\left(\frac{1}{2} - \frac{MG}{c^2 r}\right)\dot{t}^2 + \frac{m}{2}\left(\frac{1}{2} + \frac{MG}{c^2 r}\right)\left(\dot{r}^2 + r^2\dot{\varphi}^2\right)\,. \tag{18}
$$

From this one can again derive conserved quantities:

$$
\alpha = r^2\dot{\varphi}\left(1 + \frac{2MG}{c^2 r}\right)\,, \tag{19a}
$$

$$
\gamma = \dot{t}\left(1 - \frac{2MG}{c^2 r}\right)\,, \tag{19b}
$$

$$
c^2 = c^2\dot{t}^2\left(1 - \frac{2MG}{c^2 r}\right) - \left(\dot{r}^2 + r^2\dot{\varphi}^2\right)\left(1 + \frac{2MG}{c^2 r}\right)\,. \tag{19c}
$$

Combining these, and using the coordinate $u = 1/r$ and $\frac{du}{d\varphi} = \frac{\dot{u}}{\dot{\varphi}}$, one finally obtains the differential equation

$$
\left(\frac{du}{d\varphi}\right)^2 = -u^2 + \frac{1}{c^2\alpha^2}\left(\gamma^2 - 1 + 2\frac{MGu}{c^2}\right)\frac{1 + 2MGu/c^2}{1 - 2MGu/c^2}\,. \tag{20}
$$

Taylor-expand the right-hand side for small $\frac{MGu}{c^2}$ up to second order and use your result from Problem 1.2 Part d) to calculate the perihelion precession $(\Delta\varphi)_{\text{Tensor}}$ in this theory.

Compare this result with the (correct) result from general relativity given in Problem 1.2 Part d).

*c) To check whether the theory predicts the bending of light, we have to study the coupling to massless particles like photons. **+1pt(s)**

For an incoming photon with the 4-momentum $k_{\text{in}}^{\mu} = \frac{dz_{\text{in}}^{\mu}}{d(\tau/m)} = (\omega, 0, 0, \omega)$, the deflection angle is given (for small deflections) by the ratio

$$\Delta\varphi = \left(\frac{k_x}{k_z}\right)_{\text{final}} . \tag{21}$$

A straightforward calculation yields for the Euler-Lagrange equations of $L = L_{\text{particle}} + L_{\text{int}}$ (valid for arbitrary masses)

$$\frac{dK_\alpha}{d(\tau/m)} \equiv \frac{d}{d(\tau/m)}\left[(\eta_{\mu\alpha} + h_{\mu\alpha})k^\mu\right] = \frac{1}{2}h_{\mu\nu,\alpha}k^\mu k^\nu . \tag{22}$$

We are now interested in the x-component of this equation. Use the gravitational field of the sun Eq. (17) to calculate the deflection angle $\Delta\varphi$ for light.

Hint: Use $\frac{dK_x}{d(\tau/m)} = k^z \frac{dK_x}{dz}$ (with $z^\mu = (ct, x, y, z)$) and approximate $k^\mu = k_{\text{in}}^\mu$. Note that $k_x = K_x$ for $z \to \pm\infty$.

*d) Last but not least we want demonstrate that the theory is inconsistent. **+1pt(s)**

First, show that ${T^{\mu\nu}}_{,\nu} = \frac{\partial}{\partial x^\nu}T^{\mu\nu}$ can be rewritten as

$${T^{\mu\nu}}_{,\nu} = +mc \int d\tau\, \ddot{z}^\mu \delta^{(4)}(x - z(\tau)) , \tag{23}$$

which implies that ${T^{\mu\nu}}_{,\nu}$ can only be zero if $\ddot{z}^\mu = 0$.

On the other hand, use the field equations derived in a) to show that it must be ${T^{\mu\nu}}_{,\nu} = 0$.

What does this imply for the theory?

Note: The reason for this self-inconsistency is that the interaction term in this theory is *linear*. If one tries to fix this issue, one is forced to add non-linearities, and eventually ends up with the correct equations of general relativity!

For details on this procedure, see `https://doi.itp3.info/10.1007/bf00759198`.

In summary, you have shown that the linear tensor gravity theory on flat Minkowksi space Eq. (11) (correctly) predicts the bending of light in a gravitational field. However, its prediction for the perihelion precession deviate from observations (and that of general relativity). Even worse, the theory is inconsistent. The reason for this inconsistency is that the theory is linear (which, as argued in the lecture, a relativistic theory of gravity cannot be).