3.5. The metric tensor

A differentiable manifold $M$ does not automatically allow us to measure the length of curves, the angles of intersecting lines, or the area/volume of subsets of the manifold; to do so, we need a metric on $M$ (which is an additional piece of information). While the continuity structure (an atlas) that comes with $M$ determines its topology, the metric determines its geometry (shape). The same manifold $M$ can be equipped with different metrics; this corresponds to different geometries of the same topology (a potato and an egg both have the topology of a sphere, nonetheless they are geometrically distinct).

A differentiable manifold together with a (pseudo-)metric is called a (pseudo-)Riemannian manifold. In special relativity and general relativity, spacetime is modeled by such (pseudo-)Riemannian manifolds where the metric is used to represent spatial and temporal distances between events.

Motivation:

On linear spaces $V$, it is convenient to define an inner product (like in quantum mechanics where you consider Hilbert spaces and use their inner product to compute probabilities and transition amplitudes).

Recall the definition of a (real) inner product:

\[ \langle \cdot | \cdot \rangle : V \times V \rightarrow \mathbb{R} \quad \text{with} \quad \]

\[ \text{Symmetry: } \langle x | y \rangle = \langle y | x \rangle \]

\[ \text{(Bi)linearity: } \langle ax + by | z \rangle = a \langle x | z \rangle + b \langle y | z \rangle \]

\[ \text{Positive-definiteness: } x \neq 0 \Rightarrow \langle x | x \rangle > 0 \]

Once you have an inner product, you get a norm, and subsequently a metric for free:

\[ \| x \| := \sqrt{\langle x | x \rangle} \]

\[ d(x, y) := \| x - y \| \]

Thus an inner product is a rather versatile structure and nice to have!

Problem: We cannot define an inner product on the manifold directly because $M$ is not a linear space.

However: We can introduce an inner product on each of its tangent spaces $T_p M$ !

Riemannian (Pseudo-)Metric $d s^2 := \text{Symmetric, non-degenerate (0, 2)-tensor field:}$

\[ d s_p^2 \text{ bilinear } \Rightarrow d s_p^2 \in T_p^* M \otimes T_p^* M \]

\[ d s_p^2 = \sum_{i,j=1}^{D} g_{ij}(x) \, d x^i \otimes d x^j \equiv g_{ij}(x) \, d x^i d x^j \]

with $g_{ij} = g_{ji}$ (symmetry) and $g = \det(g_{ij}) \neq 0$ (non-degeneracy).
The tensor product is non-commutative: \( \text{d}x^i \otimes \text{d}x^j \neq \text{d}x^j \otimes \text{d}x^i \). However, you can always decompose a tensor product as

\[
\text{d}x^i \otimes \text{d}x^j = \frac{1}{2} (\text{d}x^i \otimes \text{d}x^j + \text{d}x^j \otimes \text{d}x^i) + \frac{1}{2} (\text{d}x^i \otimes \text{d}x^j - \text{d}x^j \otimes \text{d}x^i) \quad (3.47)
\]

with the symmetrized tensor product \( \text{d}x^i \vee \text{d}x^j \) and the anti-symmetrized tensor product \( \text{d}x^i \wedge \text{d}x^j \) (\( \wedge \) wedge product).

Since \( g_{ij} \) is assumed to be symmetric, only the symmetric component survives:

\[
g_{ij}(x) \text{d}x^i \otimes \text{d}x^j = g_{ij}(x) \text{d}x^i \vee \text{d}x^j = g_{ij}(x) \text{d}x^i \text{d}x^j \quad (3.48)
\]

This means that when writing \( \text{d}x^i \text{d}x^j \) in the above formula, you can be sloppy and either mean \( \text{d}x^i \otimes \text{d}x^j \) or, equivalently, \( \text{d}x^i \vee \text{d}x^j \). You will find both conventions in the literature. I will use \( \text{d}x^i \text{d}x^j \equiv \text{d}x^i \vee \text{d}x^j \) so that \( \text{d}x^i \text{d}x^j = \text{d}x^j \text{d}x^i \).

- It would be more appropriate to write \( g = g_{ij} \text{d}x^i \text{d}x^j \) for the metric \((0,2)\)-tensor; it is conventional, however, to reserve \( g \) for the determinant \( \det(g_{ij}) \) so that we are stuck with \( \text{d}s^2 \) for the metric. Note that the \( \text{d} \) in \( \text{d}s^2 \) does not refer to an \( \uparrow \) exterior derivative, it is purely symbolical.

- To define a proper \( \uparrow \) inner product on \( T_p M \), we should demand \( \downarrow \) positive-definiteness instead of non-degeneracy. This, however, is often (for example in relativity) too restrictive; as it turns out, non-degeneracy is all we need for an isomorphism between \( T_p M \) and \( T_p^* M \) ("pulling indices up and down", \( \rightarrow \) below). This is why negative eigenvalues of \( g_{ij} \) are fine for many purposes, and motivates the concept of a \( \rightarrow \) signature:

\[
\text{Signature:}
\]

Since \( g_{ij}(x) = g_{ji}(x) \) and \( \det(g_{ij}(x)) \neq 0 \)

\( \rightarrow \) \( g_{ij}(x) \) has \( r \) positive and \( s \) negative real eigenvalues for all \( p \in M \)

Since \( \det(g_{ij}(x)) \neq 0 \), these numbers must be the same for all \( p \in M \).

\( \rightarrow (r, s): \uparrow \) Signature of the metric \( \text{d}s^2 \)

This classification does not depend on the coordinate basis (\( \uparrow \) Sylvester’s law of inertia).

- \((r > 0, s = 0)\)

\( \rightarrow \) \( \text{d}s^2 \): Riemannian metric \( \rightarrow \) (\( M, \text{d}s^2 \)): \( \uparrow \) Riemannian manifold

I.e., \( g_{ij} \) has only positive eigenvalues for all \( p \in M \) and is therefore \( \uparrow \) positive-definite. This produces a true, positive-definite inner product on \( T_p M \).

- \((r > 0, s > 0)\)

\( \rightarrow \) \( \text{d}s^2 \): pseudo-Riemannian metric \( \rightarrow \) (\( M, \text{d}s^2 \)): \( \uparrow \) pseudo-Riemannian manifold

I.e., \( g_{ij} \) has both positive and negative eigenvalues and is therefore \( \downarrow \) indefinite.

- \((r > 0, s = 1)\) or \((r = 1, s > 0)\):

\( \rightarrow \) \( \text{d}s^2 \): Lorentzian metric \( \rightarrow \) (\( M, \text{d}s^2 \)): \( \uparrow \) Lorentzian manifold

In relativity we are only interested in metric tensors with one positive and three negative eigenvalues (equivalently: three positive and one negative eigenvalue). Mathematically speaking, spacetime is then a four-dimensional Lorentzian manifold and a special case of a pseudo-Riemannian manifold.
Example: (Details: Problemset 4)

\( i \)  \( \mathcal{M} = \mathbb{R}^2 \)

\( \mathcal{M} \) is Euclidean space \( E_2 \equiv (\mathbb{R}^2, ds^2_\mathcal{E}) \)

The Euclidean metric in Cartesian coordinates reads:

\[
    ds^2_\mathcal{E} := dx^2 + dy^2 = g_{ij}(x) \, dx^i \, dx^j \quad \text{with} \quad (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

This is consistent with the notion of \( dx \) and \( dy \) as infinitesimal shifts in coordinates and \( ds^2 \) as the infinitesimal distance (squared) that corresponds to this shift:

\[
    \mathcal{M} = \mathbb{R}^2
\]

\( ii \)  We can now transition to a new chart, namely polar coordinates \( \tilde{x}^1 = r \) and \( \tilde{x}^2 = \theta \). The induced basis change on the cotangent space is given by the total differential of the coordinate functions Eq. (3.14):

\[
    \varphi^{-1} : \begin{cases} 
    x = r \cos(\theta) \\
    y = r \sin(\theta)
    \end{cases} \quad \Rightarrow \quad \begin{aligned}
    dx &= \cos(\theta) \, dr - r \sin(\theta) \, d\theta \\
    dy &= \sin(\theta) \, dr + r \cos(\theta) \, d\theta
    \end{aligned}
\]

\( iii \)  We find the components of the metric tensor field in the new basis \( \{d\tilde{x}^1 = dr, d\tilde{x}^2 = d\theta\} \):

\[
    ds^2 = dr^2 + r^2 d\theta^2 = \tilde{g}_{ij}(\tilde{x}) \, d\tilde{x}^i \, d\tilde{x}^j \quad \text{with} \quad (\tilde{g}_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.
\]

This expression is again compatible with infinitesimal shifts in the (new) coordinates \( r \) and \( \theta \):

\[
    \mathcal{M} = \mathbb{R}^2
\]

• The Euclidean plane \( E_2 \) is therefore an example for a Riemannian manifold with metric signature \((2, 0)\); its distinctive feature is that it is flat.
• Note that here we compute *the same* infinitesimal length in different coordinates (with the same result)! We did not change the *metric*, only the *coordinates* and thereby the coordinate basis in which we express the metric tensor. This is flat Euclidean space in \( \uparrow \text{curvilinear coordinates} \). By contrast, later in \textit{general relativity} we will study curved (non-flat, non-Euclidean) metric tensors, i.e., we will modify the geometry of space(time) itself.

29 | Since the metric \( ds^2 \) is a \((0, 2)\)-tensor field:

\[
\tilde{g}_{ij}(\tilde{x}) d\tilde{x}^i d\tilde{x}^j = ds^2 = g_{ij}(x) dx^i dx^j
\]

Eq. (3.14) \( \rightarrow \)

\[
\tilde{g}_{ij}(\tilde{x}) = \frac{\partial x^i}{\partial \tilde{x}^m} \frac{\partial x^j}{\partial \tilde{x}^n} g_{mn}(x)
\]

The metric (components) transforms as any other \((0, 2)\) tensor. Nothing special!

\text{Side note:}

Let \( g := \det(g_{ij}) \) and \( \tilde{g} := \det(\tilde{g}_{ij}) \)

\[
\sqrt{|\tilde{g}|} = \left| \det \left( \frac{\partial x}{\partial \tilde{x}} \right) \right| \sqrt{|g|}
\]

\( \rightarrow \sqrt{|\tilde{g}|} \) is a \textit{pseudo} scalar tensor density of weight \( w = +1 \). The “pseudo” indicates that the absolute value of the Jacobian determinant shows up, cf. Eq. (3.37).

\(< g < 0 \) \( \rightarrow \int \sqrt{-g} \) is a scalar (\( \rightarrow \text{later} \))!

30 | Length of curves on \( M \):

One immediate benefit of having a Riemannian manifold is that we can now compute the length of curves \( \gamma(t) \) on \( M \) (parametrized by \( t \in [a, b] \) and given in some chart):

\[
L[\gamma] = \int_y ds := \int_a^b \sqrt{g_{ij}(\gamma(t)) \frac{dy^i(t)}{dt} \frac{dy^j(t)}{dt}} \, dt
\]

\( \rightarrow \frac{1}{2R} \) is a \textit{true} pseudo metric (i.e., \( g_{ij} \) has at least one negative eigenvalue), one must make sure that the chosen curve \( \gamma \) does not produce negative values under the square root. In \textit{relativity} these will be \( \uparrow \text{time-like} \) curves.

\textit{Example:}

Let \( \gamma \) be the circle with radius \( R \) in the Euclidean plane \( E_2 \). A possible parametrization in Cartesian coordinates (with origin in the center of the circle) is \( \tilde{\gamma}_{xy}(t) = (x(t), y(t)) = (R \cos(t), R \sin(t)) \) with \( 0 \leq t < 2\pi \) so that one finds for the circumference:

\[
L = \int_0^{2\pi} \sqrt{\dot{x}^2 + \dot{y}^2} \, dt = 2\pi R
\]
The same length can of course be calculated with the parametrization \( \gamma_r^\theta(t) = (r(t), \theta(t)) = (R, t) \) and \( 0 \leq t < 2\pi \) in polar coordinates:

\[
L = \int \sqrt{dr^2 + r^2d\theta^2} = \int_0^{2\pi} \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} \, dt \approx 2\pi R
\]  

(3.58)

Details: → Problemset 4

Besides computing lengths of curves (and other geometric quantities, → later), there is another benefit of having a metric tensor:

**Pulling indices down:**

\[
\tilde{T}^{i_1 \ldots i_p} \overset{\square}{\ldots} \overset{\square}{i_q} := g_{i^k} T^{i_1 \ldots \overset{\square}{k} \ldots \overset{\square}{i_p} \overset{\square}{\ldots} \square} \overset{\square}{\ldots} \overset{\square}{j_1} \ldots \overset{\square}{j_q} 
\]

(3.59)

\( \to \tilde{T} \) is a tensor of type \((p - 1, q + 1)\)

- In Eq. (3.59) we indicate “empty” slots for indices by \( \square \) to emphasize that in each index “column” an index can either be up (contravariant) or down (covariant). It is conventional to omit the \( \square \)-markers. Note that this explains why you never should write two indices directly above each other (except for special cases, → below).

Furthermore, since \( g \) is fixed, it makes sense to label \( \tilde{T} \) again by \( T \) (note that the difference between the original tensor and the new one is manifest in the different index patterns!):

\[
\tilde{T}^{i_1 \ldots i_p} \overset{\square}{\ldots} \overset{\square}{i_q} \Rightarrow T^{i_1 \ldots \overset{\square}{i_p} \overset{\square}{\ldots} \square j_1} \ldots \overset{\square}{j_q} 
\]

(3.60)

Example:

\[
A^{i \overset{\square}{k} j} \overset{\square}{l} := g_{jm} A^{imk} \overset{\square}{l} 
\]

(3.61)

- This convention matches perfectly with the computation of an inner product (which is determined by the metric tensor \( g \)) of two contravariant vectors:

\[
\langle A, B \rangle \overset{\text{def}}{=} g_{ij} A^i B^j \overset{\text{def}}{=} \underbrace{A_i B_i}_{\text{Scalar}} 
\]

(3.62)

**Pulling indices up:**

We would like to have a \((2, 0)\)-tensor \( g^{ij} \) with the property

\[
g^{ij} T^j = T^k \overset{1}{=} g^{ki} T_i \overset{\text{def}}{=} g^{ki} g_{ij} T^j .
\]

(3.63)

\( g^{ij} \) allows us to revert the pulling-down of indices defined by the metric \( g_{ij} \). Note that \( g^{ij} \) is a different tensor than \( g_{ij} \), we could call it \( \tilde{g}^{ij} \); however, it is conventional to denote it with the same label due to the following close relationship with \( g \):

\[
g^{ki} g_{ij} \overset{1}{=} \delta^k_i 
\]

(3.64)

This is an implicit equation for \( g^{ki} \)!
\( g^{ij} \) is the inverse matrix of \( g_{ij} \).

Which always exists because \( ds^2 \) is non-degenerate: \( \det(g_{ij}) \neq 0 \).

\( \rightarrow \) In general:

\[
\tilde{T}^{i_1 \ldots i_p \square \ldots j \square \ldots j_q} := g^{jk} T^{-i_1 \ldots i_p \square \ldots j \square \ldots j_q}_{k\ldots k \ldots j_q} \quad (3.65)
\]

\( \tilde{T} \) is a tensor of type \((p + 1, q - 1)\)

- Again we relabel \( \tilde{T} \) to \( T \) and omit the \( \square \)-markers:

\[
\tilde{T}^{i_1 \ldots i_p \square \ldots j \square \ldots j_q} \rightarrow T^{i_1 \ldots i_p \quad j \quad \ldots j_q} \quad (3.66)
\]

- Example:

\[
A^{ijkl} := g^{lm} A^{ij}^{\quad kl} \quad (3.67)
\]

- With these new definitions, we can now raise and lower contractions:

\[
A^i B_i = A^i \delta^i_k B_k = A^i g_{ik} g^{kj} B_j = A^i g_{ik} B^k = A_k B^k = A_i B^i \quad (3.68)
\]

- What happens if you pull the indices of the Kronecker symbol up or down?

\[
\delta_{ij} := g^{ik} \delta_{k}^i = g^{ij} \quad \text{and} \quad \delta_{ij} := g_{ik} \delta^k_j = g_{ij} \quad (3.69)
\]

\( \delta_{ij} \) and \( \delta_{ij} \) denote the metric and its inverse!

\( \rightarrow \) We never use the notation \( \delta_{ij} \) and \( \delta_{ij} \) to prevent confusion!

- Note that in general

\[
g^{ik} T^i_k = T^{ij} \neq T^{ji} = g^{jk} T^j_k \quad . \quad (3.70)
\]

This means that the “column” in which the index is located is important, and notations like \( T^i_k \) are ill defined (if you pull \( k \) up by \( g^{ik} \), do you get \( T^{ij} \) or \( T^{ji} \)?). However, if the tensor is symmetric, \( T^{ij} = T^{ji} \); this does not matter and you can get away with the sloppy notation \( T^i_k \).

This explains why writing \( \delta_{ik} \) for the Kronecker symbol is fine: \( g^{ij} = g^{jk} \delta_{ik} \) is symmetric.

### Mathematical side note:

“Pulling indices up and down” is mathematically the application of an isomorphism between \( T_p M \) and \( T_p^* \):

\[
g(\bullet, \bullet) : T_p M \ni A \mapsto g(A, \bullet) \in T_p^* M \quad (3.71)
\]

This has nothing to do with differential geometry or manifolds in particular; it is a general feature of non-degenerate bilinear forms on vector spaces. In differential geometry, this canonical isomorphism between the tangent bundle \( TM \) and the cotangent bundle \( T^* M \) is known as musical isomorphism.

For example, you are using the same kind of isomorphism all the time in quantum mechanics, namely whenever you “dagger” a ket \( \ket{\Psi} \) to obtain a bra \( \bra{\Psi} \):

\[
(\bullet)^\dagger : \mathcal{H} \ni \ket{\Psi} \mapsto \bra{\Psi} \ni \bra{\Psi}^\dagger \in \mathcal{H}^* \quad \text{with} \quad \bra{\Psi} \ket{\Phi} = (\bra{\Phi}, \Phi)^\dagger \quad \text{for all} \quad \Phi \in \mathcal{H} \quad . \quad (3.72)
\]
Note how the bra bra $\langle \Psi |$ associated to the ket $| \Psi \rangle$ is defined via the inner product $\langle \bullet | \bullet \rangle$ (and therefore metric) of the Hilbert space (Kronecker representation theorem)!

This leads to a nice dictionary between concepts in tensor calculus (and therefore Relativity) and the bra-ket formalism of quantum mechanics:

<table>
<thead>
<tr>
<th>Relativity (fixed $p \in M$)</th>
<th>Quantum mechanics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner product space</td>
<td>$T_p M$</td>
</tr>
<tr>
<td>Basis</td>
<td>${ \partial_i }$</td>
</tr>
<tr>
<td>Vector</td>
<td>$A = A^i \partial_i$</td>
</tr>
<tr>
<td>Dual space</td>
<td>$T^*_p M$</td>
</tr>
<tr>
<td>Dual basis</td>
<td>${ dx^i }$</td>
</tr>
<tr>
<td>Covector</td>
<td>$B = B_i dx^i$</td>
</tr>
<tr>
<td>Inner product</td>
<td>$g(A_1, A_2) = g_{ij} A^i_1 A^j_2$</td>
</tr>
<tr>
<td>Tensor</td>
<td>$A = A^{ij} \partial_i \otimes \partial_j$</td>
</tr>
<tr>
<td>Operator</td>
<td>$T = T^i_\partial \partial_i \otimes dx^j$</td>
</tr>
<tr>
<td>Trace</td>
<td>$T^i_\partial$</td>
</tr>
<tr>
<td>Scalar</td>
<td>$BA = B_i A^i = g_{ij} B^j A^i$</td>
</tr>
</tbody>
</table>

Pulling indices down

| $A_i = g_{ij} A^j$ |

Pulling indices up

| $A^i = g^{ij} A_j$ |

### 3.6. Differentiation of tensor fields

Remember: $\partial_i \Phi$ is covariant vector if $\Phi$ is scalar. However:

#### < Contravariant vector $A^i$:

$$\tilde{A}^i_{\;k} = \frac{\partial \tilde{A}^i}{\partial x^k} = \frac{\partial x^m}{\partial \tilde{x}^k} \frac{\partial}{\partial x^m} \left[ \frac{\partial \tilde{x}^i}{\partial x^l} A^l \right] = \frac{\partial^2 \tilde{x}^i}{\partial x^m \partial x^l} + \frac{\partial x^m}{\partial \tilde{x}^k} \frac{\partial x^l}{\partial \tilde{x}^i} A^l = \frac{\partial x^m}{\partial \tilde{x}^k} \frac{\partial x^l}{\partial \tilde{x}^i} A^l$$

(3.73)

Here we used the transformation of $\tilde{A}^i$ [Eq. (3.8)] and $\tilde{\partial}_k$ [Eq. (3.5)] and the product rule.

#### \rightarrow In general: $\frac{\partial A^i}{\partial x^k}$ is not a tensor!

#### 35 | How to define a derivative of tensor fields that again transforms as a tensor?

To solve this problem, we first need a new field:

#### \rightarrow \textit{Christoffel symbols (of the second kind)}:

$$\Gamma^i_{\;kl} := \frac{1}{2} g^{im} \left( g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$$

(3.74)

- The Christoffel symbols are symmetric in the lower two indices: $\Gamma^i_{\;kl} = \Gamma^i_{\;lk}$

- ! Despite the index notation, the Christoffel symbols are not tensors:

$$\tilde{\Gamma}^i_{\;kl} = \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^m}{\partial \tilde{x}^k} \frac{\partial x^p}{\partial \tilde{x}^l} \Gamma^m_{\;np} - \frac{\partial x^m}{\partial \tilde{x}^k} \frac{\partial x^p}{\partial \tilde{x}^l} \frac{\partial^2 \tilde{x}^i}{\partial x^m \partial x^p}$$

(3.75)

No tensor!
This is why they are called “symbols” and not “tensors”!

- There are also Christoffel symbols of the first kind:

\[ \Gamma_{ikl} := g^{ij} \Gamma_{jkl} = \frac{1}{2} \left( g_{ik,j} + g_{il,k} - g_{kl,i} \right) \]  

(3.76)

- Mathematically, the Christoffel symbols are the coefficients (in some basis) of the Levi-Civita connection which is determined by the metric tensor \( g^{ij} \) (→ later).

\[ \mathcal{A}^i_{:k} + \Gamma^i_{k:p} \mathcal{A}^p = \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial x^l}{\partial \bar{x}^i} \left[ A^l_{:m} + \Gamma^l_{m:p} A^p \right] \]  

(3.77)

Idea: Add Eq. (3.73) and Eq. (3.77) to cancel the problematic term:

\[ (1, 1)-\text{tensor} \]

36 | Contravariant vector \( \mathcal{A}^i \) and contract it with \( \tilde{\Gamma}^i_{k:i} \):  

37 | This motivates the definition of the \( \star \) Covariant derivative:

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
</tr>
</thead>
</table>
| Scalar:                  | \( \Phi_{;i} := \Phi_{,i} \)  
| Contravariant vector:    | \( A^i_{;k} := A^i_{,k} + \Gamma^i_{k:l} A^l \)  
| Covariant vector:        | \( B^i_{;k} := B^i_{,k} \)  

(3.79a)

(3.79b)

(3.79c)

- With this definition, \( A^i_{;k} \) is a \((1, 1)\)-tensor and \( B^i_{;k} \) is a \((0, 2)\)-tensor!

- With this definition, the product rule is valid for the covariant derivative:

\[ (A^i B^l)_{;k} = (A^i B^l)_{,k} = A^i_{;k} B^l + A^i B^l_{;k} \]  

(3.80)

- The construction of higher-rank tensors by tensoring contra- and covariant vectors Eq. (3.32) and the definitions of the covariant derivative above Eq. (3.79) can be used to construct covariant derivatives of arbitrary tensor fields. For example:

\[ T^{i}_{j:l} := T^{i}_{j,l} + \Gamma^{i}_{m:l} T^{m}_{j,k} - \Gamma^{m}_{j:k} T^{i}_{m} \]  

(3.81)

- With this generalization, we can apply the covariant derivative multiple times. For example:

\[ A^i_{;k:l} \equiv \left( A^i_{;k} \right)_{;l} \]  

(3.82)

- The covariant derivative is not commutative in general:

\[ A^i_{;k:l} - A^i_{;l:k} \neq 0 \]  

(3.83)

→ Riemann curvature tensor → GENERAL RELATIVITY (→ later)

(This is not the case for the “normal” derivative: \( A^i_{,k:l} = A^i_{,l,k} \))
Conclusion:

If you can formulate an equation that describes a physical theory in terms of tensors, it can always be brought into the form

\[ T^I_J (x) = 0. \]  \hspace{1cm} (3.84)

(This equation is meant to hold for all values of indices \( I \) and \( J \) and all coordinate values \( x \).)

Here is an example:

The (inhomogeneous) Maxwell equations on an arbitrary (potentially curved) spacetime read:

\[ F^\mu\nu + \frac{4\pi}{c} J^\mu = 0 \]
\[ = T^\mu(x) \]  \hspace{1cm} (3.85)

with current density \( J^\mu \) and field strength tensor \( F^\mu\nu = g^{\mu\rho} g^{\nu\sigma} (A_\rho,\nu - A_\nu,\rho) \).

How does Eq. (3.84) look like in any other coordinate system \( \tilde{x} = \varphi(x) \)?

Easy:

\[ \tilde{T}^I_J (\tilde{x}) = \frac{\partial \tilde{x}^I}{\partial \tilde{x}^M} \frac{\partial \tilde{x}^M}{\partial x^I} T^M_N (x) = 0 \Leftrightarrow \tilde{T}^I_J (\tilde{x}) = 0. \]  \hspace{1cm} (3.86)

This means:

Tensor equations are automatically form-invariant under arbitrary coordinate transformations; we say they exhibit \( \boxdot \) (manifest) general covariance.

The “manifest” means that checking general covariance is just a matter of checking whether the equation “looks right,” i.e., whether it is built from tensors following the rules discussed in this chapter. If a property of an equation is manifest, you don’t have to do calculations to verify it!

In the next chapter, we take a step back and specialize the allowed coordinate transformations to the Lorentz transformations of special relativity. We can then use the form-invariance of equations built from “Lorentz tensors” to construct Lorentz covariant equations from scratch – which was our original goal!