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Problem 4.1: Tensor Calculus

[**Written** | 9 pt(s)]

ID: ex_tensor_calculus:rt2324

> **Learning objective**
>
> Tensor calculus is a crucial toolkit for special and general relativity. In this exercise, you practice calculating with tensor fields and prove some useful rules for the construction of tensor fields.

Consider a D-dimensional differentiable manifold M and an arbitrary coordinate transformation $\bar{x} = \varphi(x)$ from one chart with coordinates $x \in \mathbb{R}^D$ to another chart with coordinates $\bar{x} \in \mathbb{R}^D$.

As motivated in the lecture, we define the transformation of *contravariant* and *covariant* vector (fields) as follows:

$$\textit{Contravariant}\text{ vector field:}\quad \bar{A}^i(\bar{x}) = \sum_{k=1}^{D} \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \equiv \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \tag{1}$$

$$\textit{Covariant}\text{ vector field:}\quad \bar{B}_i(\bar{x}) = \sum_{k=1}^{D} \frac{\partial x^k}{\partial \bar{x}^i} B_k(x) \equiv \frac{\partial x^k}{\partial \bar{x}^i} B_k(x)\,, \tag{2}$$

Here we use the *Einstein sum convention*: Sums over pairs of repeated up and down indices are implied but not explicitly written.

a) Prove that the *contraction* $\Phi(x) := A^i(x)B_i(x)$ of a contravariant vector field $A^i(x)$ with a covariant vector field $B_i(x)$ is invariant under coordinate transformations; i.e., show that it transforms like a *scalar field*. **1** pt(s)

The generalization of co- and contravariant vector fields are (mixed) (p,q) *tensor fields* $T^{m_1,\dots,m_p}{}_{n_1,\dots,n_q}(x)$ with $r = p+q$ indices (called *rank*). Like vector fields, tensor fields are defined by their transformation under coordinate transformations:

$$\bar{T}^{i_1,\dots,i_p}{}_{j_1,\dots,j_q}(\bar{x}) = \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \cdots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}} \quad \frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \cdots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}} \quad T^{m_1,\dots,m_p}{}_{n_1,\dots,n_q}(x)\,, \tag{3}$$

where p and q are the number of contravariant and covariant indices, respectively.

b) Show that the following combinations of the tensor fields $A^{ij}{}_k$, $B^{ij}{}_k$, C^{ij}, $D^k{}_l$, E_m and the scalar field Φ are again tensor fields (we suppress the x-dependency): **3** pt(s)

   i) $V^{ij}{}_k := A^{ij}{}_k + B^{ij}{}_k$ ii) $W^{ij}{}_k := \Phi A^{ij}{}_k$ iii) $X^{ijk}{}_l := C^{ij} D^k{}_l$

   iv) $Y^i := A^{ij} E_j$ v) $Z^i := A^{ij}{}_j$

c) Let C_{ij} be a collection of D^2 fields $(i,j = 1,\dots,D)$. **1** pt(s)

   Prove that *if* $B_i := C_{ij}A^j$ is a covariant vector field for *any* contravariant vector field A^i, *then* C_{ij} transforms like a covariant tensor field of rank 2.

   Note: This theorem is called *quotient law*, a quite useful tool in tensor calculus.

d) Show that the covariant derivative ${A^i}_{;k}$ of a contravariant vector field A^i transforms as a mixed $(1,1)$ tensor field. **3**pt(s)

The *covariant derivative* of a contravariant vector field is defined as

$$ {A^i}_{;k} := \partial_k A^i + {\Gamma^i}_{kl} A^l \,, \tag{4} $$

with $\partial_k \equiv \frac{\partial}{\partial x^k}$ and where the *Christoffel symbol* ${\Gamma^i}_{kl}$ is defined as

$$ {\Gamma^i}_{kl} := \frac{1}{2} g^{im} \left(\partial_l g_{mk} + \partial_k g_{ml} - \partial_m g_{kl} \right) \,. \tag{5} $$

$g_{ij} = g_{ij}(x)$ is a given, symmetric ($g_{ij} = g_{ji}$) covariant tensor field of rank 2 called the *metric*.

Hint: First, prove the translation law for the Christoffel symbol

$$ {\bar{\Gamma}^i}_{kl}(\bar{x}) = \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l} {\Gamma^j}_{mn}(x) + \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} \,, \tag{6} $$

and show that

$$ \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial^2 x^m}{\partial \bar{x}^k \partial \bar{x}^l} = - \frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^n} \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^n}{\partial \bar{x}^l} \,. \tag{7} $$

Use this to derive the transformation law for the covariant derivative.

You might want to use the shortcut notations $\alpha^i_k := \frac{\partial \bar{x}^i}{\partial x^k}$ and $\beta^i_k := \frac{\partial x^i}{\partial \bar{x}^k}$ with $\alpha^i_k \beta^k_j = \delta^i_j$.

e) In the lecture, the general *Levi-Civita symbol* $\epsilon^{i_1 \dots i_D}$ was introduced. Here we want to focus on the most important case of a $D = 4$ dimensional manifold. **1**pt(s)

The Levi-Civita symbol is defined (independent of the coordinate system) as

$$ \epsilon^{ijkl} := \begin{cases} +1 & \text{if } (i,j,k,l) \text{ is an even permutation of } (0,1,2,3) \\ -1 & \text{if } (i,j,k,l) \text{ is an odd permutation of } (0,1,2,3) \\ 0 & \text{otherwise} \end{cases} \,. \tag{8} $$

Show that this definition is consistent with the transformation law for a *relative tensor* of rank 4 with weight $w = +1$ (which we call a *tensor density*), i.e., show that

$$ \epsilon^{ijkl} = \left| \frac{\partial x}{\partial \bar{x}} \right| \frac{\partial \bar{x}^i}{\partial x^a} \frac{\partial \bar{x}^j}{\partial x^b} \frac{\partial \bar{x}^k}{\partial x^c} \frac{\partial \bar{x}^l}{\partial x^d} \epsilon^{abcd} \quad \text{with \textit{Jacobian determinant}} \left| \frac{\partial \bar{x}}{\partial x} \right| . \tag{9} $$

Hint: Use that determinants can be calculated via the Leibniz formula using the Levi-Civita symbol:

$$ \left| \frac{\partial \bar{x}}{\partial x} \right| = \epsilon^{ijkl} \frac{\partial \bar{x}^0}{\partial x^i} \frac{\partial \bar{x}^1}{\partial x^j} \frac{\partial \bar{x}^2}{\partial x^k} \frac{\partial \bar{x}^3}{\partial x^l} \,. \tag{10} $$

Problem 4.2: The metric tensor

[**Oral** | 5 pt(s)]

ID: ex_srt_metric_coordinate_transformation:rt2324

Learning objective

The metric tensor (field) is crucial for Riemannian geometry, i.e., the mathematical framework needed to described curved spaces. Both in special relativity and in general relativity, the metric tensor on the spacetime manifold determines durations (measured by clocks) and lengths (measured by rods) in spacetime.

In this exercise, you show that the components of the metric tensor indeed transform like a covariant tensor field of rank 2. To familiarize yourself with the concept, you study the simple example of Euclidean space (no time!) in two dimensions $\mathbb{R}^2$ and calculate the components of the metric tensor in polar coordinates.

Consider a D-dimensional differentiable manifold M and an arbitrary coordinate transformation $\bar{x} = \varphi(x)$ from one chart with coordinates $x \in \mathbb{R}^D$ to another chart with coordinates $\bar{x} \in \mathbb{R}^D$.

a) We start by showing that the components $g_{ij}(x)$ of the metric transform like a covariant tensor field of rank 2. To this end, use that the "line element" ds^2 is a tensor field and hence does not depend on the coordinate system: **1**pt(s)

$$g_{ij}(x)dx^i dx^j = ds^2 = \bar{g}_{ij}(\bar{x})d\bar{x}^i d\bar{x}^j \quad \text{(Einstein summation!)} \tag{11}$$

Hint: Compute the total differential $d\bar{x}^i$ and use that $\{dx^i\}$ is a basis of the cotangent space T_p^*M. The transformation law for a covariant tensor of rank 2 is given in Problem 4.1.

As an example, we consider Euclidean space $M = \mathbb{R}^2$ in $D = 2$ dimensions for the rest of this exercise. In Cartesian coordinates $x^1 = x$ and $x^2 = y$, the components of the metric tensor are

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{with line element} \quad ds^2 = g_{ij}(x)dx^i dx^j = dx^2 + dy^2 \,. \tag{12}$$

This particular metric tensor ds^2 characterizes the flat, Euclidean plane you already encountered in school.

b) We want to calculate the components of the metric tensor in polar coordinates $\bar{x}^1 = r$ and $\bar{x}^2 = \theta$. The coordinate transformation $\bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x})$ between Cartesian and polar coordinates is given by **2**pt(s)

$$\varphi^{-1} : \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} . \tag{13}$$

First, use Eq. (11) and the rule for computing the total differentials dx and dy to compute $\bar{g}_{ij}(\bar{x})$.

Then, derive the same components directly by using the transformation law for a covariant tensor of rank 2.

The physical length of a curve γ can be calculated via the metric tensor by

$$L[\gamma] = \int_\gamma ds := \int_a^b dt\,\sqrt{g_{ij}(\boldsymbol{\gamma}(t))\,\dot{\gamma}^i(t)\dot{\gamma}^j(t)}\,, \tag{14}$$

where $\boldsymbol{\gamma}(t)$ is the parametrization of the curve with $t \in [a, b]$ in some coordinate system.

c) Let γ be the circle in $\mathbb{R}^2$ with radius R and center in the origin. **2**pt(s)

Calculate the circumference $L[\gamma]$ of this circle once in Cartesian coordinates and then again in polar coordinates. Use the components of the metric tensor given above and computed in subtask b).

Problem 4.3: Covariant form of the Lagrangian of classical electrodynamics [**Oral** | 3 pt(s)]

ID: ex_covariant_em_lagrangian:rt2324

> **Learning objective**
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> Maxwell's electrodynamics is the prototypical example of a *relativistic field theory*. In this exercise you use tensor calculus [recall Problem 4.1], specialized to the inertial coordinate systems of special relativity, to show that the Lagrangian and the action of classical electrodynamics are invariant under Lorentz transformations.

Here we consider Minkowski space $M = \mathbb{R}^{1,3} \simeq \mathbb{R}^4$ as spacetime manifold and focus on inertial coordinate systems and the transformations between them (Lorentz transformations). The metric tensor is the Minkowski metric with components $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ in any inertial coordinate system. In the following, Greek indices run from 0 to 3.

The action of classical electrodynamics in vacuum (with speed of light $c = 1$) is given in some inertial coordinate system by

$$S = \int d^4x\, \mathcal{L} = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right), \tag{15}$$

with the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\pi} F_{\rho\pi}$ (where $\eta^{\mu\nu}$ denotes the inverse of the metric $\eta_{\mu\nu}$).

Here, A^μ is the gauge potential of electrodynamics; however, for this exercise you only need to know that A^μ transforms like a contravariant vector field (a $(1,0)$ tensor) and $A_\mu = \eta_{\mu\nu} A^\nu$ like a covariant vector field (a $(0,1)$ tensor).

a) Show that $F^{\mu\nu}$ is a contravariant tensor field of rank 2, and that the Lagrangian (density) $\mathcal{L}$ is a Lorentz scalar. **2**pt(s)

Hint: Use the results from Problem 4.1 and your knowledge about Lorentz transformations (which are the only coordinate transformations considered in special relativity).

b) Show that Eq. (15) is invariant under Lorentz transformations, i.e., show that **1**pt(s)

$$S = \int d^4x\, \mathcal{L} = \int d^4\bar{x}\, \bar{\mathcal{L}} = \bar{S}\,. \tag{16}$$

In summary, you have shown that the Lagrangian (density) $\mathcal{L}$ and the action given in Eq. (15) are Lorentz scalars. To prove that classical electrodynamics (the Maxwell equations) is Lorentz covariant (= the equations take the same form in all inertial systems), we still need to show that the Euler-Lagrange equations that follow from varying the action in Eq. (15) produce the Maxwell equations in their conventional form (⟶ later).

Problem 4.4: The Lorentz group

[**Oral** | 7 (+1 bonus) pt(s)]

ID: ex_lorentz_group:rt2324

Learning objective

The goal of this exercise is to get familiar with the Lorentz group and its continuous and discrete generators. Since the Lorentz group is a Lie group, we can study its Lie algebra. Knowing the Lie algebra of a group is very helpful to construct other representations of the group. This will become important in the context of relativistic quantum mechanics where we have to replace the Schrödinger equation by the Dirac equation.

The Lorentz group $O(1,3)$ is defined as group of real 4×4 matrices Λ that keep the Minkowski metric $\eta = \text{diag}(1,-1,-1,-1)$ invariant, i.e.

$$O(1,3) = \{\Lambda \in \mathbb{R}^{4\times 4} \,|\, \Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta} \iff \Lambda^T \eta \Lambda = \eta\} \,. \tag{17}$$

a) Show that $O(1,3)$ is a group [use only the definition in Eq. (17)]. **2**pt(s)

Specifically, show the following properties for any $\Lambda_1, \Lambda_2 \in O(1,3)$:

(i) $\Lambda_1 \Lambda_2 \in O(1,3)$

(ii) The inverse Λ_1^{-1} exists and is in $O(1,3)$.

Since the Lorentz group is a *Lie group* (i.e., a group that is also a differentiable manifold), we can study its *Lie algebra* (a vector space with an additional multiplication known as *Lie bracket*).

To this end, we consider an infinitesimal Lorentz transformation

$$\Lambda_{\boldsymbol{\xi}} = \exp\left(-i\xi_i X_i\right) \overset{\xi_i \ll 1}{\approx} \mathbb{1} - i\xi_i X_i \,, \tag{18}$$

where the matrices $X_i \in \mathbb{R}^{4\times 4}$ belong to the Lie algebra and are the generators of the Lorentz group; the $\xi_i \in \mathbb{R}$ are the corresponding coefficients that define the group element $\Lambda_{\boldsymbol{\xi}}$ in terms of these generators.

b) How does the defining condition of the Lorentz group $\Lambda^T \eta \Lambda = \eta$ translate to the generators X_i? **2**pt(s)

Show that we can write the generators X_i in the general form

$$(X_i)^\rho{}_\sigma = \begin{pmatrix} 0 & a & b & c \\ a & 0 & d & e \\ b & -d & 0 & f \\ c & -e & -f & 0 \end{pmatrix} , \tag{19}$$

with only 6 degrees of freedom left.

Since the Lie algebra is a vector space, we can choose a set of basis vectors (= generators) and write every element of the algebra as a linear combination of these.

A conventional basis of the Lie algebra of the Lorentz group is

$$(\mathcal{J}^{\mu\nu})^\rho{}_\sigma = i\left(\eta^{\mu\rho}\delta^\nu{}_\sigma - \delta^\mu{}_\sigma \eta^{\nu\rho}\right) \,, \tag{20}$$

with

$$\Lambda_{\boldsymbol{\omega}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}\right) , \tag{21}$$

where both, the set of coefficients $\omega_{\mu\nu}$ and the set of generators $\mathcal{J}^{\mu\nu}$ are antisymmetric in their indices.

Note: For every pair μ, ν, the object $\mathcal{J}^{\mu\nu}$ is a real 4×4 matrix, given by Eq. (20).

Convince yourself that this basis can be used to construct all matrices of the form (19) by setting one of the parameters to one and all others to zero. Therefore, the antisymmetric coefficient tensor $\omega_{\mu\nu}$ takes the place of ξ_i and encodes 6 degrees of freedom.

*c) Show that the commutator (Lie bracket) of the generators is **+1**pt(s)

$$([\mathcal{J}^{\mu\nu}, \mathcal{J}^{\alpha\beta}])^{\rho}{}_{\sigma} = i\left(\eta^{\nu\alpha}\mathcal{J}^{\mu\beta} - \eta^{\nu\beta}\mathcal{J}^{\mu\alpha} - \eta^{\mu\alpha}\mathcal{J}^{\nu\beta} + \eta^{\mu\beta}\mathcal{J}^{\nu\alpha}\right)^{\rho}{}_{\sigma} . \tag{22}$$

This is the *Lie algebra* of the Lorentz group; it has to be the same for all representations of the Lorentz group.

d) Calculate the determinant of Λ first from the definition in Eq. (17), and then from the expression in Eq. (18) using the form of the generators. **1**pt(s)

Can the exponential form $\Lambda_{\boldsymbol{\xi}}$ (equivalently: $\Lambda_{\boldsymbol{\omega}}$) be used to generate the complete group $O(1,3)$?

e) Convince yourself that time reversal T and space inversion P are part of the Lorentz group (17) but *cannot* be continuously connected to the identity $\Lambda_{\boldsymbol{\xi}=\mathbf{0}} = \mathbb{1}$: **2**pt(s)

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{23}$$

Finally, prove that T and P cannot be continuously connected to each other as well.

Interpret your result by sketching the structure of the Lorentz group.