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Problem 2.1: Transformations consistent with the relativity principle [**Oral** | 14 pt(s)]

ID: ex_derivation_lorentz_transformation:rt2324

> **Learning objective**
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> In this exercise you start from Einstein's principle of relativity – together with a few reasonable assumptions about space and time – and then derive rigorously the most general coordinate transformation between inertial systems which are consistent with these assumptions. Quite remarkably, you will find only two possible choices: The Galilei transformations of Newtonian mechanics, and the *Lorentz transformations* of relativity.

We consider two inertial systems K and K' and an arbitrary event E. Our goal is to determine the possible transformations φ which map the coordinates $[E]_K = (t, \boldsymbol{x})$ of the event in K to the coordinates $[E]_{K'} = (t', \boldsymbol{x}')$ of *the same event* in K':

$$\varphi : \ (t, \boldsymbol{x}) = x \ \mapsto \ \varphi(x) = x' = (t', \boldsymbol{x}') \,. \tag{1}$$

To derive φ, we make some (reasonable) assumptions about spacetime and the coordinate transformation itself. These are:

`[SR]` **Special Relativity:** There is no distinguished inertial system.

`[IS]` **Isotropy:** There is no distinguished direction in space.

`[HO]` **Homogeneity:** There is no distinguished place in space or point in time.

`[CO]` **Continuity:** φ is a continuous function (in the origin).

We know from experimental observations that inertial systems are related to each other via rotations $R \in \mathrm{SO}(3)$, boosts $\boldsymbol{v} \in \mathbb{R}^3$, and translations in space ($\boldsymbol{b} \in \mathbb{R}^3$) and time ($s \in \mathbb{R}$):

$$K \xrightarrow{R,\boldsymbol{v},s,\boldsymbol{b}} K' \,. \tag{2}$$

Due to the relativity principle `[SR]`, the coordinate transformation φ can only depend on these relative parameters: $\varphi = \varphi(R, \boldsymbol{v}, s, \boldsymbol{b})$.

To derive φ, we proceed in several steps:

a) First, show that the transformation has an *affine* structure, that is **3pt(s)**

$$\varphi(x) = \Lambda x + a, \quad \text{with} \quad \Lambda \in \mathbb{R}^{4\times 4},\ a \in \mathbb{R}^4. \tag{3}$$

Hint: Use homogeneity `[HO]` to argue that

$$a'(\varphi, a) := \varphi(x + a) - \varphi(x) \tag{4}$$

cannot depend on x for some given spacetime translation $a \in \mathbb{R}^4$.

Use this (and the continuity assumption `[CO]`) to show that

$$\Psi(x) := \varphi(x) - \varphi(0) \tag{5}$$

must be a *linear* function: $\Psi(x+y) = \Psi(x) + \Psi(y)$ and $\Psi(rx) = r\Psi(x)$ for $r \in \mathbb{R}$ and $y \in \mathbb{R}^4$.

In the lecture, you discussed that translations of the coordinate axes in space by $\boldsymbol{b}$ and time by s, and rotations in space by $R \in \mathrm{SO}(3)$ transform the coordinates as follows:

$$\text{Translation}: \quad t' = t - s, \quad \boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{b}, \tag{6a}$$

$$\text{Rotation}: \quad t' = t, \quad \boldsymbol{x}' = R^{-1}\boldsymbol{x}. \tag{6b}$$

With the result Eq. (3), this implies that spacetime translations are represented by $a = (-s, -\boldsymbol{b})$ and spatial rotations by matrices Λ of the special form

$$\Lambda_R = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \quad \text{with } R \in \mathrm{SO}(3)\,. \tag{7}$$

What remains open is the coordinate transformation $\Lambda_{\boldsymbol{v}}$ due to boosts $\boldsymbol{v}$.

We focus on this in the remainder of this exercise, i.e., $K \xrightarrow{\mathbb{1},\boldsymbol{v},0,\boldsymbol{0}} K'$ from now on.

b) Use isotropy `[IS]` to argue that the most general form of the boost is **3**pt(s)

$$t' = a_v\, t + b_v\, (\boldsymbol{v}\cdot\boldsymbol{x}), \tag{8a}$$

$$\boldsymbol{x}' = c_v\, \boldsymbol{x} + \frac{d_v}{v^2}\, \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{x}) + e_v\, \boldsymbol{v}t, \tag{8b}$$

where a_v, b_v, c_v, d_v and e_v are (yet unknown) scalars that can only depend on $v = |\boldsymbol{v}|$.

Hint: First, argue that `[IS]` requires that $\Lambda_{\boldsymbol{v}} x \stackrel{!}{=} \Lambda_{R^{-1}}\Lambda_{R\boldsymbol{v}}(\Lambda_R x)$. Then, rewrite this constraint as $(\Lambda_R x') \stackrel{!}{=} \Lambda_{R\boldsymbol{v}}(\Lambda_R x)$ and show this restricts the form of $\Lambda_{\boldsymbol{v}}$ to Eq. (8).

The goal of the next few steps is to determine the functions a_v, b_v, c_v, d_v, e_v.

c) Use the trajectory of the origin of K' in the system K to derive the constraint **1**pt(s)

$$0 = c_v + d_v + e_v. \tag{9}$$

d) Use *reciprocity*, that is $\Lambda_{\boldsymbol{v}}^{-1} = \Lambda_{-\boldsymbol{v}}$, to derive the additional constraints **2**pt(s)

$$c_v^2 = 1, \quad a_v^2 - e_v b_v v^2 = 1, \quad e_v^2 - e_v b_v v^2 = 1, \quad e_v(a_v + e_v) = 0, \quad b_v(a_v + e_v) = 0. \tag{10}$$

Hint: Write down the inverse transformation and plug it into Eq. (8).

Note: The assumption of reciprocity means that if you see another observer pass by with velocity $\boldsymbol{v}$, this observer sees *you* pass by with $-\boldsymbol{v}$. While this seems reasonable, it is not a trivial statement. However, it can be rigorously derived from `[SR]`, `[IS]`, and `[HO]` and is therefore not an independent assumption (see the lecture notes for a reference).

e) Use the results of the previous two subtasks to show that the boost now has the form **1pt(s)**

$$t' = a_v\, t + \frac{1-a_v^2}{v^2 a_v}\,(\boldsymbol{v}\cdot\boldsymbol{x}), \tag{11a}$$

$$\boldsymbol{x}' = \boldsymbol{x} + \frac{a_v - 1}{v^2}\,\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{x}) - a_v\,\boldsymbol{v}t. \tag{11b}$$

All that is left is to compute the form of a_v.

The relativity principle `[SR]` requires the transformation to have a group structure, that is

$$\varphi(K' \xrightarrow{R_2,\boldsymbol{v}_2,s_2,\boldsymbol{b}_2} K'') \circ \varphi(K \xrightarrow{R_1,\boldsymbol{v}_1,s_1,\boldsymbol{b}_1} K') \equiv \varphi(K \xrightarrow{R_3,\boldsymbol{v}_3,s_3,\boldsymbol{b}_3} K''). \tag{12}$$

This must be true for any transformation, in particular for two consecutive boosts in x-direction:

f) Consider two boosts in x-direction: $K \xrightarrow{v_x} K'$ and $K' \xrightarrow{u_x} K''$. Show that we have to choose **4pt(s)**

$$a_v = \frac{1}{\sqrt{1-\kappa {v_x}^2}}, \tag{13}$$

with an unknown *constant* κ of dimension $[\kappa] = \text{Velocity}^{-2}$, such that the combination of the two boosts yields another boost.

How can the velocity w_x of this new boost be computed from v_x and u_x?

Finally, argue that for a boost in an *arbitrary* direction $\boldsymbol{v}$ it must be

$$a_v = \frac{1}{\sqrt{1-\kappa v^2}} \quad \text{with } v = |\boldsymbol{v}|. \tag{14}$$

Hint: Use the relation provided in the hint of subtask b).

To wrap up, you have proven that the most general form of the coordinate transformation φ between inertial systems that is consistent with the relativity principle `[SR]` (and reasonable assumptions about space and time) has the affine form Eq. (3).

It includes spacetime translations (given by a), spatial rotations (given by Λ_R) and boosts $\Lambda_{\boldsymbol{v}}$.

The latter must have the form

$$t' = a_v\left[t - \kappa\,(\boldsymbol{v}\cdot\boldsymbol{x})\right], \tag{15a}$$

$$\boldsymbol{x}' = \boldsymbol{x} + \frac{a_v - 1}{v^2}\,\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{x}) - a_v\,\boldsymbol{v}t, \tag{15b}$$

with

$$a_v = \frac{1}{\sqrt{1-\kappa v^2}}\,. \tag{16}$$

The only parameter that remains undetermined is the constant κ. As we will discuss in the lecture, $\kappa = 0$ corresponds to *Galilei transformations* and $\kappa > 0$ to *Lorentz transformations*.

Problem 2.2: Invariant quadratic forms

[**Written** | 10 pt(s)]

ID: ex_invariant_quadratic_form:rt2324

Learning objective

In this exercise you study quadratic forms that are invariant under the general transformation derived in Problem 2.1 for different values of the undetermined constant κ. The invariant quadratic form for the Lorentz transformation plays an important role in relativity and is known as *invariant spacetime interval*. As you will visualize with spacetime diagrams, the different invariant quadratic forms suggest different geometric interpretations of the transformations.

We consider the general boost $\Lambda_{\boldsymbol{v}}$ derived in Problem 2.1 and given by

$$t' = a_v[t - \kappa(\boldsymbol{v} \cdot \boldsymbol{x})], \tag{17a}$$

$$\boldsymbol{x}' = \boldsymbol{x} + \frac{a_v - 1}{v^2}(\boldsymbol{v} \cdot \boldsymbol{x})\boldsymbol{v} - a_v\,\boldsymbol{v}t, \tag{17b}$$

with $a_v = 1/\sqrt{1 - \kappa v^2}$ and $\kappa \in \mathbb{R}$.

a) Show that you can, without loss of generality, consider the three cases of $\kappa = 0, \pm 1$. 1pt(s)

b) We define a quadratic form for each of the three cases: 3pt(s)

$$\kappa = +1: \quad D(\Delta\boldsymbol{x}, \Delta t) = (\Delta t)^2 - (\Delta\boldsymbol{x})^2, \tag{18a}$$

$$\kappa = 0 \quad: \quad D(\Delta\boldsymbol{x}, \Delta t) = (\Delta t)^2, \tag{18b}$$

$$\kappa = -1: \quad D(\Delta\boldsymbol{x}, \Delta t) = (\Delta t)^2 + (\Delta\boldsymbol{x})^2, \tag{18c}$$

where $\Delta\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ and $\Delta t = t_2 - t_1$ are differences between arbitrary events in space and time. Show that these forms are invariant under the boost Eq. (17) for the corresponding κ, i.e., show that $D(\Delta\boldsymbol{x}, \Delta t) = D(\Delta\boldsymbol{x}', \Delta t')$.

Note: The invariant quadratic form for $\kappa = +1$ (Lorentz transformations) is called the *invariant spacetime interval* in relativity. You just proved that this is a Lorentz invariant quantity: it has the same value in all inertial frames. By contrast, the invariant quadratic form for $\kappa = 0$ (Galilei transformations) tells you that time intervals between events in Newtonian mechanics are independent of the reference system; as a special case ($\Delta t = 0$), this implies that simultaneity is absolute in Newtonian mechanics.

We now want to visualize the boosts for the three cases $\kappa = 0, \pm 1$ and make a connection to their invariant quadratic forms. To this end, consider a boost in x-direction $K \xrightarrow{v_x} K'$ [that is: $\boldsymbol{v} = (v_x, 0, 0)$]. We can then restrict ourselves to the (t, x)-plane of the coordinate system K since the other coordinates are unaffected by this boost (show this!).

c) For the three cases $\kappa = 0, \pm 1$, start with the (t, x)-axes of K as an orthonormal Cartesian diagram and then fill in the axes of the coordinate system K'. 3pt(s)

Hint: The (t', x')-axes of K' are defined by $x' = 0$ and $t' = 0$.

d) Draw for the three cases a few lines of *constant* $D(\Delta x, \Delta t) = 0, 1, 2, \ldots$ with $\Delta x = x - 0$ and $\Delta t = t - 0$ [i.e., the intervals between the origin and some event $(t, \boldsymbol{x})$]. 3pt(s)

Then use that the transformations leaves these values invariant to draw the “unit ticks” on all axes.

Hint: Where on the t'-axis is the tick for $t' = 1$ if you declare a tick on the t-axis as the $t = 1$ mark?

Problem 2.3: Velocity addition

[**Written** | 4 pt(s)]

ID: ex_lorentz_velocity_addition:rt2324

Learning objective

In this exercise, you generalize the addition formula for collinear velocities, which you derived at the end of Problem 2.1, to arbitrary velocities. That is, you answer the question: Which velocity $\boldsymbol{w}$ of a signal do you observe in K if the signal has velocity $\boldsymbol{u}$ with respect to another inertial system K' which, in turn, moves with $\boldsymbol{v}$ compared to K? Intuitively (and in a Galilean world) the answer would simply be $\boldsymbol{w} = \boldsymbol{v} + \boldsymbol{u}$. In relativity, as you will show, the answer is a bit more complicated.

Consider a boost with velocity $\boldsymbol{v}$ from K to K':

$$K \xrightarrow{\boldsymbol{v}} K'. \tag{19}$$

Let there be a particle moving freely with velocity $\boldsymbol{u} = \frac{d\boldsymbol{x}'}{dt'}$ as measured in the system K'.

Use the Lorentz boost derived in Problem 2.1 for $\kappa = 1/c^2 > 0$ to compute the velocity

$$\boldsymbol{w} = \boldsymbol{v} \oplus \boldsymbol{u} := \frac{d\boldsymbol{x}}{dt} \tag{20}$$

as measured in system K.

Show that in the non-relativistic limit $\kappa \to 0 \;\Leftrightarrow\; c \to \infty$ you recover the Galilean addition of velocities: $\lim_{\kappa\to 0} \boldsymbol{v} \oplus \boldsymbol{u} = \boldsymbol{v} + \boldsymbol{u}$.

Note: The relativistic velocity addition $\oplus$ is neither commutative nor associative!