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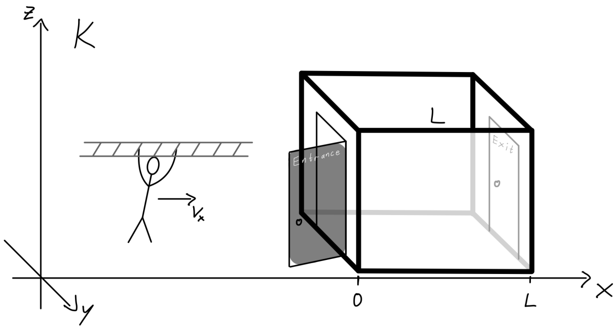
Problem 3.1: The ladder "paradox" [**Oral** | 6 pt(s)]

ID: ex_ladder_paradox:rt2324

> **Learning objective**
>
> In this exercise you study the so-called ladder "paradox." As all "paradoxes" in the context of relativity, the name reflects that this phenomenon might be counterintuitive and clash with our everyday experience at first glance; however, as you will show, it does *not* demonstrate a logical inconsistency of relativity. The goal of this exercise is to resolve the "paradox" and gain deeper insight into the thought experiment.

Consider a garage of length L with two doors: an entrance and an exit. The garage is at rest in the lab frame K. Now a relativistic athlete runs with velocity v_x towards the garage while carrying a ladder (see the great piece of art below).

In the rest frame K' of the athlete, the length of the ladder is also L. The entrance door closes when the ladder's left endpoint reaches the entrance. We call this event e_0 with $e_0 \sim (0, \mathbf{0})_K \sim (0, \mathbf{0})_{K'}$. The exit door automatically opens when the ladder's right endpoint reaches the exit; this event is called e_1.



a) Calculate the coordinates of e_1 in K and K'. Are the two events e_0 and e_1 space-like or time-like separated? **2**pt(s)

b) Draw a spacetime diagram for both inertial systems K and K'. Include the worldlines of the ladder endpoints and the garage and mark the events e_0 and e_1. Interpret the diagram and explain the "paradox." **3**pt(s)

c) Describe qualitatively from both inertial systems what happens if the exit door does *not* open, but the ladder crashes against the exit door and comes to a halt. **1**pt(s)

Problem 3.2: Symmetric spacetime diagrams (Loedel diagrams) [**Oral** | 7 pt(s)]

ID: ex_loedel_diagram:rt2324

> **Learning objective**
>
> The relativity principle of special relativity postulates that there is no distinguished inertial system. However, when drawing spacetime diagrams, we usually choose a specific inertial system, the axes of which we draw orthogonal to each other. This results in an asymmetric visualization of times (and lengths) measured in other inertial systems in relative motion. To avoid this problem, we can use Loedel

diagrams: a symmetric version of spacetime diagrams in which the symmetry between different inertial systems is manifest.

We consider two inertial systems $K \xrightarrow{v_x} K'$ which, for $t = t' = 0$, coincide at the origin $x = x' = 0$. The Loedel diagram is constructed by finding a *third* inertial system $\tilde{K}$ which is symmetric to K and K' in the sense that $K \xrightarrow{u_x} \tilde{K} \xrightarrow{u_x} K'$ for some boost velocity u_x.

a) Find the velocity u_x such that $\tilde{K}$ is symmetric to K and K'. **3**pt(s)

In the spacetime diagram of $\tilde{K}$ (with orthogonal $(\tilde{t}, \tilde{x})$-axes), draw the (t, x)-axes and the (t', x')-axes of both K and K'. What are the angles between the different axes (e.g. $t - \tilde{t}$, $t - x'$)?

In Problem 2.2 we studied the invariant spacetime interval Δs^2 that is invariant under Lorentz transformations and used it to determine the "unit ticks" on different axes. Due to the symmetry of the Loedel diagram, the unit ticks on the t- and t'-axis (x- and x'-axis) are the *same*. This has the benefit that in the spacetime diagram of $\tilde{K}$, we can directly compare times (lengths) measured in K and K'.

We can use this feature to illustrate the reciprocity of the relativistic effects of *time dilation* and *length contraction*:

b) Place clocks in the origin of both systems K and K' and draw the clocks at different times t (t') in the spacetime diagram of $\tilde{K}$. At each clock event, draw the lines of simultaneity both in K and K'. **2**pt(s)

Explain how it is possible that both observers (K and K') observe the clocks of the other observer to be slowed down. That is, demonstrate that time dilation is completely symmetric and the relativity principle is satisfied.

c) Now place rods in the origin of both systems K and K'. Draw the world lines of the endpoints of both rods in the spacetime diagram of $\tilde{K}$; add again some lines of simultaneity for K and K'. **2**pt(s)

Explain the reciprocity of length contraction as well as the "ladder paradox" in this symmetric spacetime diagram. That is, explain how it is possible that both observers see the rod of the other observer to be contracted.

Problem 3.3: The Penrose-Terrell effect

[**Oral** | 7 (+2 bonus) pt(s)]

ID: ex_penrose_terrell_effect:rt2324

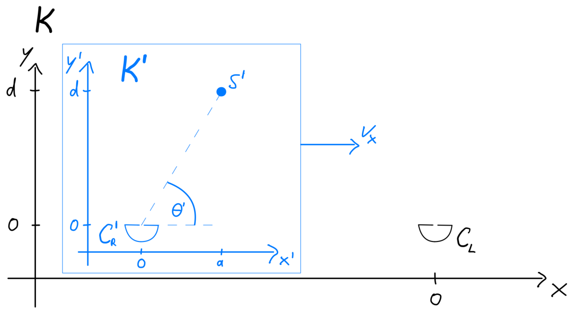
Learning objective

For many years after Einstein introduced special relativity, it was widely believed that, in principle, the Lorentz contraction could be *seen* with the eye (or a camera). For example, a fast moving *sphere* would appear as an *ellipse*. In two papers, published in 1959 independently by Penrose and Terrell, it was shown that this is *not* the case: Instead of "squeezed", relativistic objects appear *rotated*. In this exercise you derive this phenomenon known as *Penrose-Terrell effect*.

In this exercise, we consider a "camera" to be located at one spatial point which can take pictures at a certain time. A *picture* is the collection of all photons (including their spatial information, i.e., from which direction they came from) that *arrive* at the camera at the same time. You can think of a camera as an ideal pinhole, and a picture as the film on the back of the pinhole, after the pinhole was opened for a short moment.

Note: This is different from the "observer" defined in the lecture. By definition, a spatial diagram of an observer at equal time (relative to this observer) corresponds to the position of objects at that time. In the present context, such a diagram would correspond to all photons that where *emitted* simultaneously.

We consider two inertial systems $K \xrightarrow{v_x} K'$. In the rest frame K', a point-like light source is at rest at position $S' = (a, d)$ (for simplicity we ignore the z-direction throughout this exercise). In addition, a camera $\mathcal{C}'_R$ is at rest in the origin of the rest frame K'. A *second* camera $\mathcal{C}_L$ is at rest in the origin of the lab frame K (see sketch on the right).

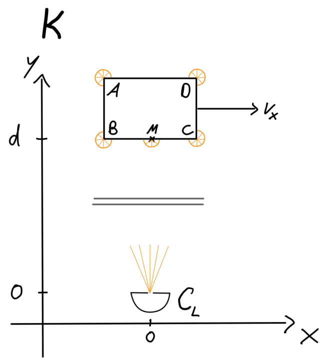


a) At the time $t = 0 = t'$ both cameras meet each other and take a picture. In the rest frame K', the photon from the light source arrives at the camera $\mathcal{C}'_R$ with an angle θ' (measured to the x'-axis). **2pt(s)**

At which angle θ does the photon arrive at the camera $\mathcal{C}_L$ in the lab frame K?

Note: The relation betwen θ and θ' is known as *relativistic aberration formula*.

Now we want to study the Penrose-Terrell effect. To this end, we replace the point-like light source by an illuminated box of rest length L_0 and height h. The box is at rest in K' and can be described by the corners $A' = (-L_0/2, d + h)$, $B' = (-L_0/2, d)$, $C' = (L_0/2, d)$ and $D' = (L_0/2, d + h)$. In addition, we consider the midpoint of the lower surface $M' = (0, 0)$.



For the sake of simplicity, we consider the special case $\theta_M = \pi/2$: When $\mathcal{C}_L$ takes a picture in the lab frame K, the photon scattered off the midpoint M arrives perpendicular to the x-axis.

b) *Where* and *when* have the photons from A, B, C been emitted such that they *arrive simultaneously* to the photon that was emitted from M at $(t = 0, x = 0, y = d)$ (this particular photon arrives perpendicular to the x-axis at the camera $\mathcal{C}_L$)? **5pt(s)**

The angles at which the photons arrive at the camera $\mathcal{C}_L$ can be calculated as $\cot\theta_i = \frac{\Delta x_i}{\Delta y_i}$, where Δx_i and Δy_i are the positions from where the photons were emitted ($i \in \{A, B, C\}$).

Compute these angles and compare your results with the picture (= angles) of a box that is *rotated* by an angle α (where both the box and the camera are at rest). For which angle α are the two pictures identical?

Hint: Assume that the distance between the box and the camera is large: $d \gg L_0$. This means: (i) Light emitted from $(\delta x, d)$ with $|\delta x| \sim L_0$ arrives at the camera $(0, 0)$ simultaneously. (ii) For $\cot\theta_i = \frac{\Delta x_i}{\Delta y_i}$ you can ignore terms of order $\mathcal{O}(1/d^2)$.

*c) In subtask b) we focused on the case $\theta_M = \pi/2$. What does the aberration formula, derived in subtask a), predict for this case? Interpret your result. **+2pt(s)**

Problem 3.4: A slower speed of light* [**Written** | 2 bonuspt(s)]

ID: ex_a_slower_speed_of_light:rt2324

> **Learning objective**
>
> In the previous exercises (and the lecture) you have studied various kinematic consequences of special relativity. In many cases, these effects are counterintuitive and incompatible with our everyday experience (or, as demonstrated by the Penrose-Terrell effect, even our naïve relativistic expectations). One reason for this is that the speed of light is so large that we do not encounter relativistic effects in our daily life. The educational game *A slower speed of light*, developed by the MIT Game Lab, lets you experience the effects a slower speed of light would have on our perception of the world.

Go to the website

http://gamelab.mit.edu/games/a-slower-speed-of-light

and download the game (available for Windows, Mac and Linux). Play the game and make a screenshot of your highscore to earn the bonus points for this exercise.

After you have finished the game, don not forget to read the "What Happened?" section of the game. Which effects in the game can be explained with the results of Problem 3.3?

Problem 3.5: Rapidities [**Written** | 4 pt(s)]

ID: ex_rapidity:rt2324

> **Learning objective**
>
> As shown in the lecture, even for collinear velocities the relativistic velocity addition formula does not have a simple form. Fortunately, one can introduce a new quantity called *rapidity* that is *additive* for collinear velocities, and thus makes the addition of velocities much easier.

The rapidity θ is defined as

$$\tanh\theta = \frac{v}{c}\,, \tag{1}$$

where tanh is the hyperbolic tangent function.

Note that for rapidities $-\infty < \theta < \infty$ the velocity is confined to the interval $-c < v < c$ (as it should be).

a) Show that for two collinear velocities v_1 and v_2, given by their rapidities θ_1 and θ_2, the relativistic addition of velocities is given by **1pt(s)**

$$c\tanh\theta_3 = v_3 = v_1 \oplus v_2 = c\tanh\left(\theta_1+\theta_2\right)\,. \tag{2}$$

That is, the rapidities are *additive* (in contrast to *velocities*).

b) Now consider the special Lorentz transformation $K \xrightarrow{v_x} K'$ with velocity v_x in x-direction. **1pt(s)**

Show that the Lorentz transformation can be written in terms of rapidities as

$$
\begin{aligned}
t' &= t\cosh\theta - \frac{x}{c}\sinh\theta \\
x' &= x\cosh\theta - ct\sinh\theta \\
y' &= y \quad \text{and} \quad z' = z\,.
\end{aligned}
\tag{3}
$$

Note: Appreciate the similarity of this form to a *rotation* in space and time (to be a proper rotation, the hyperbolic functions would have to be replaced by their trigonometric counterparts cos and sin).

c) Imagine you are preparing a space expedition from a spaceport which is at rest in some inertial system K. To accelerate your spaceship, you came up with a (clever) idea: You start by "catapulting" a large spaceship from the spaceport, such that the relative velocity from the spaceship observed by K is $v_x = 0.5c$. Now, in turn, the large spaceship catapults a smaller spaceship, such that the relative velocity from the smaller spaceship, observed by the large spaceship, is $v_x = 0.5c$ (ignore any recoil effects). We continue this process in total $N = 10$ times. **2**pt(s)

How fast is the last spaceship moving relative to the spaceport? Give an exact answer and an approximation for large N.