Dr. Nicolai Lang
Institute for Theoretical Physics III, University of Stuttgart

December 20th, 2023
WS 2023/24

Problem 6.1: Relativistic rocket

[**Oral** | 11 (+5 bonus) pt(s)]

ID: ex_relativistic_rocket:rt2324

Learning objective

In this exercise, we investigate the motion of a rocket in a relativistic setting. To make the space-trip pleasant for the passengers, the rocket accelerates with a constant acceleration in its instantaneous rest frame, such that the passengers experience a constant gravitation-like force. Among other things, you investigate if it is possible to explore the universe with this rocket in a reasonable amount of time.

Recall the definition of the *4-velocity* u^μ and the *4-acceleration* b^μ of the rocket's trajectory $x^\mu(\tau)$ (given in the coordinates x^μ of some inertial system K in which earth is at rest):

$$u^\mu = \frac{dx^\mu}{d\tau} \quad \text{and} \quad b^\mu = \frac{du^\mu}{d\tau} \,. \tag{1}$$

Here, τ denotes the *proper time* of the rocket.

a) Calculate the 4-velocity u^μ and the 4-acceleration b^μ in terms of the *coordinate velocity* $\boldsymbol{v} = \frac{d\boldsymbol{x}}{dt}$ and the coordinate acceleration $\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt}$ (measured in K). **3**pt(s)

Use your results to prove the following relations:

$$u^\mu u_\mu = c^2 \quad \text{and} \quad b^\mu u_\mu = 0 \,. \tag{2}$$

Finally, argue that the 4-acceleration b^μ can be written at any point in time in the instantaneous rest frame K_0 of the rocket as

$$b_0^\mu = \begin{pmatrix} 0 \\ \boldsymbol{a}_0 \end{pmatrix} , \tag{3}$$

where $\boldsymbol{a}_0$ is the *proper acceleration* of the rocket as measured in K_0.

In what follows we are only interested in motion and acceleration in x-direction. Therefore, we consider a $1+1$ dimensional spacetime with coordinates $x^\mu = (ct, x)$ henceforth, and the coordinate vectors $\boldsymbol{v}$, $\boldsymbol{a}$, $\boldsymbol{a}_0$ can be replaced by numbers v, a, a_0.

b) Calculate $b^2 = b_\mu b^\mu$ in the rest frame of earth K and in the instantaneous rest frame K_0 of the rocket. Use this to show that the proper acceleration a_0 is related to the coordinate acceleration a by **2**pt(s)

$$a = \frac{a_0}{\gamma^3} \,. \tag{4}$$

We now consider the situation where the rocket starts from earth at time $t = 0$ (measured in earths rest frame K) and proper time $\tau = 0$ (measured by a clock in the rocket). The rocket is accelerated with a constant proper acceleration a_0 in x-direction (so we can use $1+1$ dimensional spacetime).

c) Use Eq. (4) to calculate the velocity of the rocket $v(t)$ as a function of time t (measured in K, i.e., as observed from earth). **4**pt(s)

Use this result to calculate the distance $x(t)$ the rocket has traveled in K, the Lorentz factor $\gamma(t) \equiv \gamma_{v(t)}$, and the proper time $\tau(t)$ elapsed on the rocket.

Hint: You might find the following integrals useful:

$$\int dx \, \frac{1}{(1-x^2)^{3/2}} = \frac{x}{\sqrt{1-x^2}} \tag{5a}$$

$$\int dx \, \frac{x}{\sqrt{1+x^2}} = \sqrt{1+x^2} \tag{5b}$$

$$\int dx \, \frac{1}{\sqrt{1+x^2}} = \operatorname{arsinh}(x) \tag{5c}$$

d) Express your previous results in terms of the traveled distance $d = x(t)$, i.e., calculate $t(d)$, $v(d)$, $\tau(d)$ and $\gamma(d)$. **2**pt(s)

For interstellar travels, the usual units of time (seconds) and length (meters) are not convenient. Instead, we use the unit *year* yr for time and *light-year* ly for length. In these units, the speed of light is $c = 1\,\mathrm{ly/yr}$. Interestingly, the average gravitational acceleration on earth in these units is $g \approx 1.03\,\mathrm{ly/yr^2}$. To make the flight as pleasant as possible for the passengers, we choose a proper acceleration of $a_0 = 1\,\mathrm{ly/yr^2}$ close to earth's gravitational acceleration g.

Now calculate the (earth) time $t(D)$, the (rocket) proper time $\tau(D)$, the final velocity $v(D)$, and the Lorentz factor $\gamma(D)$ for a flight to the following destinations:

- Nearest star *Proxima Centauri*: $D = 4.24\,\mathrm{ly}$.
- Nearest (known) black hole *Gaia BH1*: $D = 1560\,\mathrm{ly}$.
- Center of our galaxy *Sagittarius A**: $D = 27\,000\,\mathrm{ly}$.
- Next nearest galaxy *Andromeda*: $D = 2.5 \times 10^6\,\mathrm{ly}$.

*e) In the current scenario we pass our destinations almost with the speed of light. To arrive with a velocity close to zero, we reverse the acceleration of the rocket at the halfway point $x = D/2$, so that we come to a halt when we reach the destination. **+3**pt(s)

Find expressions for the new total time $t'(D)$ and the new proper time $\tau'(D)$ elapsed along the full trip.

Calculate the new times $t'(D)$ and $\tau'(D)$ for the same destinations as above. How do $t'(D)$ and $\tau'(D)$ behave for large distances $D \gg c^2/a_0$ compared to the original times $t(D)$ and $\tau(D)$?

Hint: To approximate τ for large distances D, use the approximation

$$\operatorname{arsinh}(x) \overset{x\gg 1}{\approx} \ln(x) + \ln(2) + \mathcal{O}(1/x^2)\,. \tag{6}$$

*f) We now decide to never reverse or stop the constant proper acceleration of the rocket. **+2**pt(s)

Plot the distance $\Delta(\tau)$ at which an observer in the rocket sees a star with a starting distance of $\Delta(\tau = 0) = D$. Describe and interpret your result for $\tau \to \infty$.

Now consider a planet with initial distance $D < 0$ that emits a light signal at $\tau = 0$. After which proper time τ does an observer in the rocket receive the light signal?

Interpret your combined results.

Problem 6.2: Noether's theorem for the inhomogeneous Lorentz group [**Oral** | 7 pt(s)]

ID: ex_noether_srt:rt2324

> **Learning objective**
>
> In classical mechanics you have learned that to each continuous symmetry of the Lagrangian corresponds a conserved quantity; a mathematical fact known as *Noether's (first) theorem*. We already learned that the inhomogeneous Lorentz group (Poincaré group) is parametrized by 10 continuous parameters (3 boosts, 3 rotations, and 4 translations in spacetime). The goal of this exercise is to derive the corresponding conserved quantities ("Noether charges"); these quantities are conserved by every relativistic Lagrangian (which are the fundamental theories we are interested in).

We consider the trajectory γ of a relativistic particle, parametrized in some inertial system by $x^\mu(\lambda)$ with parameter $\lambda \in [\lambda_a, \lambda_b]$. Let the dynamics be given by an action

$$S[\gamma] = \int_{\lambda_a}^{\lambda_b} d\lambda\, L(x^\mu(\lambda), \dot{x}^\mu(\lambda))\,, \tag{7}$$

with $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$ and a Lagrangian $L(x^\mu, \dot{x}^\mu)$.

a) Start by calculating the variation of the action $\delta S = S[\gamma'] - S[\gamma]$ under a generic infinitesimal variation of the trajectory $\gamma \to \gamma'$. This variation of the trajectory (in some parametrization λ) is given by **2**pt(s)

$$x^\mu(\lambda) \to x'^\mu(\lambda) = x^\mu(\lambda) + w_\xi(\lambda)\delta_\xi x^\mu(x)\,, \tag{8}$$

where w_ξ are infinitesimal parameters that quantify the continuous transformation and $\delta_\xi x^\mu(x)$ are the generators of the transformations, that can depend on x. In general, the parameters w_ξ can depend on the parametrization λ.

Note: The subscript ξ indexes the different generators of the continuous transformations, and Einstein summation is implied for double occurrences of ξ.

Expand your result up to first order in w_ξ and $\frac{\partial w_\xi}{\partial \lambda}$ to bring it into the following form:

$$\delta S = S[\gamma'] - S[\gamma] = \int_{\lambda_a}^{\lambda_b} d\lambda\, w_\xi(\lambda) M_\xi(x^\mu, \dot{x}^\mu) + \frac{\partial w_\xi}{\partial \lambda}(\lambda) Q_\xi(x^\mu, \dot{x}^\mu)\,. \tag{9}$$

b) We now consider only transformations $x^\mu(\lambda) \to x'^\mu(\lambda) = x^\mu(\lambda) + w_a(\lambda)\delta_a x^\mu(x)$ that are generated by symmetries of the system (denoted by the index a instead of a generic transformation ξ). This means that for *rigid transformations* (i.e. $\frac{\partial w_a}{\partial \lambda} = 0$), the action is invariant $\delta S = 0$ for any arbitrary path γ. **1**pt(s)

Use this to argue that for symmetry transformations all terms proportional to w_a must vanish

$$M_a(x^\mu, \dot{x}^\mu) = 0\,. \tag{10}$$

Therefore, the variation of the action for a symmetry transformation $\delta_a x^\mu(x)$ (now again with trajectory-dependent parameters $w_a(\lambda)$) can be written as

$$\delta S = \int_{\lambda_a}^{\lambda_b} d\lambda \, \frac{\partial w_a}{\partial \lambda} Q_a(x^\mu, \dot{x}^\mu) \,, \tag{11}$$

with the *Noether charge*

$$Q_a = \frac{\partial L}{\partial \dot{x}^\mu} \delta_a x^\mu(x) \,. \tag{12}$$

c) Show that Q_a is conserved along a trajectory that fulfills the equations of motion. **1**pt(s)

We now apply Noether's theorem to the continuous transformations of the inhomogeneous Lorentz group, i.e., we want to find the conserved quantities Q_a associated to boosts, rotations, and spacetime translations.

To this end, we define the canonical momentum p_μ as

$$p_\mu = -\frac{\partial L}{\partial \dot{x}^\mu} \,. \tag{13}$$

Note: The minus is conventional.

d) Calculate the conserved quantities for the *inhomogeneous* part of the Lorentz group, namely for spacetime translations $x'^\mu = x^\mu + a^\mu$, with $a^\mu = (w_0, w_1, w_2, w_3)^T$. **1**pt(s)

e) Now calculate the conserved quantities associated to *homogeneous* Lorentz transformations $\Lambda^\mu{}_\nu$. Interpret the result for spatial rotations and boosts. **2**pt(s)

Hint: Remember [Problem 5.3] that the generators of homogeneous Lorentz transformations parametrized by $w_{(\alpha\beta)}$ can be written as

$$\delta_{(\alpha\beta)} x^\mu = -\frac{i}{2} (J_{\alpha\beta})^\mu{}_\nu x^\nu = \frac{1}{2} \left(\delta^\mu_\alpha \eta_{\beta\nu} - \eta_{\alpha\nu} \delta^\mu_\beta \right) x^\nu \,. \tag{14}$$

Problem 6.3: Relativistic Scattering and the Compton Effect [**Written** | 5 (+1 bonus) pt(s)]

ID: ex_relativistic_scattering:rt2324

Learning objective

Interactions between particles are fundamental to physics. In relativistic theories with translation invariance in space and time, such interactions are constrained by 4-momentum conservation (i.e., energy and momentum conservation). With only this constraint, we can already learn much about the possible result of a scattering process. An example of this is the *Compton effect*, where the change of wavelength of a photon that scatters of a charged particle (like an electron) depends only on its scattering angle. Here you derive this dependency by using the conservation of energy and momentum.

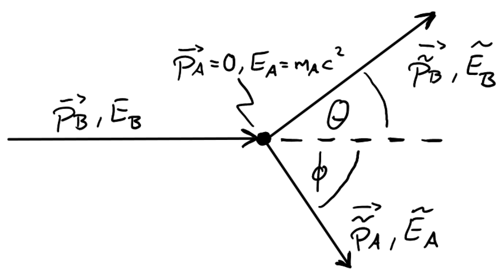
We start with a generic, elastic scattering process and specialize to the case where one of the particles is massless (e.g., a photon) later.

Consider two particles A and B of masses m_A and m_B (m_B can be possibly zero) described by their 4-momenta

$$p_A^\mu = \begin{pmatrix} E_A/c \\ \boldsymbol{p}_A \end{pmatrix}, \quad p_B^\mu = \begin{pmatrix} E_B/c \\ \boldsymbol{p}_B \end{pmatrix}. \tag{15}$$

The two particles scatter elastically of each other (via some unspecified interaction), and afterwards have new 4-momenta $\tilde{p}_A^\mu$ and $\tilde{p}_B^\mu$. We want to deduce the 4-momenta after scattering from the 4-momenta before the scattering.

In principle this can be done in an arbitrary inertial system. For simplicity, we will do our calculations in the rest frame K_A of particle A (see sketch on the right); this requires that A is a massive particle ($m_A \neq 0$).



To describe the scattering process completely, we would need to find *four* quantities $\tilde{E}_A, \tilde{E}_B, \theta, \phi$ in terms of p_A^μ and p_B^μ. However, because we know neither the exact initial positions of the particles nor the interactions between them, we cannot describe the scattering process completely. As a consequence, we can only obtain *three* of the quantities, in our case $\tilde{E}_A, \theta, \phi$, in terms of $\tilde{E}_B$, p_A^μ and p_B^μ when relying on energy and momentum conservation only (i.e., we cannot predict $\tilde{E}_B$).

a) Write down the relativistic energy and momentum conservation (in 4-vectors). Express the energy $\tilde{E}_A$ in terms of $\tilde{E}_B$, E_B and m_A. **1**pt(s)

b) Derive an expression for $\cos(\theta)$ in terms of $\tilde{E}_B$, E_B, m_A and m_B. **2**pt(s)

   Hint: Begin with $\tilde{p}_{A\mu}\tilde{p}_A^\mu$ and 4-momentum conservation.

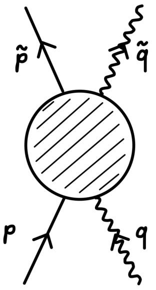
c) Find an expression for $\cos(\phi)$ in terms of $\tilde{E}_B$, E_B, m_A and m_B. **1**pt(s)

   Hint: Consider $\boldsymbol{p}_B \cdot \tilde{\boldsymbol{p}}_A$ and use momentum conservation.

We consider now the special case where particle B is a photon and therefore massless. Its 4-momentum has then the form

$$p_B^\mu = q^\mu = \begin{pmatrix} \hbar\omega/c \\ \hbar\boldsymbol{k} \end{pmatrix} \tag{16}$$

with energy $E = \hbar\omega$ and 3-momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$ (with wave vector $\boldsymbol{k}$). $p_B^2 = 0$ is then equivalent to the linear dispersion relation $\omega = c|\boldsymbol{k}|$ of light in vacuum. This is the situation of the Compton Effect (see sketch on the right).



Note: We cannot choose particle A as the photon because we have worked in the rest frame K_A.

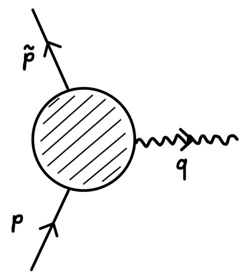
d) Use your result from part b) to show that the shift in the wavelength of the photon during this scattering process is given by 1pt(s)

$$\Delta\lambda = \tilde{\lambda} - \lambda = \frac{h}{m_A c}\left(1 - \cos\theta\right)\,, \tag{17}$$

i.e., the scattered photon has a different wavelength that only depends on its scattering angle θ (and the mass of the particle it scatters off).

Note: The quantity $\lambda_A \equiv \frac{h}{m_A c}$ is known as the *Compton wavelength* of the massive particle A.

While the Compton effect involves two photons, we could imagine a scattering process where only a single photon is emitted (or absorbed) by the massive particle (see sketch on the right).



*e) Prove, again using energy and momentum conservation, that such a process is impossible if the rest mass of the massive particle is conserved. +1pt(s)

Hint: Use 4-momentum conservation and show that $p^\mu - \tilde{p}^\mu$ is a space-like 4-vector.

Note: This means that an electron cannot emit (or absorb) a single photon.