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Problem 5.1: Hafele-Keating experiment

[**Oral** | 4 pt(s)]

ID: ex_hafele_keating_experiment:rt2324

Learning objective

Time dilation is a relativistic effect far removed from our everyday experience. It is, however, an experimentally established fact. A famous experiment measuring time dilation explicitly was the *Hafele-Keating experiment*, where portable atomic clocks were flown on commercial airliners around the world twice: once eastward and once westward. The clocks were then compared to stationary reference clocks on the ground to verify the predictions of time dilation quantitatively.

The experimental results were reported in

https://download.itp3.uni-stuttgart.de/rt2324/Hafele_Keating-Experiment.pdf

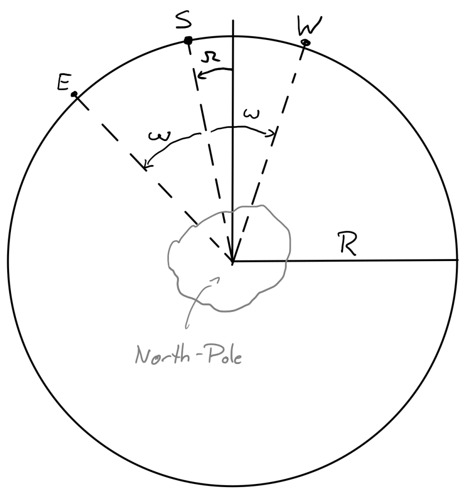
and the theory was developed in

https://download.itp3.uni-stuttgart.de/rt2324/Hafele_Keating-Theory.pdf

In this exercise, you derive the contribution of time dilation to explain the result of the experiment (which you can find in the theory paper above).

We consider the following setup:

Imagine an observer in a space station above the north pole; the space station follows earth on its orbit around the sun, but does not follow the rotation of earth (i.e., the observer sees earth slowly rotating beneath the space station). Such an observer is approximately inertial for the relevant timescales of the experiment (see sketch on the right); in particular, he is allowed to use the formalism developed in the lecture to compute the proper time along (potentially accelerated) trajectories.



Now consider three identical atomic clocks located on the equator with radius $R \approx 6.4 \times 10^6$ m. The first clock (labeled S) is stationary with respect to earth, this is our "reference clock." The second clock (E) flies eastward around earth with angular velocity ω (with respect to earth), and the third (W) westward with the same angular velocity ω (also with respect to earth). Both clocks go around earth once and meet again with the reference clock. Note that the rest system of the reference clock S is *not* inertial because earth rotates with angular velocity $\Omega = \frac{2\pi}{24\,\mathrm{h}}$.

a) Parametrize the trajectories of the three clocks $\boldsymbol{x}_S(t)$, $\boldsymbol{x}_E(t)$ and $\boldsymbol{x}_W(t)$ in the inertial system of the space station. 1pt(s)

b) The proper time τ accumulated by a clock can be calculated as shown in the lecture: **1**pt(s)

$$\tau_i = \int \sqrt{1 - \frac{\dot{\boldsymbol{x}}_i^{\,2}}{c^2}}\,\mathrm{d}t \qquad \text{for} \quad i \in \{S, E, W\}\,. \tag{1}$$

Evaluate this integral for the three clocks.

c) Calculate $\Delta\tau_E = \tau_E - \tau_S$ and $\Delta\tau_W = \tau_W - \tau_S$. **1**pt(s)

Since the angular velocities are small, expand the results in $\frac{R\Omega}{c}$ up to second order.

d) Assume the clocks travel around earth with $200\,\mathrm{m\,s^{-1}}$ relative to the ground (the speed of a typical airliner). **1**pt(s)

What are the time differences measured? Compare them to the numbers reported in the original publications.

Note: For a complete description of the experiment, effects of *general relativity* must be taken into account as well. Thus we will complete our analysis of the Hafele-Keating experiment in the next semester.

Problem 5.2: Christoffel symbols and the covariant derivative [**Oral** | 7 pt(s)]

ID: ex_christoffel_symbols:rt2324

Learning objective

This exercise is a sequel of Problem 4.2. Here you study the Christoffel symbols – induced by the metric tensor – which are needed to define the covariant derivative.

In particular, you use a simple example in flat (Euclidean) space to get familiar with the covariant derivative. You will explicitly see that the “normal” partial derivative of a vector field does not transform as a tensor whereas the covariant derivative does.

We consider the same setting as in Problem 4.2:

Our manifold is the two-dimensional Euclidean plane $M = \mathbb{R}^2$ and we consider two charts: Cartesian coordinates (x^1, x^2) and polar coordinates $(\bar{x}^1 = r, \bar{x}^2 = \theta)$. The components of the metric tensor in these charts is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \bar{g}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}\,. \tag{2}$$

We now want to study the Christoffel symbols induced by this metric.

a) As a first step, calculate the inverse metric tensor g^{ij} by using the fact that $g^{ij}g_{jk} = \delta^i_k$. **2**pt(s)

Then compute the Christoffel symbols

$$\Gamma^i{}_{kl} = \frac{1}{2} g^{im}\left(g_{mk,l} + g_{lm,k} - g_{kl,m}\right)\,. \tag{3}$$

Do this for both coordinate systems.

We see that even though the metric describes flat Euclidean space $\mathbb{R}^2$, not all Christoffel symbols vanish – depending on the coordinates we use to express the metric tensor.

We now want to get more familiar with the Christoffel symbols and the covariant derivative. To this end, we consider the vector field A with contravariant components A^i:

$$A := \partial_r = \bar{A}^i \partial_{\bar{x}^i} \quad \text{with} \quad \bar{A}^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{4}$$

b) Use the transformation law for tensors to calculate A^i (the components of A in Cartesian coordinates). Draw or plot the vector field in both coordinate systems. **1**pt(s)

c) Now compute the partial derivative $A^i{}_{,j}$ and the covariant derivative $A^i{}_{;j}$ of the vector field in both coordinate systems. **2**pt(s)

d) Explicitly check that the covariant derivative transforms like a $(1,1)$-tensor, i.e., show that **2**pt(s)

$$A^i{}_{;j} = \frac{\partial x^i}{\partial \bar{x}^k} \frac{\partial \bar{x}^l}{\partial x^j} \bar{A}^k{}_{;l} \tag{5}$$

holds. Demonstrate that this fails for the partial derivative.

Problem 5.3: The Lorentz group

[**Written** | 7 (+1 bonus) pt(s)]

ID: ex_lorentz_group:rt2324

Learning objective

The goal of this exercise is to get familiar with the Lorentz group and its continuous and discrete generators. Since the Lorentz group is a Lie group, we can study its Lie algebra. Knowing the Lie algebra of a group is very helpful to construct other representations of the group. This will become important in the context of relativistic quantum mechanics where we have to replace the Schrödinger equation by the Dirac equation.

The Lorentz group $O(1,3)$ is defined as group of real 4×4 matrices Λ that keep the Minkowski metric $\eta = \text{diag}(1,-1,-1,-1)$ invariant, i.e.

$$O(1,3) = \{\Lambda \in \mathbb{R}^{4\times 4} \,|\, \Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \iff \Lambda^T \eta \Lambda = \eta\} . \tag{6}$$

a) Show that $O(1,3)$ is a group [use only the definition in Eq. (6)]. **2**pt(s)

Specifically, show the following properties for any $\Lambda_1, \Lambda_2 \in O(1,3)$:

(i) $\Lambda_1 \Lambda_2 \in O(1,3)$

(ii) The inverse Λ_1^{-1} exists and is in $O(1,3)$.

Since the Lorentz group is a *Lie group* (i.e., a group that is also a differentiable manifold), we can study its *Lie algebra* (a vector space with an additional multiplication known as *Lie bracket*).

To this end, we consider an infinitesimal Lorentz transformation

$$\Lambda_{\boldsymbol{\xi}} = \exp\left(-i\xi_i X_i\right) \overset{\xi_i \ll 1}{\approx} \mathbb{1} - i\xi_i X_i , \tag{7}$$

where the matrices $X_i \in \mathbb{R}^{4\times 4}$ belong to the Lie algebra and are the generators of the Lorentz group; the $\xi_i \in \mathbb{R}$ are the corresponding coefficients that define the group element $\Lambda_{\boldsymbol{\xi}}$ in terms of these generators.

b) How does the defining condition of the Lorentz group $\Lambda^T \eta \Lambda = \eta$ translate to the generators X_i? **2pt(s)**

Show that we can write the generators X_i in the general form

$$(X_i)^\rho{}_\sigma = \begin{pmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & -d & 0 & f \\ c & -e & -f & 0 \end{pmatrix} , \tag{8}$$

with only 6 degrees of freedom left.

Since the Lie algebra is a vector space, we can choose a set of basis vectors (= generators) and write every element of the algebra as a linear combination of these.

A conventional basis of the Lie algebra of the Lorentz group is

$$(\mathcal{J}^{\mu\nu})^\rho{}_\sigma = i \left(\eta^{\mu\rho} \delta^\nu{}_\sigma - \delta^\mu{}_\sigma \eta^{\nu\rho} \right) , \tag{9}$$

with

$$\Lambda_{\boldsymbol{\omega}} = \exp\left(-\frac{i}{2} \omega_{\mu\nu} \mathcal{J}^{\mu\nu} \right) , \tag{10}$$

where both, the set of coefficients $\omega_{\mu\nu}$ and the set of generators $\mathcal{J}^{\mu\nu}$ are antisymmetric in their indices.

Note: For every pair μ, ν, the object $\mathcal{J}^{\mu\nu}$ is a real 4×4 matrix, given by Eq. (9).

Convince yourself that this basis can be used to construct all matrices of the form (8) by setting one of the parameters to one and all others to zero. Therefore, the antisymmetric coefficient tensor $\omega_{\mu\nu}$ takes the place of ξ_i and encodes 6 degrees of freedom.

*c) Show that the commutator (Lie bracket) of the generators is **+1pt(s)**

$$([\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}])^\rho{}_\sigma = i \left(\eta^{\nu\alpha} \mathcal{J}^{\mu\beta} - \eta^{\nu\beta} \mathcal{J}^{\mu\alpha} - \eta^{\mu\alpha} \mathcal{J}^{\nu\beta} + \eta^{\mu\beta} \mathcal{J}^{\nu\alpha} \right)^\rho{}_\sigma . \tag{11}$$

This is the *Lie algebra* of the Lorentz group; it has to be the same for all representations of the Lorentz group.

d) Calculate the determinant of Λ first from the definition in Eq. (6), and then from the expression in Eq. (7) using the form of the generators. **1pt(s)**

Can the exponential form $\Lambda_{\boldsymbol{\xi}}$ (equivalently: $\Lambda_{\boldsymbol{\omega}}$) be used to generate the complete group $O(1,3)$?

e) Convince yourself that time reversal T and space inversion P are part of the Lorentz group (6) but *cannot* be continuously connected to the identity $\Lambda_{\boldsymbol{\xi}=\mathbf{0}} = \mathbb{1}$: **2pt(s)**

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{12}$$

Finally, prove that T and P cannot be continuously connected to each other as well.

Interpret your result by sketching the structure of the Lorentz group.

Problem 5.4: Covariant form of the Lagrangian of classical electrodynamics [**Oral** | 3 pt(s)]

ID: ex_covariant_em_lagrangian:rt2324

Learning objective

Maxwell's electrodynamics is the prototypical example of a *relativistic field theory*. In this exercise you use tensor calculus [recall Problem 4.1], specialized to the inertial coordinate systems of special relativity, to show that the Lagrangian and the action of classical electrodynamics are invariant under Lorentz transformations.

Here we consider Minkowski space $M = \mathbb{R}^{1,3} \simeq \mathbb{R}^4$ as spacetime manifold and focus on inertial coordinate systems and the transformations between them (Lorentz transformations). The metric tensor is the Minkowski metric with components $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ in any inertial coordinate system. In the following, Greek indices run from 0 to 3.

The action of classical electrodynamics in vacuum (with speed of light $c = 1$) is given in some inertial coordinate system by

$$S = \int d^4x\, \mathcal{L} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \tag{13}$$

with the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\pi} F_{\rho\pi}$ (where $\eta^{\mu\nu}$ denotes the inverse of the metric $\eta_{\mu\nu}$).

Here, A^μ is the gauge potential of electrodynamics; however, for this exercise you only need to know that A^μ transforms like a contravariant vector field (a $(1,0)$ tensor) and $A_\mu = \eta_{\mu\nu} A^\nu$ like a covariant vector field (a $(0,1)$ tensor).

a) Show that $F^{\mu\nu}$ is a contravariant tensor field of rank 2, and that the Lagrangian (density) $\mathcal{L}$ is a Lorentz scalar. **2**pt(s)

   Hint: Use the results from Problem 4.1 and your knowledge about Lorentz transformations (which are the only coordinate transformations considered in special relativity).

b) Show that Eq. (13) is invariant under Lorentz transformations, i.e., show that **1**pt(s)

$$S = \int d^4x\, \mathcal{L} = \int d^4\bar{x}\, \bar{\mathcal{L}} = \bar{S}\,. \tag{14}$$

In summary, you have shown that the Lagrangian (density) $\mathcal{L}$ and the action given in Eq. (13) are Lorentz scalars. To prove that classical electrodynamics (the Maxwell equations) is Lorentz covariant (= the equations take the same form in all inertial systems), we still need to show that the Euler-Lagrange equations that follow from varying the action in Eq. (13) produce the Maxwell equations in their conventional form (⟶ later).