↓Lecture 8 [05.12.23]

3.5. The metric tensor

A differentiable manifold M does not automatically allow us to measure the length of curves, the angles of intersecting lines, or the area/volume of subsets of the manifold; to do so, we need a *metric* on M (which is an additional piece of information). While the continuity structure (an atlas) that comes with M determines its *topology*, the metric determines its *geometry* (= shape). The same manifold M can be equipped with *different* metrics; this corresponds to different geometries of the same topology (a potato and an egg both have the topology of a sphere, nonetheless they are geometrically distinct).

A differentiable manifold together with a (pseudo-)metric is called ↑ *(pseudo-)Riemannian manifold*. In SPECIAL RELATIVITY and GENERAL RELATIVITY, spacetime is modeled by such (pseudo-)Riemannian manifolds where the metric is used to represent spatial and temporal distances between events.

25 | Motivation:

On linear spaces V, it is convenient to define an ↓ *inner product* (like in quantum mechanics where you consider Hilbert spaces and use their inner product to compute probabilities and transition amplitudes).

Recall the definition of a (real) inner product:

$$\langle \bullet | \bullet \rangle : V \times V \to \mathbb{R} \quad \text{with} \ldots \tag{3.44a}$$

$$\text{Symmetry:} \quad \langle x|y\rangle = \langle y|x\rangle \tag{3.44b}$$

$$\text{(Bi)linearity:} \quad \langle ax+by|z\rangle = a\langle x|z\rangle + b\langle y|z\rangle \tag{3.44c}$$

$$\text{Positive-definiteness:} \quad x \neq 0 \;\Rightarrow\; \langle x|x\rangle > 0 \tag{3.44d}$$

Once you have an inner product, you get a norm, and subsequently a metric for free:

$$\underbrace{\langle x|y\rangle}_{\text{Inner product}} \quad \Rightarrow \quad \underbrace{\|x\| := \sqrt{\langle x|x\rangle}}_{\text{Norm}} \quad \Rightarrow \quad \underbrace{d(x,y) := \|x-y\|}_{\text{Metric}} \tag{3.45}$$

Thus an inner product is a rather versatile structure and nice to have!

Problem: We cannot define a inner product on the manifold directly because M is not a linear space.

However: We can introduce an inner product on each of its tangent spaces T_pM! →

26 | ⁂ *Riemannian (Pseudo-)Metric* $\mathrm{d}s^2$:= Symmetric, non-degenerate $(0,2)$-tensor field:

$$\mathrm{d}s^2 : M \ni p \mapsto \underbrace{\left(\mathrm{d}s^2_p : T_pM \times T_pM \to \mathbb{R}\right)}_{\text{Bilinear \& symmetric \& non-degenerate}} \tag{3.46a}$$

$$\mathrm{d}s^2_p \text{ bilinear} \;\Rightarrow\; \mathrm{d}s^2_p \in T^*_pM \otimes T^*_pM$$

$$\Rightarrow\; \mathrm{d}s^2_p = \sum_{i,j=1}^{D} g_{ij}(x)\,\mathrm{d}x^i \otimes \mathrm{d}x^j \equiv g_{ij}(x)\,\mathrm{d}x^i\mathrm{d}x^j \tag{3.46b}$$

with $g_{ij} = g_{ji}$ (symmetry) and $g = \det(g_{ij}) \neq 0$ (non-degeneracy).

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- The tensor product is *non-commutative*: $\mathrm{d}x^i \otimes \mathrm{d}x^j \neq \mathrm{d}x^j \otimes \mathrm{d}x^i$. However, you can always decompose a tensor product as

$$\mathrm{d}x^i \otimes \mathrm{d}x^j = \underbrace{\frac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}x^j + \mathrm{d}x^j \otimes \mathrm{d}x^i)}_{=:\mathrm{d}x^i \vee \mathrm{d}x^j} + \underbrace{\frac{1}{2}(\mathrm{d}x^i \otimes \mathrm{d}x^j - \mathrm{d}x^j \otimes \mathrm{d}x^i)}_{=:\mathrm{d}x^i \wedge \mathrm{d}x^j} \tag{3.47}$$

  with the symmetrized tensor product $\mathrm{d}x^i \vee \mathrm{d}x^j$ and the anti-symmetrized tensor product $\mathrm{d}x^i \wedge \mathrm{d}x^j$ (↑ *wedge product*).

  Since g_{ij} is assumed to be symmetric, only the symmetric component survives:

$$g_{ij}(x)\mathrm{d}x^i \otimes \mathrm{d}x^j = g_{ij}(x)\mathrm{d}x^i \vee \mathrm{d}x^j \equiv g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j \tag{3.48}$$

  This means that when writing $\mathrm{d}x^i\mathrm{d}x^j$ in the above formula, you can be sloppy and either mean $\mathrm{d}x^i \otimes \mathrm{d}x^j$ or, equivalently, $\mathrm{d}x^i \vee \mathrm{d}x^j$. You will find both conventions in the literature. I will use $\mathrm{d}x^i\mathrm{d}x^j \equiv \mathrm{d}x^i \vee \mathrm{d}x^j$ so that $\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}x^j\mathrm{d}x^i$.

- It would be more appropriate to write $g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ for the metric $(0,2)$-tensor; it is conventional, however, to reserve g for the determinant $\det(g_{ij})$ so that we are stuck with $\mathrm{d}s^2$ for the metric. Note that the d in $\mathrm{d}s^2$ does *not* refer to an ↑ *exterior derivative*, it is purely symbolical.

- To define a proper ↓ *inner product* on T_pM, we should demand ↓ *positive-definiteness* instead of *non-degeneracy*. This, however, is often (for example in RELATIVITY) too restrictive; as it turns out, non-degeneracy is all we need for an isomorphism between T_pM and T_p^*M ("pulling indices up and down", → *below*). This is why *negative* eigenvalues of g_{ij} are fine for many purposes, and motivates the concept of a → *signature*:

27 | Signature:

Since $g_{ij}(x) = g_{ji}(x)$ and $\det(g_{ij}(x)) \neq 0$

→ $g_{ij}(x)$ has r *positive* and s *negative* real eigenvalues for all $p \in M$

Since $\det(g_{ij}(x)) \neq 0$, these numbers must be the *same* for all $p \in M$.

→ (r,s): ⁂ *Signature* of the metric $\mathrm{d}s^2$

This classification does not depend on the coordinate basis (↑ *Sylvester's law of inertia*).

- $(r > 0, s = 0)$

  → $\mathrm{d}s^2$: *Riemannian metric* → $(M, \mathrm{d}s^2)$: ⁂ *Riemannian manifold*

  I.e., g_{ij} has only positive eigenvalues for all $p \in M$ and is therefore ↓ *positive-definite*. This produces a true, positive-definite inner product on T_pM.

- $(r > 0, s > 0)$

  → $\mathrm{d}s^2$: *pseudo-Riemannian metric* → $(M, \mathrm{d}s^2)$: ⁂ *pseudo-Riemannian manifold*

  I.e., g_{ij} has both positive and negative eigenvalues and is therefore ↓ *indefinite*.

  - $(r > 0, s = 1)$ or $(r = 1, s > 0)$:

    → $\mathrm{d}s^2$: *Lorentzian metric* → $(M, \mathrm{d}s^2)$: ⁂ *Lorentzian manifold*

    In RELATIVITY we are only interested in metric tensors with one positive and three negative eigenvalues (equivalently: three positive and one negative eigenvalue). Mathematically speaking, spacetime is then a four-dimensional Lorentzian manifold and a special case of a pseudo-Riemannian manifold.

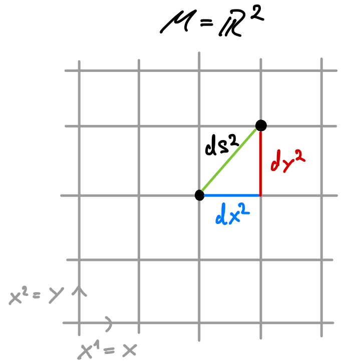
28 | Example: (Details: ➔ Problemset 4)

i | $\sphericalangle\ D = 2$ Euclidean space $E_2 \equiv (\mathbb{R}^2, \mathrm{d}s_E^2)$

The Euclidean metric in Cartesian coordinates $x^1 = x$ and $x^2 = y$ reads:

$$\mathrm{d}s_E^2 := \mathrm{d}x^2 + \mathrm{d}y^2 = g_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j \quad \text{with} \quad (g_{ij}) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\text{Signature}\\(2,0)}}\,. \tag{3.49}$$

This is consistent with the notion of $\mathrm{d}x$ and $\mathrm{d}y$ as infinitesimal shifts in coordinates and $\mathrm{d}s^2$ as the infinitesimal distance (squared) that corresponds to this shift:



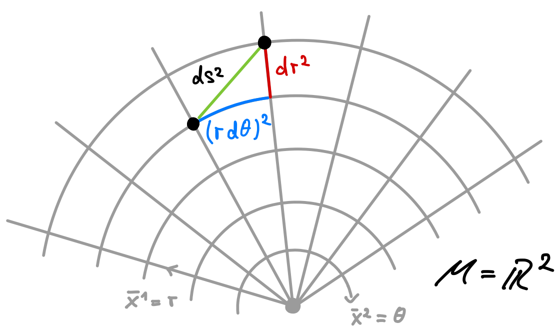
ii | We can now transition to a new chart, namely polar coordinates $\bar{x}^1 = r$ and $\bar{x}^2 = \theta$. The induced basis change on the cotangent space is given by the total differential of the coordinate functions Eq. (3.14):

$$\varphi^{-1}: \begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases} \overset{\text{Eq. (3.14)}}{\Rightarrow} \quad \begin{aligned} \mathrm{d}x &= \cos(\theta)\,\mathrm{d}r - r\sin(\theta)\,\mathrm{d}\theta \\ \mathrm{d}y &= \sin(\theta)\,\mathrm{d}r + r\cos(\theta)\,\mathrm{d}\theta \end{aligned} \tag{3.50}$$

iii | We find the components of the metric tensor field in the new basis $\{\mathrm{d}\bar{x}^1 = \mathrm{d}r, \mathrm{d}\bar{x}^2 = \mathrm{d}\theta\}$:

$$\mathrm{d}s^2 \overset{\circ}{=} \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 = \bar{g}_{ij}(\bar{x})\,\mathrm{d}\bar{x}^i\,\mathrm{d}\bar{x}^j \quad \text{with} \quad (\bar{g}_{ij}) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}}_{\substack{\text{Signature}\\(2,0)}}\,. \tag{3.51}$$

This expression is again compatible with infinitesimal shifts in the (new) coordinates r and θ:



- The Euclidean plane E_2 is therefore an example for a Riemannian manifold with metric signature $(2,0)$; its distinctive feature is that it is *flat*.

- Note that here we compute *the same* infinitesimal length in different coordinates (with the same result)! We did not change the *metric*, only the *coordinates* and thereby the coordinate basis in which we express the metric tensor. This is *flat* Euclidean space in ↑ *curvilinear coordinates*. By contrast, later in GENERAL RELATIVITY we will study curved (non-flat, non-Euclidean) metric tensors, i.e., we will modify the geometry of space(time) itself.

29 | Since the metric $\mathrm{d}s^2$ is a $(0,2)$-tensor field:

$$\bar{g}_{ij}(\bar{x})\mathrm{d}\bar{x}^i\mathrm{d}\bar{x}^j = \mathrm{d}s^2 = g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j \tag{3.52}$$

Eq. (3.14) $\overset{\circ}{\rightarrow}$

$$\bar{g}_{ij}(\bar{x}) = \frac{\partial x^l}{\partial \bar{x}^i}\frac{\partial x^m}{\partial \bar{x}^j}g_{lm}(x) \tag{3.53}$$

The metric (components) transforms as any other $(0,2)$ tensor. Nothing special!

Side note:

Let $g := \det(g_{ij})$ and $\bar{g} := \det(\bar{g}_{ij}) \xrightarrow{\text{Eq. (3.53)}}$

$$\sqrt{|\bar{g}|} = \left|\det\left(\frac{\partial x}{\partial \bar{x}}\right)\right|\sqrt{|g|} \tag{3.54}$$

$\rightarrow \sqrt{|g|}$ is a *pseudo* scalar tensor density of weight $w = +1$. The "pseudo" indicates that the absolute value of the Jacobian determinant shows up, cf. Eq. (3.37).

$\sphericalangle\, g < 0 \xrightarrow{\text{Eq. (3.39)}} \mathrm{d}^D x\sqrt{-g}$ is a scalar (→ *later*)!

30 | Length of curves on M:

One immediate benefit of having a Riemannian manifold is that we can now compute the length of curves $\gamma(t)$ on M (parametrized by $t \in [a,b]$ and given in some chart):

$$\boxed{L[\gamma] \equiv \int_\gamma \mathrm{d}s := \int_a^b \sqrt{g_{ij}(\gamma(t))\frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}\frac{\mathrm{d}\gamma^j(t)}{\mathrm{d}t}}\,\mathrm{d}t \tag{3.55}}$$

$$\boxed{\equiv \int_a^b \|\dot{\gamma}(t)\|_{\gamma(t)}\,\mathrm{d}t \tag{3.56}}$$

¡! If $\mathrm{d}s^2$ is a true *pseudo* metric (i.e., g_{ij} has at least one negative eigenvalue), one must make sure that the chosen curve γ does not produce negative values under the square root. In RELATIVITY these will be ↑ *time-like* curves.

Example:

Let γ be the circle with radius R in the Euclidean plane E_2. A possible parametrization in Cartesian coordinates (with origin in the center of the circle) is $\vec{\gamma}_{xy}(t) = (x_t, y_t) = (R\cos(t), R\sin(t))$ with $0 \leq t < 2\pi$ so that one finds for the circumference:

$$L = \int_\gamma \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \int_0^{2\pi}\sqrt{\dot{x}_t^2 + \dot{y}_t^2}\,\mathrm{d}t \overset{\circ}{=} 2\pi R \tag{3.57}$$

The same length can of course be calculated with the parametrization $\vec{\gamma}_{r\theta}(t) = (r_t, \theta_t) = (R, t)$ and $0 \leq t < 2\pi$ in polar coordinates:

$$L = \int_\gamma \sqrt{\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2} = \int_0^{2\pi} \sqrt{\dot{r}_t^2 + r_t^2 \dot{\theta}_t^2}\,\mathrm{d}t \overset{\circ}{=} 2\pi R \tag{3.58}$$

Details: ➔ Problemset 4

31 | Besides computing lengths of curves (and other geometric quantities, → *later*), there is another benefit of having a metric tensor:

Pulling indices *down*:

$$\boxed{\tilde{T}^{i_1 \dots \square \dots i_p \,\square \dots \square}{}_{\square \dots\, i\, \dots \square\, j_1 \dots j_q} := g_{ik} T^{i_1 \dots\, k\, \dots i_p \,\square \dots \square}{}_{\square \dots \square \dots \square\, j_1 \dots j_q}} \tag{3.59}$$

→ $\tilde{T}$ is a tensor of type $(p-1, q+1)$

- In Eq. (3.59) we indicate "empty" slots for indices by $\square$ to emphasize that in each index "column" an index can either be *up* (contravariant) or *down* (covariant). It is conventional to omit the $\square$-markers. Note that this explains why you never should write two indices directly above each other (except for special cases, → *below*).

  Furthermore, since g is fixed, it makes sense to label $\tilde{T}$ again by T (note that the difference between the original tensor and the new one is manifest in the different index patterns!):

$$\tilde{T}^{i_1 \dots \square \dots i_p \,\square \dots \square}{}_{\square \dots\, i\, \dots \square\, j_1 \dots j_q} \quad\mapsto\quad T^{i_1 \dots \quad \dots i_p}{}_{\quad\, i \qquad\; j_1 \dots j_q} \tag{3.60}$$

  Example:

$$A^i{}_j{}^k{}_l := g_{jm} A^{imk}{}_l \tag{3.61}$$

- This convention matches perfectly with the computation of an inner product (which is determined by the metric tensor g) of two contravariant vectors:

$$\langle A, B\rangle \overset{\text{def}}{=} g_{ij} A^i B^j \overset{\text{def}}{=} \underbrace{A^i B_i}_{\text{Scalar}} \tag{3.62}$$

32 | Pulling indices *up*:

We would like to have a $(2,0)$-tensor g^{ij} with the property

$$\delta^k_j T^j = T^k \overset{!}{=} g^{ki} T_i \overset{\text{def}}{=} g^{ki} g_{ij} T^j\,. \tag{3.63}$$

g^{ij} allows us to revert the pulling-down of indices defined by the metric g_{ij}. Note that g^{ij} is a *different* tensor than g_{ij}, we could call it $\tilde{g}^{ij}$; however, it is conventional to denote it with the same label due to the following close relationship with g:

$$\boxed{g^{ki} g_{ij} \overset{!}{=} \delta^k_j} \tag{3.64}$$

This is an implicit equation for g^{ki}!

→ g^{ij} is the *inverse matrix* of g_{ij}
(Which always exists because $\mathrm{d}s^2$ is non-degenerate: $\det(g_{ij}) \neq 0$.)

→ In general:

$$\boxed{\tilde{T}^{i_1 \dots i_p \square \dots j \dots \square}{}_{\square \dots \square j_1 \dots \square \dots j_q} := g^{jk} T^{i_1 \dots i_p \square \dots \square \dots \square}{}_{\square \dots \square j_1 \dots k \dots j_q}} \tag{3.65}$$

→ $\tilde{T}$ is a tensor of type $(p+1, q-1)$

- Again we relabel $\tilde{T}$ to T and omit the $\square$-markers:

$$\tilde{T}^{i_1 \dots i_p \square \dots j \dots \square}{}_{\square \dots \square j_1 \dots \square \dots j_q} \quad \mapsto \quad T^{i_1 \dots i_p \quad\; j}{}_{\quad\; j_1 \dots \quad \dots j_q} \tag{3.66}$$

- Example:

$$A^{ijkl} := g^{lm} A^{ijk}{}_m \tag{3.67}$$

- With these new definitions, we can now raise and lower contractions:

$$A^i B_i = A^i \delta_i^j B_j = A^i g_{ik} g^{kj} B_j = A^i g_{ik} B^k = A_k B^k = A_i B^i \tag{3.68}$$

- What happens if you pull the indices of the Kronecker symbol up or down?

$$\delta^{ij} := g^{jk} \delta^i{}_k = g^{ij} \quad \text{and} \quad \delta_{ij} := g_{ik} \delta^k{}_j = g_{ij} \tag{3.69}$$

¡! $\delta^{ij} \equiv g^{ij}$ and $\delta_{ij} \equiv g_{ij}$ denote the metric and its inverse!

→ We never use the notation δ^{ij} and δ_{ij} to prevent confusion!

- Note that in general

$$g^{jk} T^i{}_k = T^{ij} \neq T^{ji} = g^{jk} T_k{}^i \,. \tag{3.70}$$

This means that the "column" in which the index is located is *important*, and notations like T_k^i are ill defined (if you pull k up by g^{jk}, do you get T^{ij} or T^{ji}?). However, if the tensor is *symmetric*, $T^{ij} = T^{ji}$, this does not matter and you can get away with the sloppy notation T_k^i. This explains why writing δ_k^i for the Kronecker symbol is fine: $g^{ji} = g^{jk}\delta_k^i$ is symmetric.

33 | Mathematical side note:

"Pulling indices up and down" is mathematically the application of an ↓ *isomorphism* between $T_p M$ and $T_p^* M$:

$$g(\bullet, \bullet) : \; T_p M \ni A \mapsto g(A, \bullet) \in T_p^* M \tag{3.71}$$

This has nothing to do with differential geometry or manifolds in particular; it is a general feature of non-degenerate bilinear forms on vector spaces. In differential geometry, this canonical isomorphism between the tangent bundle TM and the cotangent bundle T^*M is know as ↑ *musical isomorphism*.

For example, you are using the same kind of isomorphism all the time in quantum mechanics, namely whenever you "dagger" a *ket* $|\Psi\rangle$ to obtain a *bra* $\langle\Psi|$:

$$(\bullet)^\dagger : \; \mathcal{H} \ni |\Psi\rangle \mapsto \langle\Psi| \equiv |\Psi\rangle^\dagger \in \mathcal{H}^* \quad \text{with} \quad \langle\Psi||\Phi\rangle \stackrel{!}{=} \langle\Psi|\Phi\rangle_{\mathcal{H}} \quad \text{for all } |\Phi\rangle \in \mathcal{H}. \tag{3.72}$$

Note how the bra *bra* $\langle\Psi|$ associated to the *ket* $|\Psi\rangle$ is *defined* via the inner product $\langle\bullet|\bullet\rangle_{\mathcal{H}}$ (and therefore metric) of the Hilbert space (↑ *Riesz representation theorem*)!

This leads to a nice dictionary between concepts in tensor calculus (and therefore RELATIVITY) and the bra-ket formalism of quantum mechanics:

	Relativity (fixed $p \in M$)	**Quantum mechanics**
Inner product space	$T_p M$	$\mathcal{H}$
Basis	$\{\partial_i\}$	$\{\|i\rangle\}$
Vector	$A = A^i \partial_i$	$\|\Psi\rangle = \Psi_i \|i\rangle$
Dual space	$T_p^* M$	$\mathcal{H}^*$
Dual basis	$\{\mathrm{d}x^i\}$	$\{\langle i\|\}$
…	$\mathrm{d}x^i(\partial_j) = \delta^i_j$	$\langle i\|j\rangle = \delta_{ij}$
Covector	$B = B_i \mathrm{d}x^i$	$\langle\Psi\| = \Psi_i^* \langle i\|$
Inner product	$g(A_1, A_2) = g_{ij} A_1^i A_2^j$	$\langle\Psi\|\Phi\rangle$
Tensor	$A = A^{ij} \partial_i \otimes \partial_j$	$\|\Psi\rangle \otimes \|\Phi\rangle \equiv \|\Psi\rangle\|\Phi\rangle$
…	$B = B_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$	$\langle\Psi\| \otimes \langle\Phi\| \equiv \langle\Psi\|\langle\Phi\|$
Operator	$T = T^i{}_j \partial_i \otimes \mathrm{d}x^j$	$\|\Phi\rangle \otimes \langle\Psi\| \equiv \|\Phi\rangle\langle\Psi\|$
Trace	$T^i{}_i$	$\mathrm{Tr}[\|\Phi\rangle\langle\Psi\|]$
Scalar	$BA = B_i A^i = g_{ij} B^i A^j$	$\langle\Psi\|\|\Phi\rangle = \langle\Psi\|\Phi\rangle$
Pulling indices down	$A_i = g_{ij} A^j$	$\langle\Psi\| = \|\Psi\rangle^\dagger$
Pulling indices up	$A^i = g^{ij} A_j$	$\|\Psi\rangle = \langle\Psi\|^\dagger$

3.6. Differentiation of tensor fields

34 | Remember: $\partial_i \Phi$ is covariant vector if Φ is scalar. However:

$\sphericalangle$ Contravariant vector A^i:

$$\bar{A}^i{}_{,k} \equiv \frac{\partial \bar{A}^i}{\partial \bar{x}^k} = \frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial}{\partial x^m}\left[\frac{\partial \bar{x}^i}{\partial x^l} A^l\right] = \underbrace{\frac{\partial^2 \bar{x}^i}{\partial x^m \partial x^l} \frac{\partial x^m}{\partial \bar{x}^k} A^l}_{\neq 0 \text{ (in general) ☹}} + \underbrace{\frac{\partial x^m}{\partial \bar{x}^k} \frac{\partial \bar{x}^i}{\partial x^l} \frac{\partial A^l}{\partial x^m}}_{(1,1)\text{-tensor ☺}} \tag{3.73}$$

Here we used the transformation of $\bar{A}^i$ [Eq. (3.8)] and $\bar{\partial}_k$ [Eq. (3.5)] and the product rule.

→ In general: $\frac{\partial \bar{A}^i}{\partial \bar{x}^k}$ is not a tensor!

35 | How to define a derivative of tensor fields that again transforms as a tensor?

To solve this problem, we first need a new field:

→ ⁂ *Christoffel symbols (of the second kind)*:

$$\boxed{\Gamma^i{}_{kl} := \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right)} \tag{3.74}$$

- The Christoffel symbols are symmetric in the lower two indices: $\Gamma^i{}_{kl} = \Gamma^i{}_{lk}$
- ¡! Despite the index notation, the Christoffel symbols are *not* tensors:

$$\bar{\Gamma}^i{}_{kl} \stackrel{\circ}{=} \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial x^p}{\partial \bar{x}^l} \Gamma^m{}_{np} - \underbrace{\frac{\partial x^n}{\partial \bar{x}^k} \frac{\partial x^p}{\partial \bar{x}^l} \frac{\partial^2 \bar{x}^i}{\partial x^n \partial x^p}}_{\text{No tensor!}} \tag{3.75}$$

This is why they are called "symbols" and not "tensors"!

- There are also Christoffel symbols of the *first* kind:

$$\Gamma_{ikl} := g_{ij}\Gamma^j{}_{kl} = \frac{1}{2}\left(g_{ik,l} + g_{il,k} - g_{kl,i}\right) \tag{3.76}$$

- Mathematically, the Christoffel symbols are the coefficients (in some basis) of the ↑ *Levi-Civita connection* which is determined by the metric tensor g^{ij} (→ *later*).

36 | ◁ Contravariant vector $\bar{A}^i$ and contract it with $\bar{\Gamma}^i{}_{kl}$:

$$\bar{\Gamma}^i{}_{kl}\,\bar{A}^l = \underbrace{\frac{\partial\bar{x}^i}{\partial x^m}\frac{\partial x^n}{\partial\bar{x}^k}\Gamma^m{}_{np}\underbrace{\left[\frac{\partial x^p}{\partial\bar{x}^l}\bar{A}^l\right]}_{A^p}}_{(1,1)\text{-tensor} ☺} - \underbrace{\frac{\partial x^n}{\partial\bar{x}^k}\frac{\partial^2\bar{x}^i}{\partial x^n\partial x^p}\underbrace{\left[\frac{\partial x^p}{\partial\bar{x}^l}\bar{A}^l\right]}_{A^p}}_{\text{Problematic term in Eq. (3.73)}} \tag{3.77}$$

Idea: Add Eq. (3.73) and Eq. (3.77) to cancel the problematic term:

$$\bar{A}^i{}_{,k} + \bar{\Gamma}^i{}_{kp}\,\bar{A}^p = \underbrace{\frac{\partial x^m}{\partial\bar{x}^k}\frac{\partial\bar{x}^i}{\partial x^l}\left[A^l{}_{,m} + \Gamma^l{}_{mp}\,A^p\right]}_{(1,1)\text{-tensor} ☺☺} \tag{3.78}$$

37 | This motivates the definition of the ⁂ *Covariant derivative:*

$$\text{Scalar:}\quad \Phi_{;k} := \Phi_{,k} \tag{3.79a}$$

$$\text{Contravariant vector:}\quad A^i{}_{;k} := A^i{}_{,k} + \Gamma^i{}_{kl}\,A^l \tag{3.79b}$$

$$\text{Covariant vector:}\quad B_{i;k} := B_{i,k} - \Gamma^l{}_{ik}\,B_l \tag{3.79c}$$

- With this definition, $A^i{}_{;k}$ is a $(1,1)$-tensor and $B_{i;k}$ is a $(0,2)$-tensor!
- With this definition, the product rule is valid for the covariant derivative:

$$(A^i B_i)_{;k} = (A^i B_i)_{,k} \mathrel{\hat{=}} A^i{}_{;k}B_i + A^i B_{i;k} \tag{3.80}$$

- The construction of higher-rank tensors by tensoring contra- and covariant vectors Eq. (3.32) and the definitions of the covariant derivative above Eq. (3.79) can be used to construct covariant derivatives of arbitrary tensor fields. For example:

$$T^i{}_{k;l} := T^i{}_{k,l} + \Gamma^i{}_{ml}\,T^m{}_k - \Gamma^m{}_{kl}\,T^i{}_m \tag{3.81}$$

- With this generalization, we can apply the covariant derivative multiple times. For example:

$$A^i{}_{;k;l} \equiv \left(A^i{}_{;k}\right)_{;l} \tag{3.82}$$

- The covariant derivative is *not commutative* in general:

$$A^i{}_{;k;l} - A^i{}_{;l;k} \neq 0 \tag{3.83}$$

→ Riemann curvature tensor → GENERAL RELATIVITY (→ *later*)

(This is not the case for the "normal" derivative: $A^i{}_{,k,l} = A^i{}_{,l,k}$.)

38 | Conclusion:

If you can formulate an equation that describes a physical theory in terms of tensors, it can always be brought into the form

$$T^I{}_J(x) = 0\,. \tag{3.84}$$

(This equation is meant to hold for all values of indices I and J and all coordinate values x.)

Here is an example:

The (inhomogeneous) Maxwell equations on an arbitrary (potentially curved) spacetime read:

$$\underbrace{F^{\mu\nu}{}_{;\nu} + \frac{4\pi}{c} J^\mu}_{=:T^\mu(x)} = 0 \tag{3.85}$$

with current density J^μ and field strength tensor $F^{\mu\nu} = g^{\mu\rho} g^{\nu\pi}(A_{\pi;\rho} - A_{\rho;\pi})$.

How does Eq. (3.84) look like in any other coordinate system $\bar{x} = \varphi(x)$?

Easy:

$$\bar{T}^I{}_J(\bar{x}) = \frac{\partial \bar{x}^I}{\partial x^M} \frac{\partial x^N}{\partial \bar{x}^J} \underbrace{T^M{}_N(x)}_{=0} = 0 \quad \Leftrightarrow \quad \bar{T}^I{}_J(\bar{x}) = 0\,. \tag{3.86}$$

This means:

$$\boxed{\text{Tensor equations are automatically form-invariant under \textit{arbitrary} coordinate transformations; we say they exhibit ⁂ \textit{(manifest) general covariance}.}}$$

The "manifest" means that checking general covariance is just a matter of checking whether the equation "looks right", i.e., whether it is built from tensors following the rules discussed in this chapter. If a property of an equation is manifest, you don't have to do calculations to verify it!

In the next chapter, we take a step back and specialize the allowed coordinate transformations to the Lorentz transformations of SPECIAL RELATIVITY. We can then use the form-invariance of equations built from "Lorentz tensors" to construct Lorentz covariant equations from scratch – which was our original goal!