

↓ Lecture 8 [05.12.23]

### 3.5. The metric tensor

A differentiable manifold  $M$  does not automatically allow us to measure the length of curves, the angles of intersecting lines, or the area/volume of subsets of the manifold; to do so, we need a *metric* on  $M$  (which is an additional piece of information). While the continuity structure (an atlas) that comes with  $M$  determines its *topology*, the metric determines its *geometry* (= shape). The same manifold  $M$  can be equipped with *different* metrics; this corresponds to different geometries of the same topology (a potato and an egg both have the topology of a sphere, nonetheless they are geometrically distinct).

A differentiable manifold together with a (pseudo-)metric is called ↑ *(pseudo-)Riemannian manifold*. In SPECIAL RELATIVITY and GENERAL RELATIVITY, spacetime is modeled by such (pseudo-)Riemannian manifolds where the metric is used to represent spatial and temporal distances between events.

25 | **Motivation:**

On linear spaces  $V$ , it is convenient to define an ↓ *inner product* (like in quantum mechanics where you consider Hilbert spaces and use their inner product to compute probabilities and transition amplitudes).

Recall the definition of a (real) inner product:

$$\langle \bullet | \bullet \rangle : V \times V \rightarrow \mathbb{R} \quad \text{with ...} \tag{3.44a}$$

$$\text{Symmetry: } \langle x | y \rangle = \langle y | x \rangle \tag{3.44b}$$

$$\text{(Bi)linearity: } \langle ax + by | z \rangle = a \langle x | z \rangle + b \langle y | z \rangle \tag{3.44c}$$

$$\text{Positive-definiteness: } x \neq 0 \Rightarrow \langle x | x \rangle > 0 \tag{3.44d}$$

Once you have an inner product, you get a norm, and subsequently a metric for free:

$$\underbrace{\langle x | y \rangle}_{\text{Inner product}} \Rightarrow \underbrace{\|x\| := \sqrt{\langle x | x \rangle}}_{\text{Norm}} \Rightarrow \underbrace{d(x, y) := \|x - y\|}_{\text{Metric}} \tag{3.45}$$

Thus an inner product is a rather versatile structure and nice to have!

*Problem:* We cannot define a inner product on the manifold directly because  $M$  is not a linear space.

*However:* We can introduce an inner product on each of its tangent spaces  $T_p M!$  →

26 | \* Riemannian (Pseudo-)Metric  $ds^2 :=$  Symmetric, non-degenerate (0, 2)-tensor field:

$$ds^2 : M \ni p \mapsto \underbrace{(ds_p^2 : T_p M \times T_p M \rightarrow \mathbb{R})}_{\text{Bilinear \& symmetric \& non-degenerate}} \tag{3.46a}$$

$$\begin{aligned} ds_p^2 \text{ bilinear} &\Rightarrow ds_p^2 \in T_p^* M \otimes T_p^* M \\ &\Rightarrow ds_p^2 = \sum_{i,j=1}^D g_{ij}(x) dx^i \otimes dx^j \equiv g_{ij}(x) dx^i dx^j \end{aligned} \tag{3.46b}$$

with  $g_{ij} = g_{ji}$  (symmetry) and  $g = \det(g_{ij}) \neq 0$  (non-degeneracy).

- The tensor product is *non-commutative*:  $dx^i \otimes dx^j \neq dx^j \otimes dx^i$ . However, you can always decompose a tensor product as

$$dx^i \otimes dx^j = \underbrace{\frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)}_{=: dx^i \vee dx^j} + \underbrace{\frac{1}{2}(dx^i \otimes dx^j - dx^j \otimes dx^i)}_{=: dx^i \wedge dx^j} \quad (3.47)$$

with the symmetrized tensor product  $dx^i \vee dx^j$  and the anti-symmetrized tensor product  $dx^i \wedge dx^j$  ( $\uparrow$  *wedge product*).

Since  $g_{ij}$  is assumed to be symmetric, only the symmetric component survives:

$$g_{ij}(x) dx^i \otimes dx^j = g_{ij}(x) dx^i \vee dx^j \equiv g_{ij}(x) dx^i dx^j \quad (3.48)$$

This means that when writing  $dx^i dx^j$  in the above formula, you can be sloppy and either mean  $dx^i \otimes dx^j$  or, equivalently,  $dx^i \vee dx^j$ . You will find both conventions in the literature. I will use  $dx^i dx^j \equiv dx^i \vee dx^j$  so that  $dx^i dx^j = dx^j dx^i$ .

- It would be more appropriate to write  $g = g_{ij} dx^i dx^j$  for the metric (0, 2)-tensor; it is conventional, however, to reserve  $g$  for the determinant  $\det(g_{ij})$  so that we are stuck with  $ds^2$  for the metric. Note that the  $d$  in  $ds^2$  does *not* refer to an  $\uparrow$  *exterior derivative*, it is purely symbolical.
- To define a proper  $\downarrow$  *inner product* on  $T_p M$ , we should demand  $\downarrow$  *positive-definiteness* instead of *non-degeneracy*. This, however, is often (for example in RELATIVITY) too restrictive; as it turns out, non-degeneracy is all we need for an isomorphism between  $T_p M$  and  $T_p^* M$  (“pulling indices up and down”,  $\rightarrow$  *below*). This is why *negative* eigenvalues of  $g_{ij}$  are fine for many purposes, and motivates the concept of a  $\rightarrow$  *signature*:

## 27 | Signature:

Since  $g_{ij}(x) = g_{ji}(x)$  and  $\det(g_{ij}(x)) \neq 0$

$\rightarrow g_{ij}(x)$  has  $r$  *positive* and  $s$  *negative* real eigenvalues for all  $p \in M$

Since  $\det(g_{ij}(x)) \neq 0$ , these numbers must be the *same* for all  $p \in M$ .

$\rightarrow (r, s)$ :  $\star\star$  *Signature* of the metric  $ds^2$

This classification does not depend on the coordinate basis ( $\uparrow$  *Sylvester’s law of inertia*).

- $(r > 0, s = 0)$

$\rightarrow ds^2$ : *Riemannian metric*  $\rightarrow (M, ds^2)$ :  $\star\star$  *Riemannian manifold*

I.e.,  $g_{ij}$  has only positive eigenvalues for all  $p \in M$  and is therefore  $\downarrow$  *positive-definite*. This produces a true, positive-definite inner product on  $T_p M$ .

- $(r > 0, s > 0)$

$\rightarrow ds^2$ : *pseudo-Riemannian metric*  $\rightarrow (M, ds^2)$ :  $\star\star$  *pseudo-Riemannian manifold*

I.e.,  $g_{ij}$  has both positive and negative eigenvalues and is therefore  $\downarrow$  *indefinite*.

- $(r > 0, s = 1)$  or  $(r = 1, s > 0)$ :

$\rightarrow ds^2$ : *Lorentzian metric*  $\rightarrow (M, ds^2)$ :  $\star\star$  *Lorentzian manifold*

In RELATIVITY we are only interested in metric tensors with one positive and three negative eigenvalues (equivalently: three positive and one negative eigenvalue). Mathematically speaking, spacetime is then a four-dimensional Lorentzian manifold and a special case of a pseudo-Riemannian manifold.

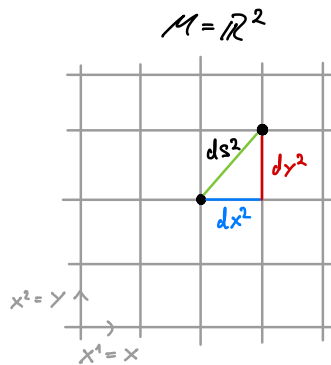
28 | Example: (Details: → Problemset 4)

i |  $\llcorner D = 2$  Euclidean space  $E_2 \equiv (\mathbb{R}^2, ds_E^2)$

The Euclidean metric in Cartesian coordinates  $x^1 = x$  and  $x^2 = y$  reads:

$$ds_E^2 := dx^2 + dy^2 = g_{ij}(x) dx^i dx^j \quad \text{with} \quad (g_{ij}) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\text{Signature} \\ (2,0)}}. \quad (3.49)$$

This is consistent with the notion of  $dx$  and  $dy$  as infinitesimal shifts in coordinates and  $ds^2$  as the infinitesimal distance (squared) that corresponds to this shift:



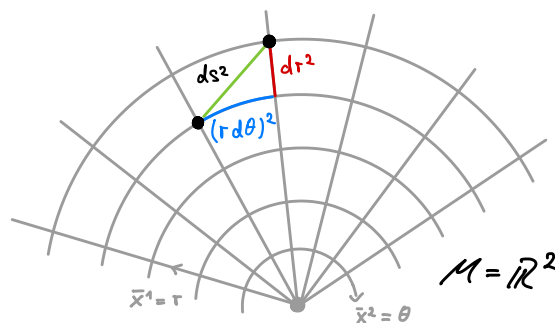
ii | We can now transition to a new chart, namely polar coordinates  $\bar{x}^1 = r$  and  $\bar{x}^2 = \theta$ . The induced basis change on the cotangent space is given by the total differential of the coordinate functions Eq. (3.14):

$$\varphi^{-1} : \begin{cases} x = r \cos(\theta) \\ y = r \sin(\theta) \end{cases} \xRightarrow{\text{Eq. (3.14)}} \begin{cases} dx = \cos(\theta) dr - r \sin(\theta) d\theta \\ dy = \sin(\theta) dr + r \cos(\theta) d\theta \end{cases} \quad (3.50)$$

iii | We find the components of the metric tensor field in the new basis  $\{d\bar{x}^1 = dr, d\bar{x}^2 = d\theta\}$ :

$$ds^2 \stackrel{\circ}{=} dr^2 + r^2 d\theta^2 = \bar{g}_{ij}(\bar{x}) d\bar{x}^i d\bar{x}^j \quad \text{with} \quad (\bar{g}_{ij}) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}_{\substack{\text{Signature} \\ (2,0)}}. \quad (3.51)$$

This expression is again compatible with infinitesimal shifts in the (new) coordinates  $r$  and  $\theta$ :



- The Euclidean plane  $E_2$  is therefore an example for a Riemannian manifold with metric signature  $(2, 0)$ ; its distinctive feature is that it is *flat*.

- Note that here we compute *the same* infinitesimal length in different coordinates (with the same result)! We did not change the *metric*, only the *coordinates* and thereby the coordinate basis in which we express the metric tensor. This is *flat* Euclidean space in ↑ *curvilinear coordinates*. By contrast, later in GENERAL RELATIVITY we will study curved (non-flat, non-Euclidean) metric tensors, i.e., we will modify the geometry of space(time) itself.

29 | Since the metric  $ds^2$  is a (0, 2)-tensor field:

$$\bar{g}_{ij}(\bar{x})d\bar{x}^i d\bar{x}^j = ds^2 = g_{ij}(x)dx^i dx^j \quad (3.52)$$

Eq. (3.14)  $\overset{\circ}{\rightarrow}$

$$\bar{g}_{ij}(\bar{x}) = \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial x^m}{\partial \bar{x}^j} g_{lm}(x) \quad (3.53)$$

The metric (components) transforms as any other (0, 2) tensor. Nothing special!

Side note:

Let  $g := \det(g_{ij})$  and  $\bar{g} := \det(\bar{g}_{ij}) \xrightarrow{\text{Eq. (3.53)}}$

$$\sqrt{|\bar{g}|} = \left| \det \left( \frac{\partial x}{\partial \bar{x}} \right) \right| \sqrt{|g|} \quad (3.54)$$

→  $\sqrt{|g|}$  is a *pseudo* scalar tensor density of weight  $w = +1$ . The “pseudo” indicates that the absolute value of the Jacobian determinant shows up, cf. Eq. (3.37).

$\triangleleft g < 0 \xrightarrow{\text{Eq. (3.39)}} d^Dx \sqrt{-g}$  is a scalar (→ later)!

30 | Length of curves on  $M$ :

One immediate benefit of having a Riemannian manifold is that we can now compute the length of curves  $\gamma(t)$  on  $M$  (parametrized by  $t \in [a, b]$  and given in some chart):

$$L[\gamma] \equiv \int_{\gamma} ds := \int_a^b \sqrt{g_{ij}(\gamma(t)) \frac{d\gamma^i(t)}{dt} \frac{d\gamma^j(t)}{dt}} dt \quad (3.55)$$

$$\equiv \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)} dt \quad (3.56)$$

! If  $ds^2$  is a true *pseudo* metric (i.e.,  $g_{ij}$  has at least one negative eigenvalue), one must make sure that the chosen curve  $\gamma$  does not produce negative values under the square root. In RELATIVITY these will be ↑ *time-like* curves.

Example:

Let  $\gamma$  be the circle with radius  $R$  in the Euclidean plane  $E_2$ . A possible parametrization in Cartesian coordinates (with origin in the center of the circle) is  $\vec{y}_{xy}(t) = (x_t, y_t) = (R \cos(t), R \sin(t))$  with  $0 \leq t < 2\pi$  so that one finds for the circumference:

$$L = \int_{\gamma} \sqrt{dx^2 + dy^2} = \int_0^{2\pi} \sqrt{\dot{x}_t^2 + \dot{y}_t^2} dt \doteq 2\pi R \quad (3.57)$$

The same length can of course be calculated with the parametrization  $\vec{\gamma}_{r\theta}(t) = (r_t, \theta_t) = (R, t)$  and  $0 \leq t < 2\pi$  in polar coordinates:

$$L = \int_{\gamma} \sqrt{dr^2 + r^2 d\theta^2} = \int_0^{2\pi} \sqrt{\dot{r}_t^2 + r_t^2 \dot{\theta}_t^2} dt \doteq 2\pi R \quad (3.58)$$

Details: → Problemset 4

- 31** | Besides computing lengths of curves (and other geometric quantities, → *later*), there is another benefit of having a metric tensor:

Pulling indices down:

$$\tilde{T} \begin{matrix} i_1 \dots \square \dots i_p \square \dots \square \\ \square \dots i \dots \square j_1 \dots j_q \end{matrix} := g_{ik} T \begin{matrix} i_1 \dots k \dots i_p \square \dots \square \\ \square \dots \square \dots \square j_1 \dots j_q \end{matrix} \quad (3.59)$$

→  $\tilde{T}$  is a tensor of type  $(p - 1, q + 1)$

- In Eq. (3.59) we indicate “empty” slots for indices by  $\square$  to emphasize that in each index “column” an index can either be *up* (contravariant) or *down* (covariant). It is conventional to omit the  $\square$ -markers. Note that this explains why you never should write two indices directly above each other (except for special cases, → *below*).

Furthermore, since  $g$  is fixed, it makes sense to label  $\tilde{T}$  again by  $T$  (note that the difference between the original tensor and the new one is manifest in the different index patterns!):

$$\tilde{T} \begin{matrix} i_1 \dots \square \dots i_p \square \dots \square \\ \square \dots i \dots \square j_1 \dots j_q \end{matrix} \mapsto T \begin{matrix} i_1 \dots \dots i_p \\ \dots i \dots j_1 \dots j_q \end{matrix} \quad (3.60)$$

Example:

$$A \begin{matrix} i \dots k \\ \dots j \dots l \end{matrix} := g_{jm} A \begin{matrix} i \dots m \dots k \\ \dots j \dots l \end{matrix} \quad (3.61)$$

- This convention matches perfectly with the computation of an inner product (which is determined by the metric tensor  $g$ ) of two contravariant vectors:

$$\langle A, B \rangle \stackrel{\text{def}}{=} g_{ij} A^i B^j \stackrel{\text{def}}{=} \underbrace{A^i B_i}_{\text{Scalar}} \quad (3.62)$$

- 32** | Pulling indices up:

We would like to have a  $(2, 0)$ -tensor  $g^{ij}$  with the property

$$\delta_j^k T^j = T^k \stackrel{!}{=} g^{ki} T_i \stackrel{\text{def}}{=} g^{ki} g_{ij} T^j. \quad (3.63)$$

$g^{ij}$  allows us to revert the pulling-down of indices defined by the metric  $g_{ij}$ . Note that  $g^{ij}$  is a *different* tensor than  $g_{ij}$ , we could call it  $\tilde{g}^{ij}$ ; however, it is conventional to denote it with the same label due to the following close relationship with  $g$ :

$$g^{ki} g_{ij} \stackrel{!}{=} \delta_j^k \quad (3.64)$$

This is an implicit equation for  $g^{ki}$ !

→  $g^{ij}$  is the *inverse matrix* of  $g_{ij}$   
(Which always exists because  $ds^2$  is non-degenerate:  $\det(g_{ij}) \neq 0$ .)

→ In general:

$$\tilde{T}^{i_1 \dots i_p \square \dots j \dots \square}_{\square \dots \square j_1 \dots \square \dots j_q} := g^{jk} T^{i_1 \dots i_p \square \dots \square}_{\square \dots \square j_1 \dots k \dots j_q} \quad (3.65)$$

→  $\tilde{T}$  is a tensor of type  $(p + 1, q - 1)$

- Again we relabel  $\tilde{T}$  to  $T$  and omit the  $\square$ -markers:

$$\tilde{T}^{i_1 \dots i_p \square \dots j \dots \square}_{\square \dots \square j_1 \dots \square \dots j_q} \mapsto T^{i_1 \dots i_p}_{j_1 \dots \dots j_q} \quad (3.66)$$

- Example:

$$A^{ijkl} := g^{lm} A^{ijk}_m \quad (3.67)$$

- With these new definitions, we can now raise and lower contractions:

$$A^i B_i = A^i \delta_i^j B_j = A^i g_{ik} g^{kj} B_j = A^i g_{ik} B^k = A_k B^k = A_i B^i \quad (3.68)$$

- What happens if you pull the indices of the Kronecker symbol up or down?

$$\delta^{ij} := g^{jk} \delta^i_k = g^{ij} \quad \text{and} \quad \delta_{ij} := g_{ik} \delta^k_j = g_{ij} \quad (3.69)$$

!  $\delta^{ij} \equiv g^{ij}$  and  $\delta_{ij} \equiv g_{ij}$  denote the metric and its inverse!

→ We never use the notation  $\delta^{ij}$  and  $\delta_{ij}$  to prevent confusion!

- Note that in general

$$g^{jk} T^i_k = T^{ij} \neq T^{ji} = g^{jk} T_k^i. \quad (3.70)$$

This means that the “column” in which the index is located is *important*, and notations like  $T^i_k$  are ill defined (if you pull  $k$  up by  $g^{jk}$ , do you get  $T^{ij}$  or  $T^{ji}$ ?). However, if the tensor is *symmetric*,  $T^{ij} = T^{ji}$ , this does not matter and you can get away with the sloppy notation  $T^i_k$ . This explains why writing  $\delta^i_k$  for the Kronecker symbol is fine:  $g^{ji} = g^{jk} \delta^i_k$  is symmetric.

### 33 | Mathematical side note:

“Pulling indices up and down” is mathematically the application of an  $\downarrow$  *isomorphism* between  $T_p M$  and  $T_p^* M$ :

$$g(\bullet, \bullet) : T_p M \ni A \mapsto g(A, \bullet) \in T_p^* M \quad (3.71)$$

This has nothing to do with differential geometry or manifolds in particular; it is a general feature of non-degenerate bilinear forms on vector spaces. In differential geometry, this canonical isomorphism between the tangent bundle  $TM$  and the cotangent bundle  $T^*M$  is known as  $\uparrow$  *musical isomorphism*.

For example, you are using the same kind of isomorphism all the time in quantum mechanics, namely whenever you “dagger” a *ket*  $|\Psi\rangle$  to obtain a *bra*  $\langle\Psi|$ :

$$(\bullet)^\dagger : \mathcal{H} \ni |\Psi\rangle \mapsto \langle\Psi| \equiv |\Psi\rangle^\dagger \in \mathcal{H}^* \quad \text{with} \quad \langle\Psi||\Phi\rangle \stackrel{!}{=} \langle\Psi|\Phi\rangle_{\mathcal{H}} \quad \text{for all } |\Phi\rangle \in \mathcal{H}. \quad (3.72)$$

Note how the bra  $\langle \Psi |$  associated to the ket  $|\Psi\rangle$  is defined via the inner product  $\langle \bullet | \bullet \rangle_{\mathcal{H}}$  (and therefore metric) of the Hilbert space ( $\uparrow$  Riesz representation theorem)!

This leads to a nice dictionary between concepts in tensor calculus (and therefore RELATIVITY) and the bra-ket formalism of quantum mechanics:

	Relativity (fixed $p \in M$ )	Quantum mechanics
Inner product space	$T_p M$	$\mathcal{H}$
Basis	$\{\partial_i\}$	$\{ i\rangle\}$
Vector	$A = A^i \partial_i$	$ \Psi\rangle = \Psi_i  i\rangle$
Dual space	$T_p^* M$	$\mathcal{H}^*$
Dual basis	$\{dx^i\}$	$\langle i $
...	$dx^i(\partial_j) = \delta_j^i$	$\langle i j\rangle = \delta_{ij}$
Covector	$B = B_i dx^i$	$\langle \Psi  = \Psi_i^* \langle i $
Inner product	$g(A_1, A_2) = g_{ij} A_1^i A_2^j$	$\langle \Psi \Phi\rangle$
Tensor	$A = A^{ij} \partial_i \otimes \partial_j$	$ \Psi\rangle \otimes  \Phi\rangle \equiv  \Psi\rangle \Phi\rangle$
...	$B = B_{ij} dx^i \otimes dx^j$	$\langle \Psi  \otimes \langle \Phi  \equiv \langle \Psi \langle \Phi $
Operator	$T = T^i_j \partial_i \otimes dx^j$	$ \Phi\rangle \otimes \langle \Psi  \equiv  \Phi\rangle\langle \Psi $
Trace	$T^i_i$	$\text{Tr}[ \Phi\rangle\langle \Psi ]$
Scalar	$BA = B_i A^i = g_{ij} B^i A^j$	$\langle \Psi \Phi\rangle = \langle \Psi \Phi\rangle$
Pulling indices down	$A_i = g_{ij} A^j$	$\langle \Psi  =  \Psi\rangle^\dagger$
Pulling indices up	$A^i = g^{ij} A_j$	$ \Psi\rangle = \langle \Psi ^\dagger$

### 3.6. Differentiation of tensor fields

34 | Remember:  $\partial_i \Phi$  is covariant vector if  $\Phi$  is scalar. However:

$\leftarrow$  Contravariant vector  $A^i$ :

$$\bar{A}^i_{,k} \equiv \frac{\partial \bar{A}^i}{\partial \bar{x}^k} = \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial}{\partial x^m} \left[ \frac{\partial \bar{x}^i}{\partial x^l} A^l \right] = \underbrace{\frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^l} \frac{\partial x^m}{\partial \bar{x}^k} A^l}_{\neq 0 \text{ (in general) } \odot} + \underbrace{\frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial A^l}{\partial x^m}}_{(1,1)\text{-tensor } \odot} \quad (3.73)$$

Here we used the transformation of  $\bar{A}^i$  [Eq. (3.8)] and  $\bar{\partial}_k$  [Eq. (3.5)] and the product rule.

$\rightarrow$  In general:  $\frac{\partial \bar{A}^i}{\partial \bar{x}^k}$  is not a tensor!

35 | How to define a derivative of tensor fields that again transforms as a tensor?

To solve this problem, we first need a new field:

$\rightarrow$   $\star\star$  Christoffel symbols (of the second kind):

$$\Gamma^i_{kl} := \frac{1}{2} g^{im} (g_{mk,l} + g_{ml,k} - g_{kl,m})$$

(3.74)

- The Christoffel symbols are symmetric in the lower two indices:  $\Gamma^i_{kl} = \Gamma^i_{lk}$
- $\downarrow$  Despite the index notation, the Christoffel symbols are *not* tensors:

$$\bar{\Gamma}^i_{kl} \stackrel{\circ}{=} \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial x^p}{\partial \bar{x}^l} \Gamma^{mnp} - \underbrace{\frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial x^p}{\partial \bar{x}^l} \frac{\partial^2 \bar{x}^i}{\partial x^n \partial x^p}}_{\text{No tensor!}} \quad (3.75)$$

This is why they are called “symbols” and not “tensors”!

- There are also Christoffel symbols of the *first* kind:

$$\Gamma_{ikl} := g_{ij} \Gamma^j_{kl} = \frac{1}{2} (g_{ik,l} + g_{il,k} - g_{kl,i}) \quad (3.76)$$

- Mathematically, the Christoffel symbols are the coefficients (in some basis) of the  $\uparrow$  *Levi-Civita connection* which is determined by the metric tensor  $g^{ij}$  ( $\rightarrow$  later).

36 |  $\leftarrow$  Contravariant vector  $\bar{A}^i$  and contract it with  $\bar{\Gamma}^i_{kl}$ :

$$\bar{\Gamma}^i_{kl} \bar{A}^l = \underbrace{\frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^k} \Gamma^m_{np}}_{(1,1)\text{-tensor } \odot} \underbrace{\left[ \frac{\partial x^p}{\partial \bar{x}^l} \bar{A}^l \right]}_{A^p} - \underbrace{\frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial^2 \bar{x}^i}{\partial x^n \partial x^p}}_{\text{Problematic term in Eq. (3.73)}} \underbrace{\left[ \frac{\partial x^p}{\partial \bar{x}^l} \bar{A}^l \right]}_{A^p} \quad (3.77)$$

*Idea:* Add Eq. (3.73) and Eq. (3.77) to cancel the problematic term:

$$\bar{A}^i_{;k} + \bar{\Gamma}^i_{kp} \bar{A}^p = \underbrace{\frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^l} \left[ A^l_{;m} + \Gamma^l_{mp} A^p \right]}_{(1,1)\text{-tensor } \odot\odot} \quad (3.78)$$

37 | This motivates the definition of the  $\ast\ast$  *Covariant derivative*:

Scalar:	$\Phi_{;k} := \Phi_{,k}$	(3.79a)
Contravariant vector:	$A^i_{;k} := A^i_{,k} + \Gamma^i_{kl} A^l$	(3.79b)
Covariant vector:	$B_{i;k} := B_{i,k} - \Gamma^l_{ik} B_l$	(3.79c)

- With this definition,  $A^i_{;k}$  is a (1, 1)-tensor and  $B_{i;k}$  is a (0, 2)-tensor!
- With this definition, the product rule is valid for the covariant derivative:

$$(A^i B_i)_{;k} = (A^i B_i)_{,k} \stackrel{\circ}{=} A^i_{;k} B_i + A^i B_{i;k} \quad (3.80)$$

- The construction of higher-rank tensors by tensoring contra- and covariant vectors Eq. (3.32) and the definitions of the covariant derivative above Eq. (3.79) can be used to construct covariant derivatives of arbitrary tensor fields. For example:

$$T^i_{k;l} := T^i_{k,l} + \Gamma^i_{ml} T^m_k - \Gamma^m_{kl} T^i_m \quad (3.81)$$

- With this generalization, we can apply the covariant derivative multiple times. For example:

$$A^i_{;k;l} \equiv \left( A^i_{;k} \right)_{;l} \quad (3.82)$$

- The covariant derivative is *not commutative* in general:

$$A^i_{;k;l} - A^i_{;l;k} \neq 0 \quad (3.83)$$

$\rightarrow$  Riemann curvature tensor  $\rightarrow$  GENERAL RELATIVITY ( $\rightarrow$  later)

(This is not the case for the “normal” derivative:  $A^i_{;k,l} = A^i_{;l,k}$ .)



**38** | Conclusion:

If you can formulate an equation that describes a physical theory in terms of tensors, it can always be brought into the form

$$T^I{}_J(x) = 0. \tag{3.84}$$

(This equation is meant to hold for all values of indices  $I$  and  $J$  and all coordinate values  $x$ .)

Here is an example:

The (inhomogeneous) Maxwell equations on an arbitrary (potentially curved) spacetime read:

$$\underbrace{F^{\mu\nu}{}_{;\nu} + \frac{4\pi}{c} J^\mu}_{=:T^\mu(x)} = 0 \tag{3.85}$$

with current density  $J^\mu$  and field strength tensor  $F^{\mu\nu} = g^{\mu\rho} g^{\nu\pi} (A_{\pi;\rho} - A_{\rho;\pi})$ .

How does Eq. (3.84) look like in any other coordinate system  $\bar{x} = \varphi(x)$ ?

Easy:

$$\bar{T}^I{}_J(\bar{x}) = \frac{\partial \bar{x}^I}{\partial x^M} \frac{\partial x^N}{\partial \bar{x}^J} \underbrace{T^M{}_N(x)}_{=0} = 0 \quad \Leftrightarrow \quad \bar{T}^I{}_J(\bar{x}) = 0. \tag{3.86}$$

This means:

Tensor equations are automatically form-invariant under *arbitrary* coordinate transformations; we say they exhibit  $\ast\ast$  (*manifest*) *general covariance*.

The “manifest” means that checking general covariance is just a matter of checking whether the equation “looks right”, i.e., whether it is built from tensors following the rules discussed in this chapter. If a property of an equation is manifest, you don’t have to do calculations to verify it!

In the next chapter, we take a step back and specialize the allowed coordinate transformations to the Lorentz transformations of SPECIAL RELATIVITY. We can then use the form-invariance of equations built from “Lorentz tensors” to construct Lorentz covariant equations from scratch – which was our original goal!