Dr. Nicolai Lang
Institute for Theoretical Physics III, University of Stuttgart

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Problem 7.1: Transformation of the Electromagnetic Fields - The Einstein Way [Oral | 4 pt(s)]

ID: ex_lorentz_transformation_E_B:rt2324

> **Learning objective**
>
> In the lecture, you have seen that the electric and magnetic fields transform under a Lorentz boost as
>
> $$\bar{\boldsymbol{E}}(\bar{\boldsymbol{x}}) = \gamma\left[\boldsymbol{E}(\boldsymbol{x}) + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{x})\right] - (\gamma-1)\frac{\boldsymbol{v}\cdot\boldsymbol{E}(\boldsymbol{x})}{v^2}\boldsymbol{v}\,, \tag{1a}$$
>
> $$\bar{\boldsymbol{B}}(\bar{\boldsymbol{x}}) = \gamma\left[\boldsymbol{B}(\boldsymbol{x}) - \frac{1}{c}\boldsymbol{v}\times\boldsymbol{E}(\boldsymbol{x})\right] - (\gamma-1)\frac{\boldsymbol{v}\cdot\boldsymbol{B}(\boldsymbol{x})}{v^2}\boldsymbol{v}\,. \tag{1b}$$
>
> This complicated transformation law was derived from the transformation of the 4-potential A^μ, which was shown to transform as a 4-vector in the lecture.
>
> In this exercise, you derive this transformation law for a boost in x-direction the way it was originally derived by Einstein in his famous publication "Zur Elektrodynamik bewegter Körper" (his discussion starts on page 907).
>
> The purpose of this exercise is twofold: First, you learn that the Lorentz covariance of Maxwell equations can be shown without formulating the theory as a gauge theory, and without using the tensor formalism. And second, since this direct approach turns out to be quite messy and opaque, you learn to appreciate the elegance of the modern tensor formalism.

We consider two inertial systems K and $\bar{K}$, connected by a boost in x-direction $K \xrightarrow{v_x} \bar{K}$.

The homogeneous Maxwell equations read

$$\text{Magnetic Gauss's law:} \qquad \nabla\cdot\boldsymbol{B} = 0\,, \tag{2a}$$

$$\text{Maxwell-Faraday law:} \qquad \nabla\times\boldsymbol{E} + \frac{1}{c}\partial_t\boldsymbol{B} = 0\,. \tag{2b}$$

The goal is to derive the transformation laws for $\boldsymbol{E}$ and $\boldsymbol{B}$ under a Lorentz boost by requiring the *form-invariance* of the homogeneous Maxwell Eq. (2).

a) Transform the homogeneous Maxwell Eq. (2) into the frame $\bar{K}$ by transforming all derivatives. Identify new quantities $\bar{\boldsymbol{E}}(\bar{x})$ and $\bar{\boldsymbol{B}}(\bar{x})$ in terms of $\boldsymbol{E}(x)$ and $\boldsymbol{B}(x)$ such that the equations in $\bar{K}$ have the same form as in K (form-invariance). 3pt(s)

   Hint: After the transformation, rearrange the derivatives such that you can again identify the divergence and curl. You need to combine two of the four equations to do this.

b) Check that the transformed fields $\bar{\boldsymbol{E}}$ and $\bar{\boldsymbol{B}}$ found in a) also satisfy the *inhomogeneous* Maxwell equations (in vacuum for simplicity) 1pt(s)

$$\text{Electric Gauss's law:} \qquad \nabla\cdot\boldsymbol{E} = 0\,, \tag{3a}$$

$$\text{Ampère's law:} \qquad \nabla\times\boldsymbol{B} - \frac{1}{c}\partial_t\boldsymbol{E} = 0\,. \tag{3b}$$

   Hint: Use that Eq. (2) and Eq. (3) map into each other via a simple substitution of fields.

Problem 7.2: Covariant formulation of the Maxwell equations

[**Oral** | 6 (+2 bonus) pt(s)]

ID: ex_covariant_maxwell_equations:rt2324

Learning objective

In this problem, you use the Lagrange formalism and the machinery of tensor calculus to derive the Maxwell equations (in vacuum) in a manifestly covariant form. To do so, you start by constructing a Lorentz- and gauge invariant Lagrangian density for the electromagnetic field. Then you evaluate the Euler-Lagrange equations to derive the inhomogeneous Maxwell equations in their manifestly covariant form. Finally, you use the dual field strength tensor to write the homogeneous Maxwell equations covariantly as well.

In the lecture we introduced the rank-2 *field strength tensor*

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{4}$$

where the four-potential is given by $A_\mu = (\varphi, -\boldsymbol{A})$, φ is the scalar potential and $\boldsymbol{A}$ is the vector potential of classical electrodynamics. In the lecture it was motivated that the field strength tensor Eq. (4) encodes all gauge-invariant information about the electromagnetic field.

We now want to construct a Lagrangian density for the electromagnetic field in vacuum. This Lagrangian density should be invariant under gauge transformations and proper orthochronous Lorentz transformations $SO^+(1,3)$. Furthermore, to recover the Maxwell equations (which are linear in the fields), the Lagrangian should be *quadratic* in the fields.

Thus, the most general Lagrangian density we can construct has the form:

$$\mathcal{L} = a\,F^{\mu\nu}F_{\mu\nu} + b\,\tilde{F}^{\mu\nu}F_{\mu\nu} + c\,\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu}\,, \tag{5}$$

where a, b and c are constants, $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is the *dual field strength tensor*, and we use the metric tensor $\eta_{\mu\nu}$ to raise and lower indices.

a) First, we want to show that $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} \propto F_{\mu\nu}F^{\mu\nu}$, so that the term can be dropped from the Lagrangian without loss of generality. **1pt(s)**

   Hint: Use the following relation for the Levi-Civita symbol:

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\mu\nu\gamma\delta} = -2(\delta^\alpha_\gamma\delta^\beta_\delta - \delta^\alpha_\delta\delta^\beta_\gamma) \tag{6}$$

b) Next, calculate the Euler-Lagrange equations for the second term of the Lagrangian density in Eq. (5) by treating the components $A_\mu(x)$ as the dynamic variables. Show that the result is identically zero. **2pt(s)**

   To understand why, show that $\tilde{F}^{\mu\nu}F_{\mu\nu}$ can be written as a four-divergence so that it leads to a surface term in the action.

   Note: The term $\tilde{F}^{\mu\nu}F_{\mu\nu}$ is known as the *θ-term*. As you have shown, it has no effect on the classical level. However, for a more general class of gauge theories known as *Yang-Mills theories*, it can have effects on the quantum level.

In conclusion, we found that only the first term in Eq. (5) is relevant. Let us then define the Lagrangian density of Maxwell theory in vacuum as

$$\mathcal{L}_{EM} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\,. \tag{7}$$

Note: The prefactor $1/16\pi$ is specific to our choice of Gaussian cgs units, and the minus sign a useful convention so that the action possesses a minimum.

c) Evaluate the Euler-Lagrange equations for Eq. (7) to derive the inhomogeneous Maxwell equations in their manifestly covariant form **1pt(s)**

$$\partial_\nu F^{\mu\nu} = 0\,. \tag{8}$$

The field strength tensor can also be expressed in terms of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad \text{or equivalently} \quad \begin{aligned} F^{i0} &= -F^{0i} = E_i \quad \text{and} \\ F^{ij} &= -\varepsilon_{ijk} B_k\,, \end{aligned} \tag{9}$$

where Latin indices run over $1, 2, 3$.

d) Check that the definition of the field strength tensor in Eq. (4) is consistent with Eq. (9). **1pt(s)**

Note: The electric and magnetic fields E_i and B_i are not four-vectors! Therefore they are neither co- nor contravariant and their index is just a label (which for consistency we always write as a subscript).

e) Write the inhomogeneous Maxwell Eq. (8) in their standard form (namely the electric Gauss law and Ampère's law) by using Eq. (9). **1pt(s)**

So far we only considered the *inhomogeneous* Maxwell equations. The *homogeneous* Maxwell equations are automatically fulfilled by expressing the field strength tensor (and thereby the electric and magnetic fields) in terms of the four-potential A_μ. However, we could also define the field strength tensor by Eq. (9) directly in terms of the electric and magnetic fields.

In this case, one can use the dual field strength tensor $\tilde{F}^{\mu\nu} := \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ to write the homogeneous Maxwell equations in manifestly covariant form:

$$\partial_\nu \tilde{F}^{\mu\nu} = 0\,. \tag{10}$$

The dual field strength tensor can be written in terms of the electric and magnetic fields as

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}, \qquad \text{or equivalently} \quad \begin{aligned} \tilde{F}^{i0} &= -\tilde{F}^{0i} = B_i \quad \text{and} \\ \tilde{F}^{ij} &= +\varepsilon_{ijk} E_k\,. \end{aligned} \tag{11}$$

Note: $F^{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ are related by the substitution $E_i \mapsto B_i$ and $B_i \mapsto -E_i$.

*f) First, check that the definition of the dual field strength tensor $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ is consistent with Eq. (11). **+2pt(s)**

Then show that the manifestly covariant homogeneous Maxwell equations (10) are equivalent to their standard form (namely the magnetic Gauss law and the Maxwell-Faraday law).

Problem 7.3: The Electromagnetic Energy-Momentum Tensor

[**Oral** | 7 pt(s)]

ID: ex_em_energy_momentum_tensor:rt2324

Learning objective

In Problem 7.2 you studied the Lagrangian density of Maxwell theory and verified that its Euler-Lagrange equations yield the Maxwell equations. In this exercise, you study the energy-momentum tensor of the electromagnetic field, i.e., the conserved Noether current that follows from the spacetime translation invariance of the Maxwell Lagrangian.

You will show that to relate the energy-momentum tensor to the standard energy and momentum density of electrodynamics, one has to modify this *canonical* energy-momentum tensor to the so called *Belinfante-Rosenfeld energy-momentum tensor*. The latter will also play a role in general relativity, as it is the source of gravity in Einstein's field equations.

The Lagrangian density for the electromagnetic field is given by

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \tag{12}$$

where A_μ is the electromagnetic four-potential.

The Noether theorem for fields tells us, that for each continuous symmetry a of the Lagrangian density, there is a conserved current given by

$$j^\mu{}_a = \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial_\rho A_\lambda - \delta^\mu_\rho \mathcal{L} \right\} \delta_a x^\rho - \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \delta_a A_\lambda \,, \tag{13}$$

which satisfies the continuity equation

$$\partial_\mu j^\mu{}_a = 0 \,. \tag{14}$$

Here, $\delta_a x^\rho$ and $\delta_a A_\lambda$ are the variations due to the symmetry a of the spacetime coordinates and the four-potential, respectively.

a) Apply Noether's theorem Eq. (13) to the continuous symmetry of spacetime translations in the direction $a = \nu$ (argue why this is a symmetry of Eq. (12)). These are generated by $\delta_\nu x^\rho = \delta^\rho_\nu$ and $\delta_\nu A_\lambda = 0$ (because the four-potential is invariant under translations). 1pt(s)

Show that the corresponding Noether currents read

$$\theta^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - \eta^{\mu\nu} \mathcal{L} \,, \tag{15}$$

and evaluate the expression for the Maxwell Lagrangian Eq. (12).

The Noether currents Eq. (15) are called *canonical energy-momentum tensor*.

Notice that $\theta^{\mu\nu}$ is neither symmetric nor gauge invariant; this is unsatisfying for several reasons:

- **Gauge dependent** quantities cannot be directly identified with physical quantities. However, we expect that the energy-momentum tensor contains information about the energy density and the momentum density of the electromagnetic field [see part e)].

- A **non-symmetric** energy-momentum tensor cannot appear in an expression like $M^{\lambda\mu\nu} = T^{\lambda\mu}x^\nu - T^{\lambda\nu}x^\mu$ which parallels the angular momentum tensor $L^{\mu\nu} = p^\mu x^\nu - p^\nu x^\mu$ of classical point mechanics [see part b) and c)]. Furthermore, in general relativity we will find that the *Hilbert energy-momentum tensor* – which acts as the source of gravity – must be symmetric.

These issues can be amended because the conserved current $\theta^{\mu\nu}$ is not unique. Indeed, it is easy to verify (do this!) that adding the four-divergence of any tensor $K^{\lambda\mu\nu}$ with the property $K^{\lambda\mu\nu} = -K^{\mu\lambda\nu}$ produces a new energy-momentum tensor $T^{\mu\nu} := \theta^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$ that still satisfies Eq. (14).

In the following, you study a systematic approach to construct a tensor $K^{\lambda\mu\nu}$ such that $T^{\mu\nu}$ is symmetric and gauge-invariant (and therefore avoids the problems listed above). This special energy-momentum tensor is known as *Belinfante-Rosenfeld tensor* and was proven to be identical to the Hilbert energy-momentum tensor that shows up in general relativity as the source of gravity.

b) For this construction, we need the angular momentum tensor. Thus use Eq. (13) to calculate the conserved Noether currents ${L^\mu}_{\alpha\beta}$ (we refer to this quantity as *canonical* angular momentum tensor) associated to *homogeneous Lorentz transformations* (i.e. rotations and boosts). **1**pt(s)

   Show that the current can be written in the form

$$ {L^\mu}_{\alpha\beta} = \frac{1}{2}\left[{\theta^\mu}_\alpha x_\beta - {\theta^\mu}_\beta x_\alpha\right] + \frac{1}{2}{S^\mu}_{\alpha\beta} \tag{16}$$

   where ${\theta^\mu}_\alpha$ is the canonical energy-momentum tensor and ${S^\mu}_{\alpha\beta}$ has to be determined.

   Hint: Remember (Problem 5.3) that the generators of homogeneous Lorentz transformations are

$$ (J_{\alpha\beta})^{\mu\nu} = i\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta\right) . \tag{17}$$

   Since the four-potential A^μ is a contravariant Lorentz vector, it transforms just like the coordinates x^μ; i.e., their infinitesimal transformations are given by

$$ \delta_a x^\rho = -\frac{i}{2}(J_{\alpha\beta})^\rho{}_\kappa x^\kappa \quad \text{and} \quad \delta_a A_\lambda = -\frac{i}{2}(J_{\alpha\beta})_\lambda{}^\kappa A_\kappa \,, \tag{18}$$

   where we replaced the label a of the different generators by the more convenient labeling $\alpha\beta$.

The additional term ${S^\mu}_{\alpha\beta}$ is called *spin tensor* and is antisymmetric ${S^\mu}_{\alpha\beta} = -{S^\mu}_{\beta\alpha}$ in the last two indices. The continuity equation (14) requires $\partial_\mu {L^\mu}_{\alpha\beta} = 0$; combined with $\partial_\mu {\theta^\mu}_\nu = 0$ this implies (check this!)

$$ \partial_\mu {S^\mu}_{\alpha\beta} = \theta_{\alpha\beta} - \theta_{\beta\alpha} \,. \tag{19}$$

This means that a non-vanishing divergence of the spin tensor makes the canonical energy-momentum tensor $\theta_{\alpha\beta}$ non-symmetric.

We can use this insight to construct the symmetric Belinfante-Rosenfeld energy-momentum tensor

$$ T^{\mu\nu} := \theta^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} \quad \text{with} \quad K^{\lambda\mu\nu} := -\frac{1}{2}\left(S^{\mu\nu\lambda} + S^{\nu\mu\lambda} - S^{\lambda\nu\mu}\right) . \tag{20}$$

c) Use Eq. (19) to show that the energy-momentum tensor $T^{\mu\nu}$ is indeed symmetric and still satisfies the continuity equation $\partial_\mu T^{\mu\nu} = 0$. **2**pt(s)

d) Calculate the symmetric energy-momentum tensor $T^{\mu\nu}$ explicitly for the electromagnetic field. Use the covariant Maxwell equations ($\partial_\mu F^{\mu\nu} = 0$) to bring $T^{\mu\nu}$ in a gauge-invariant form. **1**pt(s)

e) Finally, show that the symmetric energy-momentum tensor yields the standard electromagnetic energy density $\mathcal{E}$ and the energy flux (Poynting vector) $\boldsymbol{S}$ as its components $T^{\mu 0}$. **2**pt(s)

Hint: Remember from Problem 7.2 that the field strength tensor can be expressed in terms of the electric and magnetic field as

$$F^{i0} = E_i \quad \text{and} \quad F^{ij} = -\varepsilon_{ijk} B_k \,. \tag{21}$$