

↓ Lecture 7 [28.11.23]

7 | Since  $T_p M$  is a vector space for each point  $p$  of the manifold  $M$ , we can define *fields* on  $M$  that assign to each point  $p$  a tangent vector:

✱ *Vector field*:  $A(p) = \sum_{i=1}^D A^i(x) \partial_i$  with  $x = u(p)$

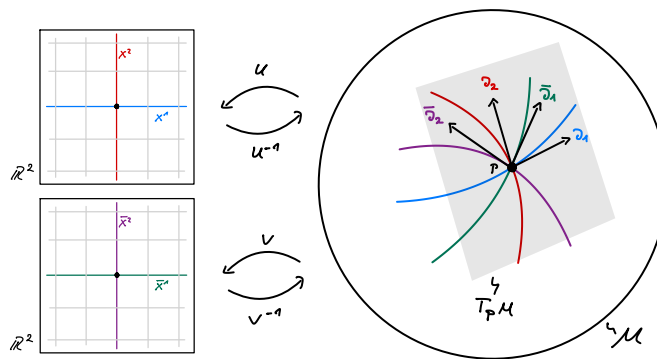
At every point  $p \in M$  the vector field yields a tangent vector  $A(p) = \sum_i A^i(u(p)) \partial_i \in T_p M$ .

8 | < Coordinate transformation  $\bar{x} = \varphi(x) \Leftrightarrow x = \varphi^{-1}(\bar{x})$

→ Chain rule:

$$\underbrace{\frac{\partial}{\partial \bar{x}^i}}_{\bar{\partial}_i} = \sum_{k=1}^D \frac{\partial x^k}{\partial \bar{x}^i} \underbrace{\frac{\partial}{\partial x^k}}_{\partial_k} \tag{3.5}$$

→ For  $x = u(p)$  and  $\bar{x} = v(p)$  this is a *basis change* on the tangent space  $T_p M$  from one coordinate basis  $\{\partial_i\}$  to another coordinate basis  $\{\bar{\partial}_i\}$  via the (invertible) matrix  $\frac{\partial x^k}{\partial \bar{x}^i}$ :



9 | < Vector field  $A$  and expand it in different coordinate bases:

$$\sum_i A^i(x) \partial_i = A(p) = \sum_i \bar{A}^i(\bar{x}) \bar{\partial}_i \tag{3.6}$$

with  $x = u(p)$  and  $\bar{x} = v(p)$ .

- ¡! The vector field  $A$  is a geometric object, just as the scalar field  $\phi$  was. That it does *not* depend on the chosen chart is the statement of this equation.
- You learned this (with different notation and without the  $x/p$ -dependency) in your first course on linear algebra: Given a vector space  $V$ , a vector  $\vec{v} \in V$ , and a basis  $\{\vec{e}_i\}$  with  $V = \text{span}\{\vec{e}_i\}$ , you can encode the vector in a basis-dependent set of numbers  $v_i$  called *components* via linear combination:  $\vec{v} = \sum_i v_i \vec{e}_i$ . The *same* vector can be encoded by *different* components  $v'_i$  in a *different* basis  $\{\vec{e}'_i\}$ :  $\vec{v} = \sum_i v'_i \vec{e}'_i$ . In our terminology, the vector  $\vec{v}$  is a “geometric object” that does not depend on your choice of basis; only its components do. In this context, the gist of the story is that  $\vec{v}$  represents something physical (like the velocity of a particle). The *components*  $v_i$  do so only indirectly because they depend on your choice of the basis  $\{\vec{e}_i\}$  – and this choice does not bear any physical meaning.

Eq. (3.6) →

$$A = \sum_i A^i(x) \partial_i \stackrel{!}{=} \sum_i \bar{A}^i(\bar{x}) \bar{\partial}_i \stackrel{\text{Eq. (3.5)}}{=} \sum_k \underbrace{\left[ \sum_i \frac{\partial x^k}{\partial \bar{x}^i} \bar{A}^i(\bar{x}) \right]}_{\stackrel{!}{=} A^k(x)} \partial_k \tag{3.7}$$

This motivates the following definition (we replace  $x \leftrightarrow \bar{x}$  and the indices  $i \leftrightarrow k$ ):

10 |  $\triangleleft$   $D$ -tuple  $\{A^i(x)\}$  of fields (in some chart with coordinates  $x$ ):

$$\text{** Contravariant vector field } \{A^i(x)\} \quad :\Leftrightarrow \quad \bar{A}^i(\bar{x}) = \sum_{k=1}^D \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \quad (3.8)$$

Contravariant vector (field) → Superscript indices!

This is a *convention* which relates syntax and semantics and is at the heart of *tensor calculus*. The idea is that whenever you are given a collection of fields  $A^i(x)$ , you immediately know that they transform like Eq. (3.8) under coordinate transformations. (Unfortunately, there are exceptions to this rule, e.g., the → *Christoffel symbols*.)

- ¡! Not every  $D$ -tuple of fields transforms as Eq. (3.8). To deserve the name “contravariant vector (field),” (and superscript indices) one has to check this transformation law explicitly!
- The rationale of Eq. (3.8) is the same as that of Eq. (3.4): Whenever we find a family of fields that transform under coordinate transformations as Eq. (3.8), we immediately know that together they encode a geometric, chart-independent object on the manifold that can be used to describe a physical quantity.

11 | (Counter)Examples:

- $\triangleleft$  Only linear coordinate transformations:  $\bar{x} = \varphi(x) = \Lambda x$   
 $\triangleleft$  Coordinate functions  $X^i(x) := x^i$  as fields:

$$\underbrace{\bar{X}^i(\bar{x})}_{\bar{x}^i} = \sum_{k=1}^D \Lambda_k^i \underbrace{X^k(x)}_{x^k} = \sum_{k=1}^D \underbrace{\frac{\partial \bar{x}^i}{\partial x^k}}_{\Lambda_k^i} X^k(x) \quad (3.9)$$

→ Coordinate functions are contravariant vectors for linear transition maps.

This is useful in SPECIAL RELATIVITY because there we only consider global Lorentz transformations (which are linear).

- $\triangleleft$   $D$  scalar fields  $\Phi^i(x)$  ( $i = 1, \dots, D$ ):

$$\text{For general } \bar{x} = \varphi(x): \quad \bar{\Phi}^i(\bar{x}) = \Phi^i(x) \neq \sum_{k=1}^D \underbrace{\frac{\partial \bar{x}^i}{\partial x^k}}_{\neq \delta_k^i} \Phi^k(x) \quad (3.10)$$

→  $\{\Phi^i(x)\}$  are *not* components of a contravariant vector field.

- You see: not every collection of  $D$  fields is a vector!
- ¡!  $\delta_k^i$  is the Kronecker symbol:  $\delta_k^i = 1$  for  $i = k$  and  $\delta_k^i = 0$  for  $i \neq k$ . The notation  $\delta_{ik}$  is *not* used in tensor calculus (→ later).

12 | Reminder: ↓ Dual spaces

**i** | Remember: Linear algebra

Consider the vector space  $V = \mathbb{R}^D$  and a column vector  $\vec{v} = (v_1, \dots, v_D)^T \in V$  (a  $1 \times D$ -matrix). Let  $\vec{w}^T = (w_1, \dots, w_D)$  be a row vector (a  $D \times 1$ -matrix). We can then perform a matrix multiplication between the vectors and interpret it as a linear map  $\vec{w}^T$  acting on the vector  $\vec{v}$  and producing a number:

$$\vec{w}^T : \vec{v} \in V \mapsto \vec{w}^T \cdot \vec{v} = (w_1 \quad \dots \quad w_D) \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_D \end{pmatrix} = \sum_i w_i v_i \in \mathbb{R}. \quad (3.11)$$

In mathematical parlance  $\vec{w}^T$  is a *linear functional* on the vector space  $V$ . All linear functionals of this form make up another vector space  $V^*$  called the  $\downarrow$  *dual space* of  $V$ . You can think of  $V^*$  as the vector space of all  $D$ -dimensional *row* vectors and  $V$  as the vector space of all  $D$ -dimensional *column* vectors. The elements of the dual space are referred to as a  $\downarrow$  *covectors*.

**ii** | Remember: Quantum mechanics

In quantum mechanics, the state of a physical system is described by  $\downarrow$  *state vectors* in some Hilbert space  $\mathcal{H}$  (which is a special kind of vector space). Vectors in this space are written as  $\downarrow$  *kets*:  $|\Psi\rangle \in \mathcal{H}$ . You can produce a  $\downarrow$  *bra*  $\langle\Psi| = |\Psi\rangle^\dagger$  by applying the complex transpose operator. As in the example above, the bra  $\langle\Psi|$  is a covector from the dual space  $\mathcal{H}^*$ ; indeed, it acts as a linear functional on state vectors via the inner product of the Hilbert space:

$$\langle\Psi||\Phi\rangle := \langle\Psi|\Phi\rangle \in \mathbb{C}. \quad (3.12)$$

This is the gist of the famous  $\downarrow$  *Dirac bra-ket notation*.

**iii** | Hopefully these examples convinced you that the dual space is just as important and useful as the vector space itself.

→ Dual space of the tangent space  $T_p M$ ?

Given a coordinate basis  $\{\partial_i\} \in T_p M$  of a vector space, there is a standard way to define a basis of the dual space  $T_p^* M$ :

$\downarrow$  *Dual basis*  $\{dx^i\}$  with

$$dx^i(\partial_j) := \delta_j^i = \frac{\partial x^i}{\partial x^j} \quad (3.13)$$

→  $\{dx^i\}$  is a basis of the  $\ast\ast$  *Cotangent space*  $T_p^* M$

$T_p^* M$  is the dual space of  $T_p M$ ; it is common to write  $T_p^* M$  and not  $(T_p M)^*$ .

**13** | Since  $T_p^* M$  is just another vector space for each point  $p$  of the manifold  $M$ , we can again define *fields* on  $M$  that map into this space:

$\ast\ast$  *Covector field*:  $B(p) = \sum_{i=1}^D B_i(x) dx^i$  with  $x = u(p)$

**14** | Just like the coordinate basis, the dual coordinate basis depends on the chart and changes under coordinate transformations:

◁ Coordinate transformation  $\bar{x} = \varphi(x)$ :

$$d\bar{x}^i = \sum_{k=1}^D \frac{\partial \bar{x}^i}{\partial x^k} dx^k \quad (3.14)$$

- Check that this is the correct transformation for the dual coordinate basis:

$$\begin{aligned} d\bar{x}^i(\bar{\partial}_j) &= \left[ \sum_k \frac{\partial \bar{x}^i}{\partial x^k} dx^k \right] \left( \sum_l \frac{\partial x^l}{\partial \bar{x}^j} \partial_l \right) \\ &= \sum_{k,l} \frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^j} \underbrace{dx^k(\partial_l)}_{\delta_l^k} = \sum_k \underbrace{\frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^k}{\partial \bar{x}^j}}_{\frac{\partial \bar{x}^i}{\partial \bar{x}^j}} = \delta_j^i \quad \odot \end{aligned} \quad (3.15)$$

- You might recognize Eq. (3.14): This is simply the rule to compute the  $\downarrow$  *total differential* of the function  $\bar{x} = \varphi(x)$ . This is no coincidence and explains why we use the differential notation  $dx^i$  for the dual vectors: The objects  $dx^i$  that we physicists like to illustrate as “infinitesimal shifts” in  $x^i$  are actually *linear functionals* ( $\uparrow$  *1-forms*).

15 | Now we can play the same game on  $T_p^*M$  as before on  $T_pM$ :

$\triangleleft$  Covector field  $B$  and expand it in different dual coordinate bases:

$$\sum_i B_i(x) dx^i = B(p) = \sum_i \bar{B}_i(\bar{x}) d\bar{x}^i \quad (3.16)$$

with  $x = u(p)$  and  $\bar{x} = v(p)$ .

! The covector field  $B$  is another geometric object, just as the vector field  $A$  was. That it does *not* depend on the chosen chart is the statement of this equation.

Eq. (3.16) →

$$B = \sum_i B_i(x) dx^i \stackrel{!}{=} \sum_i \bar{B}_i(\bar{x}) d\bar{x}^i \stackrel{\text{Eq. (3.14)}}{=} \sum_k \underbrace{\left[ \sum_i \frac{\partial \bar{x}^i}{\partial x^k} \bar{B}_i(\bar{x}) \right]}_{\stackrel{!}{=} B_k(x)} dx^k \quad (3.17)$$

This motivates the following definition (we replace  $x \leftrightarrow \bar{x}$  and the indices  $i \leftrightarrow k$ ):

16 |  $\triangleleft$   $D$ -tuple  $\{B_i(x)\}$  of fields (in some chart with coordinates  $x$ ):

$$\text{** Covariant vector field } \{B_i(x)\} \quad \Leftrightarrow \quad \bar{B}_i(\bar{x}) = \sum_{k=1}^D \frac{\partial x^k}{\partial \bar{x}^i} B_k(x) \quad (3.18)$$

Covariant vector (field) → *Subscript* indices!

The rationale of Eq. (3.18) is the same as that of Eq. (3.8): Whenever we find a family of fields that transform under coordinate transformations as Eq. (3.18), we immediately know that together they encode a geometric, chart-independent object on the manifold that can be used to describe a physical quantity. To indicate that this object is a *covariant* vector field, we use *subscript* indices.

17 | Example:

First, let us introduce an even shorter notation for partial derivatives:  $\Phi_{,i} \equiv \partial_i \Phi$

Following our index convention, the lower index in these expressions is only warranted *if* the field transforms as a covariant vector field according to Eq. (3.18). Let us check this:

$$\bar{\Phi}_{,i}(\bar{x}) = \bar{\partial}_i \bar{\Phi}(\bar{x}) \stackrel{\text{Eq. (3.4)}}{\stackrel{\text{Eq. (3.5)}}{=}} \sum_{k=1}^D \frac{\partial x^k}{\partial \bar{x}^i} \frac{\partial \Phi(x)}{\partial x^k} = \sum_{k=1}^D \frac{\partial x^k}{\partial \bar{x}^i} \Phi_{,k}(x) \quad (3.19)$$

→ The gradient of a scalar is a covariant vector field.

18 | What happens if we apply a covector field on a vector field at each point  $p \in M$ ?

$$\phi(p) := B(p)A(p) = \sum_{i,j} B_i(x)A^j(x) \underbrace{dx^i(\partial_j)}_{\delta_j^i} = \sum_i A^i(x)B_i(x) =: \Phi(x) \quad (3.20)$$

→  $\Phi(x)$  must be a scalar!

This is a good point to introduce a new (and very convenient) notation:

\*\* *Einstein sum convention*:

$$\sum_{i=1}^D A^i(x)B_i(x) \equiv \underbrace{A^i(x)B_i(x)}_{\substack{\text{** Einstein summation} \\ \text{** Contraction}}} = A^I(x)B_I(x) \quad (3.21)$$

The *Einstein sum convention* or *Einstein summation* is a syntactic convention according to which a sum is automatically implied (but not written) whenever two indices show up twice in an expression and one is up (contravariant) and one down (covariant). Note that such indices are “dummy indices” in the sense that you can rename them to whatever you want (as long as you do not use the same letter for other indices already!). The sum over one co- and one contravariant index is called a *contraction*.

With this new notation it is straightforward to check that  $\Phi$  transforms according to Eq. (3.4) by using the transformations Eq. (3.8) and Eq. (3.18):

$$\bar{\Phi}(\bar{x}) = \bar{A}^i(\bar{x})\bar{B}_i(\bar{x}) = \left[ \frac{\partial \bar{x}^i}{\partial x^k} A^k(x) \right] \left[ \frac{\partial x^l}{\partial \bar{x}^i} B_l(x) \right] \quad (3.22a)$$

$$= \underbrace{\frac{\partial \bar{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \bar{x}^i}}_{\text{Chain rule} \rightarrow \delta_k^l} A^k(x)B_l(x) = A^l(x)B_l(x) = \Phi(x) \quad (3.22b)$$

The intermediate expression contains *three* sums over the colored indices (which we don’t write)!

→ The contraction of a contra- and a covariant vector field yields a scalar field.

19 | Note on nomenclature:

- If you compare Eq. (3.18) with Eq. (3.5) you find that the *components*  $B_i$  of a covector field transform like the *basis vectors*  $\partial_i$  of the tangent space. We say the components *covary* (“vary together”) with the basis. This is why they are called *covariant*.
- A comparison of Eq. (3.8) and Eq. (3.14) shows that the components  $A^i$  of a vector field transform like the basis  $dx^i$  of the cotangent space – which is the *inverse* (“opposite”) transformation as for the basis of the tangent space  $\partial_i$ . Thus we say the components  $A^i$  *contravary* (“vary opposite to”) the basis  $\partial_i$ . This is why they are called *contravariant*.

### 3.4. Higher-rank tensors

You learned in your linear algebra course that two vector spaces  $V$  and  $W$  can be used to construct a new vector space  $V \otimes W$  called the  $\downarrow$  *tensor product*. This allows us to generalize the notion of contra- and covariant *vector* fields to *tensor* fields, all of which are geometric, chart-independent objects defined on the manifold that are needed to describe physical quantities:

20 | An  $\ast\ast$  (absolute)  $(p, q)$ -tensor (field)  $T$  of rank  $r = p + q$

$$T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} \equiv T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}(x) \quad \text{or} \quad T^I_J \equiv T^I_J(x), \quad (3.23)$$

with  $\downarrow$  multi-indices  $I = (i_1 \dots i_p)$  and  $J = (j_1 \dots j_q)$ ,

transforms like the tensor product of  $p$  contravariant and  $q$  covariant vector fields:

$$\underbrace{\bar{T}^{i_1 \dots i_p}_{j_1 \dots j_q}(\bar{x})}_{= \bar{T}^I_J(\bar{x})} = \underbrace{\left[ \frac{\partial \bar{x}^{i_1}}{\partial x^{m_1}} \dots \frac{\partial \bar{x}^{i_p}}{\partial x^{m_p}} \right]}_{=: \frac{\partial \bar{x}^I}{\partial x^M}} \underbrace{\left[ \frac{\partial x^{n_1}}{\partial \bar{x}^{j_1}} \dots \frac{\partial x^{n_q}}{\partial \bar{x}^{j_q}} \right]}_{=: \frac{\partial x^N}{\partial \bar{x}^J}} T^{m_1 \dots m_p}_{n_1 \dots n_q}(x) \quad (3.24)$$

$= T^M_N(x)$

There are  $r = p + q$  sums in this transformation rule (Einstein summation!).

- $\downarrow$  It is important that we do *not* write contra- and covariant indices above each other like so:  $T_j^i$  (at least not with additional knowledge about the tensor). This will become important below.
- Henceforth we always encode tensor fields by their chart-dependent *components*. The actual tensor field is of course chart-independent and maps each point  $p \in M$  to an element of the tensor product

$$\underbrace{T_p M \otimes \dots \otimes T_p M}_{p \text{ factors}} \otimes \underbrace{T_p^* M \otimes \dots \otimes T_p^* M}_{q \text{ factors}}. \quad (3.25)$$

like so

$$T(p) = \sum_{I, J} T^{i_1 \dots i_p}_{j_1 \dots j_q}(x) \partial_{i_1} \otimes \dots \otimes \partial_{i_p} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}. \quad (3.26)$$

- Note that while tensors (more precisely: tensor components) are indicated by upper and lower indices (corresponding to their rank), not every object that is conventionally written with upper and lower indices does encode a tensor. For example, the transformation matrices  $\frac{\partial \bar{x}^i}{\partial x^m}$ , which describe a basis change on  $T_p^* M$ , do not encode a tensor field.

21 | Examples:

$$\begin{aligned} \text{Scalar } \Phi(x) &\rightarrow (0, 0)\text{-tensor} \\ \text{Contravariant vector } A^i(x) &\rightarrow (1, 0)\text{-tensor} \\ \text{Covariant vector } B_i(x) &\rightarrow (0, 1)\text{-tensor} \\ \text{Tensor product } T^i_j(x) := A^i(x)B_j(x) &\rightarrow (1, 1)\text{-tensor (Check this!)} \end{aligned}$$

22 | Properties:

- *Equality:*

$$A = B \quad :\Leftrightarrow \quad \forall_{i_1 \dots i_p} \forall_{j_1 \dots j_q} : A^{i_1 \dots i_p}_{j_1 \dots j_q} = B^{i_1 \dots i_p}_{j_1 \dots j_q} \quad (3.27)$$

- *Symmetry:*

$$T \text{ (anti-)symmetric in } k \text{ and } l \quad :\Leftrightarrow \quad T^{\dots k \dots l \dots} = (-1) T^{\dots l \dots k \dots} \quad (3.28)$$

Every contra- or covariant rank-2 tensor can be decomposed into a sum of symmetric and antisymmetric tensors:

$$T_{ij} = \underbrace{\frac{1}{2}(T_{ij} + T_{ji})}_{=:T_{(ij)}} + \underbrace{\frac{1}{2}(T_{ij} - T_{ji})}_{=:T_{[ij]}} = T_{(ij)} + T_{[ij]}. \quad (3.29)$$

### 23 | Constructing tensors:

New tensors can be constructed from known tensors as follows (Proofs: ➔ Problemset 4):

- *Sum of  $(p, q)$ -tensors  $A$  and  $B$  yields  $(p, q)$ -tensor  $C$ :*

$$C^{i_1 \dots i_p}_{j_1 \dots j_q} := A^{i_1 \dots i_p}_{j_1 \dots j_q} + B^{i_1 \dots i_p}_{j_1 \dots j_q} \quad (3.30a)$$

$$\text{or } C^I_J := A^I_J + B^I_J \quad (3.30b)$$

- *Product of  $(p, q)$ -tensor  $A$  and scalar  $\Phi$  yields  $(p, q)$ -tensor  $C$ :*

$$C^I_J := \Phi A^I_J \quad (3.31)$$

- *Tensor product of  $(p, q)$ -tensor  $A$  and  $(r, s)$ -tensor  $B$  yields  $(p + r, q + s)$ -tensor  $C$ :*

$$C^{IK}_{JL} := A^I_J \cdot B^K_L \quad (3.32)$$

- *Contractions:*

Summing over a pair of contra- and covariant indices yields a tensor of rank  $(p - 1, q - 1)$ :

$$\tilde{A}^{i_1 \dots \bullet \dots i_p}_{j_1 \dots \bullet \dots j_q} := A^{i_1 \dots k \dots i_p}_{j_1 \dots k \dots j_q} \quad (3.33)$$

The  $\bullet$  indicates that the index summed over on the right side is missing in the list.

Proof: ➔ Problemset 4

A special case of a contraction (in combination with a tensor product) is the scalar obtained from a contra- and a covariant vector field above:

$$\Phi = C^i_i = A^i B_i. \quad (3.34)$$

- *Quotient theorem:*

$$\overline{AB} = C \text{ tensor for all tensors } B \Rightarrow A \text{ is tensor} \quad (3.35)$$

Here,  $\overline{AB}$  denotes (potentially multiple) contractions between indices of  $A$  and  $B$  (but not within  $A$  and  $B$ ).

- As an example, rewrite an arbitrary contravariant vector  $A^i$  as  $A^i = \delta_j^i A^j$  with Kronecker symbol  $\delta_j^i$ . The above theorem then implies that  $\delta_j^i$  transforms as a  $(1, 1)$ -tensor (verify this using the definition!). Hence we actually should write  $\delta^i_j$  instead of  $\delta_j^i$ . However, because the Kronecker symbol is symmetric in its indices, this simplified notation is allowed ( $\rightarrow$  later).

– Special case:

$$A_{ik} B^k = C_i \text{ covector for all vectors } B^k \Rightarrow A_{ik} \text{ is } (0, 2)\text{-tensor} \quad (3.36)$$

Proof: → Problemset 4

24 | Relative tensors:

- i | Relative tensors are a generalization of the (absolute) tensors defined above. This generalization is useful because most of the rules for computing with tensors discussed so far carry over to relative tensors.

A  $\star$  relative tensor of weight  $w \in \mathbb{Z}$  picks up an additional power  $w$  of the  $\downarrow$  Jacobian determinant under coordinate transformations:

$$\bar{R}^I{}_J(\bar{x}) = \det\left(\frac{\partial x}{\partial \bar{x}}\right)^w \frac{\partial \bar{x}^I}{\partial x^M} \frac{\partial x^N}{\partial \bar{x}^J} R^M{}_N(x) \quad \text{with weight } w \in \mathbb{Z} \quad (3.37)$$

and Jacobian determinant

$$\det\left(\frac{\partial x}{\partial \bar{x}}\right) := \sum_{\sigma \in S_D} (-1)^\sigma \prod_{i=1}^D \frac{\partial x^i}{\partial \bar{x}^{\sigma_j}}. \quad (3.38)$$

Here  $S_D$  is the group of permutations  $\sigma$  on  $D$  elements.

Since  $\bar{x} = \varphi(x)$  is invertible,  $x = \varphi^{-1}(\bar{x})$ , it is  $\frac{\partial \bar{x}}{\partial x} = \left(\frac{\partial x}{\partial \bar{x}}\right)^{-1}$  and therefore  $\det\left(\frac{\partial \bar{x}}{\partial x}\right) = \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{-1}$ .

ii | Examples:

- (Absolute) tensors  $\equiv$  Relative tensors of weight  $w = 0$
- Volume form: Relative tensor of weight  $w = -1$ :

$$d^D \bar{x} = d^D x \det\left(\frac{\partial \bar{x}}{\partial x}\right) = d^D x \det\left(\frac{\partial x}{\partial \bar{x}}\right)^{-1} \quad (3.39)$$

Remember the rule for integration by substitution with multiple variables!

- $\star$  Tensor density  $\mathcal{L}(x) :=$  Relative tensor of weight  $w = +1 \rightarrow$

$$S = \int \underbrace{d^D x \mathcal{L}(x)}_{\text{Absolute tensor}} = \int d^D \bar{x} \bar{\mathcal{L}}(\bar{x}) \quad (3.40)$$

In this example, we assume that  $\mathcal{L}(x)$  is a scalar tensor density such that its integral is a (absolute) scalar quantity.

In  $\uparrow$  relativistic field theories (like electrodynamics), the Lagrangian density  $\mathcal{L}(x)$  is a scalar tensor density such that the  $\downarrow$  action  $S$  becomes a scalar.

- Let  $i_1, i_2, \dots, i_D \in \{1, 2, \dots, D\}$  and define the  $\star$  Levi-Civita symbol as

$$\varepsilon^I \equiv \varepsilon^{i_1 i_2 \dots i_D} := \begin{cases} +1 & I \text{ even permutation of } 1, 2, \dots, D \\ -1 & I \text{ odd permutation of } 1, 2, \dots, D \\ 0 & \text{(at least) two indices equal} \end{cases} \quad (3.41)$$



An even (odd) permutation of  $1, 2, \dots, D$  is constructed by an even (odd) number of *transpositions* (= exchanges of only two indices).

→

$$\bar{\varepsilon}^I = \varepsilon^I \stackrel{\circ}{=} \det \left( \frac{\partial x}{\partial \bar{x}} \right)^{+1} \frac{\partial \bar{x}^I}{\partial x^J} \varepsilon^J \quad (3.42)$$

→  $\varepsilon^I = \varepsilon^{i_1 i_2 \dots i_D}$  is a  $(D, 0)$ -tensor density

- ¡!  $\bar{\varepsilon}^I = \varepsilon^I$  is true by definition:  $\varepsilon$  is a *symbol* defined by Eq. (3.41); this definition is independent of the coordinate system. In Eq. (3.42) we compare this trivial transformation with that of a (relative) tensor and conclude that it is equivalent to the statement that  $\varepsilon^I$  transforms as a  $(D, 0)$ -tensor density with weight  $w = +1$ . This knowledge is helpful in tensor calculus to construct covariant expressions that contain Levi-Civita symbols (→ *below*).
- To show this, note that the Levi-Civita symbol can be used to compute determinants:

$$\det \left( \frac{\partial \bar{x}}{\partial x} \right) = \sum_{\sigma \in S_D} (-1)^\sigma \prod_{i=1}^D \frac{\partial \bar{x}^i}{\partial x^{\sigma_j}} = \frac{\partial \bar{x}^1}{\partial x^{j_1}} \dots \frac{\partial \bar{x}^D}{\partial x^{j_D}} \varepsilon^{j_1 \dots j_D}. \quad (3.43)$$

Details: → Problemset 4